CONTROL POLICIES FOR INVENTORY SYSTEMS WITH PERISHABLE ITEMS

OUTSOURCING AND URGENCY CLASSES

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We consider control policies for perishable inventory systems with random input whose purpose is to mitigate the effects of unavailability. In the basic uncontrolled system, the arrival times of the items to be stored and the ones of the demands for those items form independent Poisson processes. The shelf lifetime of every item is finite and deterministic. Every demand is for a single item and is satisfied by the oldest item on the shelf, if available. The first controlled model excludes the possibility of unsatisfied demands by introducing a second source of fresh items that is completely reliable and delivers without delay whenever the system becomes empty. In the second model, there is no additional ordering option by outsourcing. However, to avoid the most adverse effects of unavailability, the demands are classified into different categories of urgency. An incoming demand is satisfied or not according to its category and the current state of the system. For both models, we determine the steady-state distribution of the virtual outdating process, which is then used to derive the relevant cost functionals: the steady-state distribution and expected value of the number of items in the system, the rate of outdatings, as well as, for model 1, the rate of special orders from the external source and, for model 2, the rate of unsatisfied demands.

1. INTRODUCTION

We consider a perishable inventory system (PIS) in which the arrival times of items to be stored and those of the demands are independent Poisson processes. The pur-

pose of this article is to study the effect of certain control policies that prevent possible serious consequences of periods of emptiness.

When uncontrolled, the PIS in question works as follows. Fresh stock arrives at the system according to a Poisson process with rate λ. Without loss of generality, we assume that the lifetime of each item is one time unit. If an item has not been consumed by demand before its expiration date, it is discarded. The demand arrival times form a Poisson process with rate μ. Demands are satisfied on a first in–first out (FIFO) basis (i.e., oldest items are issued first). A demand that arrives when the system is empty is lost. This basic model as well as several extensions have been studied in previous articles (see, e.g., [8,16,17,21–24]), in which also various applications to real-life systems such as blood banks and food storage places are exhibited. Examples of perishable products include blood portions, packaged chemicals and pharmaceuticals, fresh foods, and photographic film.

To assess the efficiency of this PIS, one has to take into account the number of items on the shelf (causing holding costs), the losses due to outdatings of unused items, and the rate of demands that arrive while the system is empty and thus have to leave unsatisfied. There are intuitively obvious trade-offs among these three random quantities; for example, an on-average well-filled shelf will usually entail a large number of outdatings and a small number of unsatisfied demands. To balance the cost, various types of intervention in the PIS are possible. Bang-bang policies in the case that the demand and the arrival process can be controlled have been studied in [19,20]. In this article, we focus on avoiding the undesired consequences of unsatisfied demands under fixed model parameters.

In particular, in blood banks the unavailability of items (blood portions) when they are demanded can cause serious, possibly disastrous, consequences. The first model we propose excludes the possibility of unsatisfied demands by introducing a second source of fresh items that is completely reliable and delivers without delay. This second source is always available in the sense that a batch of items from it arrives immediately whenever the controller places an order. To rule out unsatisfied demands, n_0 items are ordered whenever the PIS becomes empty. Thus, after an instantaneous emptiness, the shelf is refilled with n_0 items. We call this model the *never-empty PIS*.

The selection of n_0 (if there is a choice) depends, of course, on the structure of the ordering cost. Suppose that the price of a special order of n_0 items is of the form $D + dn_0$. Intuitively, a high value of the setup cost D will be an incentive for the controller to place special orders for a large number of items, but one has also to take into account that a large order size n_0 will have the undesirable consequence of causing many outdatings, indicating an inefficiently run PIS. On the other hand, if D is small compared to d, the price per item in the delivery, n_0 will probably be chosen small; in the extreme case $D = 0$, there is no point in ordering more than one item at a time. Since a change of any of the underlying parameters $(\lambda, \mu, n_0, D, d)$ gives rise to contrary cost effects, no easy monotonicity properties can be expected. In the case $n_0 = 1$, the key stochastic process in our approach will be seen to be Markovian; this property is, however, lost for $n_0 > 1$, making the analysis more complicated (and more interesting).

In our second model, there is no additional ordering option by outsourcing. To avoid the most adverse effects of unavailability, the demands are classified into different categories of urgency. An incoming demand will then be satisfied or not according to its category and the current state of the system. We will analyze a particular *PIS with urgency classification*, in which there are two categories of demands: Those of type 1 will be satisfied whenever the shelf is not empty, whereas those of type 2 will only be satisfied if there are at least m_0 items on the shelf or the residual lifetime of the oldest one is at most t_0 (with prespecified m_0 and t_0).

In general, inventory models for nondurable goods can be classified into two categories:

1. In decay and obsolescence models, the quality or utility of the whole inventory decreases over time (either continuously or by shocks). Models of this kind are surveyed in $[15,27]$. In $[1]$, the stationary distribution of the inventory level for a decay model under continuous review is derived using level crossing in a way similar to our approach. Related problems are discussed in $[26]$.

2. In perishability models, the commodities are discrete; any individual item has its own finite lifetime while being stored in the inventory. These models can be further divided into two classes.

   2a. In the first class, all items are delivered upon request by some deterministic source. Various replenishment ordering policies are discussed in the literature: Reference 4 deals with continuous review (s,S) policies in the case of zero lead times and no backorders; references 9–12 study (s,S) policies for models with backorders. Positive lead times are considered in $[5,13,14]$. For periodic review, see $[15,29,30]$. Other replenishment policies are studied in $[6,7,18,28]$.

   2b. The PISs in the second class are based on the assumption that the arrival times of the items are random and form a renewal process. In this article, we suppose a Poisson arrival stream. The connection to queuing and finite dam models, which is also basic for the approach of this article, was established in $[8]$; see also $[3]$. References 16, 17, and 21–24 deal with various ramifications such as impatient demands, handling of by-products, nonconstant lifetimes, disasters, variable input and demand rates, and external valuation of the PIS. In $[2]$, a related service system is investigated.

