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Random β-expansions

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Abstract. In this paper, random expansions to non-integer bases $\beta > 1$ are studied. For β's satisfying $\beta^2 = n\beta + k$ (with $1 \le k \le n$) and $\beta^n = \beta^{n-1} + \cdots + \beta + 1$ the ergodic properties of such expansions are described.

1. *Introduction*

1.1. *Expansions to base $\beta > 1$.* As is well known, any $x \in [0, 1)$ can be developed in a series expansion to any base $\beta > 1$:

$$x = \sum_{k=1}^{\infty} \frac{a_k}{\beta^k} = .a_1 a_2 \cdots a_n \cdots , \tag{1.1}$$

where $a_k \in \{0, 1, \ldots, \beta - 1\}$ if $\beta \in \mathbb{N}$, and $a_k \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ otherwise. The two cases $\beta > 1$ integer or non-integer behave very differently. In the first case almost every $x \in [0, 1)$ has a unique series expansion; only rationals p/q of the form m/β^n for some $n \ge 1$ and $m = 0, 1, \ldots, \beta^n - 1$ have two different expansions of the form (1.1), one of them being finite while the other expansion ends in an infinite string of $(\beta - 1)$'s. In the latter case almost every $x \in [0, 1)$ has infinitely many expansions of the form (1.1); see [**EJK, JS, DK**].

One possible way to obtain an expansion of the form (1.1) is to use the so called *greedy map* $T_\beta : [0, 1) \to [0, 1)$, defined by

$$T_\beta(x) = \beta x \ (\text{mod } 1),$$

and the digits $a_k = a_k(x),\ k \ge 1$, are given by

$$a_k = \lfloor \beta T_r^{k-1}(x) \rfloor, \quad k \ge 1,$$

where $\lfloor\xi\rfloor$ denotes the largest integer not exceeding ξ. Clearly T_β is related to the Bernoulli-shift on β symbols in the case $\beta \in \mathbb{N}$, and the Lebesgue measure λ is T_β-invariant. In the case $\beta \notin \mathbb{N}$, it was Rényi in [**R1**] who showed that

$$([0,1), \mu_\beta, T_\beta)$$

forms an ergodic system, where μ_β is a T_β-invariant probability measure equivalent to λ with density h_β, with

$$1 - \frac{1}{\beta} \le h_\beta(x) \le \frac{1}{1 - 1/\beta}.$$

Independently, Gel'fond [**G**] (in 1959) and Parry [**P1**] (in 1960) showed that

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0,T^n(1))}(x),$$

where

$$F(\beta) = \int_0^1 \left(\sum_{x<T^n(1)} \frac{1}{\beta^n} \right) dx$$

is a normalizing constant. After Parry, the ergodic properties of T_β were studied by several authors. For example, it was shown by Hofbauer [**Ho**] that μ_β is the measure of maximal entropy, and Smorodinsky [**Sm**] showed that for each non-integer $\beta > 1$ the system $([0,1), \mu_\beta, T_\beta)$ is weak-Bernoulli; see also [**DKS**]. A deep result by Friedman and Ornstein [**FO**] then yields that the natural extension of $([0,1), \mu_\beta, T_\beta)$ is a Bernoulli automorphism.

The β-expansion of $x \in [0,1)$ is also known as the *greedy expansion* of x. Since one can in fact perform the greedy algorithm for any $x \in [0, \lfloor\beta\rfloor/(\beta-1))$ (see also [**DK**]), we will extend the definition of T_β to all points in $[0, \lfloor\beta\rfloor/(\beta-1))$ by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \le x < 1, \\ \beta x - \lfloor\beta\rfloor, & 1 \le x < \lfloor\beta\rfloor/(\beta-1); \end{cases}$$

see also Figure 1. Notice that for each $x \in [0, \lfloor\beta\rfloor/(\beta-1))$ there exists a unique integer $n_0 = n_0(x)$ such that for all $n \ge n_0$ one has that $T_\beta^n(x) \in [0,1)$. In view of this we extend h_β on $[0, \lfloor\beta\rfloor/(\beta-1))$ by setting $h_\beta = 0$ on $[1, \lfloor\beta\rfloor/(\beta-1))$. The interval $[0,1)$ is the attractor for the extended map T_β. Due to this, the system

$$\left(\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right), \mu_\beta, T_\beta \right),$$

is weak-Bernoulli, since the original system on $[0,1)$ is.

An expansion which is lexicographically the 'smallest' expansion of x is the so called *lazy expansion* of x. Underlying this lazy expansion is the *lazy map* $S_\beta : (0, \lfloor\beta\rfloor/(\beta-1)] \to (0, \lfloor\beta\rfloor/(\beta-1)]$, defined by

$$S_\beta(x) = \beta x - d_1 \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left(0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right]$$

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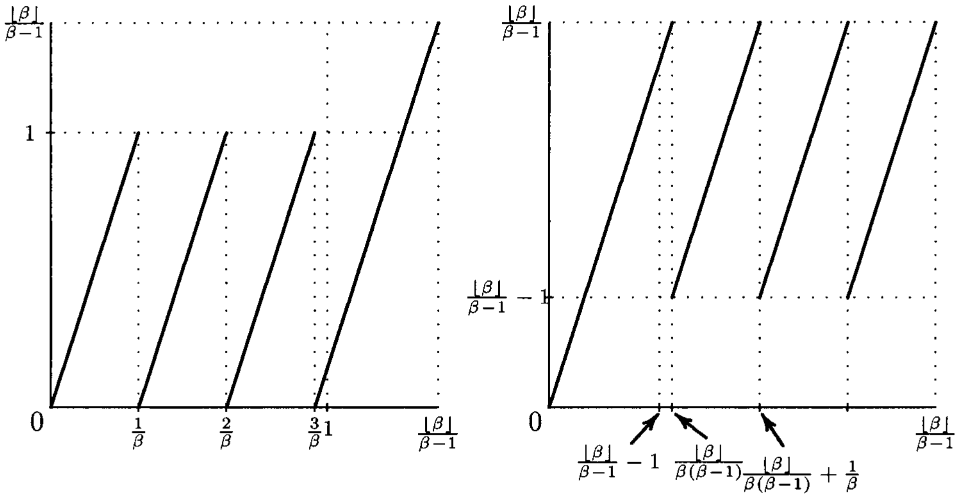

FIGURE 1. The greedy map T_β (left), and lazy map S_β (right). Here $\beta = \pi$.

and

$$\begin{aligned}\Delta(d) &= \left(\frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d + 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d}{\beta}\right] \\ &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, 2, \ldots, \lfloor\beta\rfloor\};\end{aligned}$$

see also [**DK**] for more details.

The greedy map T_β and the lazy map S_β are strongly related. If one defines the map $\psi : [0, \lfloor\beta\rfloor/(\beta-1)) \to (0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$\psi(x) = \frac{\lfloor\beta\rfloor}{\beta-1} - x,$$

then ψ is a measurable bijection, $\psi T_\beta = S_\beta \psi$ and

$$\left(\frac{\lfloor\beta\rfloor}{\beta-1} - 1, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$$

is the attractor for S_β; see Figure 1. Due to this, the system

$$\left(\left(0, \frac{\lfloor\beta\rfloor}{\beta-1}\right], \rho_\beta, S_\beta\right)$$

is weak-Bernoulli, where ρ_β is a probability measure on $[0, \lfloor\beta\rfloor/(\beta-1)]$, given by

$$\rho_\beta(A) = \mu_\beta(\psi^{-1}(A)),$$

for any Lebesgue set $A \subset [0, \lfloor\beta\rfloor/(\beta-1)]$.

