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NORI'S CONNECTIVITY THEOREM AND HIGHER CHOW GROUPS

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Abstract Nori's connectivity theorem compares the cohomology of $X \times B$ and Y_B, where Y_B is any locally complete quasiprojective family of sufficiently ample complete intersections in X. When X is the projective space, and we consider hypersurfaces of degree d, it is possible to give an explicit bound for d, sufficient to conclude that the Connectivity Theorem holds. We show that this bound is optimal, by constructing for lower d classes on Y_B not coming from the ambient space. As a byproduct we get the non-triviality of the higher Chow groups of generic hypersurfaces of degree $2n$ in $\mathbb{P}^{n+1}$.

Keywords: higher Chow groups; infinitesimal variations of Hodge structure; Connectivity Theorem

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1. Introduction

Let X be a smooth complex projective variety of dimension $n + r$, and let $L_1, \dots, L_r$ be ample line bundles on X. For $\sigma_i \in H^0(X, L_i)$, $i = 1, \dots, r$, let

$$Y := \bigcap_i V(\sigma_i) \overset{j}{\hookrightarrow} X.$$

Lefschetz's Theorem on hyperplane sections says that the restriction map

$$j^* : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$$

is an isomorphism for $k < n$ and injective for $k \leqslant n$. Let $B := \sum_i H^0(X, L_i)$. There is the natural subscheme

$$\mathcal{Y}_B := U_b Y_b \subset X \times B, \qquad Y_b = \bigcap_i V(\sigma_{i,b})$$

and if $\phi : T \to B$ is a continuous map, one can form the Cartesian product

$$\mathcal{Y}_T := \mathcal{Y}_B \times_B T \overset{j_T}{\hookrightarrow} X \times T.$$

Lefschetz's Theorem and Leray's spectral sequence then imply the following assertion.

For any continuous map $\phi : T \to B$, the restriction map

$$j_T^* : H^k(X \times T, \mathbb{Z}) \to H^k(\mathcal{Y}_T, \mathbb{Z})$$

is an isomorphism for $k < n$ and injective for $k \leqslant n$.

The theorem of Nori [**13**] improves this under the assumptions that the L_i are sufficiently ample and that the map ϕ is a submersion of smooth quasiprojective varieties. Hence it concerns essentially the cohomology of locally complete families of complete intersections of sufficiently large degree.

Theorem 1.1 (Nori). *If the L_i are sufficiently ample, for any submersive morphism $\phi : T \to B$, with T smooth quasiprojective, the restriction map*

$$j_T^* : H^k(X \times T, \mathbb{Q}) \to H^k(\mathcal{Y}_T, \mathbb{Q})$$

is an isomorphism for $k < 2n$ and injective for $k \leqslant 2n$.

It is possible to give explicit bounds for how ample the L_i must be. In particular, in the case where $r = 1$ and $X = \mathbb{P}^{n+1}$, so that we are looking at hypersurfaces Y of degree d and dimension n in projective space, we have the following theorem.

Theorem 1.2. *For $d > 2n$, and for any submersive morphism*

$$\phi : T \to B = H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(1)),$$

with T smooth quasiprojective, the restriction map

$$j_T^* : H^k(X \times T, \mathbb{Q}) \to H^k(\mathcal{Y}_T, \mathbb{Q})$$

is an isomorphism for $k < 2n$ and injective for $k \leqslant 2n$.

We will sketch below a proof of this theorem, for morphisms ϕ with value in B_0, the open set of B parametrizing smooth hypersurfaces. This proof, whose detail can be found in [**19**], is the original proof of Nori in the case of hypersurfaces in projective space. Its main interest is to emphasize the role played by infinitesimal variations of Hodge structure on the cohomology of the fibres of the families $\mathcal{Y}_T \to T$. Similar computations have been performed by Nagel [**12**] in the case of complete intersections.

One of our goals in this paper is to show the optimality of this bound, that is to exhibit when $d = 2n$, $n \geqslant 2$, classes in $H^{2n-1}(\mathcal{Y}_T, \mathbb{Q})$ which are not restrictions of classes on $\mathbb{P}^{n+1} \times T$. More importantly, we will construct such classes using the higher Chow groups $\mathrm{CH}^n(Y,1)_{\mathrm{ind}}$ for Y a generic hypersurface of degree $d = 2n$ in $\mathbb{P}^{n+1}$.

We will construct an element $z \in \mathrm{CH}^n(Y,1)$ for Y generic as above. This construction can be done in family, giving an element $z \in \mathrm{CH}^n(\mathcal{Y}_U,1)$, where U is a Zariski open set of B_0. We will show how to associate to this z a class $\alpha_z \in H^{2n-1}(\mathcal{Y}_U, \mathbb{Q})$. Our main results are the following theorems.

Theorem 1.3. *For any generically finite morphism $r : V \to U$, inducing $r : \mathcal{Y}_V \to \mathcal{Y}_U$ the class $r^*\alpha_z \in H^{2n-1}(\mathcal{Y}_V, \mathbb{Q})$ does not belong to the image of the restriction map $j_V^* : H^{2n-1}(\mathbb{P}^{n+1} \times V, \mathbb{Q}) \to H^{2n-1}(\mathcal{Y}_V, \mathbb{Q})$.*

As a consequence of this, we get the following theorem.

Theorem 1.4. *For a general hypersurface of degree $d = 2n$, $n \geqslant 2$, in $\mathbb{P}^{n+1}$, the element $z \in \mathrm{CH}^n(Y,1)$ constructed above projects to a non-torsion element in $\mathrm{CH}^n(Y,1)_{\mathrm{ind}}$.*

This theorem generalizes in any dimension results obtained by Collino [**4**] and C. Oliva (unpublished) in the case of quartic surfaces.

It also proves for $n > 2$ the existence of elements of $\mathrm{CH}^n(Y,1)$ which are annihilated by the regulator map. (For $n = 2$, the regulator map is conjectured to be injective.) Such elements have also been found on generic jacobians of curves by Collino and Fakhrudin [**5**]. The elements we construct are in fact in the deepest level of any Bloch–Beilinson type filtration on higher Chow groups, since the cohomology of an hypersurface of dimension n is algebraic, excepted in degree n.

Notice that the invariant we use to detect indecomposable higher cycles is the strict analogue of the class of the spread out cycle, considered by Nori in [**13**]. Similar but more complicated invariants have been considered by Saito (see [**15**], [**16**].)

The organization of the paper is as follows: in § 2 we recall the definition of Bloch's higher Chow groups, and show how to associate a cohomology class to a higher cycle. In § 3, we will consider hypersurfaces of degree $d = 2n$ in $\mathbb{P}^{n+1}$ and describe the construction of the element $z \in \mathrm{CH}^n(Y,1)$ we alluded to above. In § 4 we will prove Theorems 1.3 and 1.4.

We now conclude this introduction by sketching the proof of Theorem 1.2.

Proof. The first step of the proof is to reduce the theorem to a comparison statement in Dolbeault cohomology.

Proposition 1.5. *Let $U \subset V$ be a closed immersion, where U and V are smooth quasiprojective complex manifolds, and let n be a given integer. In order to show that the restriction map*

$$\mathrm{rest} : H^k(V,\mathbb{Q}) \to H^k(U,\mathbb{Q})$$

is an isomorphism for $k < 2n$ and is injective for $k = 2n$, it suffices to show that the restriction map in Dolbeault cohomology

$$\mathrm{rest}_{p,q} : H^q(V,\Omega_V^p) \to H^q(U,\Omega_U^p)$$

is an isomorphism for $p + q < 2n$, $q < n$ and is injective for $p + q \leqslant 2n$, $q \leqslant n$.

If U and V were projective, this would be a direct consequence of the Hodge decomposition and Hodge symmetry, which shows that if $\mathrm{rest}_{p,q}$ is an isomorphism, $\mathrm{rest}_{q,p}$ is also an isomorphism. In general, the proof plays on the fact that the mixed Hodge structure on the relative cohomology $H^k(V,U)$ (see [**6**]) has weights greater than or equal to $k-1$, and that if the comparison statements in Dolbeault cohomology are satisfied as in the proposition, then we have

$$F^{k-n}H^k(V,U) = 0.$$

This vanishing, together with the bound on the weights, implies that $H^k(V,U) = 0$ for $k \leqslant 2n$.

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The second step will use a basic fact about the Dolbeault cohomology of the total space of a family $\pi : \mathcal{X} \to B$ and the infinitesimal variation of Hodge structure on the cohomology of the fibres X_t of the family.

Let k be a fixed integer. Then we have on B the local system of $\mathbb{C}-$vector spaces $H^k_{\mathbb{C}} := R^k\pi_*\mathbb{C}$ and the associated flat vector bundle $\mathcal{H}^k := H^k_{\mathbb{C}} \otimes \mathcal{O}_B$ which is endowed with the flat Gauss–Manin connection ∇. There is the Hodge filtration $F^p\mathcal{H}^k \subset \mathcal{H}^k$ by holomorphic subbundles, which induces on each fibre the Hodge filtration

$$F^pH^k(X_t) \subset H^k(X_t, \mathbb{C}) = \mathcal{H}^k_t.$$

The graded pieces $\mathrm{Gr}^p_F \mathcal{H}^k$ are usually denoted by $\mathcal{H}^{p,q}$, $p + q = k$ and are naturally isomorphic to $R^q\pi_*\Omega^p_{\mathcal{X}/B}$.