In Model 1, we try to combine replenishment by random arrivals (2b) with an ordering option from an external source (2a) so as to avoid losses due to unsatisfied demands. Model 2 pursues the same goal by satisfying less urgent demands only if the inventory is well filled or the oldest item is close to outdating. The article is organized as follows. Model 1 is analyzed in detail in Section 2. We introduce the basic so-called virtual outdating time (VOT) process and derive its steady-state distribution. It turns out that all relevant cost functionals can be expressed in terms of this distribution. The VOT process is regenerative but not Markovian; so, for its

steady-state analysis, we have to exploit a certain duality, leading to a Markov process for which classical techniques can be used. A similar approach yields the cost functionals for a modification of Model 1, including lead times. In Section 3, we study the model with urgency classification.

2. MODEL 1: THE NEVER-EMPTY PIS

2.1. The Virtual Outdating Process

Demands arrive in the system according to a Poisson process of rate μ. Items enter the PIS in two kinds: regular single arrivals at Poisson times of rate λ (independent of the demand process) and special arrivals that are ordered in batches of fixed size n_0 and arrive immediately every time the shelf becomes empty (either because the last item on the shelf is taken away by a demand or its lifetime of one time unit expires).

Let $A(t)$ be the age of the oldest item (i.e. the time it has spent in the system at time t). Note that there are exactly n_0 oldest items just after a special order; their ages then start increasing in parallel and they are taken away successively when new demands arrive. The ones that are still in the system after one time unit (if any) are then outdated simultaneously. Let $\mathbf{A} = \{A(t) | t \geq 0\}$. The key process in our analysis is the *VOT process* $\mathbf{W} = \{W(t) : t \geq 0\}$ defined by $W(t) = 1 - A(t)$. Clearly, $W(t)$ can be interpreted as the time from t until the next outdating if no demands arrive after t.

The VOT process $\mathbf{W}$ is regenerative (composed of independent and identically distributed (i.i.d.) cycles) but non-Markovian. If $n_0 = 1$, $\mathbf{W}$ is a Markov process, but this property does not hold if $n_0 > 1$. A new cycle starts whenever $\mathbf{W}$ reaches its maximum value 1. If the system has just been emptied at some time t, then n_0 new items immediately start their shelf life and $W(t) = 1$. Let us describe the stochastic evolution of $\mathbf{W}$ during a cycle. First, $\mathbf{W}$ decreases linearly at rate 1 until these n_0 items have all been taken away by demands or their common lifetime expires after one time unit. As the interarrival times of demands are $\exp(\mu)$, the first jump of $\mathbf{W}$ occurs after $\min[E(\mu, n_0), 1]$ time units, where $E(\mu, n_0)$ is an Erlang-distributed random variable with parameters μ and n_0. Before this jump, new items may have arrived according to a Poisson process of rate λ. Let E_λ be the time until the first regular item arrival in the cycle. If $E_\lambda < \min[E(\mu, n_0), 1]$, this new regular item becomes the oldest one and $\mathbf{W}$ jumps up by E_λ; otherwise, the system has been emptied after $\min[E(\mu, n_0), 1]$ time units and a new special batch arrival starts the next cycle. In the first case, the cycle goes on and the next jump occurs at the arrival of the next demand (after an $\exp(\mu)$-distributed time) or when 0 is hit, whichever comes first, and the position after the jump at that time, say s, is equal to $\min[W(s) + E_\lambda', 1]$, where E_λ' is $\exp(\lambda)$ distributed. If this latter minimum is 1, a new cycle starts; otherwise, the next jump occurs after another $\exp(\mu)$-distributed time or when 0 is hit, and so on. All of the random variables mentioned in this description are independent.

A typical sample path of $\mathbf{W}$ for $n_0 = 4$ is depicted in Figure 1. The following notation is used:
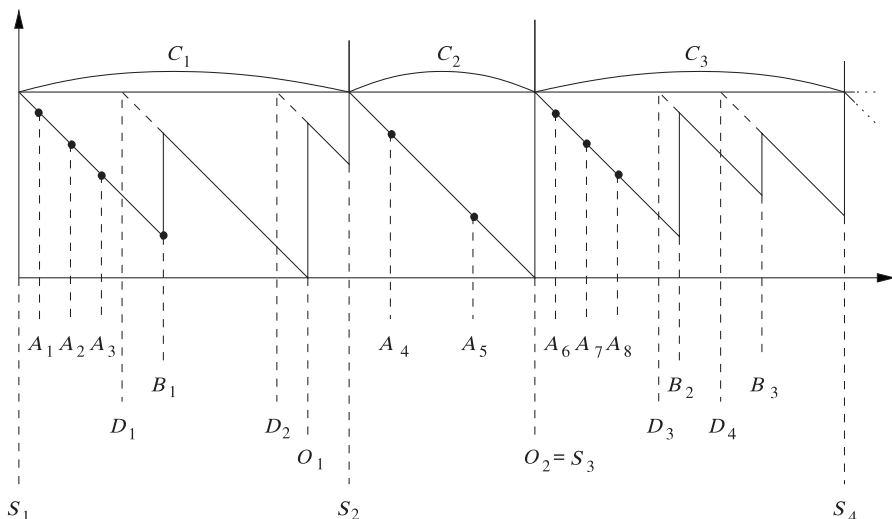
FIGURE 1. A typical sample path of the VOT process in Model 1 with $n_0 = 4$.

