

PERMUTATION-BASED PRESENTATIONS FOR BRIN'S HIGHER-DIMENSIONAL THOMPSON GROUPS nV

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(Received 16 May 2022; accepted 21 September 2022; first published online 15 November 2022)

Communicated by Michael Giudici

Abstract

The higher-dimensional Thompson groups nV , for $n \geq 2$, were introduced by Brin [*Presentations of higher dimensional Thompson groups*, *J. Algebra* **284** (2005), 520–558]. We provide new presentations for each of these infinite simple groups. The first is an infinite presentation, analogous to the Coxeter presentation for the finite symmetric group, with generating set equal to the set of transpositions in nV and reflecting the self-similar structure of n -dimensional Cantor space. We then exploit this infinite presentation to produce further finite presentations that are considerably smaller than those previously known.

2020 Mathematics subject classification: primary 20F65, 20F05; secondary 20E32.

Keywords: Thompson's groups, higher-dimensional Thompson groups, simple groups, Cantor space, presentations, generators and relations, permutations, transpositions, baker's map.

1. Introduction

A well-known result of Brouwer [6] states that a nonempty totally disconnected compact metrizable space without isolated points is homeomorphic to the Cantor space \mathbb{C} . Consequently, this space arises throughout mathematics and it is unsurprising that many groups occur among its homeomorphisms. Interesting and important examples of such groups include Grigorchuk's group of intermediate growth [12, 13], which may be naturally described as consisting of certain automorphisms of a binary rooted tree, and various generalizations, such as the Gupta–Sidki groups [15] and the multi-GGS groups (see, for example, [10]); the asynchronous rational group of Grigorchuk *et al.* [14]; and, particularly relevant to this paper, the groups F , T and V introduced by Richard J. Thompson [7, 21].

Thompson's group F is a 2-generator group with abelianization isomorphic to a free abelian group of rank 2 and such that its derived subgroup F' is simple. The

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other two groups T and V introduced by Thompson are both infinite simple groups. All three groups are finitely presented, with F having a small presentation with two generators and two relations. The presentations for T and V , as described in [7], both involve additional generators and relations to supplement those used for F . In particular, Thompson's original presentation for his group V involved four generators and fourteen relations. In work by Bleak and the author [2], we returned to possible presentations for V . We give there various presentations for this group: one involving infinitely many generators and an infinite family of relations. The generators in [2, Theorem 1.1] correspond to transpositions of certain disjoint basic open sets of Cantor space and the relations are analogous to the Coxeter presentation for a finite symmetric group, but also include what we termed 'split relations' reflecting the self-similar structure of Cantor space. The second presentation, given in [2, Theorem 1.2], is a finite presentation, essentially obtained by reducing the infinite presentation, with three generators and eight relations (which compares favourably in size to Thompson's original presentation). We then produced a two-generator presentation for V by use of Tietze transformations and our smallest presentation, obtained via computational methods, is on two generators and seven relations [2, Theorem 1.3]. One should also note a link between our infinite presentation for V and the geometric presentations given by Dehornoy [8].

In 2004, Brin [3] introduced, for each positive integer $n \geq 2$, an analogue of Thompson's group V that acts upon an n -dimensional version of Cantor space. He denotes this group by nV and, via the homeomorphism $\mathbb{C}^n \cong \mathbb{C}$, this family provides us with further groups of homeomorphisms of Cantor space. Bleak and Lanoue [1] noted that two of these groups mV and nV are isomorphic if and only if $m = n$. Brin observes in his first paper that the group $2V$ is an infinite simple group, while in [5], he shows that all the groups nV are simple. In the latter argument, he makes considerable reference to the baker's maps of \mathbb{C}^n and notes that these maps can be expressed as a product of transpositions. This observation will be particularly relevant to our proof of Theorem 1.1 in Section 2 below. Furthermore, all these groups are finitely presented: in [4, Theorem 5], Brin observes that $2V$ has a finite presentation with 8 generators and 70 relations. This method was extended by Hennig and Matucci [17] to establish a finite presentation for the groups nV involving $2n + 4$ generators and $10n^2 + 10n + 10$ relations (see [17, Theorem 25]). Indeed, each group nV is of type F_∞ , as established in [11, 19]. In this article, it is demonstrated that these groups possess infinite presentations involving elements corresponding to transpositions of disjoint basic open sets and involving relations that have a Coxeter-like shape and reflect the self-similar nature of \mathbb{C}^n (see Theorem 1.1 below). Again, these infinite presentations bear comparison with Dehornoy's geometric presentations [8] for F and V . In the final section of the paper, we demonstrate that the group nV is isomorphic to a group G with a finite presentation involving 3 generators and $2n^2 + 3n + 11$ relations. It is noteworthy that the number of generators is bounded independent of the parameter n and that the number of relations significantly improves upon the presentations in [4, 17]. In particular, the resulting finite presentation for $2V$ involves 3 generators and 25 relations.

1.1. Notation. We write \mathbb{C} for the Cantor set; that is, the collection of all infinite words from the alphabet $\{0, 1\}$. We also use the set $\{0, 1\}^*$ of finite words in this alphabet and write ε to denote the empty word. If $\alpha, \beta \in \{0, 1\}^*$, then $\alpha \leq \beta$ indicates that α is a prefix of β . The notation $\alpha \perp \beta$ denotes that $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, and we then say these words are *incomparable*. The *length* of a finite word α , denoted by $|\alpha|$, is the number of symbols from $\{0, 1\}$ occurring in α .

If n is a positive integer with $n \geq 2$, the higher-dimensional Thompson group nV is defined (see below) as consisting of certain transformations defined on n -dimensional Cantor space $\Gamma = \mathbb{C}^n$. Accordingly, we also need the set Ω of sequences $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where each $\alpha_i \in \{0, 1\}^*$ and we use the term *address* to refer to elements of Ω . These addresses are used to index the basic open subsets of Γ which in turn appear in the definition of the elements of nV . We extend the concept of incomparability to addresses by writing $\alpha \perp \beta$, for a pair of addresses $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, when $\alpha_d \perp \beta_d$ for some index d with $1 \leq d \leq n$. Similarly, for such addresses, we write $\alpha \leq \beta$ when $\alpha_d \leq \beta_d$ for all $d = 1, 2, \dots, n$.

The higher-dimensional Thompson group nV consists of certain homeomorphisms of Γ . We use right action notation throughout and so write wg for the image of $w \in \Gamma$ under $g \in nV$. Let $\Gamma(\alpha) = \{\alpha w \mid w \in \Gamma\}$ be the collection of all sequences in Γ with the address α as prefix, and this is the basic open set indexed by α . Note $\Gamma(\alpha) \cap \Gamma(\beta) = \emptyset$ if and only if $\alpha \perp \beta$ and that $\Gamma(\alpha) \supseteq \Gamma(\beta)$ if and only if $\alpha \leq \beta$. An element g of nV is then described as follows: given two partitions $\Gamma = \bigcup_{i=1}^k \Gamma(\alpha^{(i)}) = \bigcup_{i=1}^k \Gamma(\beta^{(i)})$ into the same number of disjoint basic open sets, we define the homeomorphism g of Γ by $\alpha^{(i)}w \mapsto \beta^{(i)}w$ for $i = 1, 2, \dots, k$ and any $w \in \Gamma$. Thus, each homeomorphism in nV is given by piecewise affine maps on Γ determined by two partitions of the space into the same number of basic open sets and some bijection between the parts. Figure 1 illustrates an example partition of \mathbb{C}^3 ; that is, a potential choice for domain or codomain partially determining an element of $3V$. If α and β are incomparable addresses, we call the element of nV that maps $\alpha w \mapsto \beta w$, $\beta w \mapsto \alpha w$ for all $w \in \Gamma$ and fixes all other points in Γ a *transposition*. This element has the effect of interchanging the basic open sets $\Gamma(\alpha)$ and $\Gamma(\beta)$. In what follows, we write G_∞ for the group with the presentation given in Theorem 1.1 below. The element denoted by $(\alpha \ \beta)$ that appears in that presentation corresponds, under the natural homomorphism $G_\infty \rightarrow nV$, to this transposition that interchanges $\Gamma(\alpha)$ and $\Gamma(\beta)$.

To describe an address in Ω , in theory, requires one to write a sequence of n finite words in $\{0, 1\}$. Such a sequence would appear quite cumbersome in our calculations particularly when appearing as entries in the transpositions with which we work. Accordingly, we present a more compact and useful notation. If α is some (usually explicit) finite word in $\{0, 1\}$, we write α_d for the address all of whose entries are the empty word with the exception of the d th coordinate which equals α . Thus, for example, $\mathbf{010}_d = (\varepsilon, \dots, \varepsilon, 010, \varepsilon, \dots, \varepsilon)$ where 010 occurs in the d th coordinate in this n -tuple. We particularly make use of this notation when we wish to append one (or more) letters from $\{0, 1\}$ to particular entries in an address $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. For

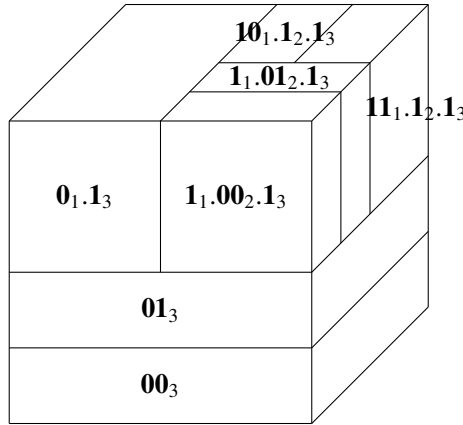


FIGURE 1. A domain or codomain partition of $\Gamma = \mathbb{C}^3$.

example, we write $\alpha.\mathbf{0}_d$ to indicate that we concatenate the addresses α and $\mathbf{0}_d$; that is, we append the symbol 0 to the d th coordinate α_d of α :

$$\alpha.\mathbf{0}_d = (\alpha_1, \dots, \alpha_{d-1}, \alpha_d 0, \alpha_{d+1}, \dots, \alpha_n).$$

(The use of the dot appearing in this notation is to demarcate the end of the first address α and the beginning of the second and is intended to achieve clarity. Indeed, according to our notation, $\alpha\mathbf{0}_d$ (without the dot) would indicate the address with a single nonempty entry $\alpha 0$ in the d th coordinate. The dot notation is unnecessary when concatenating two finite words in $\{0, 1\}$ but helps when dealing with n -tuples.) The use of this notation can be observed within what we term the ‘split relations’ appearing in the statement of Theorem 1.1 below (see Equation (1-4)) and in the addresses labelling the parts in Figure 1. An additional piece of notation that we use is that if $x \in \{0, 1\}$, then \bar{x} denotes the other element in this set and then, following our above convention, \bar{x}_d is the sequence $(\varepsilon, \dots, \varepsilon, \bar{x}, \varepsilon, \dots, \varepsilon)$, where \bar{x} occurs in the d th coordinate. Finally, ε denotes the address $(\varepsilon, \varepsilon, \dots, \varepsilon)$ all of whose entries are the empty word.

To specify the relations that define our group, we define an additional notation that encodes the partial action of transpositions in nV upon the basic open sets indexed by the addresses in Ω . To be specific, if $\alpha, \beta, \gamma \in \Omega$ with $\alpha \perp \beta$, we define a partial map by

$$\gamma \bullet (\alpha \ \beta) = \begin{cases} \beta\delta & \text{if } \gamma = \alpha\delta \text{ for some } \delta \in \Omega; \\ \alpha\delta & \text{if } \gamma = \beta\delta \text{ for some } \delta \in \Omega; \\ \gamma & \text{if both } \gamma \perp \alpha \text{ and } \gamma \perp \beta; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1-1}$$

Thus, we associate to the symbol $(\alpha \ \beta)$ the partial map on the set Ω of addresses that performs a prefix substitution that interchanges the prefix α with the prefix β .

1.2. Statement of results. In Section 2, we establish the following infinite presentation for nV .

THEOREM 1.1. *Let $n \geq 2$. Let \mathcal{A} be the set of all symbols $(\alpha \beta)$ where α and β are addresses in Ω with $\alpha \perp \beta$. Then Brin's higher-dimensional Thompson group nV has infinite presentation with generating set \mathcal{A} and all relations*

$$(\alpha \beta)^2 = 1, \quad (1-2)$$

$$(\alpha \beta)^{(\gamma \delta)} = (\alpha \bullet (\gamma \delta) \beta \bullet (\gamma \delta)), \quad (1-3)$$

$$(\alpha \beta) = (\alpha \cdot \mathbf{0}_d \beta \cdot \mathbf{0}_d) (\alpha \cdot \mathbf{1}_d \beta \cdot \mathbf{1}_d), \quad (1-4)$$

where α, β, γ and δ range over all addresses in Ω such that $\alpha \perp \beta, \gamma \perp \delta$ and such that both $\alpha \bullet (\gamma \delta)$ and $\beta \bullet (\gamma \delta)$ are defined, and d ranges over all indices with $1 \leq d \leq n$.

We refer to relations of the form in Equation (1-3) as 'conjugacy relations' and those of the form in Equation (1-4) as 'split relations' in what follows. The latter arise due to the self-similar nature of Cantor space: to exchange prefixes α and β is equivalent to exchanging both the pairs of prefixes obtained by 'splitting' the d th coordinate. Note that we use exponential notation for conjugation, writing g^h for $h^{-1}gh$, where g and h belong to some group, and this is consistent with our use of right actions.