Griffiths transversality says that

$$\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \otimes \Omega_B.$$

It can be used to define a Hodge filtration on the de Rham complex

$$\mathrm{DR}(H^k_{\mathbb{C}}) := 0 \to \mathcal{H}^k \xrightarrow{\nabla} \mathcal{H}^k \otimes \Omega_B \xrightarrow{\nabla} \mathcal{H}^k \otimes \Omega^2_B \cdots .$$

We define $F^p\,\mathrm{DR}(H^k_{\mathbb{C}})$ to be the subcomplex

$$0 \to F^p\mathcal{H}^k \xrightarrow{\nabla} F^{p-1}\mathcal{H}^k \otimes \Omega_B \xrightarrow{\nabla} F^{p-2}\mathcal{H}^k \otimes \Omega^2_B \cdots .$$

The graded piece $\mathrm{Gr}^p_F\,\mathrm{DR}(H^k_{\mathbb{C}})$ will be denoted by $\mathcal{K}^{\cdot}_{p,q}$, $p + q = k$. It takes the form

$$0 \to \mathcal{H}^{p,q} \xrightarrow{\overline{\nabla}} \mathcal{H}^{p-1,q+1} \otimes \Omega_B \xrightarrow{\overline{\nabla}} \mathcal{H}^{p-2,q+2} \otimes \Omega^2_B \cdots .$$

The differential $\overline{\nabla}$ of this complex is $\mathcal{O}_B$-linear and describes the infinitesimal variation of Hodge structure on the kth cohomology of the fibres (cf. [**19**]). The degree in this complex is the degree r of the sheaf of holomorphic forms appearing in each piece.

On the other hand, one can also do the following. We fix an integer l and study the derived sheaves

$$R\pi_*\Omega^l_{\mathcal{X}}.$$

The sheaf $\Omega^l_{\mathcal{X}}$ is endowed with the Leray filtration

$$L^r\Omega^l_{\mathcal{X}} := \pi^*\Omega^r_B \wedge \Omega^{l-r}_{\mathcal{X}}.$$

The graded pieces $\mathrm{Gr}^r_L\,\Omega^l_{\mathcal{X}}$ are canonically isomorphic to

$$\pi^*\Omega^r_B \otimes \Omega^{l-r}_{\mathcal{X}/B}.$$

The Leray filtration provides a spectral sequence abutting to $R\pi_*\Omega^l_{\mathcal{X}}$

$$E^{r,s}_1 \Rightarrow R^{r+s}\pi_*\Omega^l_{\mathcal{X}},$$
$$E^{r,s}_1 = R^{r+s}\pi_*\,\mathrm{Gr}^r_L\,\Omega^l_{\mathcal{X}} = \Omega^r_B \otimes R^{r+s}\pi_*\Omega^{l-r}_{\mathcal{X}/B} = \Omega^r_B \otimes \mathcal{H}^{l-r,r+s}.$$

The differential $d_1 : E^{r,s}_1 \to E^{r+1,s}_1$ provides for fixed l and s a complex which will be denoted by $E^{\cdot,s}_{1,l}$. We have now the basic result in Hodge theory (cf. [**19**]).

Theorem 1.6. *The complexes* $(E_{1,l}^{\cdot,s}, d_1)$ *and* $(\mathcal{K}_{l,s}^{\cdot}, \overline{\nabla})$ *are canonically isomorphic.*

We now conclude the proof. Consider the inclusion of smooth quasiprojective varieties over T:

$$\begin{array}{ccc} \mathcal{Y}_T & \overset{j}{\hookrightarrow} & X \times T \\ {\scriptstyle \pi_Y}\downarrow & & {\scriptstyle \pi_X}\downarrow \\ T & = & T \end{array}$$

We want to show that

$$j^* : H^q(X \times T, \Omega^p_{X\times T}) \to H^q(\mathcal{Y}_T, \Omega^p_{\mathcal{Y}_T})$$

is an isomorphism for $p+q < 2n$, $q < n$ and is injective for $p+q \leqslant 2n$, $q \leqslant n$. It suffices by Leray's spectral sequence to show the same result for the restriction maps

$$j^* : R^q\pi_{X*}\Omega^p_{X\times T} \to R^q\pi_{Y*}\Omega^p_{\mathcal{Y}_T}$$

in the same range. But by compatibility of the Leray filtration with the restriction map j^*, we have a morphism

$$j^* : E^{r,s}_{i,p,X\times T} \to E^{r,s}_{i,p,\mathcal{Y}_T}$$

of spectral sequences associated to the Leray filtrations on $\Omega^p_{X\times T}$ and $\Omega^p_{\mathcal{Y}_T}$ abutting respectively to $R^{r+s}\pi_{X*}\Omega^p_{X\times T}$ and $R^{r+s}\pi_{Y*}\Omega^p_{\mathcal{Y}_T}$ and it suffices to show that it is an isomorphism on E_2 for $r+s+p < 2n$, $r+s < n$, and injective for $r+s+p \leqslant 2n$, $r+s \leqslant n$. But being an isomorphism on E_2 means being a quasi-isomorphism on E_1. By the Theorem 1.6, we are then reduced to show that the complexes of infinitesimal variations of Hodge structure $\mathcal{K}^{\cdot}_{p,q,X}$ and $\mathcal{K}^{\cdot}_{p,q,Y}$ built respectively for the constant family $X \times T \to T$ and the family $\mathcal{Y}_T \to T$ are quasi-isomorphic in certain degrees.

Let us now assume that $X = \mathbb{P}^{n+1}$ and $\mathcal{Y}_T$ is a family of hypersurfaces parametrized by T. The cohomology of an hypersurface in degree less than or equal to $2n$ splits canonically as the direct orthogonal sum

$$H^*(Y,\mathbb{C}) = H^*(Y,\mathbb{C})_{\text{prim}} \oplus H^*(\mathbb{P}^{n+1},\mathbb{C}), \quad * \leqslant 2n,$$

where the primitive cohomology $H^*(Y,\mathbb{C})_{\text{prim}}$ is non-zero only in degree n.

It follows that in degrees $p+q \leqslant 2n$, the complex $\mathcal{K}^{\cdot}_{p,q,X}$ is a direct summand of the complex $\mathcal{K}^{\cdot}_{p,q,Y}$, the other summand being the complex $\mathcal{K}^{\cdot}_{p,q,Y,\text{prim}}$ of infinitesimal variation of Hodge structure on the primitive cohomology.

Hence to show that the two complexes $\mathcal{K}^{\cdot}_{p,q,X}$ and $\mathcal{K}^{\cdot}_{p,q,Y}$ are quasi-isomorphic in certain degrees, with $p+q \leqslant 2n$, is equivalent to show that the complex $\mathcal{K}^{\cdot}_{p,q,Y,\text{prim}}$ is acyclic in the same range.

The end of the proof is then the computation of the cohomology of the complexes $\mathcal{K}^{\cdot}_{p,q,Y,\text{prim}}$. It uses the identification of the fibre at $F \in T$ of these complexes (or more precisely of the dual complexes), with Koszul complexes of the jacobian ring of the hypersurface Y_F, with respect to the $S^{\cdot}T_{T,F}$-module structure

$$T_{T,F} \to \operatorname{Hom}(R^{\cdot}_F \to R^{\cdot+d}_F)$$

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coming from the differential $T_{T,F} \to S^d$ of $\phi : T \to B$ and the natural module structure of $R^{\cdot}F$ over $S^{\cdot}S^d$. This description is due to Griffiths [**10**] and Carlson and Griffiths [**3**] and follows from the description by residues of the Hodge structure on the primitive cohomology of a hypersurface.

Having identified these cohomology groups to Koszul cohomology groups, the desired vanishing of the cohomology of the complexes $\mathcal{K}^{\cdot}_{p,q,Y,\mathrm{prim}}$ in degree $l < n$, for $q + l < n$ and $d > 2n$, follows from the following result due to Green [**8**] on the syzygies of projective space.

Theorem 1.7. *Let $S = \oplus S^k$ be the polynomial ring in $n + 2$ variables. The sequence*

$$S^a \otimes \bigwedge^{l+1} S^d \xrightarrow{\delta} S^{a+d} \otimes \bigwedge^{l} S^d \xrightarrow{\delta} S^{a+2d} \otimes \bigwedge^{l-1} S^d,$$

where the Koszul differential δ is defined by

$$\delta(P \otimes A_1 \wedge \cdots \wedge A_k) = \sum_i (-1)^i PA_i \otimes A_1 \wedge \cdots \hat{A}_i \cdots \wedge A_k$$

is exact when $a \geqslant l$.

This concludes the proof of Theorem 1.2. □

2. Higher Chow groups and cohomology

In [**2**], Bloch defines the higher Chow groups $\mathrm{CH}^k(X, i)$ for X an irreducible algebraic variety defined over a field K.

Let $\Delta_i \subset \mathbb{A}^{i+1}_K$ be the affine simplex of dimension i

$$\Delta_i = \left\{ (x_1, \ldots, x_{i+1}),\ \sum_i x_i = 1 \right\}.$$

For any $j \in \{1, \ldots, i+1\}$, there is the obvious face map $l_j : \Delta_{i-1} \hookrightarrow \Delta_i$, which is the inclusion of the hyperplane $x_j = 0$. Define $\mathcal{Z}^k(X \times \Delta_i)_{\mathrm{pr}}$ to be the subgroup of the group of codimension k cycles, generated by subvarieties meeting properly all the $X \times \Delta_\sigma$ where Δ_σ is any face (of any codimension) of Δ_i. Then we have a differential

$$d : \mathcal{Z}^k(X \times \Delta_i)_{\mathrm{pr}} \to \mathcal{Z}^k(X \times \Delta_{i-1})_{\mathrm{pr}}$$
$$\mathrm{d}Z = \sum_1^{i+1} (-1)^j l_j^* Z,$$

where $l_j^* Z$ is the cycle associated to the subscheme of codimension k

$$Z \bigcap l_j(\Delta_{i-1}) \subset l_j(\Delta_{i-1}) \overset{l_j^{-1}}{\cong} \Delta_{i-1},$$

for $Z \subset X \times \Delta_i$ irreducible of codimension k.