1.2. *Random β-expansions.* Let $\beta > 1$ be a non-integer. If we superimpose the greedy map and the corresponding lazy map on $[0, \lfloor\beta\rfloor/(\beta-1)]$, we get $\lfloor\beta\rfloor$ overlapping regions

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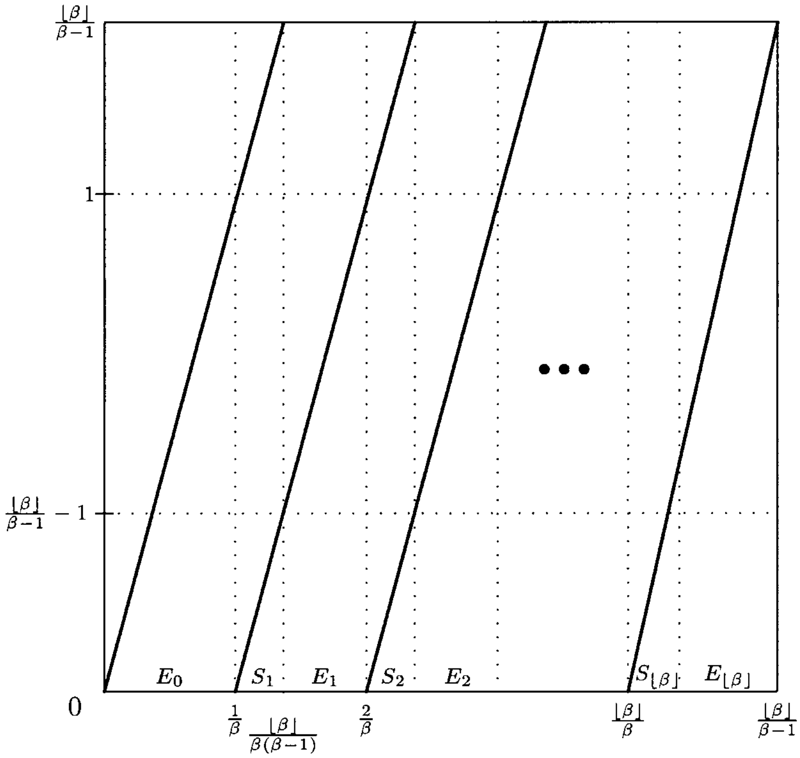

FIGURE 2. The greedy and lazy maps, and their switch regions.

of the form

$$S_\ell = \left[\frac{\ell}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\ell-1}{\beta}\right], \quad \ell = 1, \ldots, \lfloor\beta\rfloor,$$

which we will refer to as *switch regions*; see Figure 2. On S_ℓ, the greedy map assigns the digit ℓ, while the lazy map assigns the digit $\ell - 1$. Outside these switch regions both maps are identical, and hence they assign the same digits. We will now define a new random expansion in base β by randomizing the choice of the map used in the switch regions. For each switch region we assign a coin, and whenever x belongs to the ith switch region we flip the ith coin to decide which map will be applied to x, and hence which digit will be assigned.

To be more precise, we partition the interval $[0, \lfloor\beta\rfloor/(\beta-1)]$ into switch regions S_ℓ and *equality regions* E_ℓ, where

$$E_\ell = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\ell-1}{\beta}, \frac{\ell+1}{\beta}\right), \quad \ell = 1, \ldots, \lfloor\beta\rfloor - 1,$$

$$E_0 = \left[0, \frac{1}{\beta}\right) \quad \text{and} \quad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right];$$

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see Figure 2. Let

$$S = \bigcup_{\ell=1}^{\lfloor\beta\rfloor} S_\ell \quad \text{and} \quad E = \bigcup_{\ell=0}^{\lfloor\beta\rfloor} E_\ell,$$

and consider $\Omega = \{0, 1\}^{\mathbb{N}}$ with the product σ-algebra. Let $\sigma : \Omega \to \Omega$ be the left shift, i.e. if $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, then $\sigma(\omega) = (\omega_2, \omega_3, \dots)$. Define $K : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$K(\omega, x) = \begin{cases} (\omega, \beta x - \ell), & x \in E_\ell,\ \ell = 0, 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - \ell), & x \in S_\ell \text{ and } \omega_1 = 1,\ \ell = 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - \ell + 1), & x \in S_\ell \text{ and } \omega_1 = 0,\ \ell = 1, \dots, \lfloor\beta\rfloor. \end{cases} \tag{1.2}$$

The elements of Ω represent the coin tosses ('heads' = 1 and 'tails' = 0) used every time the orbit hits a switch region. Let

$$d_1 = d_1(\omega, x) = \begin{cases} \ell, & \text{if } x \in E_\ell,\ \ell = 0, 1, \dots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_\ell,\ \ell = 1, 2, \dots, \lfloor\beta\rfloor, \\ \ell - 1, & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_\ell,\ \ell = 1, 2, \dots, \lfloor\beta\rfloor, \end{cases}$$

then

$$K(\omega, x) = \begin{cases} (\omega, \beta x - d_1), & \text{if } x \in E, \\ (\sigma(\omega), \beta x - d_1), & \text{if } x \in S. \end{cases}$$

Set $d_n = d_n(\omega, x) = d_1(K^{n-1}(\omega, x))$, and let $\pi : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$ be the canonical projection onto the second coordinate. Then

$$\pi(K^n(\omega, x)) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting gives

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi(K^n(\omega, x))}{\beta^n}.$$

Since $\pi(K^n(\omega, x)) \in [0, \lfloor\beta\rfloor/(\beta-1)]$, it follows that

$$\left| x - \sum_{i=1}^{n} \frac{d_i}{\beta^i} \right| = \frac{\pi(K^n(\omega, x))}{\beta^n} \to 0 \quad \text{as } n \to \infty.$$

This shows that for all $\omega \in \Omega$ and for all $x \in [0, \lfloor\beta\rfloor/(\beta-1)]$ one has that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

2. *Ergodic properties for certain Pisot bases*

In this section we study the dynamical properties of the map K for certain Pisot values of β, namely, $\beta > 1$ satisfying $\beta^2 - n\beta - k = 0$, $1 \le k \le n$, $n \ge 1$, and $\beta^n - \beta^{n-1} - \beta^{n-2} - \cdots - \beta - 1 = 0$, $n \ge 2$. In both cases we will show that the dynamics of K is isomorphic to a mixing Markov chain.