We also need an additional relation that can be deduced immediately from Equation (1-3). On the face of it, if α and β are incomparable addresses in Ω , the symbols $(\alpha \beta)$ and $(\beta \alpha)$ are not necessarily the same element of the group with the given presentation. However, we expect them to correspond to the same transposition in nV . The 'symmetry relation'

$$(\alpha \beta) = (\beta \alpha) \quad (1-5)$$

follows from Equation (1-3) by taking $\gamma = \alpha$ and $\delta = \beta$:

$$(\alpha \beta) = (\alpha \beta)^{(\alpha \beta)} = (\alpha \bullet (\alpha \beta) \beta \bullet (\alpha \beta)) = (\beta \alpha).$$

We make use of this additional relation in Equation (1-5) throughout our arguments.

The method of proof of the above theorem is essentially to verify a family of relations for nV originally found in [17]. Let G_∞ denote the group with presentation given in Theorem 1.1. The key steps in the proof in Section 3 are to define and investigate elements in G_∞ that correspond to baker's maps. The two-dimensional baker's map is a basic object within the study of dynamical systems (see, for example, [9]) and is illustrated in Figure 2(i). In the context of n dimensions, we refer to baker's maps that arise in the domain from a cut in the first coordinate and in the codomain from a cut in the d th coordinate. Thus, we define, in G_∞ , an element $B_d(\alpha)$ that corresponds to the element of nV with support equal to $\Gamma(\alpha)$ mapping $\mathbb{C}^n \rightarrow \mathbb{C}^n$ via

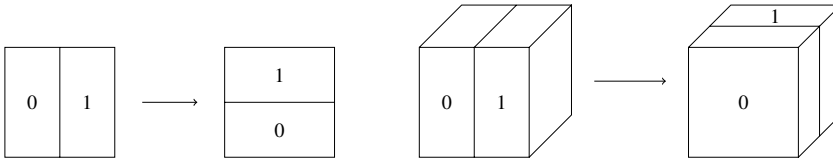


FIGURE 2. Baker's maps when $d = 2$ (left) and $d = 3$ (right).

the formula

$$w \mapsto \begin{cases} \alpha \cdot \mathbf{0}_d \cdot u & \text{if } w = \alpha \cdot \mathbf{0}_1 \cdot u \text{ for some } u \in \mathbb{C}^n, \\ \alpha \cdot \mathbf{1}_d \cdot u & \text{if } w = \alpha \cdot \mathbf{1}_1 \cdot u \text{ for some } u \in \mathbb{C}^n, \\ w & \text{otherwise (that is, if } \alpha \not\preceq w \text{).} \end{cases}$$

When we refer below to the element $B_d(\alpha)$ evaluating to an ‘index d ’ baker’s map, we mean that it evaluates to the homeomorphism of \mathbb{C}^n given by this formula. All baker’s maps arising within our work will have such a form (for some address $\alpha \in \Omega$ and some $d \geq 2$).

The elements $B_d(\alpha)$ in G_∞ are defined via the formulae expressing baker’s maps in terms of transpositions found in [5]. In Section 2, we observe that the behaviour of baker’s maps can be deduced from the relations assumed about transpositions. Lemmas 2.2–2.4 give the properties upon which we depend. In summary, while Brin [5] establishes simplicity of nV by expressing baker’s maps as products of transpositions, we use relational properties between transpositions to produce information about baker’s maps to establish our presentation in Theorem 1.1.

In Section 3, we reduce our infinite presentation to a finite presentation (the relations are those listed in Relations R1–R7 in that section).

THEOREM 1.2. *Let $n \geq 2$. Brin’s higher-dimensional Thompson group nV has a finite presentation with three generators and $2n^2 + 3n + 11$ relations.*

To prove this theorem, we begin with transpositions with entries from the set Δ of addresses $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha_d| = 2$ for $1 \leq d \leq n$. Thus, as a base point in an induction argument, we assume that we have a subgroup isomorphic to (a quotient of) the symmetric group of degree 4^n . In an induction argument, we build further transpositions by successively conjugating the transpositions constructed at a previous stage and then finally exploit the split relations in Equation (1-4) to complete the definitions. In this way, we demonstrate that the group G with the presentations provided in Theorem 1.2 is a quotient of the group G_∞ described in Theorem 1.1.

Finally, by applying Tietze transformations, we deduce the following corollary.

COROLLARY 1.3. *Let $n \geq 2$. Brin’s higher-dimensional Thompson group nV has a finite presentation with two generators and $2n^2 + 3n + 13$ relations.*

REMARKS 1.4. In common with the finite presentation given by Hennig–Matucci [17], the number of relations we use is quadratic in the dimension n . One could ask whether there are presentations for nV on two generators but where the number of relations grows at most linearly in n . In our case, the quadratic function arises from the family of Relations R5, which is used to ensure the well definedness of transpositions $(\alpha \beta)$ where two coordinates of α have length 3 and all remaining coordinates of α and all those of β have length 2. Although the growth in the number of relations relative to the dimension of the space acted upon seems reasonable, surprising results such as that of Guralnick *et al.* [16] stand in contrast to expectations. The arguments used in [16] employ a process that has been termed the *Burnside procedure* and presented in detail in the appendix of [20]. This process can be used to reduce some large presentation of a group to a much smaller one. Potentially, it could be applied to the infinite presentation found in Theorem 1.1 and one might wonder how the result would compare with Theorem 1.2. It seemed to the author that the most direct application of the Burnside procedure (if successful) would likely result in more relations. Nevertheless, it remains an interesting question whether smaller presentations exist for Brin's groups nV .

2. The infinite presentation for nV

We devote this section to establishing Theorem 1.1. Accordingly, we define G_∞ to be the group presented by the generators $\mathcal{A} = \{(\alpha \beta) \mid \alpha, \beta \in \Omega, \alpha \perp \beta\}$ subject to the family of Relations (1-2)–(1-4). In this context, we shall use the term *transposition* for any element $(\alpha \beta)$ appearing in the generating set \mathcal{A} . It was observed by Brin [5] that the group nV is generated by the corresponding transpositions of basic open sets of Γ . It is readily verified that these homeomorphisms satisfy the relations listed in Theorem 1.1. Hence, there exists a surjective homomorphism $\phi: G_\infty \rightarrow nV$ that maps $(\alpha \beta)$ to the corresponding transposition in nV . In what follows, we speak of *evaluating* a product g in nV to mean the effect of applying the homomorphism ϕ to the element $g \in G_\infty$.

We can extend the definition appearing in Equation (1-1) to a product g of transpositions, say $g = g_1 g_2 \dots g_k$, where each $g_i \in \mathcal{A}$, by defining $\alpha \bullet g$ to equal the value obtained by successively applying Equation (1-1) with each transposition g_i . Note that this is, strictly speaking, a function of the word in \mathcal{A} representing g rather than depending upon g as an element of G_∞ . With this extended definition, if $\alpha, \beta \in \Omega$ with $\alpha \perp \beta$ and $g \in G_\infty$ is expressed as a product of transpositions such that both $\alpha \bullet g$ and $\beta \bullet g$ are defined, then it follows by repeated use of the conjugacy relations in Equation (1-3) that

$$(\alpha \beta)^g = g^{-1} (\alpha \beta) g = (\alpha \bullet g \beta \bullet g).$$

Note that $\alpha \bullet g$, when it is defined, coincides with the value obtained if the product g is evaluated as an element of the Brin's higher-dimensional Thompson group nV and then $\alpha \bullet g$ is calculated via the natural partial action of nV upon the addresses Ω . The only difference is that there may exist some addresses α for which $\alpha \bullet g$ is not defined

for our given word representing g but for which the corresponding transformation in nV does have an action defined upon α . However, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and *provided* the words α_i are sufficiently long, then $\alpha \bullet g$ is defined and hence coincides with the value obtained via the partial action of nV upon Ω .

We establish that G_∞ is isomorphic to the group nV by demonstrating that a family of relations found within Hennig–Matucci’s work [17] can be deduced from our defining relations. We define elements of our group G_∞ that correspond to the family of generators that are used in [17]. Some are readily constructed using transpositions $(\alpha \ \beta)$ but others depend upon building analogues of the baker’s maps. Brin’s paper [5] describes how to construct a baker’s map from transpositions in his Lemma 3. By following this recipe we are able to define the required elements of G_∞ . Moreover, it then follows that the products of transformations we define evaluate to the required baker’s maps in nV and hence we can determine the value of $\alpha \bullet g$ for such products g provided the coordinates of α are sufficiently long.

If $\alpha, \beta \in \Omega$ with $\alpha \perp \beta$ and d is an index with $2 \leq d \leq n$, we define

$$A_d(\alpha, \beta) = (\alpha.0_1 \ \beta.0_d) (\alpha.1_1 \ \beta.1_d) (\alpha \ \beta). \tag{2-1}$$

We define further elements of G_∞ in terms of this product as follows:

$$\begin{aligned} \hat{B}_d(\mathbf{0}_1, \mathbf{0}_d) &= A_d(\mathbf{0}_1, \gamma) (\mathbf{0}_1.0_1 \ \mathbf{0}_1.1_0) A_d(\gamma, \mathbf{0}_1.1_d) \\ \hat{B}_d(\mathbf{0}_1, \mathbf{1}_d) &= A_d(\mathbf{0}_1, \gamma) (\mathbf{0}_1.0_1 \ \mathbf{0}_1.1_0) A_d(\gamma, \mathbf{0}_1.0_d) \\ \hat{B}_d(\mathbf{1}_1, \mathbf{0}_d) &= A_d(\mathbf{1}_1, \gamma) (\mathbf{1}_1.0_1 \ \mathbf{1}_1.1_0) A_d(\gamma, \mathbf{1}_1.1_d) \\ \hat{B}_d(\mathbf{1}_1, \mathbf{1}_d) &= A_d(\mathbf{1}_1, \gamma) (\mathbf{1}_1.0_1 \ \mathbf{1}_1.1_0) A_d(\gamma, \mathbf{1}_1.0_d), \end{aligned} \tag{2-2}$$

where $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ is an address in Ω satisfying $|\gamma_1|, |\gamma_d| \geq 1$ and the additional condition that $\gamma \perp \mathbf{0}_1$ in the first two definitions and that $\gamma \perp \mathbf{1}_1$ in the last two definitions in Equation (2-2). Further, we then define:

$$\begin{aligned} B_d(\mathbf{0}_1) &= \hat{B}_d(\mathbf{0}_1.0_d) \hat{B}_d(\mathbf{0}_1.1_d) (\mathbf{0}_1.0_1 \ \mathbf{0}_1.1_0) \\ B_d(\mathbf{1}_1) &= \hat{B}_d(\mathbf{1}_1.0_d) \hat{B}_d(\mathbf{1}_1.1_d) (\mathbf{1}_1.0_1 \ \mathbf{1}_1.1_0) \end{aligned} \tag{2-3}$$

These three types of element are the analogues of the maps arising within the proof of [5, Lemma 3] and their definition precisely follows that proof. Consequently, the product $A_d(\alpha, \beta)$ evaluates in the group nV to the composite of an ‘index d ’ baker’s map with support $\Gamma(\alpha)$ and the inverse of an ‘index d ’ baker’s map with support $\Gamma(\beta)$. The subsequent elements $\hat{B}_d(\alpha)$ and $B_d(\alpha)$ both evaluate to the ‘index d ’ baker’s map with support $\Gamma(\alpha)$. The difference is that the address $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ that we have first defined them upon satisfies $|\alpha_1| = 1$ for both products, but the \hat{B}_d version requires $|\alpha_d| = 1$ while B_d permits α_d to be empty. One notes that to define a baker’s map on the whole space $\Gamma = \mathbb{C}^n$ (that is, with address ε) requires a further such definition. As this is (up to choice of index d) a single element in G_∞ , we delay the definition of this element, which appears as $C_{0,d}$ below (see Equation (2-5)).

To extend the baker’s maps to arbitrary addresses, we use another convenient notation. If g is an element of G_∞ and $\delta \in \Omega$, we write $\delta.g$ for the element of G_∞ obtained by inserting δ as a prefix in both entries of every transposition appearing in the product g . Since Relations (1-2)–(1-4) are closed under performing such insertions, it follows that (i) $\delta.g$ is a well-defined element of G_∞ and (ii) if $u = v$ is a relation that holds in G_∞ , then $\delta.u = \delta.v$ also holds in G_∞ . We use the latter observation to reduce the number of calculations required.

In terms of this prefix notation, observe that

$$\delta.A_d(\alpha, \beta) = A_d(\delta\alpha, \delta\beta)$$

for any addresses α, β and δ with $\alpha \perp \beta$. For our baker’s maps, we define $\hat{B}_d(\alpha)$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ satisfies $|\alpha_1|, |\alpha_d| \geq 1$, by writing $\alpha = \delta\beta$ where $\beta \in \{\mathbf{0}_1.\mathbf{0}_d, \mathbf{0}_1.\mathbf{1}_d, \mathbf{1}_1.\mathbf{0}_d, \mathbf{1}_1.\mathbf{1}_d\}$ and then setting

$$\hat{B}_d(\alpha) = \delta.\hat{B}_d(\beta),$$

where $\hat{B}_d(\beta)$ is as defined in Equation (2-2). Similarly, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ satisfies $|\alpha_1| \geq 1$, by writing $\alpha = \delta\beta$ where $\beta \in \{\mathbf{0}_1, \mathbf{1}_1\}$, we define

$$B_d(\alpha) = \delta.B_d(\beta).$$

Note that inserting this prefix δ into the definition in Equation (2-3) yields

$$B_d(\alpha) = \hat{B}_d(\alpha.\mathbf{0}_d) \hat{B}_d(\alpha.\mathbf{1}_d) (\alpha.\mathbf{01}_d \ \alpha.\mathbf{10}_d)$$

for any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha_1| \geq 1$. Now, if α is an address such that both $\hat{B}_d(\alpha)$ and $B_d(\alpha)$ are defined, then they evaluate to the same baker’s map in the group nV . However, the former is not defined for addresses $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with α_d empty.