Bloch defines $\mathrm{CH}^k(X,i)$ to be the homology group $H_i(\mathcal{Z}^k(X \times \Delta_*), d)$.

It is immediate to see that $\mathrm{CH}^k(X,0) = \mathrm{CH}^k(X)$. Indeed $X \times \Delta_0 = X$ and the boundaries are generated by applying d to codimension $k-1$ subvarieties W of $X \times \Delta_1$. By normalizing, we get normal varieties $\tilde{W}$ together with a generically finite proper map $n : W \to X$, and a meromorphic function ϕ on $\tilde{W}$ which is non-zero on any component of $\tilde{W}$: then the boundary $\mathrm{d}W$ is equal to $n_*(\tilde{W}_1 - \tilde{W}_0)$, where $\tilde{W}_\alpha := \phi^{-1}(\alpha), \alpha \in \mathbb{P}^1$. But, changing ϕ to $\psi = \phi/(1-\phi)$, $\tilde{W}_1 - \tilde{W}_0$ becomes the divisor of ψ so that these cycles $n_*(\tilde{W}_1 - \tilde{W}_0)$ generate in fact the subgroup of codimension k cycles rationally equivalent to 0 on X.

We shall be interested in the group $\mathrm{CH}^k(X,1)$. Arguing as before, we see that it has the following description: up to cycles coming from $\mathcal{Z}^k(X)$, which are easily seen to be boundaries, elements of $\mathcal{Z}^k(X \times \Delta_1)_{\mathrm{pr}}$ can be seen as combinations $Z = \sum_i m_i(Z_i, n_i, \phi_i)$, where Z_i is normal of dimension equal to $\dim X + 1 - k$, $n_i : Z_i \to X$, is a generically finite proper map, and ϕ_i is a meromorphic function on Z_i. Changing ϕ_i to $\psi_i = \phi_i/(1-\phi_i)$, the condition that $\mathrm{d}Z = 0$ is translated into the condition

$$\sum_i m_i n_{i*} \operatorname{div} \psi = 0$$

as a cycle of codimension k on X.

The boundaries are generated by 'tame symbols': let W be a normal variety of dimension equal to $\dim X + 2 - k$, $n : W \to X$, be a generically finite to one proper map, and let ϕ_1, ϕ_2 be two meromorphic functions on W. Then the tame symbol (cf. [**1**])

$$T(\phi_1, \phi_2) \in \bigoplus_{\substack{\operatorname{codim} D = 1 \\ D \subset W}} K(D)^*$$

is defined as

$$\sum_{\mathrm{D}} (-1)^{\nu_{\mathrm{D}}(\phi_1)\nu_{\mathrm{D}}(\phi_2)} \frac{\phi_1^{\nu_{\mathrm{D}}(\phi_2)}}{\phi_2^{\nu_{\mathrm{D}}(\phi_1)}}_{|D}.$$

If ϕ_1 and ϕ_2 have no common divisor, this is simply

$$\phi_{1|\operatorname{div}\phi_2} + \left(\frac{1}{\phi_2}\right)_{|\operatorname{div}\phi_1},$$

where, on the negative part of $\operatorname{div}\phi_2$, the restriction of ϕ_1 is defined to be $1/\phi_1$.

One checks easily that modulo cycles coming from $\mathcal{Z}^k(X)$, the boundary $d(\mathcal{Z}^k(X \times \Delta_2))$ is exactly generated by the $(T(\phi_1, \phi_2), n)$, for W, n, ϕ_1, ϕ_2 as above. Here for $T(\phi_1, \phi_2) = \sum_i \phi_i, \phi_i \in K(D_i)^*$, we define $(T(\phi_1, \phi_2), n)$ to be $\sum_i (D_i, n_{|D_i}, \phi_i)$, where in the last sum, only those i for which $n_{|D_i}$ remains generically finite appear.

Remark 2.1. One checks easily that in the quotient group $\mathrm{CH}^n(X,1)$, a cycle

$$Z = \sum_i m_i(Z_i, n_i, \psi_i), \qquad \sum_i m_i n_{i*} \operatorname{div} \phi_i = 0,$$

has the same image as $\sum_i (Z_i, n_i, \psi_i^{m_i})$. It has also the same image as

$$\sum_i (Z_i', n_i', Nm(\psi_i^{m_i})),$$

where Z_i' is the normalization of the image of Z_i in X under n_i, $Nm(\psi_i^{m_i})$ is the trace of $\psi_i^{m_i}$ under the map $n_i : Z_i \to Z_i'$, and $n_i' : Z_i' \to X$ is the induced map.

For these reasons, the relations $(T(\phi_1, \phi_2), n) = 0$ in $\mathrm{CH}^k(X, 1)$ above imply as well the relations $n_* T(\phi_1, \phi_2) := \sum_i (n(D_i), Nm\phi_i) = 0$ in $\mathrm{CH}^k(X, 1)$.

This description of the group $\mathrm{CH}^k(X, 1)$ shows that there is a natural map

$$\mathrm{CH}^{k-1}(X) \otimes K^* \to \mathrm{CH}^k(X, 1).$$

Indeed, if Z is a cycle of codimension $k - 1$ on X and α is a non-zero constant, we see α as a nowhere zero function $\alpha_{|Z}$ on Z. (If $Z = \sum_i n_i Z_i$, this function takes the value α^{n_i} on Z_i.) This function has obviously an empty divisor, which gives a representative for an element of $\mathrm{CH}^k(X, 1)$. If furthermore Z is rationally equivalent to zero, so that there exist normal varieties $W_i \xrightarrow{n_i} X$ together with rational functions ϕ_i on them, such that $Z = \sum_i n_{i*} \operatorname{div} \phi_i$, we have

$$\alpha_{|Z} = \sum_i n_{i*} T(\alpha, \phi_i).$$

Hence by the remark above $\alpha_{|Z}$ goes to 0 in $\mathrm{CH}^k(X, 1)$ and the above constructed map

$$\mathcal{Z}^{k-1}(X) \otimes K^* \to \mathrm{CH}^k(X, 1)$$

factors through $\mathrm{CH}^{k-1}(X) \otimes K^*$. Notice that, more generally, we have a map

$$\mathrm{CH}^{k-1}(X) \otimes H^0(X, \mathcal{O}_X^*) \to \mathrm{CH}^k(X, 1), \tag{2.1}$$

which is defined exactly in the same way, replacing constants by nowhere vanishing functions, and induces the previous one on

$$\mathrm{CH}^{k-1}(X) \otimes K^* \subset \mathrm{CH}^{k-1}(X) \otimes H^0(X, \mathcal{O}_X^*).$$

Definition 2.2. The elements of the image of the map $\mathrm{CH}^{k-1}(X) \otimes K^* \to \mathrm{CH}^k(X, 1)$ are called decomposable. The quotient

$$\mathrm{CH}^k(X, 1)_{\mathrm{ind}} := \mathrm{CH}^k(X, 1) / \operatorname{Im} \mathrm{CH}^{k-1}(X) \otimes K^*$$

is called the indecomposable part of the group $\mathrm{CH}^k(X, 1)$.

Let now X be a smooth complex quasiprojective variety of dimension n and let $Z = \sum_i (Z_i, n_i, \phi_i)$, $\sum_i n_{i*} \operatorname{div} \phi_i = 0$ represent an element $z \in \mathrm{CH}^k(X, 1)$. We show how to associate a class $\alpha_z \in H^{2k-1}(X, \mathbb{Z})/\text{torsion}$ to z. For each i, the proper map $n_i : Z_i \to X$, with $\dim Z_i = \dim X + 1 - k$ together with the meromorphic function ϕ_i

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provides a current D_i acting on differentiable forms of degree $2n-2k+1$ with compact support, given by

$$D_i(\eta) = \frac{1}{2\mathrm{i}\pi}\int_{Z_i} n_i^*\eta \wedge \frac{\mathrm{d}\phi_i}{\phi_i}.$$

(Indeed, the form $n_i^*\eta$ has compact support, and forms with logarithmic poles are known to be integrable on analytic spaces.)

Lelong's formula (cf. [**19**]) says now that the boundary δD_i of D_i in the sense of distributions, is given by

$$\delta D_i(\eta) = \int_{\operatorname{div}\phi_i} n_i^*\eta.$$

Denoting $D_Z = \sum_i D_i$, it follows that the condition $\sum_i n_{i*}\operatorname{div}\phi_i = 0$ translates into the equation

$$\delta D_Z = 0.$$

The current D_Z being closed has a cohomology class $[D_Z] \in H^{2k-1}(X,\mathbb{C})$. We now have the following proposition.

Proposition 2.3.