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2.1. *The case $\beta^2 - n\beta - k = 0$, $1 \le k \le n$, $n \ge 1$.* The behavior of K in the case $k = 1$ is slightly different than the behavior for all other values of k with $2 \le k \le n$. We will analyse the case $k \ge 2$, indicating the adjustments needed in case $k = 1$. In this case it is easy to see that $\lfloor\beta\rfloor = n$, and if we denote by $\bar{I}$ the closure of an interval I, then the image of any switch region S_i under the greedy map is the interval

$$E_0^{(1)} = \left[0, \frac{n}{\beta(\beta-1)} - \frac{k}{\beta^2}\right],$$

which is a subset of E_0 (note that for $k = 1$ one has that $E_0^{(1)} = \bar{E}_0$). Since

$$T_\beta\left(\frac{n}{\beta(\beta-1)} - \frac{k}{\beta^2}\right) = \frac{n}{\beta(\beta-1)} + \frac{n-k}{\beta},$$

which is the right endpoint of the switch region S_{n-k+1}, we find for $i = 1, \dots, n$, that

$$T_\beta^2(S_i) = T_\beta(E_0^{(1)}) = E_0 \cup S_1 \cup \dots \cup E_{n-k} \cup S_{n-k+1}.$$

Similarly, the image of any switch region S_i under the lazy map S_β is the interval

$$E_n^{(1)} = \left[1, \frac{n}{\beta-1}\right],$$

which is a subset of E_n (note that for $k = 1$ one has that $E_n^{(1)} = \bar{E}_n$), and we find for $i = 1, \dots, n$, that

$$S_\beta^2(S_i) = S_\beta(E_n^{(1)}) = S_k \cup E_k \cup \dots \cup S_n \cup E_n.$$

This suggests that in order to find the Markov chain underlying the map K, we need to subdivide both E_0 and E_n into two intervals $E_0^{(1)}$ and $E_0^{(2)}$, and $E_n^{(1)}$ and $E_n^{(2)}$, respectively, where

$$E_0^{(2)} = E_0 \setminus E_0^{(1)} \quad \text{and} \quad E_n^{(2)} = E_n \setminus E_n^{(1)},$$

(note that in the case $k = 1$ both $E_0^{(2)}$ and $E_n^{(2)}$ are empty sets). Notice that the partition

$$\{E_0^{(1)}, E_0^{(2)}, S_1, E_1, \dots, S_{n-1}, E_{n-1}, E_n^{(2)}, E_n^{(1)}\}$$

satisfies the following property: the interior of any of the partition elements is mapped to the interior of a union of partition elements; see Figure 3.

We now assume that on each switch region S_ℓ we flip a coin with P(heads) $= p_\ell$, $\ell = 1, \dots, n$. To incorporate the randomness of the expansions which takes places in the switch regions $S_1, S_2, \dots, S_n$ we consider the Markov chain on $2n + 3$ states

$$e_0^{(1)}, e_0^{(2)}, s_1, e_1, s_2, \dots, e_{n-1}, s_n, e_n^{(2)}, e_n^{(1)}.$$

(in the case $k = 1$ we only need $2n+1$ states $e_0, s_1, e_1, s_2, \dots, e_{n-1}, s_n, e_n$). e_i corresponds to the case when the orbit under K is in region E_i, $i = 2, \dots, n-1$, while $e_i^{(j)}$ corresponds to the case when the orbit is in $E_i^{(j)}$, $i = 0, n$, and $j = 1, 2$, and finally s_i corresponds to the case when the orbit is in S_i. We will show that for $k \ge 2$ the map K

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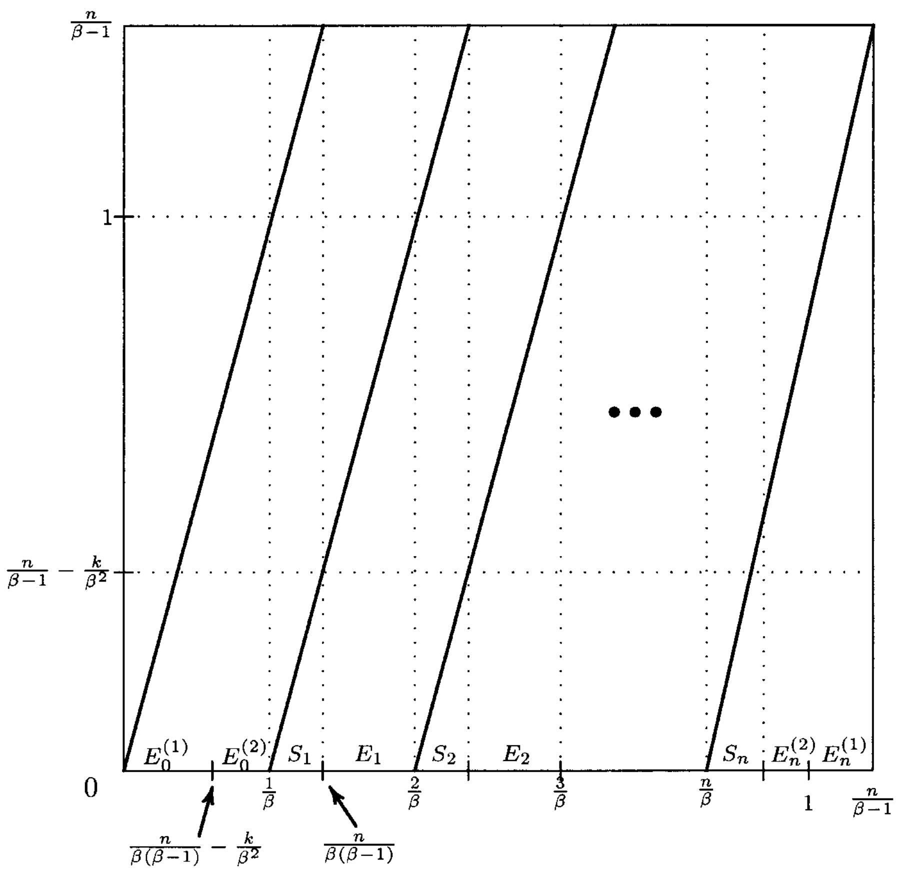

FIGURE 3. Refinement of the regions E_0 and E_n.

given by (1.2) is isomorphic modulo sets of measure zero to the Markov chain with states $e_0^{(1)}, e_0^{(2)}, s_1, e_1, s_2, \ldots, e_{n-1}, s_n, e_n^{(2)}, e_n^{(1)}$, and transition probabilities given by

$$
\begin{aligned}
p_{e_0^{(1)}e_0^{(1)}} &= \frac{1}{\beta} = p_{e_n^{(1)}e_n^{(1)}}, \\
p_{e_0^{(1)}e_0^{(2)}} &= \frac{k-1}{\beta(\beta-k)} = p_{e_n^{(1)}e_n^{(2)}}, \\
p_{e_0^{(1)}s_i} &= \frac{1}{\beta^2} = p_{e_n^{(1)}s_{n-i+1}}, && i = 1, 2, \ldots, n-k+1, \\
p_{e_0^{(1)}e_i} &= \frac{2\beta-2-n}{\beta(\beta-k)} = p_{e_n^{(1)}e_{n-i}}, && i = 1, 2, \ldots, n-k, \\
p_{e_0^{(2)}e_i} &= \frac{2\beta-2-n}{\beta(k-1)} = p_{e_n^{(2)}e_{n-i}}, && i = n-k+1, \ldots, n-1, \\
p_{e_0^{(2)}s_i} &= \frac{\beta-k}{\beta(k-1)} = p_{e_n^{(2)}s_{n-i+1}}, && i = n-k+2, \ldots, n,
\end{aligned}
$$

$$
\begin{aligned}
p_{e_0^{(2)} e_n^{(2)}} &= \frac{1}{\beta} = p_{e_n^{(2)} e_0^{(2)}}, & & \\
p_{s_i e_0^{(1)}} &= p_i = 1 - p_{s_i e_n^{(1)}}, & & i = 1, \ldots, n, \\
p_{e_i s_j} &= \frac{\beta - k}{\beta^2(2\beta - 2 - n)}, & & i = 1, \ldots, n-1, \ j = 1, \ldots, n, \\
p_{e_i e_j} &= \frac{1}{\beta}, & & i, j = 1, \ldots, n-1, \\
p_{e_i e_0^{(2)}} &= \frac{k-1}{\beta(2\beta - 2 - n)} = p_{e_i e_n^{(2)}}, & & i = 1, \ldots, n-1.
\end{aligned}
$$

2 Note that these transition probabilities are obtained using Lebesgue measure and the dynamics of K.

The case $k = 1$ follows easily from the case $2 \leq k \leq n$, by putting $e_0 = e_0^{(1)}$, $e_0 = e_0^{(1)}$, and removing the states $e_0^{(2)}$ and $e_n^{(2)}$ (and their corresponding transition probabilities).