LEMMA 2.1. *Let $\alpha, \beta, \gamma, \delta \in \Omega$ be addresses with $\alpha \perp \beta$ and $\gamma \perp \delta$. Let d, d' be indices in the range $2 \leq d, d' \leq n$. Then:*

- (i) $A_d(\alpha, \beta)^{(\gamma \ \delta)} = A_d(\alpha \bullet (\gamma \ \delta), \beta \bullet (\gamma \ \delta))$, whenever both $\alpha \bullet (\gamma \ \delta)$ and $\beta \bullet (\gamma \ \delta)$ are defined;
- (ii) if every pair from $\{\alpha, \beta, \gamma, \delta\}$ are incomparable, then $A_d(\alpha, \beta)$ and $A_d(\gamma, \delta)$ commute;
- (iii) $A_d(\alpha, \beta) = (\alpha.\mathbf{01}_1 \ \alpha.\mathbf{10}_1) A_d(\alpha.\mathbf{0}_1, \beta.\mathbf{0}_1) A_d(\alpha.\mathbf{1}_1, \beta.\mathbf{1}_1) (\beta.\mathbf{01}_1 \ \beta.\mathbf{10}_1)$;
- (iv) if $d' \neq d$, then $A_d(\alpha, \beta) = A_d(\alpha.\mathbf{0}_{d'}, \beta.\mathbf{0}_{d'}) A_d(\alpha.\mathbf{1}_{d'}, \beta.\mathbf{1}_{d'})$.

PROOF. Part (i) follows immediately by Equation (1-3). We deduce part (ii) by noting that Equation (2-1) expresses $A_d(\alpha, \beta)$ as a product of transpositions each of which, by part (i), commutes with $A_d(\gamma, \delta)$.

(iii) We expand the right-hand side using the definition of the terms A_d and collect the transpositions using the conjugacy relations in Equation (1-3):

$$\begin{aligned}
 & (\alpha.01_1 \ \alpha.10_1) A_d(\alpha.0_1, \beta.0_1) A_d(\alpha.1_1, \beta.1_1) (\beta.01_1 \ \beta.10_1) \\
 &= (\alpha.01_1 \ \alpha.10_1) (\alpha.00_1 \ \beta.0_1.0_d) (\alpha.01_1 \ \beta.0_1.1_d) (\alpha.0_1 \ \beta.0_1) \\
 &\quad \cdot (\alpha.10_1 \ \beta.1_1.0_d) (\alpha.11_1 \ \beta.1_1.1_d) (\alpha.1_1 \ \beta.1_1) (\beta.01_1 \ \beta.10_1) \\
 &= (\alpha.00_1 \ \beta.0_1.0_d) (\alpha.01_1 \ \alpha.10_1) (\alpha.10_1 \ \beta.1_1.0_d) (\alpha.01_1 \ \beta.0_1.1_d) \\
 &\quad \cdot (\alpha.11_1 \ \beta.1_1.1_d) (\alpha.0_1 \ \beta.0_1) (\alpha.1_1 \ \beta.1_1) (\beta.01_1 \ \beta.10_1) \\
 &= (\alpha.00_1 \ \beta.0_1.0_d) (\alpha.01_1 \ \beta.1_1.0_d) (\alpha.10_1 \ \beta.0_1.1_d) (\alpha.11_1 \ \beta.1_1.1_d) \\
 &\quad \cdot (\alpha.01_1 \ \alpha.10_1) (\alpha.0_1 \ \beta.0_1) (\alpha.1_1 \ \beta.1_1) (\beta.01_1 \ \beta.10_1) \\
 &= (\alpha.0_1 \ \beta.0_d) (\alpha.1_1 \ \beta.1_d) (\alpha.01_1 \ \alpha.10_1) (\alpha \ \beta) (\beta.01_1 \ \beta.10_1) \quad \text{(by Equation (1-4))} \\
 &= (\alpha.0_1 \ \beta.0_d) (\alpha.1_1 \ \beta.1_d) (\alpha \ \beta) (\beta.01_1 \ \beta.10_1)^2 = A_d(\alpha, \beta).
 \end{aligned}$$

(iv) Apply the split relation in Equation (1-4) in the d' th coordinate to each transposition appearing in $A_d(\alpha, \beta)$. For example, $(\alpha.0_1 \ \beta.0_d) = (\alpha.0_{d'}.0_1 \ \beta.0_{d'}.0_d) (\alpha.1_{d'}.0_1 \beta.1_{d'}.0_d)$ and similarly for the other transpositions. Every pair of addresses from $\alpha.0_{d'}$, $\alpha.1_{d'}$, $\beta.0_{d'}$ and $\beta.1_{d'}$ are incomparable so rearranging the resulting formula for $A_d(\alpha, \beta)$ yields the claimed result. \square

LEMMA 2.2. *Let d, d' be indices in the range $2 \leq d, d' \leq n$.*

- (i) *If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega$ satisfies $|\alpha_1|, |\alpha_d| \geq 1$ and if $\zeta, \eta \in \Omega$ are such that every pair from $\{\alpha, \zeta, \eta\}$ are incomparable, then $\hat{B}_d(\alpha)$ commutes with $(\zeta \ \eta)$.*
- (ii) *If $\alpha = \delta.x_1.y_d$ for some $\delta \in \Omega$ and some $x, y \in \{0, 1\}$, then*

$$\hat{B}_d(\alpha) = A_d(\delta.x_1, \gamma) (\delta.x_1.01_d \ \delta.x_1.10_d) A_d(\gamma, \delta.x_1.\bar{y}_d) \tag{2-4}$$

for any address $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ such that $\gamma \perp \delta.x_1$ and $|\gamma_1|, |\gamma_d| \geq 1$.

- (iii) *If $y \in \{0, 1\}$, then $\hat{B}_d(0_1.y_d)^{(0_1 \ 1_1)} = \hat{B}_d(1_1.y_d)$.*
- (iv) *If $y \in \{0, 1\}$, then $\hat{B}_d(0_1.y_d)^{(0_1 \ 10_1)} = \hat{B}_d(10_1.y_d)$ and $\hat{B}_d(0_1.y_d)^{(0_1 \ 11_1)} = \hat{B}_d(11_1.y_d)$.*
- (v) *If $x \in \{0, 1\}$, then $\hat{B}_d(x0_1.1_d)$ and $\hat{B}_d(x1_1.0_d)$ commute.*
- (vi) *If $x, y \in \{0, 1\}$, then $\hat{B}_d(x_1.y_d) = (x01_1.y_d \ x10_1.y_d) \hat{B}_d(x0_1.y_d) \hat{B}_d(x1_1.y_d)$.*
- (vii) *If $d' \neq d$, then $\hat{B}_d(x_1.y_d) = \hat{B}_d(x_1.y_d.0_{d'}) \hat{B}_d(x_1.y_d.1_{d'})$.*

If we take $\delta = \varepsilon$ in part (ii), then it tells us that the definitions in Equation (2-2) are independent of the choice of address γ used. Consequently, $\hat{B}_d(\alpha)$ does depend only on the address α .

PROOF. (i) Repeatedly apply the split relation in Equation (1-4) to express $(\zeta \ \eta)$ as a product of transpositions $(\zeta' \ \eta')$ having entries with sufficiently long coordinates that the values $\zeta' \bullet \hat{B}_d(\alpha)$ and $\eta' \bullet \hat{B}_d(\alpha)$ are defined. These values therefore coincide with those obtained when the corresponding baker’s map in nV is applied to ζ' and η' . Since ζ and η are both incomparable with α , we conclude $\zeta' \bullet \hat{B}_d(\alpha) = \zeta'$ and

$\eta' \bullet \hat{B}_d(\alpha) = \eta'$. It then follows, by the conjugacy relation in Equation (1-3), that $\hat{B}_d(\alpha)$ commutes with all such $(\zeta' \eta')$ and hence also with their product $(\zeta \eta)$.

(ii) One formula of the form given in Equation (2-4) for $\hat{B}_d(\delta.x_1.y_d)$ is obtained by inserting δ as prefix into the entries of the transpositions in the definition of $\hat{B}_d(x_1.y_d)$ in Equation (2-2). Start with one such valid Equation (2-4) involving a particular address γ and consider another potential address $\gamma' = (\gamma'_1, \gamma'_2, \dots, \gamma'_n)$ satisfying $\gamma' \perp \delta.x_1$ and $|\gamma'_d| \geq 1$. If $\gamma' \perp \gamma$, conjugate by $(\gamma \gamma')$ and use part (i) to produce the required formula for $\hat{B}_d(\delta.x_1.y_d)$. If γ and γ' are not incomparable, then as $|\gamma_d|, |\gamma'_d| \geq 1$ there is another address ζ , with nonempty d th coordinate, that is incomparable with all three of γ, γ' and $\delta.x_1$. Now conjugate by the product $(\gamma \zeta)(\gamma' \zeta)$, again using part (i), to produce the required Equation (2-4) involving the address γ' .

Part (iii) follows immediately from the definition of the \hat{B}_d elements.

(iv) We establish here the first equation in the case when $y = 0$. The other formulae are similar. First, use part (ii) to assume that in the expression of Equation (2-2) for $\hat{B}_d(\mathbf{0}_1.\mathbf{0}_d)$, the address $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ satisfies $11 \leq \gamma_1$. Consequently, $\gamma \bullet (\mathbf{0}_1 \mathbf{1}_1) = \gamma$ and $\gamma = \mathbf{1}_1.\delta$, where $\delta \perp \mathbf{0}_1$. Hence, using Lemma 2.1(i),

$$\begin{aligned} \hat{B}_d(\mathbf{0}_1.\mathbf{0}_d)^{(\mathbf{0}_1 \mathbf{1}_1)} &= A_d(\mathbf{1}_1, \gamma) (\mathbf{1}_1.\mathbf{01}_d \mathbf{1}_1.\mathbf{10}_d) A_d(\gamma, \mathbf{1}_1.\mathbf{1}_d) \\ &= \mathbf{1}_1.(A_d(\mathbf{0}_1, \delta) (\mathbf{0}_1.\mathbf{01}_d \mathbf{0}_1.\mathbf{10}_d) A_d(\delta, \mathbf{0}_1.\mathbf{1}_d)) = \mathbf{1}_1.\hat{B}_d(\mathbf{0}_1.\mathbf{0}_d) = \hat{B}_d(\mathbf{1}_1.\mathbf{0}_d). \end{aligned}$$

(v) Using part (ii), we can assume that

$$\begin{aligned} \hat{B}_d(x\mathbf{0}_1.\mathbf{1}_d) &= A_d(x\mathbf{0}_1, \gamma) (x\mathbf{0}_1.\mathbf{01}_d x\mathbf{0}_1.\mathbf{10}_d) A_d(\gamma, x\mathbf{0}_1.\mathbf{0}_d) \\ \hat{B}_d(x\mathbf{1}_1.\mathbf{0}_d) &= A_d(x\mathbf{1}_1, \gamma') (x\mathbf{1}_1.\mathbf{01}_d x\mathbf{1}_1.\mathbf{10}_d) A_d(\gamma', x\mathbf{1}_1.\mathbf{1}_d), \end{aligned}$$

where γ and γ' are addresses with $\gamma \perp \gamma'$ and $x_1 \perp \gamma, \gamma'$. These commute by Lemma 2.1(i) and (ii).