A_1, A_2 = arrival times of demands that are satisfied without jumps

B_1, B_2 = arrival times of demands that are satisfied with jumps

C_1, C_2 = cycles

D_1, D_2 = Poisson item arrival times

O_1, O_2 = outdating times

S_1, S_2 = times of special orders

The first cycle shown in Figure 1 starts with the arrival of a special order in an empty system at time S_1. The first three special items are taken away by demands at times A_1, A_2, and A_3. At time B_1, the fourth special item leaves. The regular item that arrived at time D_1 becomes the only item on the shelf; it is outdated at time O_1. A second regular item arrived before, at time D_2; this item satisfies a demand at time S_2 and the system becomes empty. Thus, a new cycle starts with the instantaneous arrival of four special items. Two of them are taken away by demands and the other two are outdated at time $O_2 = S_3$. Since there were no Poisson item arrivals in the meantime, a new cycle begins.

A little reflection shows that $\mathbf{W}$ has the same distribution as the content process of a special modified finite dam, which can be defined as follows:

a. The dam has capacity 1 (overflows being immediately lost) and releases water at rate 1.

b. The dry periods are deleted and the wet ones are glued together.

c. The inputs are $\exp(\lambda)$ distributed.

d. Whenever the dam is completely filled, the time until the next input is Erlang distributed with parameters μ and n_0, *truncated at 1*, whereas the subsequent interarrival times between inputs are $\exp(\mu)$ distributed.

The long-run cost of operating a never-empty PIS of the above type will depend on the following characteristics:

   i. the number of items in the system in steady state (causing holding costs);
  ii. the rate λ^* of outdatings;
 iii. the rate μ^* of special orders.

For a long-run average cost analysis and optimization, one can, for example, try to minimize a linear combination

$$C = a\mathbb{E}(K) + b\lambda^* + c\mu^*, \tag{2.1}$$

where a, b, and c are positive constants and K is a random variable indicating the number of items in the system in steady state. Decision variables can be the arrival rates λ and μ within certain ranges (if they are controllable) and the batch size n_0 of the special orders. For example, if the ordering cost per batch is of the form $D + n_0 d$, where D is some setup cost and d is the extra cost per unit, then the long-run average contribution of special orders to the total cost will be $(D + n_0 d)\mu^*$. The three components $\mathbb{E}(K)$, λ^*, and μ^* of C depend on λ, μ, and n_0 in a complicated way. The following monotonicity properties seem obvious: λ^* is increasing in λ and n_0 and decreasing in μ; μ^* is decreasing in λ and n_0 and increasing in μ. The dependence of $\mathbb{E}(K)$ on λ, μ, and n_0 is not clear. For example, a larger value of λ gives rise to more random item arrivals but decreases the number of special orders, so the overall change of $\mathbb{E}(K)$ is hard to predict. One may conjecture that if $n_0 > 1$, the expected inventory level, as a function of λ, first decreases and then increases: When $\lambda = 0$, there are only special orders, which increase the inventory level from zero to n_0 every time they are placed. As λ increases starting from zero, the number of special orders will get smaller. For moderately large values of λ, the reduction in inventory due to the decrease of their frequency is likely to outweigh the increase due to random arrivals. However, as λ gets large, the more and more frequent random arrivals will probably outweigh the decrease of K due to fewer special orders.

As a scenario in which λ and μ can be controlled, consider a blood bank receiving donations from and serving hospitals in certain neighboring areas. Suppose that the arrival times of donations and demands from different neighborhoods are independent. Then by selecting or excluding certain regions, the total rates of arrivals of items and demands are at least partially controllable.

We will derive the steady-state density f_W of the VOT process **W** and then express the rates λ^* and μ^* and the distribution of K in terms of f_W. As **W** is not Markovian, it seems difficult to determine f_W directly. Our approach is to construct a certain dual process that turns out to be Markovian, so that its steady-state distribution can be obtained by well-known methods (mainly level crossing techniques and PASTA). The basic arguments of this approach have been expounded in [25].

2.2. The Dual Process and the Steady-State Density

We now construct the sample paths of the dual process $\mathbf{V} = \{V(t)\,|\,t \geq 0\}$ from those of $\mathbf{A}$ by the following steps:

1. Replace any jump of $\mathbf{A}$ by a linear decreasing piece of trajectory with slope -1 on an interval whose length is equal to the jump size.
2. Replace the increasing pieces of $\mathbf{A}$ between its jumps by positive jumps whose sizes are equal to the lengths of the linear segments.

Thus, the process $\mathbf{V}$ results from a certain "reversal" of jumps and linear segments of the sample paths. In Figure 2, it is shown how the sample path of $\mathbf{W}$ displayed in Figure 1 is transformed first into the corresponding path of $\mathbf{A}$ and then to that of $\mathbf{V}$.

$\mathbf{V}$ can be interpreted as the content level process of a special regenerative Markovian finite dam of the $M/G/1$ type having the following features:

i. The underlying dam of $\mathbf{V}$ has capacity 1 (so that the jumps due to inflows will be truncated at level 1) and $\mathbf{V}$ decreases at rate 1 (between positive jumps).
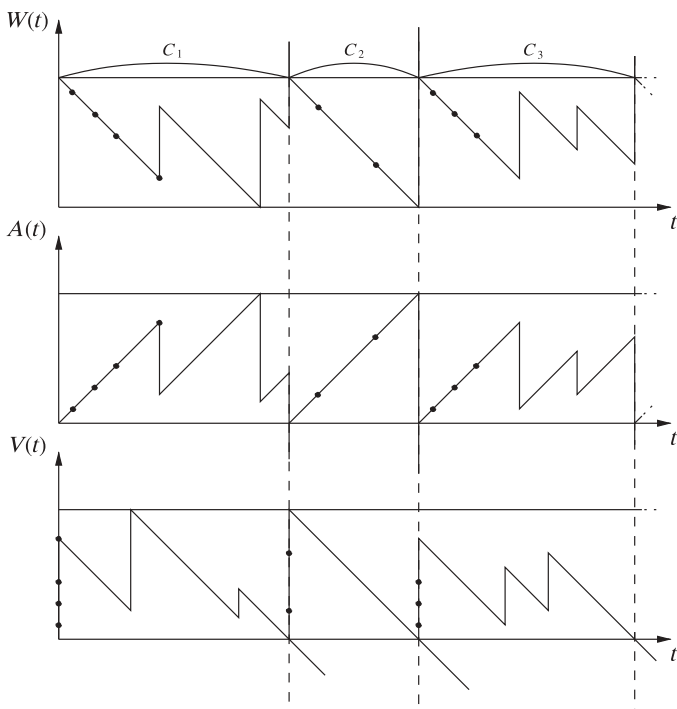