To find the stationary distribution $\pi = (\pi_{e_0^{(1)}}, \pi_{e_0^{(2)}}, \pi_{s_1}, \pi_{e_1}, \ldots, \pi_{s_n}, \pi_{e_n^{(2)}}, \pi_{e_n^{(1)}})$ of this Markov chain one has to distinguish three cases:

(1) $k - 1 < n - k + 1$;
(2) $k - 1 > n - k + 1$;
(3) $k - 1 = n - k + 1$ (in this case n is even).

For instance, for $k = 1$ one finds that the stationary distribution $\pi = (\pi_{e_0}, \pi_{s_1}, \pi_{e_1}, \ldots, \pi_{s_n}, \pi_{e_n})$ is given by

$$
\begin{aligned}
\pi_{e_0} &= \frac{\beta(p_1 + \cdots + p_n)}{(\beta - 1)(\beta^2 + 1)}, \\
\pi_{s_k} &= \frac{1}{\beta^2 + 1}, \quad k = 1, 2, \ldots, n, \\
\pi_{e_k} &= \frac{\beta - 1}{\beta^2 + 1}, \quad k = 1, 2, \ldots, n-1, \\
\pi_{e_n} &= \frac{(n - (p_1 + \cdots + p_n))\beta}{(\beta - 1)(\beta^2 + 1)}.
\end{aligned}
$$

Notice that if $p_i = 1$ for all $i = 1, \ldots, n$, one gets the Parry measure, and if $p_i = 0$ for all $i = 1, \ldots, n$, one gets the lazy measure described in §1.1.

We now show that almost any sequence of e_i's and s_i's generated by the above Markov chain corresponds to a random expansion in base β generated by iterating the map K. Let Y be the set of all sequences generated by the Markov chain, i.e. with symbols in the set $\{e_0^{(1)}, e_0^{(2)}, s_1, e_1, \ldots, s_n, e_n^{(2)}, e_n^{(1)}\}$, transition matrix $\mathbf{P}$ and stationary distribution π. We denote by $\mathcal{F}$ the σ-algebra generated by the cylinders, and let μ be the stationary measure given by $\mathbf{P}$ and π, which is σ_Y-invariant, where σ_Y is the left-shift on Y.

To every $y \in Y$ we can associate a point

$$
x \in \left[0, \frac{n}{\beta - 1}\right]
$$

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as follows. Define

$$b_i = \begin{cases} 0, & \text{if } y_i \in \{e_0^{(1)}, e_0^{(2)}\}, \\ n, & \text{if } y_i \in \{e_n^{(1)}, e_n^{(2)}\}, \\ j, & \text{if } y_i = e_j,\ 1 \le j \le n-1, \\ j, & \text{if } y_i = s_j,\ y_{i+1} = e_0^{(1)},\ 1 \le j \le n, \\ j-1, & \text{if } y_i = s_j,\ y_{i+1} = e_n^{(1)},\ 1 \le j \le n, \end{cases}$$

and let

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}, \tag{2.1}$$

then

$$x \in \left[0, \frac{n}{\beta - 1}\right].$$

However, to capture the dynamics of K, we define a measurable isomorphism

$$\varphi : (Y, \sigma_Y, \mu) \to \left(\Omega \times \left[0, \frac{n}{\beta - 1}\right], K, \rho\right),$$

where $\rho = \mu \circ \varphi^{-1}$, as follows. Let

$$Y' = \{y = (y_1, y_2, \dots) \in Y :\ y_i = s_\ell \text{ for finitely many } i\text{'s and } \ell = 1, \dots, n\}$$

then $\mu(Y') = 0$, and define

$$\varphi : Y \setminus Y' \to \Omega \times \left[0, \frac{n}{\beta - 1}\right]$$

as follows. To a point $y = (y_1, y_2, \dots) \in Y \setminus Y'$ we associate a point

$$(\omega, x) \in \Omega \times \left[0, \frac{n}{\beta - 1}\right].$$

To do this, we first locate the indices $n_i = n_i(y)$ where the realization y of the Markov chain is in state s_ℓ for some $\ell \in \{1, \dots, n\}$. That is, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_\ell$ for some $\ell = 1, \dots, n$. Define

$$\omega_j = \begin{cases} 1, & \text{if } y_{n_j+1} = e_0^{(1)}, \\ 0, & \text{if } y_{n_j+1} = e_n^{(1)}, \end{cases}$$

then $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Now, set $\varphi(y) = (\omega, x)$, where x is as given in (2.1).

Remark 1. In the case $k \ge 2$, in $Y \setminus Y'$ the only realization corresponding to i/β is

$$(s_i, e_n^{(1)}, s_k, e_n^{(1)}, s_k, \dots);$$

under φ this is mapped to $(\omega^{(0)}, 1/\beta)$, with $\omega^{(0)} = (0, 0, \dots)$. Similarly, the only realization corresponding to $[n/\beta(\beta - 1)] + [(i - 1)/\beta]$ is

$$(s_i, e_0^{(1)}, s_{n-k+1}, e_0^{(1)}, s_{n-k+1}, \dots),$$

which is mapped under φ to $(\omega^{(1)}, [n/\beta(\beta - 1)] + [(i - 1)/\beta])$, where $\omega^{(1)} = (1, 1, \dots)$.

The following lemma shows that the dynamics of K behaves essentially in the same way as the Markov chain.

LEMMA 1. *Let* $y \in Y \setminus Y'$ *and* $\varphi(y) = (\omega, x)$. *Then:*
(i) $y_1 = e_i^{(j)} \Rightarrow x \in E_i^{(j)}$ *for* $i = 0, n$ *and* $j = 1, 2$*;*
(ii) $y_1 = e_i \Rightarrow x \in E_i$ *for* $i = 1, \dots, n-1$*;*
(iii) $y_1 = s_i,\ y_2 = e_0^{(1)} \Rightarrow x \in S_i$ *and* $\omega_1 = 1$ *for* $i = 1, \dots, n$*; and*
(iv) $y_1 = s_i,\ y_2 = e_n^{(1)} \Rightarrow x \in S_i$ *and* $\omega_1 = 0$ *for* $i = 1, \dots, n$.

Proof. (i) We study each subcase separately. If $y_1 = e_0^{(1)}$, then by construction $b_1 = 0$. The realization in the Markov chain leading to the largest possible digits is $(e_0^{(1)}, s_{n-k+1}, e_n^{(1)}, e_n^{(1)}, \dots)$, yielding

$$\begin{aligned} 0 \le x &\le \frac{0}{\beta} + \frac{n-k}{\beta^2} + \frac{n}{\beta^3} + \frac{n}{\beta^4} + \cdots \\ &\le \left(\frac{n}{\beta^2} + \frac{n}{\beta^3} + \cdots\right) - \frac{k}{\beta^2} \\ &= \frac{n}{\beta(\beta-1)} - \frac{k}{\beta^2}. \end{aligned}$$

Equality on the right-hand side is achieved since

$$y^* = (e_0^{(1)}, s_{n-k+1}, e_0^{(1)}, s_{n-k+1}, \dots) \in Y \setminus Y',$$

and by (2.1) y^* corresponds to $[n/\beta(\beta-1)] - (k/\beta^2)$.