(vi) We present the case $x = y = 0$, with the other cases established by similar calculations. Recall that $\gamma \perp \mathbf{0}_1$ in our definition by Equation (2-2) of $\hat{B}_d(\mathbf{0}_1.\mathbf{0}_d)$. In the following calculation, we begin by applying Lemma 2.1(iii) to the terms appearing in the definition of $\hat{B}_d(\mathbf{0}_1.\mathbf{0}_d)$:

$$\begin{aligned} \hat{B}_d(\mathbf{0}_1.\mathbf{0}_d) &= (\mathbf{001}_1 \mathbf{010}_1) A_d(\mathbf{00}_1, \gamma.\mathbf{0}_1) A_d(\mathbf{01}_1, \gamma.\mathbf{1}_1) (\gamma.\mathbf{01}_1 \gamma.\mathbf{10}_1) (\mathbf{0}_1.\mathbf{01}_d \mathbf{0}_1.\mathbf{10}_d) \\ &\quad \cdot (\gamma.\mathbf{01}_1 \gamma.\mathbf{10}_1) A_d(\gamma.\mathbf{0}_1, \mathbf{00}_1.\mathbf{1}_d) A_d(\gamma.\mathbf{1}_1, \mathbf{01}_1.\mathbf{1}_d) (\mathbf{001}_1.\mathbf{1}_d \mathbf{010}_1.\mathbf{1}_d) \\ &= (\mathbf{001}_1 \mathbf{010}_1) A_d(\mathbf{00}_1, \gamma.\mathbf{0}_1) A_d(\mathbf{01}_1, \gamma.\mathbf{1}_1) (\mathbf{00}_1.\mathbf{01}_d \mathbf{00}_1.\mathbf{10}_d) \\ &\quad \cdot (\mathbf{01}_1.\mathbf{01}_d \mathbf{01}_1.\mathbf{10}_d) A_d(\gamma.\mathbf{0}_1, \mathbf{00}_1.\mathbf{1}_d) A_d(\gamma.\mathbf{1}_1, \mathbf{01}_1.\mathbf{1}_d) (\mathbf{001}_1.\mathbf{1}_d \mathbf{010}_1.\mathbf{1}_d) \\ &= (\mathbf{001}_1 \mathbf{010}_1) A_d(\mathbf{00}_1, \gamma.\mathbf{0}_1) (\mathbf{00}_1.\mathbf{01}_d \mathbf{00}_1.\mathbf{10}_d) A_d(\gamma.\mathbf{0}_1, \mathbf{00}_1.\mathbf{1}_d) \\ &\quad \cdot A_d(\mathbf{01}_1, \gamma.\mathbf{1}_1) (\mathbf{01}_1.\mathbf{01}_d \mathbf{01}_1.\mathbf{10}_d) A_d(\gamma.\mathbf{1}_1, \mathbf{01}_1.\mathbf{1}_d) (\mathbf{001}_1.\mathbf{1}_d \mathbf{010}_1.\mathbf{1}_d) \\ &= (\mathbf{001}_1 \mathbf{010}_1) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d) (\mathbf{001}_1.\mathbf{1}_d \mathbf{010}_1.\mathbf{1}_d) \quad (\text{using part (ii)}) \\ &= (\mathbf{001}_1.\mathbf{0}_d \mathbf{010}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d). \end{aligned}$$

(vii) This follows by applying Lemma 2.1(iv) and the split relation in Equation (1-4) to terms appearing in Equation (2-2), and then rearranging in a similar way to the proof of Lemma 2.1(iv). □

LEMMA 2.3. *Let d, d' be indices in the range $2 \leq d, d' \leq n$.*

- (i) *If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega$ satisfies $|\alpha_1| \geq 1$ and if $\zeta, \eta \in \Omega$ are such that every pair from $\{\alpha, \zeta, \eta\}$ is incomparable, then $B_d(\alpha)$ commutes with $(\zeta \ \eta)$. In particular, for any $x \in \{0, 1\}$, the element $B_d(x_1)$ commutes with any element of the form \bar{x}_1 .g.*
- (ii) $B_d(\mathbf{0}_1)^{(0_1 \ 1_1)} = B_d(\mathbf{1}_1)$.
- (iii) $B_d(\mathbf{0}_1)^{(0_1 \ 10_1)} = B_d(\mathbf{10}_1)$ and $B_d(\mathbf{0}_1)^{(0_1 \ 11_1)} = B_d(\mathbf{11}_1)$.
- (iv) *If $x \in \{0, 1\}$, then $B_d(x_1) = (x\mathbf{01}_1 \ x\mathbf{10}_1) B_d(x\mathbf{0}_1) B_d(x\mathbf{1}_1)$.*
- (v) $B_d(\mathbf{0}_1)$ and $B_{d'}(\mathbf{1}_1)$ commute.
- (vi) *If $d' \neq d$, then $B_d(x_1) = B_d(x_1.\mathbf{0}_{d'}) B_d(x_1.\mathbf{1}_{d'})$.*

PROOF. The first section of part (i) is established by the same argument as used in Lemma 2.2(i) and the remainder follows immediately. Parts (ii) and (iii) follow using parts (iii) and (iv), respectively, of Lemma 2.2, while part (vi) is an extension of Lemma 2.2(vii) that is established similarly.

We establish part (iv) in the case that $x = 0$. First, apply Lemma 2.2(vi) to the two \hat{B}_d terms in the definition of Equation (2-3) and then use parts (i) and (v) of Lemma 2.2 to rearrange terms:

$$\begin{aligned}
 B_d(\mathbf{0}_1) &= (\mathbf{001}_1.\mathbf{0}_d \ \mathbf{010}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d) (\mathbf{001}_1.\mathbf{1}_d \ \mathbf{010}_1.\mathbf{1}_d) \\
 &\quad \cdot \hat{B}_d(\mathbf{00}_1.\mathbf{1}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{1}_d) (\mathbf{0}_1.\mathbf{01}_d \ \mathbf{0}_1.\mathbf{10}_d) \\
 &= (\mathbf{001}_1.\mathbf{0}_d \ \mathbf{010}_1.\mathbf{0}_d) (\mathbf{001}_1.\mathbf{1}_d \ \mathbf{010}_1.\mathbf{1}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{1}_d) \\
 &\quad \cdot \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{1}_d) (\mathbf{0}_1.\mathbf{01}_d \ \mathbf{0}_1.\mathbf{10}_d) \\
 &= (\mathbf{001}_1 \ \mathbf{010}_1) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{1}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{1}_d) \\
 &\quad \cdot (\mathbf{00}_1.\mathbf{01}_d \ \mathbf{00}_1.\mathbf{10}_d) (\mathbf{01}_1.\mathbf{01}_d \ \mathbf{01}_1.\mathbf{10}_d) \\
 &= (\mathbf{001}_1 \ \mathbf{010}_1) \hat{B}_d(\mathbf{00}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{00}_1.\mathbf{1}_d) (\mathbf{00}_1.\mathbf{01}_d \ \mathbf{00}_1.\mathbf{10}_d) \\
 &\quad \cdot \hat{B}_d(\mathbf{01}_1.\mathbf{0}_d) \hat{B}_d(\mathbf{01}_1.\mathbf{1}_d) (\mathbf{01}_1.\mathbf{01}_d \ \mathbf{01}_1.\mathbf{10}_d) \\
 &= (\mathbf{001}_1 \ \mathbf{010}_1) B_d(\mathbf{00}_1) B_d(\mathbf{01}_1).
 \end{aligned}$$

Finally, it then follows that $B_d(\mathbf{0}_1) = \mathbf{0}_1.((\mathbf{01}_1 \ \mathbf{10}_1) B_d(\mathbf{0}_1) B_d(\mathbf{1}_1))$ commutes with $B_{d'}(\mathbf{1}_1)$ using part (i). Hence, we have established the remaining part (v). □

We now consider one of the presentations for Brin’s higher-dimensional Thompson group nV given by Hennig–Matucci [17]. They define generators $X_{m,d}$ (for $m \geq 0$ and $1 \leq d \leq n$), $C_{m,d}$ (for $m \geq 0$ and $2 \leq d \leq n$), π_m (for $m \geq 0$) and $\bar{\pi}_m$ (for $m \geq 0$) and describe eighteen families of relations (numbered (1)–(18) on pages 59 and 60 of [17]). They observe in [17, Theorem 23] that these do indeed give a presentation for nV . Since we write our maps on the right, we convert the relations to our setting by

reversing each one and record these now for reference. We have also changed some of the labels on Hennig–Matucci’s generators appearing in the lists so that our arguments can be unified when establishing Proposition 2.5 below. In the families of Relations (HM1)–(HM7), one should assume that $1 \leq d, d' \leq n$:

$$X_{m,d} X_{q,d'} = X_{q+1,d'} X_{m,d} \quad \text{for } m < q, \quad \text{(HM1)}$$

$$X_{m,d} \pi_q = \pi_{q+1} X_{m,d} \quad \text{for } m < q, \quad \text{(HM2)}$$

$$X_{m,d} \pi_m = \pi_{m+1} \pi_m X_{m+1,d} \quad \text{for } m \geq 0, \quad \text{(HM3)}$$

$$X_{q,d} \pi_m = \pi_m X_{q,d} \quad \text{for } q > m + 1, \quad \text{(HM4)}$$

$$X_{m,d} \bar{\pi}_q = \bar{\pi}_{q+1} X_{m,d} \quad \text{for } m < q, \quad \text{(HM5)}$$

$$X_{m,1} \bar{\pi}_m = \bar{\pi}_{m+1} \pi_m \quad \text{for } m \geq 0, \quad \text{(HM6)}$$

$$X_{m,d'} X_{m+1,d'} X_{m,d} = \pi_{m+1} X_{m,d} X_{m+1,d} X_{m,d'} \quad \text{for } m \geq 0 \text{ and } d \neq d'. \quad \text{(HM7)}$$

The second collection of relations is as below. Note that we have adjusted the range of the parameters in Equation (HM8) to bring it into line with the relations given by Brin (see [4, Equation (22)]).

$$\pi_m \pi_q = \pi_q \pi_m \quad \text{for } |m - q| \geq 2, \quad \text{(HM8)}$$

$$\pi_m \pi_{m+1} \pi_m = \pi_{m+1} \pi_m \pi_{m+1} \quad \text{for } m \geq 0, \quad \text{(HM9)}$$

$$\pi_m \bar{\pi}_q = \bar{\pi}_q \pi_m \quad \text{for } q \geq m + 2, \quad \text{(HM10)}$$

$$\pi_m \bar{\pi}_{m+1} \pi_m = \bar{\pi}_{m+1} \pi_m \bar{\pi}_{m+1} \quad \text{for } m \geq 0, \quad \text{(HM11)}$$

$$\pi_m^2 = 1 \quad \text{for } m \geq 0, \quad \text{(HM12)}$$

$$\bar{\pi}_m^2 = 1 \quad \text{for } m \geq 0. \quad \text{(HM13)}$$

Finally, in the families of Equations (HM14)–(HM18), $2 \leq d \leq n$ and $1 \leq d' \leq n$, unless otherwise indicated:

$$X_{m,d} \bar{\pi}_m = \bar{\pi}_{m+1} \pi_m C_{m+1,d} \quad \text{for } m \geq 0, \quad \text{(HM14)}$$

$$X_{m,d'} C_{q,d} = C_{q+1,d} X_{m,d'} \quad \text{for } m < q, \quad \text{(HM15)}$$

$$X_{m,1} C_{m,d} = \pi_{m+1} C_{m+2,d} X_{m,d} \quad \text{for } m \geq 0, \quad \text{(HM16)}$$

$$C_{q,d} \pi_m = \pi_m C_{q,d} \quad \text{for } q > m + 1, \quad \text{(HM17)}$$

$$C_{m+2,d'} X_{m,d'} C_{m,d} = \pi_{m+1} C_{m+2,d} X_{m,d} C_{m,d'} \quad \text{for } m \geq 0, 1 < d' < d \leq n. \quad \text{(HM18)}$$

We now define the elements of our group G_∞ that correspond to the above generators. In the following, d is an index with $2 \leq d \leq n$. First, we set

$$\begin{aligned} X_{0,1} &= (\mathbf{0}_1 \ \mathbf{1}_1) (\mathbf{0}_1 \ \mathbf{10}_1) (\mathbf{10}_1 \ \mathbf{11}_1), & X_{0,d} &= X_{0,1} B_d(\mathbf{1}_1), \\ \pi_0 &= (\mathbf{01}_1 \ \mathbf{1}_1), & \bar{\pi}_0 &= (\mathbf{0}_1 \ \mathbf{1}_1), \\ C_{0,d} &= (\mathbf{01}_1 \ \mathbf{10}_1) B_d(\mathbf{0}_1) B_d(\mathbf{1}_1), & C_{1,d} &= B_d(\mathbf{0}_1). \end{aligned} \quad \text{(2-5)}$$

These are extended, for positive integers $m \geq 1$, to

$$\begin{aligned}
 X_{m,d} &= \mathbf{0}_1^m \cdot X_{0,d} && \text{for } 1 \leq d \leq n, \\
 \pi_m &= \mathbf{0}_1^m \cdot \pi_0, \\
 \bar{\pi}_m &= \mathbf{0}_1^m \cdot \bar{\pi}_0, \\
 C_{m,d} &= \mathbf{0}_1^{m-1} \cdot C_{1,d} && \text{for } m \geq 2 \text{ and } 2 \leq d \leq n,
 \end{aligned}
 \tag{2-6}$$

where $\mathbf{0}_1^m$ denotes $\mathbf{00} \dots \mathbf{0}_1 = (00 \dots 0, \varepsilon, \dots, \varepsilon)$ with the word $00 \dots 0$ having length m .

Note that, by Lemma 2.3(iv), $C_{1,d} = \mathbf{0}_1 \cdot C_{0,d}$. Consequently, the definition of $C_{m,d}$ can be extended to include the case $m = 1$; that is, $C_{m,d} = \mathbf{0}_1^m \cdot C_{0,d}$ for all $m \geq 1$ and all indices d in the range $2 \leq d \leq n$. This will enable us to treat these baker’s maps in a uniform manner.

LEMMA 2.4. *Let d and d' be indices in the range $2 \leq d, d' \leq n$.*

- (i) $B_d(\mathbf{0}_1)^{C_{0,d'}} = \mathbf{0}_{d'} \cdot C_{0,d}$ and $B_d(\mathbf{1}_1)^{C_{0,d'}} = \mathbf{1}_{d'} \cdot C_{0,d}$.
- (ii) If $d \neq d'$, then $C_{0,d} = (\mathbf{0}_{d'} \cdot C_{0,d}) (\mathbf{1}_{d'} \cdot C_{0,d})$.