FIGURE 2. A typical sample path of the VOT process and its transformation to the corresponding path of $\mathbf{A}$ and to that of $\mathbf{V}$.

ii. Instantenuous inflows arrive according to a Poisson process with rate λ. The jump sizes are $\exp(\mu)$ distributed.

iii. Every dry period of $\mathbf{V}$ is replaced by an Erlang-distributed jump with parameters μ and n_0, which is truncated at 1.

Thus, the dry periods are deleted, whereas the wet periods, forming the cycles of $\mathbf{V}$, are glued together and start with special jumps. The cycle lengths have the same law as the times between truncated overshoots of $\mathbf{W}$ (or the times between hittings of 0 of $\mathbf{A}$). In Theorem 1, we relate the steady-state laws of $\mathbf{V}$ and $\mathbf{W}$.

THEOREM 1: *Let f_W and f_V be the steady-state densities of $\mathbf{W}$ and $\mathbf{V}$, respectively. Then for all $x \in [0,1]$,*

$$f_V(x) = f_W(1 - x)$$

$$= \lambda \int_0^x e^{-\mu(x-v)} f_V(v) \, dv + f_V(0) \sum_{j=0}^{n_0-1} \frac{e^{-\mu x}(\mu x)^j}{j!}. \tag{2.2}$$

Hence, f_V is given by

$$f_V(x) = c e^{(\lambda-\mu)x} \left[1 + \sum_{i=0}^{n_0-2} (\mu/\lambda)^{i+1} \left(1 - e^{-\lambda x} \sum_{k=0}^{i} \frac{(\lambda x)^k}{k!} \right) \right], \tag{2.3}$$

where the empty sum is defined to be 0 and

$$c = \left(\int_0^1 e^{(\lambda-\mu)x} \left[1 + \sum_{i=0}^{n_0-2} (\mu/\lambda)^{i+1} \left(1 - e^{-\lambda x} \sum_{k=0}^{i} \frac{(\lambda x)^k}{k!} \right) \right] dx \right)^{-1}.$$

PROOF: By level crossing theory (LCT), the steady-state densities $f_W(x)$ and $f_V(x)$ are given by the long-run average rates of $\mathbf{W}$ and $\mathbf{V}$, respectively, of downcrossing level x (see, e.g., [25]). Let $0 < T_1 < T_2 < \cdots$ be the times of jumps of $\mathbf{A}$ and let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be the times of jumps of $\mathbf{V}$. By construction, $A(T_n-) = V(\tau_{n-1}+)$ and $A(T_n+) = V(\tau_n-)$, and if T_n is a time at which $\mathbf{A}$ hits 0, then $T_n = \tau_n$. Thus, the long-run average downcrossing rates at any level x are the same for $\mathbf{A}$ and $\mathbf{V}$, so that their steady-state densities coincide. Since $W(t) = 1 - A(t)$ for all t, we obtain the first equation in (2.2). To prove that the right-hand side of (2.2) is equal to $f_V(x)$, we note that there are two types of arrival determining the jumps of $\mathbf{V}$. The first are Poisson arrivals of rate λ. For them, the conditional rate of upcrossings of level x given $V = v$ is $\lambda e^{-\mu(x-v)}$, since their jump sizes are $\exp(\mu)$ distributed. By deconditioning on $V = v$ and using PASTA, we immediately recognize the first term on the right-hand side of (2.2) as the upcrossing rate at x due to the Poisson arrivals. Jumps of the second type occur whenever $\mathbf{V}$ downcrosses level $0+$. By LCT, $f_V(0)$ is the long-run average rate of downcrossings of level 0. Any jump following a

hit of 0 is the first jump in some cycle of $\mathbf{V}$, and an upcrossing of level x takes place if this jump is greater than x, an event that has probability $\sum_{j=0}^{n_0-1} e^{-\mu x}(\mu x)^j/j!$ because the first jump of the cycle is $\text{Erl}(\mu, n_0)$.

To solve the integral equation (2.2), let

$$g(x) = e^{\mu x} f_V(x), \qquad \beta(x) = \sum_{j=0}^{n_0-1} (\mu x)^j/j!$$

and take derivatives. We obtain

$$g'(x) = \lambda g(x) + g(0)\beta'(x),$$

whose general solution is

$$g(x) = g(0)e^{\lambda x}\left[\int_0^x e^{-\lambda u}\beta'(u)\, du + C\right],$$

where C is some constant. For f_V, this yields

$$f_V(x) = e^{(\lambda-\mu)x} f_V(0)\left[\int_0^x e^{-\lambda u}\beta'(u)\, du + C\right]. \tag{2.4}$$

Setting $x = 0$ shows that $C = 1$. Moreover,

$$\int_0^x e^{-\lambda u}\beta'(u)\, du = \sum_{i=0}^{n_0-2} \frac{\mu^{i+1}}{i!}\int_0^x u^i e^{-\lambda u}\, du. \tag{2.5}$$

The functions $a_i(x) = \int_0^x u^i e^{-\lambda u}\, du$ appearing in (2.5) satisfy the recursion

$$a_i(x) = \lambda^{-1}\left[i a_{i-1}(x) - x^i e^{-\lambda x}\right], \qquad i \geq 1,$$

$$a_0(x) = \lambda^{-1}(1 - e^{-\lambda x})$$

and are thus given in closed form by

$$a_i(x) = \lambda^{-i-1} i!\left(1 - e^{-\lambda x}\sum_{k=0}^i \frac{(\lambda x)^k}{k!}\right). \tag{2.6}$$

Inserting (2.6) in (2.5) and the resulting expression in (2.4) yields (2.3). Finally, $f_V(0)$ can be computed from the normalizing condition $\int_0^1 f_V(x)\, dx = 1$. The proof is complete. ∎

2.3. Outdatings, Special Orders, and the Number of Items in the System

It turns out that the relevant functionals can all be computed from the steady-state density of the key process $\mathbf{W}$. Let us first determine the rates of outdatings and of special orders.