If $y_1 = e_0^{(2)}$, then the path yielding the largest digits is given by

$$(e_0^{(2)}, e_n^{(2)}, e_{k-1}, e_n^{(2)}, e_{k-1}, \dots) \in Y',$$

implying

$$x \le \frac{n}{\beta^2} + \frac{k-1}{\beta^3} + \frac{n}{\beta^4} + \frac{k-1}{\beta^5} + \cdots.$$

Since

$$\frac{n}{\beta^2} + \frac{k-1}{\beta^3} = \frac{1}{\beta} - \frac{1}{\beta^3},$$

it follows that

$$x \le \left(\frac{1}{\beta} - \frac{1}{\beta^3}\right) + \left(\frac{1}{\beta^3} - \frac{1}{\beta^5}\right) + \cdots = \frac{1}{\beta}.$$

However, equality is never attained by an element from $Y \setminus Y'$ starting with $e_0^{(2)}$. The 'smallest digits path' is given by

$$(e_0^{(2)}, e_{n-k+1}, e_0^{(2)}, e_{n-k+1}, e_0^{(2)}, \dots) \in Y'.$$

Using

$$\frac{1}{\beta^\ell} = \frac{n}{\beta^{\ell+1}} + \frac{k}{\beta^{\ell+2}},$$

we get

$$\begin{aligned} x &\geq \frac{n-k+1}{\beta^2} + \frac{n-k+1}{\beta^4} + \frac{n-k+1}{\beta^6} + \cdots \\ &= \frac{n}{\beta^2} - \frac{k}{\beta^2} + \frac{n}{\beta^3} - \frac{k}{\beta^3} + \frac{n}{\beta^4} - \frac{k}{\beta^4} + \cdots \\ &= \frac{n}{\beta^2} + \frac{n}{\beta^3} + \frac{n}{\beta^4} + \cdots - \frac{k}{\beta^2} = \frac{n}{\beta(\beta-1)} - \frac{k}{\beta^2}. \end{aligned}$$

Again, no point in $Y \setminus Y'$ starting with $e_0^{(2)}$ corresponds to $[n/\beta(\beta-1)] - (k/\beta^2)$. Thus

$$x \in E_0^{(2)} = \left(\frac{n}{\beta(\beta-1)} - \frac{k}{\beta^2}, \frac{1}{\beta} \right).$$

If $y_1 = e_n^{(1)}$, then $b_1 = n$, and the path with the largest digits is $(e_n^{(1)}, e_n^{(1)}, \dots) \in Y'$, corresponding to 1. The path with the smallest digits is

$$(e_n^{(1)}, s_k, e_0^{(1)}, e_0^{(1)}, e_0^{(1)}, \dots) \in Y',$$

which corresponds to $n/(\beta-1)$. We remark that the point 1 corresponds to an element $y \in Y \setminus Y'$, starting with $e_n^{(1)}$, namely $(e_n^{(1)}, s_k, e_n^{(1)}, s_k, \dots)$. Thus $x \in E_n^{(1)}$.

If $y_1 = e_n^{(2)}$, then $b_1 = n$. The only paths yielding the largest digits and the smallest digits are, respectively,

$$(e_n^{(2)}, e_{k-1}, e_n^{(2)}, e_{k-1}, \dots) \quad \text{and} \quad (e_n^{(2)}, e_0^{(2)}, e_{n-k+1}, e_0^{(1)}, e_{n-k+1}, \dots),$$

which belong to Y'. The corresponding points are 1 and $[(n-1)/\beta] + [n/\beta(\beta-1)]$, and it follows that $x \in E_n^{(2)}$.

(ii) Let $y_1 = e_i$ for $i = 1, \dots, n-1$. Then by construction $b_1 = i$, and the path giving the largest digits is

$$(e_i, e_n^{(2)}, e_{k-1}, e_n^{(2)}, e_{k-1}, \dots) \in Y',$$

corresponding to $(i+1)/\beta$. The path in Y leading to the smallest digits is

$$(e_i, e_0^{(2)}, e_{n-k+1}, e_0^{(2)}, e_{n-k+1}, e_0^{(2)}, \dots) \in Y',$$

corresponding to $[n/\beta(\beta-1)]+[(i-1)/\beta]$. No point in $Y \setminus Y'$ starting with e_i corresponds to either of these end points. Therefore $x \in E_i$.

(iii) Let $y_1 = s_i$ and $y_2 = e_0^{(1)}$, where $i = 1, \dots, n$. Then $b_1 = i$, and the path yielding the largest digits is

$$(s_i, e_0^{(1)}, s_{n-k+1}, e_n^{(1)}, e_n^{(1)}, e_n^{(1)}, \dots) \in Y',$$

corresponding to $[n/\beta(\beta-1)] + [(i-1)/\beta]$. The path leading to the smallest digits is

$$(s_i, e_0^{(1)}, e_0^{(1)}, e_0^{(1)}, e_0^{(1)}, \dots) \in Y',$$

corresponding to i/β. Note that the point i/β does not correspond to an element in $Y \setminus Y'$, starting with $s_i, e_0^{(1)}$, while $[n/\beta(\beta-1)] + [(i-1)/\beta]$ corresponds to

$$(s_i, e_0^{(1)}, s_{n-k+1}, e_0^{(1)}, s_{n-k+1}, \dots) \in Y \setminus Y'.$$

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Thus

$$\frac{i}{\beta} < x \le \frac{n}{\beta(\beta-1)} + \frac{i-1}{\beta}.$$

Hence, $x \in S_i$, and definition of φ we have $\omega_1 = 1$.

(iv) Finally, let $y_1 = s_i$ and $y_2 = e_n^{(1)}$, where $i = 1, \dots, n$. Then $b_1 = i - 1$, and the path yielding the largest digits is

$$(s_i, e_n^{(1)}, e_n^{(1)}, \dots) \in Y',$$

which corresponds to $[n/\beta(\beta-1)] + [(i-1)/\beta]$. The path leading to the smallest digits is

$$(s_i, e_n^{(1)}, s_k, e_0^{(1)}, e_0^{(1)}, \dots) \in Y',$$

corresponding to i/β. Similar to (iii), the point $[n/\beta(\beta-1)] + [(i-1)/\beta]$ does not correspond to a point in $Y \setminus Y'$ starting with $s_i, e_n^{(1)}$, while the point i/β corresponds to

$$(s_i, e_n^{(1)}, s_k, e_n^{(1)}, s_k, \dots) \in Y \setminus Y'.$$

This shows that

$$\frac{i}{\beta} \le x < \frac{n}{\beta(\beta-1)} + \frac{i-1}{\beta},$$

hence $x \in S_i$, and by the definition of φ we have $\omega_1 = 0$. □

Remark 2. (1) It follows from the above lemma that if

$$x = \sum_{i=1}^{\infty} b_i/\beta^i$$

is as defined by the Markov chain, and if we look at the random expansion of x under K, then $b_1 = d_1(\omega, x)$.

(2) In the case $k = 1$, the statement of Lemma 1 only contains three cases; two cases are (iii) and (iv) from Lemma 1, while one case can be formulated as

$$y_1 = e_\ell \Rightarrow x \in E_\ell \quad \text{for } \ell = 0, 1, \dots, n.$$

The proofs for the case $k = 1$ are similar to those for the case $2 \le k \le n$ as described above.