By inserting the prefix $\mathbf{1}_1$ into the entries of transpositions in part (i) of this lemma, it follows with use of Lemma 2.3(iv) that:

$$B_d(\mathbf{10}_1)^{B_{d'}(\mathbf{1}_1)} = B_d(\mathbf{1}_1 \cdot \mathbf{0}_{d'}) \quad \text{and} \quad B_d(\mathbf{11}_1)^{B_{d'}(\mathbf{1}_1)} = B_d(\mathbf{1}_1 \cdot \mathbf{1}_{d'}).
 \tag{2-7}$$

PROOF. (i) First note that, by suitable choice of γ appearing in the definition of Equation (2-2), we can express $C_{0,d'}$ as a product of transpositions whose entries have nonempty coordinates only for index 1 and index d' . Consequently, provided the index 1 and index d' coordinates of an address ζ are sufficiently long, $\zeta \bullet C_{0,d'}$ is defined. Furthermore, $C_{0,d'}$ evaluates in nV to the (primary) baker’s map with full support on Γ , so this value $\zeta \bullet C_{0,d'}$ coincides with that obtained when we act with the baker’s map. Now if $x \in \{0, 1\}$, then $(x_1 \cdot \zeta) \bullet C_{0,d'}$ is also defined (as the required coordinates are sufficiently long) and equals $x_{d'} \cdot \zeta$ in view of how the baker’s map acts.

If $(\alpha \beta)$ is any transposition, apply Equation (1-4) repeatedly to express it as a product of transpositions $(\zeta \eta)$ with the index 1 and index d coordinates of ζ and η sufficiently long that the partial action of $C_{0,d'}$ upon them is defined. By inserting x_1 as a prefix into all the transpositions involved, we express $(x_1 \cdot \alpha \ x_1 \cdot \beta)$ as a product of transpositions $(x_1 \cdot \zeta \ x_1 \cdot \eta)$. Since $x_1 \cdot \zeta \bullet C_{0,d'} = x_{d'} \cdot \zeta$ by our above observation and similarly for $x_1 \cdot \eta$, we deduce

$$(x_1 \cdot \zeta \ x_1 \cdot \eta)^{C_{0,d'}} = (x_{d'} \cdot \zeta \ x_{d'} \cdot \eta).$$

We now recombine the resulting transpositions using Equation (1-4) to conclude

$$(x_1 \cdot \alpha \ x_1 \cdot \beta)^{C_{0,d'}} = (x_{d'} \cdot \alpha \ x_{d'} \cdot \beta).$$

Now if g is any element of G_∞ , then it is a product of transpositions and it follows from the above calculation that

$$(x_1 \cdot g)^{C_{0,d'}} = x_{d'} \cdot g$$

for any $x \in \{0, 1\}$. The claimed equations now follow by taking $g = C_{0,d}$ and noting, by Lemma 2.3(iv), that $B_d(x_1) = x_1 \cdot C_{0,d}$.

Part (ii) is an extension of Lemma 2.3(vi) established by a similar argument. □

PROPOSITION 2.5. *The elements $X_{m,d}$, $C_{m,d}$, π_m and $\bar{\pi}_m$ of G_∞ defined in Equations (2-5) and (2-6) satisfy Relations (HM1)–(HM18).*

Establishing this proposition completes the proof of Theorem 1.1 since it establishes that the surjective homomorphism $\phi: G_\infty \rightarrow nV$ has trivial kernel.

PROOF. First, as observed earlier, if $u = v$ is a relation in G_∞ , then so is $\mathbf{0}_1 \cdot u = \mathbf{0}_1 \cdot v$. Consequently, it suffices to establish each of Relations (HM1)–(HM18) only when $m = 0$. Relations (HM8)–(HM13) are the most straightforward to verify and follow directly from the assumed relations involving transpositions (that is Equations (1-2) and (1-3)). Relation (HM6) is established in a similar manner. When $q > 1$, observe that both $X_{q,d}$ and $C_{q,d}$ is a product of transpositions all of whose entries have $\mathbf{00}_1$ as a prefix. These transpositions are therefore disjoint from π_0 and Relations (HM4) and (HM17) follow. We now describe the details involved in the other relations.

Note that $\mathbf{00}_1 \bullet X_{0,1} = \mathbf{0}_1$. Hence, for any $g \in G_\infty$,

$$X_{0,1}^{-1}(\mathbf{00}_1 \cdot g)X_{0,1} = \mathbf{0}_1 \cdot g.$$

This establishes Relations (HM1), (HM2) and (HM5) in the case when $m = 0$ and $d = 1$, and it also establishes Relation (HM15) in the case when $m = 0$ and $d' = 1$. We extend the first three to $d \geq 2$ by use of Lemma 2.3(i) to tell us that $B_d(\mathbf{1}_1)$ commutes with each of $X_{q,d'}$, π_q and $\bar{\pi}_q$ for $q > 0$. Similarly, we extend Relation (HM15) to the case when $d' \geq 2$ by using the same fact to show $B_{d'}(\mathbf{1}_1)$ commutes with $C_{q,d}$ for $q \geq 2$ and, by Lemma 2.3(v), also with $C_{1,d}$.

Relation (HM3) when $m = 0$ and $d = 1$ is established by collecting transpositions using Equation (1-3) and one application of Equation (1-4):

$$\begin{aligned} \pi_1 \pi_0 X_{1,1} &= (\mathbf{001}_1 \ \mathbf{01}_1) (\mathbf{01}_1 \ \mathbf{1}_1) (\mathbf{00}_1 \ \mathbf{01}_1) (\mathbf{00}_1 \ \mathbf{010}_1) (\mathbf{010}_1 \ \mathbf{011}_1) \\ &= (\mathbf{001}_1 \ \mathbf{01}_1) (\mathbf{00}_1 \ \mathbf{1}_1) (\mathbf{00}_1 \ \mathbf{10}_1) (\mathbf{10}_1 \ \mathbf{11}_1) (\mathbf{01}_1 \ \mathbf{1}_1) \\ &= (\mathbf{00}_1 \ \mathbf{1}_1) (\mathbf{01}_1 \ \mathbf{11}_1) (\mathbf{00}_1 \ \mathbf{10}_1) (\mathbf{10}_1 \ \mathbf{11}_1) (\mathbf{01}_1 \ \mathbf{1}_1) \\ &= (\mathbf{00}_1 \ \mathbf{1}_1) (\mathbf{0}_1 \ \mathbf{1}_1) (\mathbf{10}_1 \ \mathbf{11}_1) (\mathbf{01}_1 \ \mathbf{1}_1) \\ &= (\mathbf{0}_1 \ \mathbf{1}_1) (\mathbf{0}_1 \ \mathbf{10}_1) (\mathbf{10}_1 \ \mathbf{11}_1) (\mathbf{01}_1 \ \mathbf{1}_1) = X_{0,1} \pi_0. \end{aligned}$$

The case when $d \geq 2$ now follows using Lemma 2.3(i)–(iii).

We establish Relation (HM7) first in the case when $d' = 1$ and $m = 0$. If $d \geq 2$, then:

$$\begin{aligned} X_{0,1} X_{1,1} X_{0,d} &= X_{0,1} X_{1,1} X_{0,1} B_d(\mathbf{1}_1) \\ &= X_{0,1} X_{1,1} X_{0,1} (\mathbf{101}_1 \ \mathbf{110}_1) B_d(\mathbf{10}_1) B_d(\mathbf{11}_1) \\ &= (\mathbf{001}_1 \ \mathbf{01}_1) X_{0,1} X_{1,1} X_{0,1} B_d(\mathbf{10}_1) B_d(\mathbf{11}_1) \quad \text{by repeated use of Equation (1-3)} \\ &= \pi_1 X_{0,1} X_{1,1} B_d(\mathbf{01}_1) B_d(\mathbf{1}_1) X_{0,1} \quad \text{by Lemma 2.3(i)–(iii) and Equation (1-3)} \\ &= \pi_1 X_{0,1} B_d(\mathbf{1}_1) X_{1,1} B_d(\mathbf{01}_1) X_{0,1} \quad \text{by Lemma 2.3(i) to move } B_d(\mathbf{1}_1) \\ &= \pi_1 X_{0,d} X_{1,d} X_{0,1}. \end{aligned}$$

We can interchange the roles of d and d' in Relation (HM7) by multiplying on the left by π_{m+1} . Hence, it remains to establish the relation in the cases when both $d, d' \geq 2$. This is achieved as follows:

$$\begin{aligned} X_{0,d'} X_{1,d'} X_{0,d} &= X_{0,d'} X_{1,d'} X_{0,1} B_d(\mathbf{1}_1) \\ &= \pi_1 X_{0,1} X_{1,1} X_{0,d'} B_d(\mathbf{1}_1) \quad \text{by the case } d = 1 \\ &= \pi_1 X_{0,1} X_{1,1} X_{0,1} B_{d'}(\mathbf{1}_1) B_d(\mathbf{1}_1 \cdot \mathbf{0}_{d'}) B_d(\mathbf{1}_1 \cdot \mathbf{1}_{d'}) \quad \text{by Lemma 2.3(vi)} \\ &= \pi_1 X_{0,1} X_{1,1} X_{0,1} B_d(\mathbf{10}_1) B_d(\mathbf{11}_1) B_{d'}(\mathbf{1}_1) \quad \text{by Equation (2-7)} \\ &= \pi_1 X_{0,1} B_d(\mathbf{1}_1) X_{1,1} B_d(\mathbf{01}_1) X_{0,d'} \quad \text{by Lemma 2.3 and Equation(1-3)} \\ &= \pi_1 X_{0,d} X_{1,d} X_{0,d'}. \end{aligned}$$

Relation (HM14) is established by using formulae about conjugation by $(\mathbf{0}_1 \ \mathbf{1}_1)$:

$$\begin{aligned} X_{0,d} \bar{\pi}_0 &= (\mathbf{0}_1 \ \mathbf{1}_1) (\mathbf{0}_1 \ \mathbf{10}_1) (\mathbf{10}_1 \ \mathbf{11}_1) B_d(\mathbf{1}_1) (\mathbf{0}_1 \ \mathbf{1}_1) \\ &= (\mathbf{00}_1 \ \mathbf{1}_1) (\mathbf{00}_1 \ \mathbf{01}_1) B_d(\mathbf{0}_1) = (\mathbf{00}_1 \ \mathbf{01}_1) (\mathbf{01}_1 \ \mathbf{1}_1) B_d(\mathbf{0}_1) = \bar{\pi}_1 \pi_0 C_{1,d}. \end{aligned}$$

For Relation (HM16), we collect the transpositions comprising $X_{0,1}$ to the right:

$$\begin{aligned} X_{0,1} C_{0,d} &= X_{0,1} (\mathbf{01}_1 \ \mathbf{10}_1) B_d(\mathbf{0}_1) B_d(\mathbf{1}_1) \\ &= (\mathbf{001}_1 \ \mathbf{01}_1) B_d(\mathbf{00}_1) X_{0,1} B_d(\mathbf{1}_1) = \pi_1 C_{2,d} X_{0,d}. \end{aligned}$$

Finally, consider Relation (HM18) in the case when $m = 0$. We calculate

$$\begin{aligned} C_{2,d'} X_{0,d'} C_{0,d} &= B_{d'}(\mathbf{00}_1) X_{0,1} B_{d'}(\mathbf{1}_1) C_{0,d} \\ &= X_{0,1} B_{d'}(\mathbf{0}_1) B_{d'}(\mathbf{1}_1) C_{0,d} \quad \text{by Equation (1-3) and Lemma 2.3(i), (iii)} \\ &= X_{0,1} C_{0,d} C_{0,d'} \quad \text{by Lemma 2.4(i), (ii)} \\ &= X_{0,1} (\mathbf{01}_1 \ \mathbf{10}_1) B_d(\mathbf{0}_1) B_d(\mathbf{1}_1) C_{0,d'} \\ &= (\mathbf{001}_1 \ \mathbf{01}_1) B_d(\mathbf{00}_1) X_{0,1} B_d(\mathbf{1}_1) C_{0,d'} \quad \text{by Equation (1-3) and Lemma 2.3(i), (iii)} \\ &= \pi_1 C_{2,d} X_{0,d} C_{0,d'}, \end{aligned}$$

as required. This completes the proof of the proposition and the work of this section. □

3. Finite presentations for nV

In this section, we prove Theorem 1.2. In the course of our argument, we refer to two particular subsets of the collection Ω of addresses. Write $\Delta = (X^2)^n$ for the set of addresses $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ satisfying $|\alpha_i| = 2$ for $i = 1, 2, \dots, n$. Also define Ω^* to be the set of addresses $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ satisfying $|\alpha_i| \geq 2$ for $i = 1, 2, \dots, n$.

We define a presentation for a group G from three generators a, b and c , and $2n^2 + 3n + 11$ relations. The starting point is a presentation for the symmetric group of degree 4^n on two generators. According to [16, Theorem A], this can be achieved using merely eight relations (independent of the value of n). A recent article by Huxford [18] presents corrections to the arguments and the relations given in [16]. Note, however, that the number of relations is unaffected and the statement of [16, Theorem A] remains valid.

Our first family (Relation R1 below) of relations is sufficient to ensure that the generators a and b satisfy all relations that hold in the symmetric group of degree 4^n . Moreover, all the relations that we assume are satisfied if one maps a, b and c to the corresponding elements of nV (where for c , we interpret Relation R7 in nV). In particular, the resulting homomorphism $G \rightarrow nV$ induces a homomorphism from $H = \langle a, b \rangle$ onto the above symmetric group. Consequently, $H \cong \text{Sym}(\Delta)$ and we may interpret the elements in H as defining permutations of Δ . We therefore use the symbol $(\alpha \beta)$, where $\alpha, \beta \in \Delta$ with $\alpha \perp \beta$, to denote certain elements of the subgroup H and more generally refer to *permutations* of Δ by which we mean the corresponding elements of this subgroup. This also means that we can speak of the support of an element $g \in H$ and use the notation $\gamma \bullet g$ to denote the effect of applying g to some address $\gamma \in \Delta$. (In some sense, this extends the notation given in Section 1.)