(i) *The class $[D_Z]$ is integral, that is in the image of the natural map*

$$H^{2k-1}(X,\mathbb{Z}) \to H^{2k-1}(X,\mathbb{C}).$$

(ii) *The class $[D_Z]$ belongs to $F^kH^{2k-1}(X)$, where F is the Hodge filtration of the mixed Hodge structure on $H^{2k-1}(X,\mathbb{C})$ (cf. [**19**, I]).*

(iii) *The class $[D_Z]$ is trivial if $z = 0$ in $\mathrm{CH}^k(X,1)$.*

Proof. (i) We note that for each i the integrable form

$$\frac{1}{2\mathrm{i}\pi}\frac{\mathrm{d}\phi_i}{\phi_i}$$

on Z_i, or on a desingularization $\tilde{Z}_i$ of Z_i defines an integral class γ_i in $H^1(\tilde{Z}_i - \operatorname{div}\phi_i,\mathbb{Z})$. The natural composed map $\tilde{n}_i : \tilde{Z}_i \to X$ induces a proper map

$$\tilde{n}_i : \tilde{Z}_i - \tilde{n}_i^{-1}(\tilde{n}_i(\operatorname{div}\phi_i)) \to X - \tilde{n}_i(\operatorname{div}\phi_i).$$

The Gysin map gives then a class

$$\tilde{n}_{i*}\gamma_i \in H^{2k-1}(X - \tilde{n}_i(\operatorname{div}\phi_i),\mathbb{Z}).$$

Hence we get a class

$$\sum_i \tilde{n}_{i*}\gamma_i \in H^{2k-1}(X - \cup_i\tilde{n}_i(\operatorname{div}\phi_i),\mathbb{Z}).$$

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It follows from the definition of $[D_Z]$ that the image of this class in the homology group $H^{2k-1}(X-\cup_i\tilde{n}_i(\operatorname{div}\phi_i),\mathbb{C})$ is equal to the image by restriction of the class $[D_Z]\in H^{2k-1}(X,\mathbb{C})$. Since the restriction map

$$H^{2k-1}(X,\mathbb{C})\to H^{2k-1}(X-\cup_i\tilde{n}_i(\operatorname{div}\phi_i),\mathbb{C})$$

is injective for dimension reasons, it follows that $[D_Z]$ is rational. Since the cokernel of the restriction map above in integral cohomology has no torsion, it is in fact integral.

(ii) This follows from the definition of the Hodge filtration on $H^{2k-1}(X)$ (cf. [**19**, I]), the fact that the currents D_i are of Hodge type $(k,k-1)$, that is annihilate forms of degree (p,q), $(p,q)\neq(n-k,n-k+1)$, and that furthermore, they also have 'logarithmic growth at infinity', with respect to any compactification of X with a divisor with normal crossings at infinity.

(iii) Since the boundaries are generated by tame symbols, it suffices to show that if W is normal of dimension $n-k+2$, $n:W\to X$ is a generically finite to one proper morphism, ϕ_1, ϕ_2 are two meromorphic functions on W, which we assume without common divisor for simplicity, with boundary $Z=(\operatorname{div}\phi_1,n_{|\operatorname{div}\phi_1},\phi_{2|\operatorname{div}\phi_1})-(\operatorname{div}\phi_2,n_{|\operatorname{div}\phi_2},\phi_{1|\operatorname{div}\phi_2})$, the current D_Z associated to Z is of class 0. But it follows again from Lelong's formula that it is the boundary of the current

$$\eta\mapsto\left(\frac{1}{2\mathrm{i}\pi}\right)^2\int_W n^*\eta\wedge\frac{\mathrm{d}\phi_1}{\phi_1}\wedge\frac{\mathrm{d}\phi_2}{\phi_2}.$$

□

The proposition allows us to define the integral class associated to the cycle $z\in\mathrm{CH}^k(X,1)$ to be

$$\alpha_z:=[D_Z]\in H^{2k-1}(X,\mathbb{Z})/\text{torsion}$$

for any representative Z of z.

Remark 2.4. With a little more work we could have defined α_z to be in $H^{2k-1}(X,\mathbb{Z})$.

Remark 2.5. The invariant α_z is in fact part of the image of z under the regulator map ρ, but it is usually not considered. The regulator map will associate to z an element of the Deligne cohomology group $H^{2k-1}_{\mathrm{D}}(X,\mathbb{Z}(k))$. The Deligne complex

$$\mathbb{Z}(k)=0\to\mathbb{Z}\to\mathcal{O}_X\to\Omega_X\to\cdots\to\Omega_X^{k-1}\to0$$

admits the obvious map o to the complex $\mathbb{Z}$ supported in degree 0. Hence there is an induced map

$$o:H^{2k-1}_{\mathrm{D}}(X,\mathbb{Z}(k))\to H^{2k-1}(X,\mathbb{Z}).$$

One can show that

$$\alpha_z=o(\rho(z)).$$

The reason why α_z is usually not considered, is the fact that when X is projective, the map o is identically 0, so that in fact $\alpha_z=0$. Indeed for a smooth projective variety

$$F^kH^{2k-1}(X)\bigcap H^{2k-1}(X,\mathbb{Z})=0,$$

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as shown by the Hodge decomposition and the Hodge symmetry. On the other hand,

$$\operatorname{Im} o = \{u \in H^{2k-1}(X, \mathbb{Z}),\ u_{\mathbb{C}} \in F^k H^{2k-1}(X)\}.$$

In the sequel, we will use the invariant α_z for the non-projective smooth variety $\mathcal{Y}_U$, where U is an open set of $H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ and $\mathcal{Y}_U$ is the universal hypersurface parametrized by U. In the case we will be considering, the invariant α_z will be the only interesting invariant of our cycle z. It is the exact analogue of the class of a cycle $\mathcal{Z}$ in a family of varieties $\mathcal{Y}$, which is used in [**13**], and happens to control in some cases the more refined invariants of the cycles Z_t in the fibres Y_t.

3. Hypersurfaces of degree $2n$ in P^{n+1}

Let $Y \subset \mathbb{P}^{n+1}$ be a generic hypersurface of degree $d = 2n$, $n \geqslant 2$. We want to construct an element $z \in \mathrm{CH}^n(Y, 1)$. Following § 2, it will be represented by a sum $Z = \sum_i (C_i, n_i, \phi_i)$ such that $\sum_i n_{i*} \operatorname{div} \phi_i = 0$ in $\mathcal{Z}^n(Y)$, where C_i are smooth curves, $n_i : C_i \to Y$ are morphisms, and ϕ_i are non-zero meromorphic functions on C_i.

In order to construct such Z, we start with the following easy facts concerning the geometry of such hypersurfaces.

Lemma 3.1. *Y being generic, the set*

$$C_Y := \left\{ y \in Y \mid \exists \operatorname{line} \Delta \subset \mathbb{P}^{n+1},\ \Delta \bigcap Y = \mathrm{d}y \textit{ or } \Delta \subset Y \right\}$$

is a smooth non-empty projective curve.

Proof. Let $\pi : P \to \mathbb{P}^{n+1}$ be the projective bundle parametrizing pairs (y, Δ) where Δ is a line and $y \in \Delta$. We have $\dim P = 2n + 1$. There is a vector bundle $\mathcal{E}'_d$ on P, whose fibre at (y, Δ) is equal to $H^0(\mathcal{O}_{\mathbb{P}^1}(d))/H^0(\mathcal{O}_{\mathbb{P}^1}(d)(-\mathrm{d}y))$. We have $rk\mathcal{E}'_d = d = 2n$.

For each equation $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$, there is an associated section $\tilde{\sigma}$ of $\mathcal{E}'_d$ given by

$$\tilde{\sigma}(y, \Delta) = \sigma_{|\Delta} \bmod H^0(\mathcal{O}_{\mathbb{P}^1}(d)(-\mathrm{d}y)).$$

If Y is defined by σ, and $D_Y \subset P$ is the zero set of $\tilde{\sigma}$, it is clear that the image of D_Y under π is equal to C_Y. The proof follows then from the following lemma.

Lemma 3.2.

(i) $\pi : D_Y \to \mathbb{P}^{n+1}$ *is an isomorphism on its image C_Y for generic Y.*

(ii) D_Y *is a smooth non-empty projective curve for generic Y.*

Proof. (ii) follows from the fact that the bundle $\mathcal{E}'_d$ is generated by the sections $\tilde{\sigma}$. Hence the variety

$$\mathcal{D} := \{(\sigma, (x, \Delta)) \in H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d)) \times P \mid \tilde{\sigma}(x, \Delta) = 0\}$$

is smooth of codimension $2n$ in $H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d)) \times P$. By Sard's Theorem, it follows that the generic fibre $\mathrm{pr}_1^{-1}(\sigma) = D_Y$, $Y = V(\sigma)$ is smooth of dimension 1, projective since pr_1

is proper. The fact that it is non-empty is shown by exhibiting one point in $\mathcal{D}$ where pr_1 is of maximal rank.

(i) is proved by a dimension count. We introduce the variety

$$P' = P \times_{\mathbb{P}^{n+1}} P - \text{diagonal}$$

parametrizing a point together with two distinct lines through it, and we consider inside $H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d)) \times P'$ the variety

$$\mathcal{D}' := \{(\sigma, x, \Delta_1, \Delta_2) \mid \tilde{\sigma}(x, \Delta_1) = 0,\ \tilde{\sigma}(x, \Delta_2) = 0\}.$$

One shows by the same argument as above that $\mathcal{D}'$ is smooth of dimension strictly less than the dimension of $H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$. It follows that $\mathcal{D}'$ does not dominate $H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$ by the first projection. Since clearly the fibre $D'_Y = \mathrm{pr}_1^{-1}(\sigma)$, $Y = V(\sigma)$ is equal to $D_Y \times_{\mathbb{P}^{n+1}} D_Y -$ diagonal, it follows that for generic Y, the map $\pi : D_Y \to \mathbb{P}^{n+1}$ is one to one on its image.

One shows by a similar argument that the map $\pi : D_Y \to \mathbb{P}^{n+1}$ has injective differential, for generic Y. This concludes the proof of Lemma 3.2 and hence of Lemma 3.1. □

We shall also use the following fact in the next section.

Lemma 3.3.

(i) *The generic hypersurface Y does not contain a line.*

(ii) *Let $B_\Delta \subset B := H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$ be the set of equations σ defining a hypersurface containing a line. Then $B_\Delta \subset B$ is a hypersurface, and for generic $\sigma \in B_\Delta$, the hypersurface Y defined by σ contains only one line.*

Remark 3.4. The last fact is obviously false if $n = 1$, as is the Theorem 1.4.