LEMMA 2. *For $y \in Y \setminus Y'$,*

$$\varphi \circ \sigma_Y(y) = K \circ \varphi(y).$$

Proof. Let $y = (Y_i)_{i=1}^{\infty} \in Y \setminus Y'$, and suppose that $\varphi(y) = (\omega, x)$ and $\varphi(\sigma_Y(y)) = (\omega^*, x^*)$, where

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i} \quad \text{and} \quad x^* = \sum_{i=1}^{\infty} \frac{b_i^*}{\beta^i}.$$

By the definition of φ we see that $b_i^* = b_{i+1}$ and $d_1(\omega, x) = b_1$, and therefore

$$x^* = \sum_{i=1}^{\infty} \frac{b_{i+1}}{\beta^i} = \beta x - d_1.$$

We now show that

$$\omega^* = \begin{cases} \omega, & \text{if } y_1 = e_\ell,\ \ell = 1, \dots, n-1, \\ & \text{or } y_1 = e_i^{(j)},\ i = 0, n,\ j = 1, 2, \\ \sigma(\omega), & \text{if } y_1 = s_\ell,\ \ell = 1, \dots, n. \end{cases}$$

To this end, let us first assume that $y_1 = e_\ell$ for some $\ell \in \{1, \dots, n-1\}$ or $y_1 = e_i^{(j)}$ for $i = 0, n,\ j = 1, 2$. Then, the successive times

$$n_1(\sigma_Y(y)) < n_2(\sigma_Y(y)) < \cdots$$

that the realization $\sigma_Y(y)$ of the Markov chain is in state s_k are given by $n_i(\sigma_Y(y)) = n_i(y) - 1$. Hence,

$$(\sigma_Y(y))_{n_i(\sigma_Y(y))} = (y)_{n_i(\sigma_Y(y))+1} = y_{n_i(y)},$$

which implies that $\omega_i^* = \omega_i$ for all i, i.e. $\omega^* = \omega$.

Next, assume that $y_1 = s_k$ for some $k \in \{1, \dots, n\}$. In this case $n_i(\sigma_Y(y)) = n_{i+1}(Y) - 1$, and it follows that

$$(\sigma_Y(y))_{n_i(\sigma_Y(y))} = (y)_{n_i(\sigma_Y(y))+1} = (y)_{n_{i+1}(y)},$$

which implies that $\omega^* = \omega_{i+1}$ for all i, i.e. $\omega^* = \sigma(\omega)$. Thus we find that $(\omega^*, x^*) = K(\omega, x)$ and $\varphi(\sigma_Y(y)) = K(\varphi(y))$. □

Remark 3. It follows from Lemma 2 that $\varphi \circ \sigma_Y^n = \sigma_Y^n \circ \varphi$ for all $n \geq 0$, and since $b_1 = d_1(\omega, x)$ we find that

$$b_n = d_n(\omega, x) = d_1(K^{n-1}(\omega, x)) \quad \text{for all } n \in \mathbb{N}.$$

Let $Z = \varphi(Y \setminus Y')$. From the above, one has that

$$Z = \{(\omega, x) : K^n(\omega, x) \in \Omega \times S \text{ infinitely often}\}.$$

On $\Omega \times [0, n/(\beta - 1)]$ we consider the completion $\mathcal{C}$ of the σ-algebra $\sigma(\bigvee_{n=0}^{\infty} \mathcal{P}_n)$, where

$$\mathcal{P}_0 = \{\Omega \times E_i : i = 1, 2, \dots, n-1\} \cup \{\Omega \times E_i^{(j)} : i = 0, 1,\ j = 1, 2\} \\ \cup \{\{\omega_1 = i\} \times S_\ell : i = 0, 1,\ \ell = 1, \dots, n\}$$

and $\mathcal{P}_n = \mathcal{P}_0 \vee K^{-1}\mathcal{P}_0 \vee \cdots \vee K^{n-1}\mathcal{P}_0$. It is easy to check that the inverse image under φ of an element in $\mathcal{P}_n$ is a cylinder in X. Thus φ is $(\mathcal{F}, \mathcal{C})$-measurable. In fact, if we consider the σ-algebra $\mathcal{A}$ on $\Omega \times [0, n/(\beta - 1)]$ which is the product of the σ-algebra on Ω and the Borel σ-algebra on $[0, n/(\beta - 1)]$, then the inverse image of any cylinder in $\mathcal{A}$ that is not an element of $\mathcal{C}$ is the empty set, hence φ is also $(\mathcal{F}, \mathcal{A})$-measurable. Define a measure ρ on $\Omega \times [0, n/(\beta - 1)]$ by $\rho = \mu \circ \varphi^{-1}$. Then ρ is K-invariant and $\rho(Z) = 1$. Further, φ is a factor map by the above. We now show that φ is in fact an isomorphism.

THEOREM 1. *The map*

$$\varphi : (Y, \mathcal{F}, \mu, \sigma_Y) \to \left(\Omega \times \left[0, \frac{n}{\beta - 1}\right], \mathcal{C}, \rho, K\right)$$

is a measurable isomorphism.

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Proof. Since φ is a factor map, it is enough to show that $\varphi : Y \setminus Y' \to Z$ is invertible. To this end, let $\psi : Z \to Y \setminus Y'$ be given by $\psi(\omega, x) = y = (y_i)$, where

$$y_i = \begin{cases} e_\ell, & \text{if } \pi(K^{i-1}(\omega, x)) \in E_\ell, \text{ for some } \ell = 1, \dots, n-1, \\ e_i^{(j)}, & \text{if } \pi(K^{i-1}(\omega, x)) \in E_i^{(j)}, \text{ for some } i = 0, n, \; j = 1, 2, \\ s_\ell, & \text{if } \pi(K^{i-1}(\omega, x)) \in S_\ell, \text{ for some } \ell = 1, \dots, n. \end{cases}$$

Since K on Z has the same allowed transitions as the Markov chain on $Y \setminus Y'$, we see that $y \in Y \setminus Y'$. We show that $\varphi(y) = (\omega, x)$, and hence $\psi = \varphi^{-1}$. To this end, notice that if $n_1 < n_2 < \cdots$ are the indices for which $y_{n_i} = s_\ell$ for some $\ell \in \{1, \dots, n\}$, then by construction $\pi(K^{n_i - 1}(\omega, x)) \in S_\ell$ and hence

$$\pi(K^{n_i}(\omega, x)) \in \begin{cases} E_0^{(1)}, & \text{if } \omega_i = 1, \\ E_n^{(1)}, & \text{if } \omega_i = 0, \end{cases}$$

i.e.

$$y_{n_i+1} = \begin{cases} e_0^{(1)}, & \text{if } \omega_i = 1, \\ e_n^{(1)}, & \text{if } \omega_i = 0. \end{cases}$$

Suppose that $\varphi(y) = (\omega^*, x^*)$, then

$$\omega^* = \begin{cases} 1, & \text{if } y_{n_i+1} = e_0^{(1)}, \\ 0, & \text{if } y_{n_i+1} = e_n^{(1)}, \end{cases}$$

and we see that $\omega = \omega^*$. If b_i denotes the ith digit as defined in (2.1), then by Remark 3 $b_i = d_i(\omega^*, x^*)$, and by construction

$$b_i = \begin{cases} j, & \text{if } \pi(K^{i-1}(\omega, x)) \in E_j, \; j = 0, 1, \dots, n, \\ & \text{or } \pi(K^{i-1}(\omega, x)) \in S_j, \text{ and } \omega_i = 1, \; j = 1, \dots, n, \\ j-1, & \text{if } \pi(K^{i-1}(\omega, x)) \in S_j \text{ and } \omega_i = 0, \; j = 1, \dots, n. \end{cases}$$

from which $b_i = d_i(\omega, x) = d_i(\omega^*, x^*)$, and we see that

$$x^* = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i} = x.$$

Hence $\varphi(y) = (\omega, x)$ and thus $\psi = \varphi^{-1}$. This proves that φ is invertible. □