LEMMA 1: *The rate μ^* of special orders is given by*

$$\mu^* = \mu \int_0^1 e^{-\lambda v} f_V(v)\, dv + f_V(1) e^{-\lambda}.$$

PROOF: The rate of special orders is the rate of truncated overshoots of level 1 by **W**. By conditioning on W and then deconditioning, it is seen that the rate of upcrossings of level 1 by **W** is $\mu \int_0^1 e^{-\lambda(1-w)} f_W(w)\, dw + f_W(0) e^{-\lambda}$. By Theorem 1, $f_W(w) = f_V(1-w)$.   ∎

LEMMA 2: *The rate λ^* of outdatings is given by*

$$\lambda^* = \lambda - \mu + n_0 \mu^*.$$

PROOF: Since the Poisson arrivals and the arrivals of special orders have rates λ and μ^*, respectively, and every special order is for n_0 items, the long-run average input of items is $\lambda + n_0 \mu^*$. This equals the output rate, which is the sum of the demand rate μ plus the outdating rate λ^*.   ∎

Let K be the number of items in the system in steady state. The third important functional is the expected value $\mathbb{E}(K)$. We now derive the generating function of K. Define the event

$$B = \{\text{The oldest item present comes from a special order}\}.$$

To determine $\mathbb{P}(B)$, consider the time interval I between two consecutive special orders. This interval can be split into the part during which items from the special order launched at the beginning of I are still present and the ensuing part during which only regular items are present (this second part may have length 0). The expected length of the first part of I is $\mathbb{E}(\min[E(\mu, n_0), 1])$, whereas the expected length of I is, of course, $1/\mu^*$. The sequence of these split intervals clearly forms an alternating renewal process. Thus, it follows from the renewal theorem that

$$\mathbb{P}(B) = \mu^* \mathbb{E}(\min[E(\mu, n_0), 1]).$$

Hence,

$$\begin{aligned}
\mathbb{E}(z^K) &= \mathbb{E}(z^K | B) \mu^* \mathbb{E}(\min[E(\mu, n_0), 1]) \\
&\quad + \mathbb{E}(z^K | B^c)(1 - \mu^* \mathbb{E}(\min[E(\mu, n_0), 1])), \qquad |z| \le 1.
\end{aligned}$$

Given B^c, all of the items in the system are of the Poisson arrival type. Since the event $\{K = k\}$ means that $k - 1$ Poisson arrivals entered during the age of the oldest item and $f_W(1 - v) = f_V(v)$, we get

$$\begin{aligned}
\mathbb{E}(z^K | B^c) &= \sum_{k=1}^{\infty} z^k \int_0^1 \frac{e^{-\lambda v}(\lambda v)^{k-1}}{(k-1)!} f_W(1-v)\, dv \\
&= z \mathbb{E}(e^{-\lambda(1-z)V_{eq}}),
\end{aligned} \tag{2.7}$$

where V_{eq} is a random variable having the steady-state distribution of $\mathbf{V}$ (i.e., its density is f_V).

Given B, there may be several oldest items, all coming from some special order; all other items present, if any, are of the Poisson arrival type. The event $\{K = k\}$ then means that for some $i \in \{1, \ldots, n_0 - 1\}$, exactly i out of n_0 oldest items have been taken away by i demands and there have been $k - (n_0 - i)$ Poisson item arrivals after the last special order. Note that in this situation, $n_0 - i - 1$ oldest items are still on the shelf and $k - n_0 + i$ Poisson arrivals arrived during the age of the oldest items. Let $E_{\mu, i}$ and E_μ be two independent random variables that have the $\text{Erl}(\mu, i)$ and the $\exp(\mu)$ distribution, respectively. Then we obtain

$$
\mathbb{E}(z^K|B) = \sum_{k=1}^{\infty} z^k \sum_{i=0}^{\min[n_0-1,k]} \int_0^1 f_W(1-v) \mathbb{P}(E_{\mu,i} \leq v < E_{mu,i} + E_\mu)
$$

$$
\times \frac{e^{-\lambda v}(\lambda v)^{k-(n_0-i)}}{(k-(n_0-i))!} \, dv
$$

$$
= \sum_{k=1}^{\infty} z^k \sum_{i=0}^{\min[n_0-1,k]} \int_0^1 f_V(v) \left(\int_0^\infty \mathbb{P}(v - x < E_{\mu,i} \leq v) \mu e^{-\mu x} \, dx \right)
$$

$$
\times \frac{e^{-\lambda v}(\lambda v)^{k-n_0+i}}{(k-n_0+i)!} \, dv.
$$

We have proved the following result.