Proof. Let $G = \mathrm{Grass}(1, n)$ be the Grassmannian of lines in $\mathbb{P}^{n+1}$. Let $\mathcal{E}_d$ be the vector bundle of rank $d+1 = 2n+1$ on G with fibre $H^0(\mathcal{O}_\Delta(d))$ at $\Delta \in G$. Any $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$ defines by evaluation a section $\tilde{\sigma}$ of $\mathcal{E}_d$ on G. Since the sections $\tilde{\sigma}$ generate $\mathcal{E}_d$, it follows that the variety $\mathcal{D}'' \subset B \times G$

$$\mathcal{D}'' = \{(\sigma, \Delta) \mid \tilde{\sigma}(\Delta) = 0\}$$

is smooth of codimension $2n + 1$. Since $\dim G = 2n$ it follows that $\mathcal{D}''$ cannot dominate B by the first projection. This proves (i).

To prove (ii) we note that the above shows that $\dim \mathcal{D}'' = \dim B - 1$, so it suffices to show that for $n \geqslant 2$ the subvariety $\mathcal{D}'''$ of $B \times G \times G$ made of triples $(\sigma, \Delta_1, \Delta_2)$, $\Delta_1 \neq \Delta_2$, such that $\tilde{\sigma}(\Delta_1) = \tilde{\sigma}(\Delta_2) = 0$, has dimension strictly less than $\dim B - 1$, since we have

$$\mathcal{D}''' = \mathcal{D}'' \times_B \mathcal{D}''.$$

This fact is proved by similar arguments as before. □

We now construct the announced Z. Let H be a hyperplane in $\mathbb{P}^{n+1}$ in general position. Let $H \cap C_Y = \{x_1, \ldots, x_N\}$, $N = d^0 C_Y$. We also fix a line Δ and put $h := \Delta \cap Y \in \mathcal{Z}_0(Y)$.

For each $i = 1, \ldots, N$, let Δ_i be the line osculating Y at x_i. We choose in $\mathbb{P}^{n+1}$ a family $(\Delta_t)_{t \in \mathbb{P}^1}$ of lines parametrized by $\mathbb{P}^1$, such that $\Delta_0 = \Delta_i$ and $\Delta_\infty = \Delta$. Let $\phi_i : S_i \to \mathbb{P}^1$ be the ruled surface so obtained, and $n_i : S_i \to \mathbb{P}^{n+1}$ be the natural morphism. Let C_i be the normalization of $n_i^{-1}(Y)$. Then by restriction, we have

$$n_i : C_i \to Y, \qquad \phi_i : C_i \to \mathbb{P}^1$$

such that

$$n_{i*}(\operatorname{div} \phi_i) = \mathrm{d}x_i - h. \tag{3.1}$$

Next we do the following. First of all, we choose a pencil of hypersurfaces $(Y_t)_{t \in \mathbb{P}^1}$ such that $Y_0 = Y$, and Y_∞ is smooth and the variety C_{Y_∞} is a curve meeting properly Y. We assume furthermore that the equations F_t defining Y_t satisfy

$$\left(\frac{\mathrm{d}F_t}{\mathrm{d}t}\right)_{|t=0} = X^d \bmod F_0, \tag{3.2}$$

where X is a defining equation for the hyperplane H. Let then S_0 be the surface $\cup_t C_{Y_t}$, or more precisely the irreducible component of it dominating $\mathbb{P}^1$. We have the natural morphisms

$$n_0 : S_0 \to \mathbb{P}^{n+1}$$

and

$$\phi_0 : S_0 \to \mathbb{P}^1.$$

Let C_0 be the normalization of the union of the irreducible components of $n_0^{-1}(Y)$ dominating $\mathbb{P}^1$. Then we have by restriction the two morphisms

$$n_0 : C_0 \to Y, \qquad \phi_0 : C_0 \to \mathbb{P}^1.$$

By construction $n_{0*}(\phi_0^{-1}(\infty)) = C_{Y_\infty} \cap Y$. We now have the following lemma.

Lemma 3.5. *$n_{0*}(\phi_0^{-1}(0))$ is the cycle $d \sum_{i=1}^{i=N} x_i$.*

Proof. Let t be a local coordinate on $\mathbb{P}^1$ near 0 such that F_t is proportional to $F_0 + tX^d$, which exists by equation (3.2). Since each curve C_{Y_t} is contained in the hypersurface Y_t, we conclude that the section $n_0^* F_0 + t n_0^* X^d$ of $n_0^* * \mathcal{O}(d)$ is identically 0 on S_0 near the central fibre. Since the curve $n_0^{-1}(Y)$ is defined by the equation $n_0^* F_0$, it is as well defined near the central fibre C_Y by the equation $t n_0^* X^d$. Hence it is defined by the equation $n_0^* X^d$ away from the fibre C_Y which is defined by $t = 0$. Since C_Y is a component of $n_0^{-1}(Y)$ which does not dominate $\mathbb{P}^1$, the curve C_0 is, near $t = 0$, the normalization of the curve defined by $n_0^* X^d$, which proves the lemma. □

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From this lemma we conclude that

$$n_{0*}(\operatorname{div}\phi_0) = \mathrm{d}H \bigcap C_Y - C_{Y_\infty} \bigcap Y \in \mathcal{Z}_0(Y). \tag{3.3}$$

To finish we now choose in $\mathbb{P}^{n+1}$ a 'deformation of the curve C_{Y_∞} to $N\Delta$' parametrized by $\mathbb{P}^1$. In other words, we have a surface

$$S_0' \xrightarrow{n_0'} \mathbb{P}^{n+1}$$

together with a morphism

$$S_0' \xrightarrow{\phi_0'} \mathbb{P}^1$$

such that

$$n'_{0*}(\operatorname{div}\phi_0') = C_{Y_\infty} - N\Delta.$$

Again, let C_0' be the normalization of the union of the components of ${n_0'}^{-1}(Y)$ dominating $\mathbb{P}^1$. Then we have the morphisms

$$n_0' : C_0' \to Y, \qquad \phi_0' : C_0' \to \mathbb{P}^1$$

such that

$$n'_{0*}(\operatorname{div}\phi_0') = C_{Y_\infty} \bigcap Y - Nh \in \mathcal{Z}_0(Y). \tag{3.4}$$

We combine now the three relations: let

$$Z = \sum_i (C_i, n_i, \phi_i) - (C_0, n_0, \phi_0) - (C_0', n_0', \phi_0').$$

Summing up the equations (3.1), (3.3) and (3.4), we get

$$\sum_{i=1}^{i=N} n_{i*}(\operatorname{div}\phi_i) - n_{0*}(\operatorname{div}\phi_0) - n'_{0*}(\operatorname{div}\phi_0') = 0 \in \mathcal{Z}_0(Y).$$

This gives the desired cycle.

Remark 3.6. It is easy to see that the class of the cycle Z in $\mathrm{CH}^n(Y,1)$ depends on the choices made only up to the image of the composed map

$$\mathrm{CH}^{n-1}(\mathbb{P}^{n+1}) \otimes \mathbb{C}^* \to \mathrm{CH}^{n-1}(Y) \otimes \mathbb{C}^* \to \mathrm{CH}^n(Y,1),$$

where the last map was defined in § 2. Indeed, all the rational equivalence relations we have exhibited in Y come from rational equivalence relations in $\mathbb{P}^{n+1}$. Hence a change of choice would modify our cycle Z by the restriction of a cycle in $\mathbb{P}^{n+1}$ (given by a sum of surfaces together with rational functions of them such that the sum of their divisors is equal to 0). So the remark follows from the easy-to-prove fact that

$$\mathrm{CH}^n(\mathbb{P}^{n+1},1) = \mathrm{CH}^{n-1}(\mathbb{P}^{n+1}) \otimes \mathbb{C}^*.$$

4. Proof of Theorems 1.3 and 1.4

Let $B = H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$, and $B_0 \subset B$ be the open set parametrizing smooth hypersurfaces. The construction of $z \in \mathrm{CH}^n(Y, 1)$ described in the previous section can be done in family, over a Zariski open set U of B_0. Let $\mathcal{C}_{B_0} \to B_0$ be the family of varieties C_Y introduced in the previous section. The intersection $\mathcal{C}_{B_0} \cap H \times B_0$ is a codimension n subvariety of $\mathcal{Y}_{B_0}$, which is étale over a Zariski open set U of B_0. By definition of the curves C_Y, there is a codimension n subvariety Δ_1 of $\mathbb{P}^{n+1} \times U$, whose fibre over σ is the union over i of the lines Δ_i osculating Y_σ at $x_i \in H \cap C_{Y_\sigma}$. Hence as codimension n cycles in $\mathcal{Y}_U$, we have

$$\Delta_1 \bigcap \mathcal{Y}_U = \mathrm{d}\mathcal{C}_{B_0} \bigcap (H \times B_0).$$

Shrinking U if necessary, we can choose now a family $n_1 : \Sigma_1 \to \mathbb{P}^{n+1} \times U$ of surfaces in $\mathbb{P}^{n+1}$ parametrized by U, together with a meromorphic function ϕ_1 such that $n_{1*} \operatorname{div} \phi_1 = \Delta_1 - N\Delta \times U$. Intersecting it with $\mathcal{Y}_U$, we get $n_1 : \mathcal{Z}_1 \to \mathcal{Y}_U$ and a meromorphic function ϕ_1 on $\mathcal{Z}_1$ such that

$$n_{1*} \operatorname{div} \phi_1 = \mathrm{d}\mathcal{C}_U \bigcap (H \times U) - Nh,$$

as a codimension n cycle in $\mathcal{Y}_U$, where $h = \Delta \times U \cap \mathcal{Y}_U$. On the other hand, shrinking U if necessary, we can obviously do in family the construction of the curves $n_0 : C_0 \to Y$, $n'_0 : C'_0 \to Y$ together with the meromorphic functions ϕ_0, ϕ'_0 on them. Hence we get a cycle

$$\mathcal{Z} = (\mathcal{Z}_1, n_1, \phi_1) - (\mathcal{Z}_0, n_0, \phi_0) - (\mathcal{Z}'_0, n'_0, \phi'_0),$$
$$n_{1*} \operatorname{div} \phi_1 - n_{0*} \operatorname{div} \phi_0 - n'_{0*} \operatorname{div} \phi'_0 = 0,$$

with class $z \in \mathrm{CH}^n(\mathcal{Y}_U, 1)$.