Remark 4. In the case $k = 1$ the above proof needs some small adjustments. Let $\vartheta^{(0)} = (0, 1, 0, 1, \dots)$, and $\vartheta^{(1)} = (1, 0, 1, 0, \dots)$. Note that for $i = 1, \dots, n$ one has that

$$y^{(i)} := (s_i, e_n, s_1, e_n, s_1, \dots) \in Y \setminus Y' \quad \text{and} \quad \varphi(y^{(i)}) = \left(\vartheta^{(0)}, \frac{i}{\beta}\right).$$

However,

$$\psi\left(\vartheta^{(0)}, \frac{i}{\beta}\right) = (s_i, s_n, s_1, s_n, s_1, \dots) \notin Y.$$

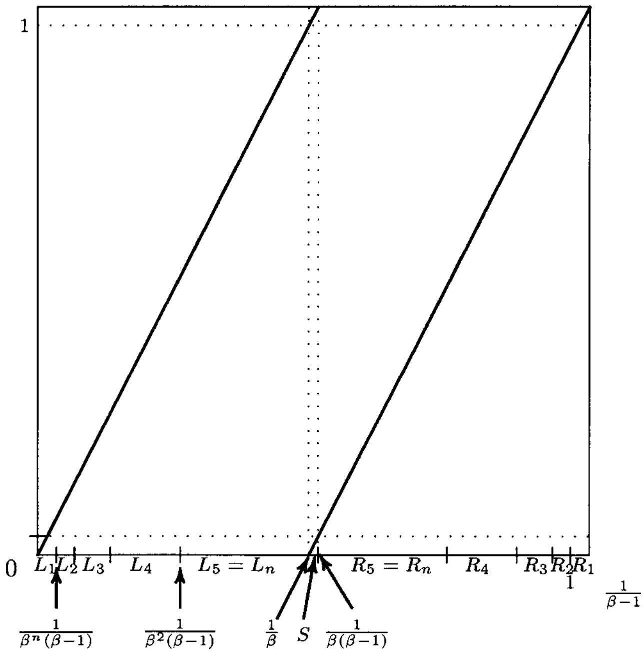

FIGURE 4. The greedy and lazy maps and their switch region S, in the case $\beta^n = \beta^{n-1} + \cdots + \beta + 1$ (here $n = 5$).

Similarly,

$$z^{(i)} := (s_i, e_0, s_1, e_0, s_1, \dots) \in Y \setminus Y' \quad \text{and} \quad \varphi(z^{(i)}) = \left(\vartheta^{(1)}, \frac{n}{\beta(\beta-1)} + \frac{i-1}{\beta}\right),$$

while

$$\psi\left(\vartheta^{(1)}, \frac{n}{\beta(\beta-1)} + \frac{i-1}{\beta}\right) = (s_i, s_1, s_n, s_1, s_n, \dots) \notin Y.$$

Hence, for the proof that φ is invertible we need to remove from $Y \setminus Y'$ all points whose orbit under σ_Y eventually equals $y^{(i)}$ or $z^{(i)}$ for some $i = 1, \dots, n$, and their corresponding images under φ in Z (this is a set of μ measure 0).

If one calculates the entropy of the Markov chain Y (and hence the entropy of K) one finds that for $k = 1$

$$h_\rho(K) = \log \beta - \sum_{i=1}^{n} \frac{p_i \log p_i + (1-p_i)\log(1-p_i)}{1+\beta^2}.$$

2.2. *The case* $\beta^n = \beta^{n-1} + \cdots + \beta + 1$, $n \geq 2$. (See Figure 4.) First notice that the case $n = 2$, which is the case when β equals the 'golden mean', has already been dealt with in the previous section. Therefore, throughout this section we will assume that $n \geq 3$. It is easily checked that one has $\lfloor \beta \rfloor = 1$. In this case we have one switch region

$$S = \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right],$$

which is mapped under the greedy map T_β to the interval $[0, 1/(\beta^n(\beta-1))]$, which is a subinterval of $[0, 1/\beta)$. Further, for $i = 0, 1, \dots, n$ one has that

$$T_\beta^i\left(\left[0, \frac{1}{\beta^n(\beta-1)}\right]\right) = \left[0, \frac{1}{\beta^{n-i}(\beta-1)}\right].$$

Notice that

$$T_\beta^n\left(\frac{1}{\beta(\beta-1)}\right) = \frac{1}{\beta(\beta-1)}.$$

Let

$$L_1 = \left[0, \frac{1}{\beta^n(\beta-1)}\right), \quad L_n = \left[\frac{1}{\beta^2(\beta-1)}, \frac{1}{\beta}\right),$$
$$L_i = \left[\frac{1}{\beta^{n-i+2}(\beta-1)}, \frac{1}{\beta^{n-i+1}(\beta-1)}\right), \quad \text{for } i = 2, \dots, n-1,$$

and

$$R_1 = \left(1, \frac{1}{\beta-1}\right], \quad R_n = \left(\frac{1}{\beta(\beta-1)}, \frac{1}{\beta} + \frac{1}{\beta^2}\right],$$
$$R_i = \left(\frac{1}{\beta} + \dots + \frac{1}{\beta^{n-i+1}}, \frac{1}{\beta} + \dots + \frac{1}{\beta^{n-i+2}}\right], \quad \text{for } i = 2, \dots, n-1,$$

then, under the greedy map T_β we have

$$T_\beta(S) = \bar{L}_1, \quad T_\beta(L_1) = L_1 \cup L_2, \quad T_\beta(L_i) = L_{i+1}, \quad i = 2, \dots, n-2,$$
$$T_\beta(\bar{L}_{n-1}) = L_n \cup S, \quad T_\beta(L_n) = R_2 \cup \dots \cup R_n,$$

and under the lazy map S_β we have

$$S_\beta(S) = \bar{R}_1, \quad S_\beta(R_1) = R_1 \cup R_2, \quad S_\beta(R_i) = R_{i+1}, \quad i = 2, \dots, n-2,$$
$$S_\beta(R_{n-1}) = R_n \cup S, \quad S_\beta(R_n) = L_2 \cup \dots \cup L_n.$$

Motivated by this, we will study the following Markov chain in order to study the dynamics of the map K. Consider the Markov chain with state space

$$\{\ell_1, \ell_2, \dots, \ell_n, s, r_n, r_{n-1}, \dots, r_1\}$$

and transition probabilities

$$p_{\ell_1\ell_1} = \frac{1}{\beta} = p_{r_1r_1} = p_{\ell_nr_n} = p_{r_n\ell_n}, \quad p_{\ell_1\ell_2} = \frac{\beta-1}{\beta} = p_{r_1r_2},$$
$$p_{\ell_i\ell_{i+1}} = 1 = p_{r_ir_{i+1}}, \qquad i = 2, \dots, n-2,$$
$$p_{\ell_{n-1}\ell_n} = \frac{\beta^2-\beta-1}{\beta-1} = p_{r_{n-1}r_n}, \quad p_{\ell_{n-1}s} = \frac{2\beta-\beta^2}{\beta-1} = p_{r_{n-1}s}$$

and

$$p_{\ell_nr_i} = \frac{\beta-1}{\beta^{n-i+1}(\beta^2-\beta-1)} = p_{r_n\ell_i}, \quad i = 2, \dots, n-1.$$