LEMMA 3: *The number K of items in the system in steady state has the generating function*

$$
\mathbb{E}(z^K) = \mu^* \mathbb{E}(\min[E(\mu, n_0), 1]) z \mathbb{E}(e^{-\lambda(1-z)V_{eq}})
$$

$$
+ (1 - \mu^* \mathbb{E}(\min[E(\mu, n_0), 1]))
$$

$$
\times \left[\sum_{k=1}^{n_0-1} z^k \sum_{i=0}^{k} \int_0^1 f_V(v) \left(\int_0^\infty \left(\int_{(v-x)_+}^v e^{-\mu y} \frac{\mu^i y^{i-1}}{(i-1)!} \, dy \right) \mu e^{-\mu x} \, dx \right) \right.
$$

$$
\times \frac{e^{-\lambda v}(\lambda v)^{k-n_0+i}}{(k-n_0+i)!} \, dv + \sum_{k=n_0}^{\infty} z^k \sum_{i=0}^{n_0-1} \int_0^1 f_V(v)
$$

$$
\times \left. \left[\int_0^\infty \left(\int_{(v-x)_+}^v e^{-\mu y} \frac{\mu^i y^{i-1}}{(i-1)!} \, dy \right) \mu e^{-\mu x} \, dx \right] \frac{e^{-\lambda v}(\lambda v)^{k-n_0+i}}{(k-n_0+i)!} \, dv \right].
$$

$$
(2.8)
$$

By Lemmas 1–3, the long-run average cost function $a\mathbb{E}(K) + b\lambda^* + c\mu^*$ introduced in (2.1) can now be completely expressed in terms of the steady-state density f_V, which, in turn, has been computed in Theorem 1.

2.4. A Model Variant with General Lead Times

Suppose now that as soon as the system becomes empty, the controller places a special order of n_0 items for which it takes a random lead time to arrive. However, due to their high cost, such special orders are canceled immediately if a regular item arrives at the system before the specially ordered one. For example, if $D + n_0 d$ is the price of a special order, at least the setup cost D can be saved by this cancellation. In this variation of Model 1, there are times when the shelf is empty and demands have to leave unsatisfied.

Let G be the common distribution function of the lead times and let $G^*(\alpha) = \int_0^\infty e^{-\alpha x} dG(x)$ be its Laplace transform. Let $\widetilde{\mathbf{W}} = \{\widetilde{W}(t)\,|\,t \geq 0\}$ be the VOT process of the modified PIS. Unlike $\mathbf{W}$, the process $\widetilde{\mathbf{W}}$ is not limited to being less than or equal to 1. When $\widetilde{\mathbf{W}}$ hits zero, it jumps to $1 + L$, where L is a generic random variable independent of the past and has distribution function G; then $\widetilde{\mathbf{W}}$ decreases linearly at slope -1 until it hits 1 (i.e., the ordered items enter the system) or a regular item arrives. In this latter case, $\widetilde{\mathbf{W}}$ immediately jumps down to 1; the special order is canceled. Inside $(0,1]$, $\widetilde{\mathbf{W}}$ evolves like the VOT process $\mathbf{W}$ above.

It is assumed here that the lifetime of an item starts when it arrives at the inventory system. Thus, we do not take into account reductions of the lifetimes of items before their arrival at the PIS due to transportation.

It is not difficult to see that $\widetilde{\mathbf{W}}$ is a Markov process if $n_0 = 1$ (i.e., if every special order is for one item). In this case, the steady-state density can be derived by level crossing. If $n_0 > 1$, neither $\widetilde{\mathbf{W}}$ nor its dual constructed as in Section 2.2 are Markovian, so our approach is not applicable. In the following, we assume that $n_0 = 1$.

LEMMA 4: *Let Y be the length of a time interval of emptiness and let $\eta(\alpha)$ be the Laplace transform of Y. Then*

$$\mathbb{P}(Y > x) = e^{-\lambda x}[1 - G(x)], \qquad x \geq 0,$$

and

$$\eta(\alpha) = \frac{\lambda + \alpha G^*(\lambda + \alpha)}{\lambda + \alpha}.$$

PROOF: Clearly, $Y = \min(E_\lambda, V)$, where $E_\lambda \sim \exp(\lambda)$, V has distribution function G, and E_λ and V are independent. Thus,

$$\mathbb{P}(Y > x) = \mathbb{P}(E_\lambda > x)\mathbb{P}(V > x) = e^{-\lambda x}[1 - G(x)]$$

and

$$\frac{1 - \eta(\alpha)}{\alpha} = \int_0^\infty e^{-\alpha x}\mathbb{P}(Y > x)\,dx = \int_0^\infty e^{-\alpha x}e^{-\lambda x}[1 - G(x)]\,dx$$

$$= \int_0^\infty \frac{1 - e^{-(\alpha+\lambda)u}}{\alpha + \lambda}\,dG(u) = \frac{1 - G^*(\lambda + \alpha)}{\alpha + \lambda}. \qquad \blacksquare$$

Now, we derive a Pollaczek–Khintchine-type equation for the steady-state density $\tilde{f}$ of $\widetilde{\mathbf{W}}$.

THEOREM 2: *The density $\tilde{f}$ satisfies*

$$\tilde{f}(x) = \begin{cases} \mu \int_0^x e^{-\lambda(x-w)}\tilde{f}(w)\,dw + \tilde{f}(0)e^{-\lambda x}, & 0 \le x \le 1 \\[2mm] e^{-\lambda x}[1 - G(x-1)]\left(\tilde{f}(0) + \mu \int_0^1 e^{\lambda w}\tilde{f}(w)\,dw\right), & x > 1, \end{cases} \tag{2.9}$$

and is thus given by

$$\tilde{f}(x) = \begin{cases} \tilde{f}(0)e^{(\mu-\lambda)x}, & 0 \le x \le 1 \\[2mm] \tilde{f}(0)e^{\mu-\lambda x}(1 - G(x-1)), & x > 1, \end{cases} \tag{2.10}$$

where

$$\tilde{f}(0) = \left[\frac{e^{\mu-\lambda} - 1}{\mu - \lambda} + \frac{e^{\mu-\lambda}[1 - G^*(\lambda)]}{\lambda}\right]^{-1}. \tag{2.11}$$

PROOF: In this model, there is no need for duality. The value of the density $\tilde{f}$ at x is equal to the long-run average number of downcrossings of level x by $\widetilde{\mathbf{W}}$. Thus, it is sufficient to show that the right-hand side of (2.9) gives the long-run average number of upcrossings of x. First, let $0 \le x \le 1$. Then there are two upcrossing possibilities: (i) jumps at moments of Poisson demand arrivals and (ii) jumps at outdating times. For upcrossings of type (i), note that the arrival rate of these jumps is μ, and when jumping upwards from level $w \in (0, x)$, the probability to upcross x is $e^{-\lambda(x-w)}$. In case (ii), the VOT has to reach level 0 and then to jump above x (which happens with probability $e^{-\lambda x}$). This leads to the two terms on the right-hand side of (2.9) for $x \in [0,1]$.