The third generator c will be used to construct further transpositions $(\alpha \beta)$ for other addresses $\alpha, \beta \in \Omega^*$ with $\alpha \perp \beta$. The details of this construction will be described later in this section, together with appropriate verifications that the resulting elements of G are well defined and satisfy the relations in Equations (1-2)–(1-4) listed in Theorem 1.1. For the remaining discussion, prior to explaining the construction, we assume the existence of the various transpositions $(\alpha \beta)$, each of which will be expressed as some product in G involving the generators a, b and c .

To specify the relations that we assume, we fix certain elements of H . Let $\delta^{(0)}, \delta^{(1)}, \dots, \delta^{(4^n-1)}$ be an enumeration of the addresses in Δ and define the following elements of H :

$$\begin{aligned}
 p &= (\delta^{(n+1)} \delta^{(n+2)} \dots \delta^{(4^n-1)}) \\
 q_d &= (\delta^{(0)} \delta^{(1)} \dots \delta^{(d-1)} \delta^{(d+1)} \dots \delta^{(n)} \delta^{(n+2)} \dots \delta^{(4^n-1)}).
 \end{aligned}
 \tag{3-1}$$

The relations that we assume are the following:

- R1. eight relations involving a and b that define the symmetric group of degree 4^n (from [16, Theorem A]);
- R2. $[c, p] = [c, (\delta^{(n+2)} \delta^{(n+3)})] = 1$;

- R3. $[q_d, (\delta^{(d)} \cdot x_d \ \delta^{(n+1)})] = 1$ for $d = 1, 2, \dots, n$ and $x \in \{0, 1\}$;
- R4. $[c, (\delta^{(n+2)} \cdot x_d \ \delta^{(n+3)})] = 1$ for $d = 1, 2, \dots, n$ and $x \in \{0, 1\}$;
- R5. $(\delta^{(0)} \cdot x_d \ \delta^{(1)})^{(\delta^{(0)} \ \delta^{(2)} \cdot y_{d'})} = (\delta^{(0)} \cdot y_{d'} \ \delta^{(1)})^{(\delta^{(0)} \ \delta^{(2)} \cdot x_d)}$
for all choices of distinct indices $d, d' \in \{1, 2, \dots, n\}$ and all pairs $x, y \in \{0, 1\}$;
- R6. $(\delta^{(0)} \ \delta^{(1)}) = (\delta^{(0)} \cdot \mathbf{0}_d \ \delta^{(1)} \cdot \mathbf{0}_d) (\delta^{(0)} \cdot \mathbf{1}_d \ \delta^{(1)} \cdot \mathbf{1}_d)$ for $d = 1, 2, \dots, n$; and
- R7. $c = (\delta^{(0)} \ \delta^{(1)} \cdot \mathbf{0}_1) (\delta^{(0)} \ \delta^{(1)} \cdot \mathbf{1}_1) (\delta^{(0)} \ \delta^{(2)} \cdot \mathbf{0}_2) (\delta^{(0)} \ \delta^{(2)} \cdot \mathbf{1}_2)$
 $\dots (\delta^{(0)} \ \delta^{(n)} \cdot \mathbf{0}_n) (\delta^{(0)} \ \delta^{(n)} \cdot \mathbf{1}_n).$

We establish Theorem 1.2 by showing that the group G , generated by a, b and c subject to the $2n^2 + 3n + 11$ relations listed in Relations R1–R7, is isomorphic to the group nV .

The first part of the proof of Theorem 1.2 involves the definition of all transpositions $(\alpha \ \beta)$, for $\alpha, \beta \in \Omega^*$ with $\alpha \perp \beta$, as elements of the group G , verifying that these transpositions are well defined, and that they satisfy all those relations in Equations (1-2)–(1-4) listed in Theorem 1.1 involving only addresses in Ω^* . (In this part of the proof, we do not consider any address $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with one or more coordinates satisfying $|\alpha_i| \leq 1$.) We also verify explicitly the relation listed in Equation (1-5) since we depend upon this to reduce the number of cases when verifying Equation (1-3) (see Lemma 3.1 below). The overall process is an induction argument based on the number of coordinates of an address α having some specified length. To make this precise, we set $m(\alpha) = \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|\}$ and define $k(\alpha)$ to be the number of coordinates α_i satisfying $|\alpha_i| = m(\alpha)$. The *weight* of α is then the ordered pair $\text{wt}(\alpha) = (m(\alpha), k(\alpha))$. This pair then measures the longest coordinate of α and counts the number of longest coordinates in the address. Thus, for example, Δ consists of all addresses of weight $(2, n)$.

We order weights lexicographically, so $\text{wt}(\alpha) < \text{wt}(\beta)$ means either $m(\alpha) < m(\beta)$, or $m(\alpha) = m(\beta)$ and $k(\alpha) < k(\beta)$; that is, either the coordinates of α are all shorter than the longest of β or that the greatest length is the same but α has fewer of these longest coordinates than β .

In each step of the induction, we assume that we already have defined transpositions whose entries have weight less than (m, k) and verified all Equations (1-2)–(1-4), and also Equation (1-5), involving such transpositions. Our first stage is then to define transpositions $(\alpha \ \beta)$, where $\text{wt}(\alpha) = (m, k)$ and $\beta \in \Delta$, or *vice versa*. We then verify that our definitions make sense and that all the required relations involving the newly defined transpositions are satisfied. At the second stage, we perform the same definitions and verifications for the remaining transpositions $(\alpha \ \beta)$ with $\text{wt}(\alpha), \text{wt}(\beta) \leq (m, k)$. The new transpositions will always be conjugates of transpositions from $\text{Sym}(\Delta)$ and consequently Equation (1-2) will always hold and we do not verify it explicitly.

A significant part of our argument is verifying the conjugacy relations in Equation (1-3). The following lemma describes which instances of this relation are needed at each step of the induction.

LEMMA 3.1. *Assume that all transpositions defined at some stage of the induction satisfy $(\alpha \beta) = (\beta \alpha)$ and $(\alpha \beta)^2 = 1$. To verify all required conjugacy relations in Equation (1-3), it is sufficient to establish them in the following cases.*

(A) *Every pair from $\{\alpha, \beta, \gamma, \delta\}$ is incomparable. The required relation then has the form*

$$[(\alpha \beta), (\gamma \delta)] = 1.$$

(B) *$\gamma \leq \alpha$ and every pair from $\{\beta, \gamma, \delta\}$ is incomparable. Then, $\alpha = \gamma\eta$ for some $\eta \in \Omega$ and the required relation is*

$$(\gamma\eta \beta)^{(\gamma \delta)} = (\delta\eta \beta).$$

(C) *$\gamma < \alpha, \delta \leq \beta$ and $\gamma \perp \delta$. Then, $\alpha = \gamma\eta$ and $\beta = \delta\theta$ for some $\eta, \theta \in \Omega$ and the required relation is*

$$(\gamma\eta \delta\theta)^{(\gamma \delta)} = (\gamma\theta \delta\eta).$$

(D) *$\gamma < \alpha, \beta$ and $\gamma \perp \delta$. Then, $\alpha = \gamma\eta$ and $\beta = \gamma\theta$ for (necessarily nonempty) $\eta, \theta \in \Omega$ and the required relation is*

$$(\gamma\eta \gamma\theta)^{(\gamma \delta)} = (\delta\eta \delta\theta).$$

PROOF. In order that $\alpha \bullet (\gamma \delta)$ be defined, we require (i) $\gamma \perp \alpha$ or $\gamma \leq \alpha$, and (ii) $\delta \perp \alpha$ or $\delta \leq \alpha$. A similar pair of conditions apply when $\beta \bullet (\gamma \delta)$ is defined. We analyse the four resulting conditions. Furthermore, we note, for example, that if $\gamma \leq \alpha$ or $\delta \leq \alpha$, then by exploiting the symmetry $(\gamma \delta) = (\delta \gamma)$ we may assume that in fact $\gamma \leq \alpha$. This reduces the four conditions to the configurations described in the statement of the lemma. □

3.1. Base step. As the base step of the induction, we rely upon the assumed Relation R1 to ensure that the transpositions in $H = \langle a, b \rangle$ exist and satisfy all the relations in the symmetric group of degree 4^n ; that is, we may use all transpositions $(\alpha \beta)$, where $\alpha, \beta \in \Omega^*$ satisfy $\alpha \perp \beta$ and $\text{wt}(\alpha) = \text{wt}(\beta) = (2, n)$, and all relations in Equations (1-2), (1-3) and (1-5) between them. (There are no relations of the form of Equation (1-4) involving only transpositions from H .)

3.2. Induction, stage 1. Let us assume that for some fixed weight $(m, k) \geq (3, 1)$, we have already defined all transpositions $(\alpha \beta)$, where $\alpha, \beta \in \Omega^*$ satisfy $\alpha \perp \beta$ and $\text{wt}(\alpha), \text{wt}(\beta) < (m, k)$ and verified all relations in Equations (1-2)–(1-5) involving such transpositions. We now define those transpositions with one entry of weight (m, k) and one entry from Δ . The definitions and argument actually depend upon whether $(m, k) = (3, 1)$ or $(m, k) > (3, 1)$. Consequently, we split into these cases.

Suppose then that $(m, k) = (3, 1)$. First, set

$$(\delta^{(d)} .x_d \delta^{(n+1)}) := (\delta^{(0)} \delta^{(n+1)})^{c^{2d+x-1}}, \tag{3-2}$$

where $d = 1, 2, \dots, n$ and $x \in \{0, 1\}$. (Since d and x are nonnegative integers, the sum $2d + x - 1$ is defined.) We now define transpositions $(\alpha \ \beta)$, where $\alpha, \beta \in \Omega^*$ satisfy $\alpha \perp \beta$, $\text{wt}(\alpha) = (3, 1)$ and $\beta \in \Delta$. Then, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $|\alpha_d| = 3$ for some index d and $|\alpha_i| = 2$ for all other indices i . Write $\alpha = \hat{\alpha}.x_d$, where $\hat{\alpha} \in \Delta$, and choose a permutation $\sigma \in H$ that moves $\delta^{(d)}$ to $\hat{\alpha}$ and $\delta^{(n+1)}$ to β . We then define

$$(\alpha \ \beta) := (\delta^{(d)}.x_d \ \delta^{(n+1)})^\sigma \tag{3-3}$$

in terms of the transposition defined in Equation (3-2). To achieve the symmetry required by Equation (1-5), we also define $(\beta \ \alpha) := (\alpha \ \beta)$ for such α and β . To verify that these definitions are independent of the choice of permutation $\sigma \in H$, we use the following lemma.

LEMMA 3.2

- (i) *If $\sigma \in H$ is a permutation of Δ with support disjoint from $\delta^{(0)}, \delta^{(1)}, \dots, \delta^{(n)}$, then $[c, \sigma] = 1$.*
- (ii) *Let $d = 1, 2, \dots, n$ and $x \in \{0, 1\}$. Then, $(\delta^{(d)}.x_d \ \delta^{(n+1)})$ commutes with every permutation in H that has support disjoint from $\delta^{(d)}$ and $\delta^{(n+1)}$.*

PROOF. (i) Note that our permutation p , defined in Equation (3-1), and the transposition $(\delta^{(n+2)} \ \delta^{(n+3)})$ generate all the permutations of $\Delta \setminus \{\delta^{(0)}, \delta^{(1)}, \dots, \delta^{(n)}\}$. Hence, all permutations of the latter set commute with c by our assumed Relation R2.

(ii) The disjoint transpositions $(\delta^{(0)} \ \delta^{(n+1)})$ and $(\delta^{(n+2)} \ \delta^{(n+3)})$ commute. Consequently, $(\delta^{(d)}.x_d \ \delta^{(n+1)}) = (\delta^{(0)} \ \delta^{(n+1)})^{c^{2d+x-1}}$ commutes with $(\delta^{(n+2)} \ \delta^{(n+3)})$ by Relation R2. The definition in Equation (3-1) of the cycle q_d ensures that it and the transposition $(\delta^{(n+2)} \ \delta^{(n+3)})$ generate all permutations in H with support disjoint from $\delta^{(d)}$ and $\delta^{(n+1)}$. Hence, $(\delta^{(d)}.x_d \ \delta^{(n+1)})$ commutes with all such permutations with use of Relation R3. □

We now show that the definition of $(\alpha \ \beta)$ in Equation (3-3) does not depend on the choice of σ . For if σ_1 and σ_2 are two permutations of Δ that both move $\delta^{(d)}$ to $\hat{\alpha}$ and $\delta^{(n+1)}$ to β , then $\sigma_1\sigma_2^{-1}$ fixes both $\delta^{(d)}$ and $\delta^{(n+1)}$, so commutes with $(\delta^{(d)}.x_d \ \delta^{(n+1)})$ by part (ii) of the lemma. Therefore, $(\delta^{(d)}.x_d \ \delta^{(n+1)})^{\sigma_1} = (\delta^{(d)}.x_d \ \delta^{(n+1)})^{\sigma_2}$, establishing that $(\alpha \ \beta)$ is well defined.

There are no split relations in Equation (1-4) to verify at this stage, since we have not yet introduced any transpositions $(\alpha \ \beta)$, where both α and β have coordinates of length 3. Hence, we only need to check conjugacy relations in Equation (1-3) involving the transpositions that we have introduced.