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The stationary distribution $\pi = (\pi_{\ell_1}, \dots, \pi_{\ell_n}, \pi_s, \pi_{r_n}, \dots, \pi_{r_1})$ is given by

$$\pi_s = \frac{\beta - 1}{\beta^{n+1} - n},$$
$$\pi_{\ell_1} = \frac{p\beta}{\beta^{n+1} - n},$$
$$\pi_{r_1} = \frac{(1-p)\beta}{\beta^{n+1} - n},$$
$$\pi_{\ell_n} = \frac{(\beta^2 - \beta - 1)(p + \beta^n - 1)}{\beta(\beta^{n+1} - n)},$$
$$\pi_{r_n} = \frac{(\beta^2 - \beta - 1)(\beta^n - p)}{\beta(\beta^{n+1} - n)},$$
$$\pi_{\ell_i} = \frac{p\beta^{i-1}(\beta^{n-i+1} - 1) + \beta^n(\beta^{i-1} - 1)}{\beta^n(\beta^{n+1} - n)}, \quad i = 2, \dots, n-1,$$

and

$$\pi_{r_i} = \frac{\beta^{i-1}(\beta^n - 1) - p\beta^{i-1}(\beta^{n-i+1} - 1)}{\beta^n(\beta^{n+1} - n)}, \quad i = 2, \dots, n-1.$$

If $p = \frac{1}{2}$, then one gets $\pi_{\ell_i} = \pi_{r_i}$, $i = 1, 2, \dots, n$, as expected.

Let Y be the set of all sequences generated by the above Markov chain, σ_Y be the left shift on Y, and let μ be the invariant measure generated by P and π. As in the previous case, for each $y \in Y$ we associate a point

$$x \in \left[0, \frac{1}{\beta - 1}\right]$$

as follows. Define

$$b_i = \begin{cases} 0, & \text{if } y_i = \ell_i \text{ for some } i = 1, \dots, n \\ & \text{or } y_i = s \text{ and } y_{i+1} = r_1, \\ 1, & \text{if } y_i = r_i \text{ for some } i = 1, \dots, n \\ & \text{or } y_i = s \text{ and } y_{i+1} = \ell_1, \end{cases}$$

and set

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}, \tag{2.2}$$

so we have that

$$x \in \left[0, \frac{1}{\beta - 1}\right].$$

To show that the random expansion under K can be symbolically described by the above Markov chain, we consider a map

$$\varphi : Y \to \Omega \times \left[0, \frac{1}{\beta - 1}\right]$$

defined almost everywhere as follows. Let

$$Y' = \{y = (y_1, y_2, \dots) \in Y : y_i = s \text{ for finitely many } i\text{'s}\}.$$

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For $y = (y_i)_{i=1}^{\infty} \in Y \setminus Y'$, denote by $n_1 < n_2 < \cdots$ the indices for which $y_{n_i} = s$. Setting

$$\omega_i = \begin{cases} 1, & \text{if } y_{n_i+1} = \ell_1, \\ 0, & \text{if } y_{n_i+1} = r_1, \end{cases}$$

then $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Now define $\varphi(y)$ for each $y \in Y$ by $\varphi(y) = (\omega, x)$, with x given by (2.2). Let $\mathcal{C}$ and ρ be defined as in the case $\beta^2 = n\beta + k$. Then we have the following theorem.

THEOREM 2. *The map*

$$\varphi : (Y, \mathcal{F}, \mu, \sigma_Y) \to \left(\Omega \times \left[0, \frac{1}{\beta - 1}\right], \mathcal{C}, \rho, K\right)$$

is a measurable isomorphism.

The proof that

$$\varphi : (Y, \mu, \sigma_Y) \to \left(\Omega \times \left[0, \frac{1}{\beta - 1}\right], \rho, K\right)$$

is a measurable isomorphism, with $\rho = \mu \circ \varphi^{-1}$, is similar to the case $\beta^2 = n\beta + 1$. In this case, ψ is not the inverse of φ at the points $(\omega^{(1)}, 1/\beta(\beta - 1))$ and $(\omega^{(0)}, 1/\beta)$, with $\omega^{(i)} = (i, i, \dots)$, $i = 0, 1$. To see this, observe that the sequence

$$\alpha = (s, \ell_1, \ell_2, \dots, \ell_{n-1}, s, \ell_1, \dots, \ell_{n-1}, s, \dots)$$

belongs to $Y \setminus Y'$ with

$$\varphi(\alpha) = \left(\omega^{(1)}, \frac{1}{\beta(\beta - 1)}\right),$$

while

$$\psi\left(\omega^{(1)}, \frac{1}{\beta(\beta - 1)}\right) = (s, \ell_2, \ell_3, \dots, \ell_n, s, \ell_2, \dots, \ell_n, s, \dots) \notin Y.$$

Similarly,

$$\beta = (s, r_1, r_2, \dots, r_{n-1}, s, r_1, \dots, r_{n-1}, s, \dots) \in Y \setminus Y',$$

with

$$\varphi(\beta) = \left(\omega^{(0)}, \frac{1}{\beta}\right),$$

while

$$\psi\left(\omega^{(0)}, \frac{1}{\beta}\right) = (s, r_2, r_3, \dots, r_n, s, r_2, \dots, r_n, s, \dots) \notin Y.$$

So apart from Y' we need to remove from Y all points whose orbit under σ_Y eventually equals α or β, and their corresponding images in Z.

2.3. *Final remarks.* The approach of the previous sections also works for many $\beta > 1$ for which the expansion of 1 in base β is either finite or eventually periodic. For example, if $\beta = \frac{1}{2}(3+\sqrt{5}) = 2.6183\ldots$, one has that the greedy expansion of 1 in base β is given by

$$1 = \frac{2}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} + \cdots + \frac{1}{\beta^n} + \cdots .$$

Setting $g = \frac{1}{2}(-1+\sqrt{5}) = 0.6183\ldots$, $G = g+1$ and hence† $\beta = G^2 = G+1$, one finds that there are two switch regions

$$S_1 = \left[\frac{1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right] = [g^2, 4g-2] \quad \text{and} \quad S_2 = \left[\frac{2}{\beta}, \frac{1}{\beta} + \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right] = [2g^2, 3g-1],$$

and three equality regions:

$$E_0 = [0, g^2), \quad E_1 = (4g-2, 2g^2) \quad \text{and} \quad E_2 = E_{\lfloor\beta\rfloor} = (3g-1, 2g].$$

As in §2.1 we need to subdivide E_0 and E_2:

$$E_0^{(1)} = [0, 2g-1), \quad E_0^{(2)} = E_0 \setminus E_0^{(1)} \quad \text{and} \quad E_2^{(1)} = (1, 2g], \quad E_2^{(2)} = E_2 \setminus E_2^{(1)}.$$

E_1 also needs to be subdivided into two equal parts:

$$E_1^{(\ell)} = (4g-2, g) \quad \text{and} \quad E_1^{(r)} = (g, 2g^2);$$

note that g is a fixed point under K. As in the previous sections, one can show that the dynamics of K is isomorphic to a mixing Markov chain. We leave the details to the reader.

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† The number G is the so-called *golden mean*, g is the small golden mean.