Now let $x > 1$. For a jump upcrossing x the following has to happen:

a. The VOT is at some level $w \in [0,1]$ and an item leaves because it is outdated or a demand takes it away.

b. The system becomes empty (i.e., no item has arrived during the last $1 - w$ time units) and there is no regular (Poisson) item arrival during the next $x - 1$ time units (which, together, has probability $e^{-\lambda(x-w)}$).

c. A special order is launched and will arrive after V time units (where V has distribution G) unless it is canceled due to a regular arrival earlier.

d. We have $V > x - 1$ (which has probability $1 - G(x - 1)$).

These arguments immediately lead to the right-hand side of (2.9) for any $x > 1$. The solution of (2.9) is straightforward, yielding (2.10) and (2.11). ∎

In this model, the rate μ^* of unsatisfied demands is given by

$$\mu^* = \mu \left[\tilde{f}(0) \int_1^\infty e^{\mu - \lambda x}(1 - G(x - 1))\, dx \right] = \tilde{f}(0) \frac{\mu}{\lambda} e^{\mu - \lambda}[1 - G^*(\lambda)].$$

For the rate of launching special orders, we obtain

$$\mu \int_0^1 e^{-\lambda(1 - w)} \tilde{f}(w)\, dw + \tilde{f}(0) e^{-\lambda} = \tilde{f}(0) e^{\mu - \lambda},$$

and for that of cancelling special orders,

$$\tilde{f}(0)[e^{\mu - \lambda} - e^{-\lambda}][1 - G^*(\lambda)].$$

3. MODEL 2: DIFFERENT CATEGORIES OF URGENCY

In this section, we assume that there is no possibility of additional ordering. The incoming demands are classified into different categories of urgency. For simplicity, assume that there are two such categories whose demand arrival times form independent Poisson processes of intensities μ_1 and μ_2, respectively. One possible policy is then to satisfy high-urgency (type 1) demands whenever possible (i.e., if the PIS is not empty) and demands of the less urgent type 2 only if there are at least, say, $m_0 > 1$ items on the shelf. However, it has the undesirable effect of not taking the lifetime of the oldest item into account. For example, under this policy, the oldest item will not be used for a less urgent demand even if its residual lifetime is very short, so its outdating is imminent. To avoid this drawback, we propose to consider the following policy. Fix $\gamma \in (0,1)$ and an integer $m_0 > 1$. A demand of type 1 is satisfied if and only if the PIS is not empty; a demand of type 2 is satisfied if and only if there are at least m_0 items in the system or the shelf age of the oldest item is at least $1 - \gamma$. Any demand (type 1 or 2) that is not immediately satisfied is lost. Let $\overline{\mathbf{W}} = \{\overline{W}(t) | t \geq 0\}$ be the VOT process of this PIS in steady state. We now derive its steady-state density $\bar{f}$ using level crossing.

THEOREM 3: *We have*

$$\bar{f}(x) = (\mu_1 + \mu_2) \int_0^x e^{-\lambda(x-w)} \bar{f}(w)\, dw + \bar{f}(0) e^{-\lambda x}, \qquad 0 \le x \le \gamma, \tag{3.1}$$

$$\bar{f}(x) = \mu_1 \int_0^x e^{-\lambda(x-w)} \bar{f}(w)\, dw + \mu_2 \int_0^{\gamma} e^{-\lambda(x-w)} \bar{f}(w)\, dw$$

$$+ \mu_2 \int_{\gamma}^x \left[e^{-\lambda(x-w)} - e^{-\lambda(1-w)} \sum_{i=0}^{m_0-2} \frac{(\lambda(1-x))^i}{i!} \right] \bar{f}(w)\, dw$$

$$+ \bar{f}(0) e^{-\lambda x}, \qquad \gamma < x \le 1, \tag{3.2}$$

$$\bar{f}(x) = \mu_1 \int_0^1 e^{-\lambda(x-w)} \bar{f}(w)\, dw + \mu_2 \int_0^{\gamma} e^{-\lambda(x-w)} \bar{f}(w)\, dw + \bar{f}(0) e^{-\lambda x}, \qquad x > 1.$$

$$\tag{3.3}$$

PROOF: Again, we have to show that the right-hand sides of (3.1)–(3.3) are the long-run average upcrossing rates of x, this time distinguishing three different cases.

Case 1: $0 < x \le \gamma$. Given that $\overline{W}(t) \le \gamma$, a jump of $\overline{W}$ occurs in the infinitesimal interval $[t, t + dt]$ with probability $(\mu_1 + \mu_2) dt$, since every incoming demand is satisfied. In order to lead to an upcrossing of level x ($\le \gamma$), the jump size has to exceed $x - \overline{W}(t)$. By PASTA, the latter probability is equal to $\int_0^x e^{-\lambda(x-w)} \bar{f}(w)\, dw$. Since $\bar{f}(0)$ is the long-run average rate of reaching level 0, the product $\bar{f}(0) e^{-\lambda x}$ is the rate of upcrossings of x by $\overline{W}$ after hittings of 0.