LEMMA 3.3

- (i) *Let $\beta, \gamma \in \Delta$, $x \in \{0, 1\}$ and d be an index with $1 \leq d \leq n$. If $\sigma \in \text{Sym}(\Delta)$, then*

$$(\beta.x_d \ \gamma)^\sigma = ((\beta \bullet \sigma).x_d \ \gamma \bullet \sigma),$$

where, as above, $\beta \bullet \sigma$ and $\gamma \bullet \sigma$ denote the images of β and γ under the action of $\sigma \in H$.

(ii) If $\beta \in \Delta \setminus \{\delta^{(0)}, \delta^{(1)}, \dots, \delta^{(n)}\}$, $1 \leq d \leq n$ and $x \in \{0, 1\}$, then

$$(\delta^{(0)} \beta)^{c^{2d+x-1}} = (\delta^{(d)} .x_d \beta).$$

PROOF. (i) This follows immediately from the definition of transpositions of the form $(\beta .x_d \gamma)$ and their well definedness.

(ii) We must establish the result when $\beta \neq \delta^{(n+1)}$. Take $\sigma = (\delta^{(n+1)} \beta)$. This commutes with c by Lemma 3.2(i). Hence,

$$(\delta^{(0)} \beta)^{c^{2d+x-1}} = (\delta^{(0)} \delta^{(n+1)})^\sigma c^{2d+x-1} = (\delta^{(0)} \delta^{(n+1)})^{c^{2d+x-1} \sigma} = (\delta^{(d)} .x_d \delta^{(n+1)})^\sigma,$$

which produces the required result using part (i). □

Part (i) of Lemma 3.3 establishes any instance of the conjugacy relation in Equation (1-3) when $(\gamma \delta) \in H$. We consider now the remaining instances of Equation (1-3) involving transpositions defined at this stage of the induction (via Equation (3-3) above) and we split into the cases listed in Lemma 3.1.

(A): Consider four incomparable addresses $\alpha, \beta, \gamma, \delta$, where we may assume $\text{wt}(\alpha) = \text{wt}(\gamma) = (3, 1)$ and $\beta, \delta \in \Delta$. Write $\alpha = \hat{\alpha} .x_d$ and $\gamma = \hat{\gamma} .y_{d'}$ for $x, y \in \{0, 1\}$, indices d and d' , and some $\hat{\alpha}, \hat{\gamma} \in \Delta$. Suppose $\hat{\alpha} \neq \hat{\gamma}$, so that $\hat{\alpha}, \beta, \hat{\gamma}$ and δ are distinct addresses in Δ . From Lemma 3.3(i), we know $(\delta^{(n+2)} .x_d \delta^{(n+3)})$ commutes with $(\delta^{(0)} \delta^{(n+1)})$ and it commutes with c by Relation R4. Hence, conjugating by $c^{2d'+y-1}$, we conclude that

$$[(\delta^{(n+2)} .x_d \delta^{(n+3)}), (\delta^{(d')} .y_{d'} \delta^{(n+1)})] = 1.$$

Finally, conjugate by a permutation σ in H that maps $\delta^{(n+2)}$ to $\hat{\alpha}$, $\delta^{(n+3)}$ to β , $\delta^{(d')}$ to $\hat{\gamma}$ and $\delta^{(n+1)}$ to δ to establish $[(\alpha \beta), (\gamma \delta)] = 1$, as required.

However, if $\hat{\alpha} = \hat{\gamma}$, then in order that $\alpha \perp \gamma$, we know that $d = d'$ and, after suitable relabelling, we can assume $x = 0$ and $y = 1$. We make use of Lemma 3.3(ii) to observe:

$$\begin{aligned} (\delta^{(d)} .0_d \delta^{(n+1)}) &= (\delta^{(0)} \delta^{(n+1)})^{c^{2d-1}} \\ (\delta^{(d)} .1_d \delta^{(n+2)}) &= (\delta^{(0)} \delta^{(n+2)})^{c^{2d}} = (\delta^{(1)} .0_1 \delta^{(n+2)})^{c^{2d-1}}. \end{aligned}$$

By Lemma 3.3(i), $(\delta^{(0)} \delta^{(n+1)})$ commutes with $(\delta^{(1)} .0_1 \delta^{(n+2)})$. Hence, upon conjugating by c^{2d-1} , we deduce

$$[(\delta^{(d)} .0_d \delta^{(n+1)}), (\delta^{(d)} .1_d \delta^{(n+2)})] = 1.$$

We establish the required relation $[(\alpha \beta), (\gamma \delta)] = 1$ by finally conjugating by a permutation σ in H that moves $\delta^{(d)}$ to $\hat{\alpha}$, $\delta^{(n+1)}$ to β and $\delta^{(n+2)}$ to δ .

(B): In view of Lemma 3.3(i) and the symmetry between γ and δ in the conjugacy relation, we may assume, in this case, that $\text{wt}(\gamma) = (3, 1)$ and $\alpha = \gamma$. The remaining addresses β and δ must be from Δ . Write $\gamma = \hat{\gamma} .x_d$ for some $x \in \{0, 1\}$ and some

index d . Conjugate the equation $(\delta^{(0)} \ \delta^{(n+1)})^{(\delta^{(0)} \ \delta^{(n+2)})} = (\delta^{(n+1)} \ \delta^{(n+2)})$ by c^{2d+x-1} to produce

$$(\delta^{(d)} \cdot x_d \ \delta^{(n+1)})^{(\delta^{(d)} \cdot x_d \ \delta^{(n+2)})} = (\delta^{(n+1)} \ \delta^{(n+2)})$$

by use of Lemmas 3.2(i) and 3.3(ii). Finally, conjugate by a permutation σ of Δ moving $\delta^{(d)}$ to $\hat{\gamma}$, $\delta^{(n+1)}$ to β and $\delta^{(n+2)}$ to δ , using Lemma 3.3(i), to establish the required relation.

(C)/(D): Conjugacy relations of the form (C) occur only at this stage when $\gamma, \delta \in \Delta$, which all then hold by Lemma 3.3. There are no type (D) relations to verify. This completes the verifications required for stage 1 of the induction when $(m, k) = (3, 1)$.

Now assume $(m, k) > (3, 1)$. Consider a pair of incomparable addresses $\alpha, \beta \in \Omega^*$ with $\text{wt}(\alpha) = (m, k)$ and $\beta \in \Delta$. To define the transposition $(\alpha \ \beta)$, first choose d to be the index of one of the coordinates of α that has length m and write $\alpha = \hat{\alpha} \cdot x_d$, where $x \in \{0, 1\}$. Then, $\hat{\alpha}$ is an address with $\text{wt}(\hat{\alpha}) < (m, k)$ and, since every coordinate of β has length 2, it follows $\hat{\alpha} \perp \beta$ also. Choose $\zeta \in \Delta$ incomparable with both $\hat{\alpha}$ and β . Then, $\text{wt}(\zeta \cdot x_d) = (3, 1) < (m, k)$ and, by induction, both transpositions $(\zeta \cdot x_d \ \beta)$ and $(\zeta \ \hat{\alpha})$ have been constructed. We then define

$$(\alpha \ \beta) = (\hat{\alpha} \cdot x_d \ \beta) := (\zeta \cdot x_d \ \beta)^{(\zeta \ \hat{\alpha})}. \tag{3-4}$$

To achieve Equation (1-5), we also set $(\beta \ \alpha) := (\alpha \ \beta)$. We must verify that the above definition is independent of the choice of the address ζ and of the index d .

With the above assumptions, we make the following observations.

LEMMA 3.4

(i) Suppose that ζ and η are distinct addresses in Δ that are incomparable with both $\hat{\alpha}$ and β . Then,

$$(\zeta \cdot x_d \ \beta)^{(\zeta \ \hat{\alpha})} = (\eta \cdot x_d \ \beta)^{(\eta \ \hat{\alpha})}.$$

(ii) Suppose that d and d' are both indices of coordinates of α of length m . Write $\alpha = \gamma \cdot x_d \cdot y_{d'}$ for some $x, y \in \{0, 1\}$. If ζ is an address in Δ incomparable with both β and γ , then

$$(\zeta \cdot x_d \ \beta)^{(\zeta \ \gamma \cdot y_{d'})} = (\zeta \cdot y_{d'} \ \beta)^{(\zeta \ \gamma \cdot x_d)}.$$

PROOF. (i) As distinct addresses in Δ , certainly ζ and η are incomparable. All addresses appearing in the following calculation have weight $< (m, k)$ and so, by induction,

$$(\zeta \cdot x_d \ \beta)^{(\zeta \ \hat{\alpha})(\eta \ \hat{\alpha})} = (\zeta \cdot x_d \ \beta)^{(\eta \ \zeta)(\zeta \ \hat{\alpha})} = (\eta \cdot x_d \ \beta)^{(\zeta \ \hat{\alpha})} = (\eta \cdot x_d \ \beta)$$

and the required equation then follows.

(ii) Note that our assumption that $\alpha \perp \beta$ and $\beta \in \Delta$ implies that $\beta \perp \gamma$. Now consider first the case when $(m, k) = (3, 2)$, so that $\gamma \in \Delta$. We simply conjugate

Relation R5 by a permutation $\sigma \in \text{Sym}(\Delta)$ that moves $\delta^{(0)}$ to ζ , $\delta^{(1)}$ to β and $\delta^{(2)}$ to γ (using relations established in the weight (3, 1) stage) to yield the required formula.

Now consider the case when $(m, k) > (3, 2)$. Choose $\delta \in \Delta$ that is incomparable with each of β , γ and ζ . Note that $\text{wt}(\delta.x_d.y_{d'}) = (3, 2)$ and so the transpositions with this address as an entry in the following calculation were constructed at an earlier stage. Then,

$$(\zeta.x_d \beta)^{\zeta \gamma.y_{d'}}(\gamma \delta) = (\zeta.x_d \beta)^{\zeta \delta.y_{d'}} = (\delta.x_d.y_{d'} \beta)$$

and similarly $(\zeta.y_{d'} \beta)^{\zeta \gamma.x_d}(\gamma \delta) = (\delta.x_d.y_{d'} \beta)$. It therefore follows that the left-hand sides of these equations are equal, from which we deduce our required formula. \square

It follows from part (i) of this lemma that our definition in Equation (3-4) of $(\alpha \beta)$ is independent of the choice of ζ . Then, part (ii) shows the definition is also independent of the choice of index d . In conclusion, the transpositions $(\alpha \beta)$, where $\text{wt}(\alpha) = (m, k)$ and $\beta \in \Delta$ or *vice versa*, are well defined. The remaining work in this part of the induction is to establish the four types of conjugacy relations in Equation (1-3) and then the split relation in Equation (1-4) when they involve such transpositions.

(A): Consider two transpositions $(\alpha \beta)$ and $(\gamma \delta)$, at least one of which was defined as in Equation (3-4) and the other possibly arriving at an early stage in the induction, such that every pair of addresses from $\{\alpha, \beta, \gamma, \delta\}$ is incomparable. Exploiting the symmetry relation in Equation (1-5), we can suppose without loss of generality that one of the following sets of conditions holds:

- (A.i) $\text{wt}(\alpha) = (m, k)$, $\beta \in \Delta$ and $\text{wt}(\gamma), \text{wt}(\delta) < (m, k)$; or
- (A.ii) $\text{wt}(\alpha) = \text{wt}(\gamma) = (m, k)$ and $\beta, \delta \in \Delta$.

In Case (A.i), write $\alpha = \hat{\alpha}.x_d$ as above. Choose $\zeta \in \Delta$ that is incomparable with each of $\hat{\alpha}$, β , γ and δ and use this, together with Lemma 3.4, in the definition in Equation (3-4) of $(\alpha \beta)$. By induction, $(\gamma \delta)$ commutes with both $(\zeta.x_d \beta)$ and $(\zeta \hat{\alpha})$ and hence with $(\alpha \beta) = (\zeta.x_d \beta)^{\zeta \hat{\alpha}}$, as required. Case (A.ii) is similar, but we now write $\gamma = \hat{\gamma}.y_{d'}$ for some suitable index d' and choose $\eta \in \Delta$ incomparable with each address in $\{\hat{\alpha}, \beta, \hat{\gamma}, \delta, \zeta\}$. We then observe that, by induction, each transposition used in the definition $(\gamma \delta) = (\eta.y_{d'} \delta)^{\eta \hat{\gamma}}$ commutes with each one used to construct $(\alpha \beta)$. This establishes all type (A) conjugacy relations at this stage.

(B): Consider a conjugacy relation of Equation (1-3), where $\alpha = \gamma\eta$ as in Lemma 3.1(B). At least one address in the relation has weight (m, k) and the other entry in a transposition involving such an address must be in Δ . Exploiting the symmetry present, there are four possibilities:

- (B.i) $\text{wt}(\beta) = (m, k)$, $\gamma, \delta \in \Delta$ and $\eta = \varepsilon$;
- (B.ii) $\text{wt}(\gamma\eta) = (m, k)$, $\beta \in \Delta$ and $\text{wt}(\gamma), \text{wt}(\delta), \text{wt}(\delta\eta) < (m, k)$;
- (B.iii) $\text{wt}(\gamma\eta) = \text{wt}(\delta\eta) = (m, k)$, $\beta \in \Delta$ and $\text{wt}(\gamma), \text{wt}(\delta) < (m, k)$; or
- (B.iv) $\text{wt}(\gamma) = \text{wt}(\gamma\eta) = (m, k)$, $\beta, \delta \in \Delta$ and $\text{wt}(\delta\eta) < (m, k)$.