By remark 3.6, this element depends on the choices made only up to the image of

$$\mathrm{CH}^{n-1}(\mathbb{P}^{n+1}) \otimes \mathcal{O}^*_U \to \mathrm{CH}^n(\mathcal{Y}_U, 1).$$

Let $\alpha_z \in H^{2n-1}(\mathcal{Y}_U, \mathbb{Z})$ modulo torsion be the associated cohomology class. Our first goal is to prove the following theorem.

Theorem 4.1. *For any generically finite dominating morphism $V \xrightarrow{r} U$ with V smooth quasiprojective, inducing $r : \mathcal{Y}_V \to \mathcal{Y}_U$, the class $r^*\alpha_z \in H^{2n-1}(\mathcal{Y}_V, \mathbb{Q})$ does not belong to the image of the restriction map*

$$j^*_V : H^{2n-1}(\mathbb{P}^{n+1} \times V, \mathbb{Q}) \to H^{2n-1}(\mathcal{Y}_V, \mathbb{Q}).$$

The proof will split in two steps. Recall from Lemma 3.3 the hypersurface $B_\Delta \subset B_0$ parametrizing smooth hypersurfaces containing a line. We will consider only the (smooth and dense since $n \geqslant 2$) open set $B_{\Delta,0}$ parametrizing hypersurfaces Y_σ containing exactly one line l_σ.

Let

$$\mathcal{Y}_{\Delta,0} \to B_{\Delta,0}$$

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be the family of hypersurfaces parametrized by $B_{\Delta,0}$. $\mathcal{Y}_{\Delta,0}$ is a smooth hypersurface in $\mathcal{Y}_{B_0}$. There is the family of lines

$$\mathcal{L} = \cup_\sigma l_\sigma \subset \mathcal{Y}_{\Delta,0}$$

which is a codimension $n-1$ smooth subvariety. Let

$$[\mathcal{L}] \in H^{2n-2}(\mathcal{Y}_{\Delta,0}, \mathbb{Z})$$

be its cohomology class.

We shall show the following propositions.

Proposition 4.2. *The residue*

$$\mathrm{Res}_{\mathcal{Y}_{\Delta,0}}(\alpha_z) \in H^{2n-2}(\mathcal{Y}_{\Delta,0}, \mathbb{Z})/\mathrm{torsion}$$

is equal to $[\mathcal{L}]$.

Proposition 4.3. *For any generically finite cover* $r : V' \to B_{\Delta,0}$, *inducing* $r : \mathcal{Y}_{V'} \to \mathcal{Y}_{\Delta,0}$, *the class* $r^*[\mathcal{L}] \in H^{2n-2}(\mathcal{Y}_{V'}, \mathbb{Q})$ *does not belong to the image of the restriction map*

$$j^*_{V'} : H^{2n-2}(\mathbb{P}^{n+1} \times V', \mathbb{Q}) \to H^{2n-2}(\mathcal{Y}_{V'}, \mathbb{Q}).$$

These two propositions imply Theorem 4.1. Indeed, let $r : V \to U$ be a generically finite morphism, with V smooth quasiprojective. We can extend it to a proper morphism

$$r : \bar{V} \to B_0,$$

with $\bar{V}$ smooth quasiprojective. Since r is proper, there exists an irreducible hypersurface $V' \subset \bar{V}$ such that $r(V') = B_\Delta$, so that $r : V' \to B_\Delta$ is a generically finite dominating morphism. Let V'' be the smooth part of $r^{-1}(B_{\Delta_0})$, and let l be the ramification index of r along V''. We shall also denote by $r' : \mathcal{Y}_{V''} \to \mathcal{Y}_{B_{\Delta,0}}$ the induced morphism. Then we have

$$\mathrm{Res}_{\mathcal{Y}_{V''}} r^*\alpha_z = l r'^* \,\mathrm{Res}_{\mathcal{Y}_{B_{\Delta,0}}} \alpha_z \in H^{2n-2}(\mathcal{Y}_{V''}, \mathbb{Z})/\mathrm{torsion}. \tag{4.1}$$

By Proposition 4.2, equation (4.1) gives

$$\mathrm{Res}_{\mathcal{Y}_{V''}} r^*\alpha_z = l r'^*[\mathcal{L}] \in H^{2n-2}(\mathcal{Y}_{V''}, \mathbb{Q}). \tag{4.2}$$

By Proposition 4.3, the right-hand side does not belong to the image of the restriction map

$$j^*_{V''} : H^{2n-2}(\mathbb{P}^{n+1} \times V'', \mathbb{Q}) \to\in H^{2n-2}(\mathcal{Y}_{V''}, \mathbb{Q}).$$

This implies that $r^*\alpha_z \in H^{2n-1}(\mathcal{Y}_V, \mathbb{Q})$ does not belong to the image of the restriction map

$$j^*_V : H^{2n-1}(\mathbb{P}^{n+1} \times V, \mathbb{Q}) \to H^{2n-1}(\mathcal{Y}_V, \mathbb{Q}).$$

Indeed, if $r^*\alpha_z = j^*_V(\gamma)$, we have also

$$\mathrm{Res}_{\mathcal{Y}_{V''}} r^*\alpha_z = j^*_{V''}(\mathrm{Res}_{\mathbb{P}^{n+1}\times V''}\gamma).$$

$\square$

It remains to prove the propositions.

Proof of Proposition 4.2. We start with the following observation: let $\pi : \mathcal{Z} \to B'$ be a family of reduced curves, with smooth total space, where B' is a Zariski open set of B containing a non-empty open set of $B_{\Delta,0}$. Let ϕ be a meromorphic function on $\mathcal{Z}$. Let $\operatorname{div}\phi = R_h + R_v$ be the decomposition of $\operatorname{div}\phi$ into horizontal and vertical part, where, horizontal means dominating B' and vertical means supported over a hypersurface. Since we are interested in the generic point of $B_{\Delta,0}$ we may up to shrinking B' assume that R_v is supported over $B_{\Delta,0}$. Let $S := R_v \cap R_h \subset \mathcal{Z}_{\Delta,0}$. Let now $B'' := B' - B_{\Delta,0}$. The 1-form

$$\frac{1}{2\mathrm{i}\pi}\frac{\mathrm{d}\phi}{\phi}$$

defines a cohomology class α in $H^1(\mathcal{Z}_{B''} - R_h, \mathbb{Z})/\text{torsion}$. We can compute its residue along the smooth part of $\mathcal{Z}_{\Delta,0} - S$, which gives a class

$$\operatorname{Res}\alpha \in H^0(\mathcal{Z}_{\Delta,0} - S - \operatorname{Sing}\mathcal{Z}_{\Delta,0}, \mathbb{Z}).$$

Now this group is naturally isomorphic to the free abelian group generated by the components of $\mathcal{Z}_{\Delta,0}$ and we have by Cauchy's formula, the following fact.

Under this isomorphism, $\operatorname{Res}\alpha$ *identifies to* R_v*, which is also an element of the free group generated by the components of* $\mathcal{Z}_{\Delta,0}$.

Recall now our cycle in $\mathcal{Y}_U$

$$\left.\begin{aligned} \mathcal{Z} = (\mathcal{Z}_1, n_1, \phi_1) - (\mathcal{Z}_0, n_0, \phi_0) - (\mathcal{Z}'_0, n'_0, \phi'_0), \\ n_{1*}\operatorname{div}\phi_1 - n_{0*}\operatorname{div}\phi_0 - n'_{0*}\operatorname{div}\phi'_0 = 0. \end{aligned}\right\} \tag{4.3}$$

Coming back to its definition, it is immediate to see that we can extend over B_0 along a Zariski open set of $B_{\Delta,0}$ the families of curves $\mathcal{Z}_0$, $\mathcal{Z}'_0$, $\mathcal{Z}_1$, together with the proper morphisms n_0, n'_0, n_1 and the meromorphic functions ϕ_0, ϕ'_0, ϕ_1, loosing however the equality (4.3). Let $K \subset \mathcal{Y}_U$ be the union of the algebraic sets $n_i(\operatorname{div}\phi_i)$. Then the restriction to $\mathcal{Y}_U - K$ of the class α_z is the sum of the classes $n_{i*}\alpha_i$, where

$$n_{i*} : H^1(\mathcal{Z}_i - (n_i)^{-1}(K), \mathbb{Q}) \to H^{2n-1}(\mathcal{Y}_U, \mathbb{Q})$$

is the Gysin morphism, and α_i is the restriction to $\mathcal{Z}_i - (n_i)^{-1}(K)$ of the class of the 1-form

$$\frac{1}{2\mathrm{i}\pi}\frac{\mathrm{d}\phi_i}{\phi_i}.$$

(Here $i = 0, 0', 1$.)