Case 2: $\gamma < x \le 1$. The arrival rate of demands of the first category is μ_1 and such a demand arrival causes an upcrossing of x with probability $\int_0^x e^{-\lambda(x-w)} \bar{f}(w)\, dw$. Now, assume that at time t, a demand of type 2 is arriving. There are two subcases. If $\overline{W}(t) < \gamma$, the demand is satisfied and an upcrossing of x occurs with probability $e^{-\lambda(x-w)}$. However, if $\gamma \le \overline{W}(t) = w < x$, the demand is satisfied if and only if there are at least m_0 items on the shelf just prior to time t. Moreover, to upcross level x, the process $\overline{W}$ has to jump from below x to $[x,1)$; this means that there have been no item arrivals in the time interval $(t - (1 - w), t - (1 - x))$ and at least $m_0 - 1$ arrivals in $[t - (1 - x), t)$, and the intersection of these two events has probability $e^{-\lambda(x-w)}(1 - \sum_{i=0}^{m_0-2} e^{-\lambda(1-x)}[\lambda(1 - x)]^i/i!)$. As earlier, the term $\bar{f}(0) e^{-\lambda x}$ is the long-run average rate of upcrossings of level 0 following hittings of 0 by $\overline{W}$.

Case 3: $x > 1$. Jumps upcrossing x can start from any level $w \in (0,1)$ of $\overline{W}$ for type 1 demands, any level $w \in (0, \gamma]$ for type 2 demands, and, of course, from level 0 (when the system becomes empty and there have been no item arrivals for more than x time units). ∎

To solve (3.1)–(3.3), define $G(x) = \int_0^x e^{\lambda w} \bar{f}(w)\, dw$. This function satisfies

$$G'(x) = (\mu_1 + \mu_2)G(x) + \bar{f}(0), \qquad 0 \le x \le \gamma, \tag{3.4}$$

$$G'(x) = (\mu_1 + \mu_2)G(x) + \bar{f}(0)$$

$$- \mu_2 e^{\lambda(x-1)} \sum_{i=0}^{m_0-2} \frac{(\lambda(1-x))^i}{i!}[G(x) - G(\gamma)], \qquad \gamma < x \le 1, \tag{3.5}$$

$$G'(x) = \mu_1 G(1) + \mu_2 G(\gamma) + \bar{f}(0), \qquad x > 1. \tag{3.6}$$

It follows from (3.4) that

$$G(x) = \bar{f}(0)\, \frac{e^{(\mu_1+\mu_2)x} - 1}{\mu_1 + \mu_2}, \qquad 0 \le x \le \gamma.$$

In particular,

$$G(\gamma) = \bar{f}(0)\, \frac{e^{(\mu_1+\mu_2)\gamma} - 1}{\mu_1 + \mu_2}.$$

On $[\gamma, 1]$, we solve (3.5) and find that

$$G(x) = \exp\left\{ \int_\gamma^x (\mu_1 + \mu_2 - \mu_2 h(u))\, du \right\}$$

$$\times \left[G(\gamma) + \int_\gamma^x (\bar{f}(0) + \mu_2 G(\gamma)h(v)) \right.$$

$$\left. \times \exp\left\{ -\int_\gamma^v (\mu_1 + \mu_2 - \mu_2 h(u))\, du \right\} dv \right], \qquad \gamma \le x \le 1,$$

where

$$h(x) = e^{-\lambda(1-x)} \sum_{i=0}^{m_0-2} \frac{(\lambda(1-x))^i}{i!}.$$

On $[1, \infty)$, we have, by (3.6),

$$G(x) = G(1) + (\mu_1 G(1) + \mu_2 G(\gamma) + \bar{f}(0))x, \qquad x > 1.$$

The density $\bar{f}$ is then obtained by $\bar{f}(x) = e^{-\lambda x}G'(x)$. The constant $\bar{f}(0)$ can be calculated from the normalizing condition $\int_0^\infty \bar{f}(x)\, dx = 1$.

Now, consider the cost functionals. Clearly, $\bar{f}(0)$ is the outdating rate. The number of items in the system is $n \ge 1$ if and only if the number of arrivals during the sojourn time of the oldest item is $n - 1$. Thus, the generating function of the number K of items in steady state is

$$\mathbb{E}(z^K) = \int_1^\infty \bar{f}(w)\,dw + \sum_{k=1}^\infty z^k \int_0^1 \frac{e^{-\lambda(1-w)}[\lambda(1-w)]^{k-1}}{(k-1)!}\,\bar{f}(w)\,dw$$

$$= \int_1^\infty \bar{f}(w)\,dw + z\int_0^1 e^{-\lambda(1-z)u}\bar{f}(u)\,du, \qquad |z| \le 1.$$

Finally, the rate of unsatisfied demands is

$$\mu_1 \int_1^\infty \bar{f}(w)\,dw + \mu_2 \int_\gamma^\infty \bar{f}(w)\,dw.$$

4. POSSIBLE EXTENSIONS

In many applications (e.g., to blood banks), it seems realistic that the PIS is filled by random arrivals as well as by scheduled orders. Our Model 1 is a first attempt to bridge the gap between the two kinds of model dealing exclusively with only one type of replenishment. The control policy analyzed in this article is restrictive because special orders are only placed when the system gets empty and because it assumes that the external source never runs out of stock. An extension to policies ordering as soon as some critical low level is reached and to the situation when it may happen that special orders are not delivered would be of much interest.

A second shortcoming is, of course, the neglecting of lead times for the special orders. The modification considered in Subsection 2.4 tries to deal with this point. However, it is based on the cancellation of special orders once a new random arrival takes place. Without this assumption, the VOT process is no longer Markovian and duality also does not seem to work. The derivation of its steady-state distribution is an open problem even for exponential lead times.

Model 2 can be generalized by introducing more than two different categories of urgency. The approach of Theorem 3 again leads to a solvable system of differential equations for the steady-state density.

The closed-form results derived in this article can form the basis of a sensitivity analysis and an optimization of the cost functionals with respect to the system parameters λ, μ, and n_0. Due to the complexity of the explicit formulas, this will require a considerable numerical effort.

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