It follows from this that the restriction to the open set

$$\mathcal{Y}_{\Delta,0} - \bar{K}\bigcap \mathcal{Y}_{\Delta,0} - \cup_i n_i(\operatorname{Sing}\mathcal{Z}_{i\Delta,0})$$

of the class $\operatorname{Res}\alpha_z$ is equal to

$$\sum_i m_{i*}\operatorname{Res}_{\mathcal{Z}_{i\Delta,0}}\alpha_i,$$

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where m_i is the restriction of n_i to $\mathcal{Z}_{i\Delta,0} - n_i^{-1}(K \cup_i n_i(\mathrm{Sing}\,\mathcal{Z}_{i\Delta,0}))$. Applying the fact explained above to each α_i, we conclude that the restriction to

$$\mathcal{Y}_{\Delta,0} - \bar{K} \bigcap \mathcal{Y}_{\Delta,0} - \cup_i n_i(\mathrm{Sing}\,\mathcal{Z}_{i\Delta,0})$$

of $\mathrm{Res}_{\mathcal{Y}_{\Delta,0}} \alpha_z$ is equal to the restriction to the same open set of the cohomology class of the algebraic cycle

$$\sum_i n_{i*}((\mathrm{div}\,\phi_i)_{\mathrm{vert}}),$$

where again the subscript 'vert' means the part supported over $\mathcal{Z}_{i,\Delta,0}$. Since by dimension reasons the restriction map

$$H^{2n-2}(\mathcal{Y}_{\Delta,0}, \mathbb{Q}) \to H^{2n-2}\Big(\mathcal{Y}_{\Delta,0} - \bar{K} \bigcap \mathcal{Y}_{\Delta,0} - \cup_i n_i(\mathrm{Sing}\,\mathcal{Z}_{i\Delta,0}), \mathbb{Q}\Big)$$

is injective, it suffices now, to conclude the proof of Proposition 4.2, to prove that

$$\sum_i n_{i*}((\mathrm{div}\,\phi_i)_{\mathrm{vert}}) = \mathcal{L}.$$

But it suffices for that to recall the constructions of the $(\mathcal{Z}_i, \phi_i)$. It is immediate to see that $\mathcal{Z}_0$ and $\mathcal{Z}_0'$ do not have a vertical component in the divisor of the function ϕ_0, ϕ_0', since this divisor is finite over any generic point of $B_{\Delta,0}$.

However, there is a vertical part of the divisor of ϕ_1 which is exactly equal to $\mathcal{L}$. Indeed recall that $\mathcal{Z}_1$ is the intersection with $\mathcal{Y}_{B'}$ of a surface in $\mathbb{P}^{n+1} \times B'$ admitting a meromorphic function ϕ_1 such that

$$\mathrm{div}\,\phi_1 = \Delta_1 - N\Delta \times B'. \tag{4.4}$$

Here the fibre at each point $F \in U$ of Δ_1 is the union over i of the osculating lines meeting Y_F at the intersection $C_{Y_F} \cap H$. Now, over $B_{\Delta,0}$, the universal line $\mathcal{L}$ becomes one reduced component of the family of curves $\mathcal{C}_{B_{\Delta,0}}$ and the intersection $\mathcal{L} \cap H \times B_{\Delta,0}$ is a section $\sigma \mapsto x_\sigma$ of $\mathcal{L} \to B_{\Delta,0}$. For each point $\sigma \in B_{\Delta,0}$, the osculating line Δ_{x_σ} is equal to l_σ hence is contained in Y_σ. It follows that we have the inclusion $\mathcal{L} \subset \Delta_1 \cap \mathcal{Y}_{B'}$, which identifies $\mathcal{L}$ to an irreducible component of $\Delta_1 \cap \mathcal{Y}_{\Delta,0}$. On the other hand, for any $\sigma \in B_{\Delta,0}$ the other osculating lines Δ_i at the points $x_i \neq x_\sigma$ meet Y_σ exactly along $\mathrm{d}x_i$. Hence, intersecting equation (4.4) with $\mathcal{Y}_{\Delta,0}$, we find that $\mathrm{div}\,\phi_1$ on $\mathcal{Z}_1$ has for unique vertical component $\mathcal{L}$. □

Proof of Proposition 4.3. Notice that it suffices to prove the result when the morphism $r : V \to B_{\Delta,0}$ is the inclusion of a Zariski open set. Indeed, given any generically finite morphism $r : V \to B_{\Delta,0}$ with V smooth, there exist Zariski open sets $B_{\Delta,0,0} \subset B_{\Delta,0}$, $V_0 \subset V$ such that the restriction r_0 of r to V_0 is a proper finite morphism of degree m from V_0 to $B_{\Delta,0,0}$. We denote also by $r_0 : \mathcal{Y}_{V_0} \to \mathcal{Y}_{B_{\Delta,0,0}}$ the induced morphism. Now suppose there exists $\gamma \in H^{2n-2}(\mathbb{P}^{n+1} \times V, \mathbb{Q})$ such that

$$j_V^* \gamma = r^*[\mathcal{L}] \in H^{2n-2}(\mathcal{Y}_V, \mathbb{Q}).$$

Then we have as well, with $\gamma' = \gamma_{|\mathbb{P}^{n+1}\times V_0}$

$$j_{V_0}^*\gamma' = r_0^*[\mathcal{L}_0] \in H^{2n-2}(\mathcal{Y}_{V_0}, \mathbb{Q}),$$

where $\mathcal{L}_0 \subset \mathcal{Y}_{B_{\Delta,0,0}}$ is the universal line over $B_{\Delta,0,0}$. But we can apply r_{0*} to this equality: if $\gamma_0 := r_{0*}\gamma' \in H^{2n-2}(\mathbb{P}^{n+1} \times B_{\Delta,0,0}, \mathbb{Q})$, this gives

$$j_{B_{\Delta,0,0}}^*\gamma_0 = m[\mathcal{L}_0] \in H^{2n-2}(\mathcal{Y}_{B_{\Delta,0,0}}, \mathbb{Q}).$$

Hence it suffices to prove the result for the open set $B_{\Delta,0,0} \subset B_{\Delta,0}$.

Now let $V \subset B_{\Delta,0}$ be Zariski open and assume there exists $\gamma \in H^{2n-2}(\mathbb{P}^{n+1} \times V, \mathbb{Q})$ with

$$j_V^*\gamma = [\mathcal{L}_V] \in H^{2n-2}(\mathcal{Y}_V, \mathbb{Q}). \tag{4.5}$$

We fix a hyperplane

$$\mathbb{P}^n \subset \mathbb{P}^{n+1}$$

and we choose $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d, and a point $O \in \mathbb{P}^{n+1} - \mathbb{P}^n$. For any $x \in X$, the set E_x of equations $\sigma \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$ vanishing on X and on the line $\Delta_x = \langle O, x\rangle$ is a vector space of fixed dimension, so that we have a vector bundle $\mathcal{E}$ on X with fibre E_x over x. Any meromorphic section ν of $\mathcal{E}$ provides a rational map

$$\phi : X \dashrightarrow B_\Delta$$

which to x associates the equation $\nu(x)$, which vanishes on the line Δ_x and on X.

Choosing ν generically, it is clear that we can arrange that $\operatorname{Im}\phi$ meets V. Hence there is a non-empty Zariski open set X_0 of X on which ϕ is defined and takes value in V.

Consider the family $\mathcal{Y}_{X_0}$ obtained as the Cartesian product

$$\begin{array}{ccc} \mathcal{Y}_{X_0} & \longrightarrow & \mathcal{Y}_V \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{\phi} & V \end{array}$$

There is a natural inclusion

$$i : X \times X_0 \subset \mathcal{Y}_{X_0},$$

since each equation $\phi(x)$ vanishes on X. In fact, since the hypersurfaces defined by the $\phi(x)$ are smooth, they cannot contain $\mathbb{P}^n$ hence they cut exactly X on $\mathbb{P}^n$. Hence we have in fact

$$X \times X_0 = (\mathbb{P}^n \times X_0) \bigcap \mathcal{Y}_{X_0}, \tag{4.6}$$

where the intersection is inside $\mathbb{P}^{n+1} \times X_0$.

We take now the pullback under ϕ of equation (4.5): denoting

$$\gamma_{X_0} := (\mathrm{Id} \times \phi)^*\gamma \in H^{2n-2}(\mathbb{P}^{n+1} \times X_0, \mathbb{Q}),$$

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we find that

$$j_{X_0}^*\gamma_{X_0} = [\mathcal{L}_{X_0}] \in H^{2n-2}(\mathcal{Y}_{X_0}, \mathbb{Q}).$$

We restrict now this equality to $X \times X_0 \subset \mathcal{Y}_{X_0}$. Using the equality (4.6), and denoting by $\gamma'_{X_0} \in H^{2n-2}(\mathbb{P}^n \times X_0, \mathbb{Q})$ the restriction of γ_{X_0} to $\mathbb{P}^n \times X_0$, we find

$$\operatorname{rest} \gamma'_{X_0} = \Big[\mathcal{L}_{X_0} \bigcap (X \times X_0)\Big] \in H^{2n-2}(X \times X_0, \mathbb{Q}), \tag{4.7}$$

where the restriction on the left refers to the inclusion $X \times X_0 \subset \mathbb{P}^n \times X_0$ and the (cycle theoretic) intersection on the right refers to the inclusion $X \times X_0 \subset \mathcal{Y}_{X_0}$.

Now we note that for each $x \in X_0$, the line Δ_x meets $X \subset Y_x$ exactly at x and transversally, and it follows that we have the scheme theoretic and cycle theoretic equality

$$\mathcal{L}_{X_0} \bigcap (X \times X_0) = \Delta_X \bigcap (X \times X_0),$$

where $\Delta_X \subset X \times X$ is the diagonal of X. Hence we conclude that

$$\operatorname{rest} \gamma'_{X_0} = [\Delta_X]_{|X \times X_0} \in H^{2n-2}(X \times X_0). \tag{4.8}$$

We use now the fact that $n \geqslant 2$. The variety X is then of dimension $n-1 \geqslant 1$ and since $d = 2n$, it has non-zero holomorphic forms $\beta \in H^0(\Omega_X^{n-1}) \subset H^{n-1}(X, \mathbb{C})$. If $j : X \hookrightarrow \mathbb{P}^n$ is the inclusion, we deduce from equation (4.8) the equality

$$\mathrm{pr}_{2*}(\mathrm{pr}_1{}^*\beta \cup [\Delta_X]_{|X \times X_0}) = \mathrm{pr}_{2*}(\mathrm{pr}_1{}^* j_*\beta \cup \gamma'_{X_0}) \in H^{n-1}(X_0, \mathbb{C}),$$

where in the left-hand side the pr_i are the projections of $X \times X_0$ on its factors, while on the left-hand side the pr_i are the projections of $\mathbb{P}^n \times X_0$ on its factors.

Since $j_*\beta = 0$ in $H^{n+1}(\mathbb{P}^n, \mathbb{C})$, the right-hand side vanishes. On the other hand, the left-hand side is equal to $\beta_{|X_0}$. Hence we conclude that $\beta_{|X_0} = 0$ in $H^{n-1}(X_0, \mathbb{C})$. But the class of a non-zero holomorphic form of degree greater than 0 does not vanish on any non-empty Zariski open set, which is a contradiction. □

We conclude with the following theorem.

Theorem 4.4. *For general $t \in U$, the induced cycle $z_t \in \mathrm{CH}^n(Y_t, 1)$ projects to a non-torsion element in $\mathrm{CH}^n(Y_t, 1)_{\mathrm{ind}}$.*

Proof. We shall show the following proposition.

Proposition 4.5. *Let $\mathcal{X} \to U$ be a smooth family of complex projective varieties and let $\mathcal{Z}$ be a cycle representing a element $z \in \mathrm{CH}^n(\mathcal{X}, 1)$. Then if for a general $t \in U$ a multiple $m_t z_t$, $m_t \in \mathbb{Z}^*$ of the induced cycle $z_t \in \mathrm{CH}^n(X_t, 1)$ is decomposable, there exists a generically finite dominating morphism $r : T \to U$, inducing $r : \mathcal{X}_T \to \mathcal{X}_U$, and an integer $m \neq 0$ such that mr^*z belongs to the image of the map (2.1)*

$$\mathrm{CH}^{n-1}(\mathcal{X}_T) \otimes H^0(T, \mathcal{O}_T^*) \to \mathrm{CH}^n(\mathcal{X}_T, 1).$$

Assuming the proposition, we conclude the proof of the theorem as follows. Suppose a multiple of our cycle z_t is decomposable for general $t \in U$. Applying our proposition, we find a generically finite map

$$r : T \to U,$$

which we may assume to be étale, with T smooth quasiprojective, and a non-zero integer m such that

$$mr^* z \in \mathrm{CH}^n(\mathcal{Y}_T, 1)$$

is decomposable, that is lies in the image of the map

$$\mathrm{CH}^{n-1}(\mathcal{X}_T) \otimes H^0(T, \mathcal{O}_T^*) \to \mathrm{CH}^n(\mathcal{X}_T, 1).$$

This means that $r^* z$ can be represented by a cycle of the form

$$\sum (\mathcal{W}_i, n_i, \phi_i)$$

with $n_i : \mathcal{W}_i \to \mathcal{Y}_T$ proper, $\mathcal{W}_i$ normal and ϕ_i an invertible function on $\mathcal{W}_i$ which comes from T, i.e. $\phi_i = \pi_i^* \psi_i$, $\pi_i := \pi \circ n_i$.

Recall now the definition of the cohomology class $mr^* \alpha_z = \alpha_{mr^* z}$. Given a representative $\sum (\mathcal{W}_i, n_i, \phi_i)$ of $mr^* z \in \mathrm{CH}^n(\mathcal{Y}_T, 1)$, $mr^* \alpha_z$ will be the cohomology class of the current

$$\eta \mapsto \sum_i \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{W}_i} n_i^* \eta \wedge \frac{\mathrm{d}\phi_i}{\phi_i}.$$

But clearly if $\phi_i = \pi_i^* \psi_i$, $\psi_i \in H^0(\mathcal{O}_T^*)$, this class is equal to

$$\sum_i [n_{i*} \mathcal{W}_i] \cup \pi^* \frac{1}{2\mathrm{i}\pi} \left[\frac{\mathrm{d}\psi_i}{\psi_i} \right],$$

where $[n_{i*} \mathcal{W}_i] \in H^{2n-2}(\mathcal{Y}_T, \mathbb{Z})$ is the cohomology class of the closed analytic space $n_i(\mathcal{W}_i)$ (counted with multiplicity) and

$$\frac{1}{2\mathrm{i}\pi} \left[\frac{\mathrm{d}\psi_i}{\psi_i} \right] \in H^1(T, \mathbb{Z}) \bmod \mathrm{torsion}$$

is the cohomology class of the closed form

$$\frac{1}{2\mathrm{i}\pi} \frac{\mathrm{d}\psi_i}{\psi_i}.$$

Hence, under our assumptions, the class $r^* \alpha_z$ belongs to the image of the natural map

$$H^{2n-2}(\mathcal{Y}_T, \mathbb{Q}) \otimes H^1(T, \mathbb{Q}) \to H^{2n-1}(\mathcal{Y}_T, \mathbb{Q}). \tag{4.9}$$

On the other hand, the proof sketched in the introduction of the explicit connectivity theorem for hypersurfaces also gives the following theorem.

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Theorem 4.6. *For $d \geqslant 2n$, and for any submersive morphism*

$$T \to B = H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)),$$

the restriction map

$$H^{2n-2}(\mathbb{P}^{n+1} \times T, \mathbb{Q}) \to H^{2n-2}(\mathcal{Y}_T, \mathbb{Q})$$

is an isomorphism.

Now we have the following commutative diagram

$$\begin{array}{ccc} H^{2n-2}(\mathbb{P}^{n+1} \times T, \mathbb{Q}) \otimes H^1(T, \mathbb{Q}) & \longrightarrow & H^{2n-1}(\mathbb{P}^{n+1} \times T, \mathbb{Q}) \\ \downarrow & & \downarrow \text{rest} \\ H^{2n-2}(\mathcal{Y}_T, \mathbb{Q}) \otimes H^1(T, \mathbb{Q}) & \xrightarrow{(4.9)} & H^{2n-1}(\mathcal{Y}_T, \mathbb{Q}) \end{array}$$

where the first vertical map is surjective. Hence if $r^*\alpha_z$ belongs to the image of the map (4.9), it also belongs to the image of the restriction map rest. But this contradicts Theorem 4.1. □

Proof of Proposition 4.5. There are countably many quasiprojective varieties $T_i \xrightarrow{r_i} T$ parametrizing decomposable cycles in the fibres X_y of our family, since such a cycle is given by a combination of subvarieties together with a constant function on them. Each T_i carries a universal cycle $\mathcal{T}_i$ with class $t_i \in \mathrm{CH}^n(\mathcal{X}_{T_i}, 1)$. By construction t_i belongs to the image of the map

$$\mathrm{CH}^{n-1}(\mathcal{X}_{T_i}) \otimes \mathcal{O}^*_{T_i} \to \mathrm{CH}^n(\mathcal{X}_{T_i}, 1).$$

For each i, and for each integer m, we consider now the subset $T_{i,m} \subset T_i$,

$$T_{i,m} = \{t \in T_i,\ t_{i,t} = mz_{\pi_i(t)} \text{ in } \mathrm{CH}^n(X_{\pi_i(t)}, 1)\}.$$

Since the cycles which are trivial in the group $\mathrm{CH}^n(X_t, 1)$ are described by the tame symbols, which themselves are parametrized by countably many quasiprojective varieties, it is clear that the $T_{i,m}$ are countable unions of locally closed algebraic subsets $T_{i,m,j}$ of T_i. Furthermore, for each i, m, j, the cycles $t_{i|\mathcal{X}_{T_{i,m,j}}}$ and $\pi_i^* mz_{|\mathcal{X}_{T_{i,m,j}}}$ coincide in $\mathrm{CH}^n(\mathcal{X}_{T_{i,m,j}}, 1)$.

The subset $\cup_{i,m,j}\pi_i(T_{i,m,j})$ is by construction equal to the subset of T consisting of those points t for which a multiple mz_t is decomposable. On the other hand, it is a countable union of algebraic subsets of T. Since our field is uncountable, we conclude that either for general t, no multiple of z_t is decomposable, or at least one of the maps

$$\pi_i : T_{i,m,j} \to T$$

is dominating. Replacing $T_{i,m,j}$ by its smooth part, and eventually by some algebraic subset $\tilde{T}$ of it where π_i remains dominating but becomes generically finite, and denoting

$$\tilde{t} = t_{i|\mathcal{X}_{T_{i,m,j}}} \in \mathrm{CH}^n(\mathcal{X}_{\tilde{T}}, 1),$$

we have then found a generically finite dominating map $\pi_i : \tilde{T} \to T$, such that

$$\pi_i^* mz = \tilde{t} \in \mathrm{CH}^n(\mathcal{X}_{\tilde{T}}, 1).$$

On the other hand, we know that

$$\tilde{t} \in \operatorname{Im} \mathrm{CH}^{n-1}(\mathcal{X}_{\tilde{T}}, 1) \otimes \mathcal{O}^*_{\tilde{T}} \to \mathrm{CH}^n(\mathcal{X}_{\tilde{T}}, 1),$$

since this is the case for the cycle t_i on $\mathcal{X}_{T_i}$. □

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