(It is impossible that $\text{wt}(\gamma) = \text{wt}(\gamma\eta) = \text{wt}(\delta\eta) = (m, k)$: if this situation were to happen, then there would be some index d , where the d th coordinate of $\delta\eta$ has length m . Then the d th coordinate of $\gamma\eta$ has length m , but that of γ is shorter, contradicting $\text{wt}(\gamma) = \text{wt}(\gamma\eta)$.)

In case (B.i), write $\beta = \hat{\beta}.x_d$ and choose $\zeta \in \Delta$ incomparable with each of $\hat{\beta}$, γ and δ to define $(\gamma \beta) = (\gamma \zeta.x_d)^{(\zeta \hat{\beta})}$ and similarly for $(\delta \beta)$. Then, by induction,

$$(\gamma\eta \beta)^{(\gamma \delta)} = (\gamma \beta)^{(\gamma \delta)} = (\gamma \zeta.x_d)^{(\zeta \hat{\beta})^{(\gamma \delta)}} = (\delta \zeta.x_d)^{(\zeta \hat{\beta})} = (\delta \beta).$$

In case (B.ii), note that there is an index d such that the d th coordinate of $\gamma\eta$ has length m and that of γ is shorter. Therefore, we can write $\eta = \hat{\eta}.x_d$ and choose $\zeta \in \Delta$ incomparable with each of β , γ and δ to define $(\gamma\eta \beta) = (\zeta.x_d \beta)^{(\zeta \gamma\hat{\eta})}$. Then,

$$(\gamma\eta \beta)^{(\gamma \delta)} = (\zeta.x_d \beta)^{(\zeta \gamma\hat{\eta})^{(\gamma \delta)}} = (\zeta.x_d \beta)^{(\zeta \delta\hat{\eta})} = (\delta\hat{\eta}.x_d \beta) = (\delta\eta \beta).$$

For case (B.iii), there are two possibilities. If there is some d such that the d th coordinate of $\gamma\eta$ and $\delta\eta$ both have length m , but those of γ and δ are shorter, then we use the same argument as for case (B.ii), but now the last step in the calculation is actually the definition of $(\delta\eta \beta)$. Otherwise, there are d and d' such that the d th coordinate of $\gamma\eta$ has length m and that of γ is shorter and the d' th coordinate of $\delta\eta$ has length m and that of δ is shorter. Moreover, by hypothesis, the d th coordinate of γ must be longer than that of δ , so has length at least 3. Choose distinct addresses $\zeta, \theta \in \Delta$ incomparable with each of β , γ and δ . We use ζ when employing Equation (3-4) to define $(\gamma\eta \beta)$ and $(\delta\eta \beta)$, exploiting the coordinates of indices d and d' , respectively, having written $\eta = \hat{\eta}.x_d.y_{d'}$. Thus, $(\gamma\eta \beta) = (\zeta.x_d \beta)^{(\zeta \gamma\hat{\eta}.y_{d'})}$ and similarly for $(\delta\eta \beta)$ (as in the second set of calculations below). Furthermore, $\text{wt}(\theta\eta) < (m, k)$ since the d th coordinate of $\theta\eta$ is shorter than that of $\gamma\eta$. We therefore compute:

$$\begin{aligned} (\gamma\eta \beta)^{(\gamma \delta)^{(\gamma \theta)}} &= (\zeta.x_d \beta)^{(\zeta \gamma\hat{\eta}.y_{d'})^{(\gamma \delta)^{(\gamma \theta)}}} \\ &= (\zeta.x_d \beta)^{(\zeta \theta\hat{\eta}.y_{d'})^{(\theta \delta)}} \\ &= (\theta\hat{\eta}.x_d.y_{d'} \beta)^{(\theta \delta)} = (\theta\eta \beta)^{(\theta \delta)} \\ (\delta\eta \beta)^{(\theta \delta)} &= (\zeta.y_{d'} \beta)^{(\zeta \delta\hat{\eta}.x_d)^{(\theta \delta)}} \\ &= (\zeta.y_{d'} \beta)^{(\zeta \theta\hat{\eta}.x_d)} \\ &= (\theta\eta \beta). \end{aligned}$$

Hence, $(\gamma\eta \beta)^{(\gamma \delta)^{(\gamma \theta)}} = (\delta\eta \beta)$ and, with use of our already verified type (A) conjugacy relation, we conclude $(\gamma\eta \beta)^{(\gamma \delta)} = (\delta\eta \beta)^{(\gamma \theta)} = (\delta\eta \beta)$.

Finally, in case (B.iv), let d be the index of a coordinate of γ of length m . Write $\gamma = \hat{\gamma}.x_d$. Note that the d th coordinate of η is empty. Choose distinct addresses $\zeta, \theta \in \Delta$ that are incomparable with each of β , $\hat{\gamma}$ and δ . The first is used in the construction of $(\gamma \delta)$ and $(\gamma\eta \beta)$. One observes that $\text{wt}(\theta\eta.x_d) < (m, k)$. We calculate

$$(\gamma \delta)^{(\hat{\gamma} \theta)} = (\zeta.x_d \delta)^{(\zeta \hat{\gamma})^{(\hat{\gamma} \theta)}} = (\zeta.x_d \delta)^{(\hat{\gamma} \theta)^{(\zeta \theta)}} = (\zeta.x_d \delta)^{(\zeta \theta)} = (\theta.x_d \delta)$$

(since $(\zeta.x_d \delta)$ and $(\hat{\gamma} \theta)$ commute) so that

$$\begin{aligned} (\gamma\eta \beta)^{(\gamma \delta)(\hat{\gamma} \theta)} &= (\zeta.x_d \beta)^{(\zeta \hat{\gamma}\eta)(\gamma \delta)(\hat{\gamma} \theta)} = (\zeta.x_d \beta)^{(\zeta \theta\eta)(\theta.x_d \delta)} \\ &= (\theta\eta.x_d \beta)^{(\theta.x_d \delta)} = (\delta\eta \beta). \end{aligned}$$

Hence, by a type (A) conjugacy relation, $(\gamma\eta \beta)^{(\gamma \delta)} = (\delta\eta \beta)^{(\hat{\gamma} \theta)} = (\delta\eta \beta)$. This establishes all the type (B) conjugacy relations.

(C): In the notation of Lemma 3.1(C), if it were the case that $\text{wt}(\gamma\eta) = \text{wt}(\gamma\theta) = (m, k)$, then at this stage $\delta, \delta\eta, \delta\theta \in \Delta$, which would force $\eta = \theta = \varepsilon$. The conjugacy relation would reduce to one form $g^g = g$ that holds in any group. Consequently, upon exploiting the symmetry in the relation, we must verify Equation (1-3) in the following two cases:

- (C.i) $\text{wt}(\gamma\eta) = (m, k), \delta \in \Delta, \theta = \varepsilon$ and $\text{wt}(\gamma) < (m, k)$; or
- (C.ii) $\text{wt}(\gamma\eta) = \text{wt}(\delta\eta) = (m, k), \gamma, \delta \in \Delta$ and $\theta = \varepsilon$.

Both are dealt with in the same manner, namely by an argument similar to case (B.ii) above.

(D)/Split: Note that, in the notation of Lemma 3.1(D), the addresses η and θ must be nonempty. However, since all the transpositions introduced via Equation (3-4) have one entry from Δ , we conclude that no new conjugacy relations of type (D) must be verified at this stage. Similarly, there are no split relations in Equation (1-4) to verify at this stage. In conclusion, we have verified all the required relations involving the transpositions that have been introduced.

3.3. Induction, stage 2. At the second stage of the induction, we assume that, for the fixed weight $(m, k) \geq (3, 1)$, we have already defined all transpositions $(\alpha \beta)$, where $\alpha, \beta \in \Omega^*$ satisfy $\alpha \perp \beta$ and either $\text{wt}(\alpha), \text{wt}(\beta) < (m, k)$, or $\text{wt}(\alpha) = (m, k)$ and $\beta \in \Delta$ (or *vice versa*). The former case holds by the inductive assumption and the latter by the completion of stage 1. We also assume that we have verified all relations in Equations (1-2)–(1-5) involving these transpositions. We now define the remaining transpositions with entries of weight at most (m, k) .

Assume then that α and β are incomparable addresses in Ω^* of which one has weight (m, k) and the other has weight at most (m, k) and is not from Δ . Choose $\zeta \in \Delta$ incomparable with both α and β and define

$$(\alpha \beta) := (\alpha \zeta)^{(\beta \zeta)}. \tag{3-5}$$

At least one of the transpositions on the right-hand side is defined via stage 1, while the other (in the case that the relevant entry has weight $< (m, k)$) may have been constructed earlier in the inductive process. We first verify that this definition is independent of the choice of ζ .

LEMMA 3.5. *Let α, β, ζ and η be incomparable addresses in Ω^* with $\text{wt}(\alpha), \text{wt}(\beta) \leq (m, k)$ and $\zeta, \eta \in \Delta$. Then:*

- (i) $(\alpha \zeta)^{(\beta \zeta)} = (\alpha \eta)^{(\beta \eta)}$;
- (ii) $(\alpha \zeta)^{(\beta \zeta)} = (\beta \zeta)^{(\alpha \zeta)}$.

PROOF. (i) In the following calculation, and indeed for many used during this stage, all the transpositions we manipulate involve one entry from Δ . Hence, the relations we rely upon hold by induction or were established in Stage 1. Observe

$$(\alpha \zeta)^{(\beta \zeta)(\beta \eta)} = (\alpha \zeta)^{(\beta \eta)(\eta \zeta)} = (\alpha \zeta)^{(\eta \zeta)} = (\alpha \eta),$$

from which the claimed equation follows.

(ii) Choose $\eta \in \Delta$ that is incomparable with each of α, β and ζ . We then calculate

$$(\alpha \zeta)^{(\beta \zeta)(\beta \eta)} = (\alpha \zeta)^{(\eta \zeta)} = (\alpha \eta) \quad \text{and} \quad (\beta \zeta)^{(\alpha \zeta)(\beta \eta)} = (\eta \zeta)^{(\alpha \zeta)} = (\eta \alpha),$$

so that $(\alpha \zeta)^{(\beta \zeta)(\beta \eta)} = (\beta \zeta)^{(\alpha \zeta)(\beta \eta)}$, from which the claimed equation follows. □

Part (i) of this lemma tells us that the definition in Equation (3-5) of $(\alpha \beta)$ is independent of the choice of $\zeta \in \Delta$. Part (ii) tells us that $(\alpha \beta) = (\beta \alpha)$; that is, Equation (1-5) holds for the transpositions defined via Equation (3-5).

Before verifying the remaining relations, we observe that

$$(\alpha \beta) = (\alpha \zeta)^{(\beta \zeta)}$$

holds for every triple α, β and ζ of pairwise incomparable addresses in Ω^* with $\text{wt}(\alpha), \text{wt}(\beta) \leq (m, k)$ and $\zeta \in \Delta$. When one or both of α and β have weight (m, k) , this is the definition in Equation (3-5) combined with Lemma 3.5(i). When they both have weight $< (m, k)$, it follows by induction.

The four cases (A)–(D) of conjugacy relations in Equation (1-3) described in Lemma 3.1 are all established by the same method. We illustrate this for case (B), namely we establish

$$(\gamma \eta \beta)^{(\gamma \delta)} = (\delta \eta \beta)$$

for incomparable addresses $\beta, \gamma, \delta \in \Omega^*$ and some (possibly empty) $\eta \in \Omega$ such that the addresses appearing in the formula all have weight $\leq (m, k)$. Choose distinct $\zeta, \theta \in \Delta$ incomparable with each of β, γ and δ , so that $(\gamma \eta \beta) = (\gamma \eta \zeta)^{(\beta \zeta)}$ and $(\gamma \delta) = (\gamma \theta)^{(\delta \theta)}$. In the following calculation, all the transpositions manipulated have second entry either ζ or θ (selected from Δ):

$$\begin{aligned} (\gamma \eta \beta)^{(\gamma \delta)} &= (\gamma \eta \zeta)^{(\beta \zeta)(\delta \theta)(\gamma \theta)(\delta \theta)} = (\gamma \eta \zeta)^{(\beta \zeta)(\gamma \theta)(\delta \theta)} \\ &= (\theta \eta \zeta)^{(\beta \zeta)(\delta \theta)} = (\delta \eta \zeta)^{(\beta \zeta)} \end{aligned}$$

and the latter is equal to $(\delta \eta \beta)$. Cases (A), (C) and (D) of the conjugacy relations are established similarly.

Finally, we establish all split relations in Equation (1-4) for this stage. If $(m, k) = (3, 1)$, then an arbitrary split relation has the form

$$(\alpha \beta) = (\alpha.0_d \beta.0_d)(\beta.1_d \beta.1_d)$$

for incomparable $\alpha, \beta \in \Delta$. This is deduced from Relation R6 by conjugating by a permutation $\sigma \in \text{Sym}(\Delta)$ that moves $\delta^{(0)}$ to α and $\delta^{(1)}$ to β . For the case when $(m, k) > (3, 1)$, choose $\zeta, \eta \in \Delta$ such that every pair from $\{\alpha, \beta, \zeta, \eta\}$ is incomparable. We have just established that $(\zeta \ \eta) = (\zeta.0_d \ \eta.0_d)(\zeta.1_d \ \eta.1_d)$ and we deduce the general case of Equation (1-4) by conjugating by $(\zeta \ \alpha)(\eta \ \beta)$ and using the conjugacy relations in Equation (1-3) that we have already established.

3.4. Transpositions with short coordinates. We have now constructed all transpositions $(\alpha \ \beta)$ with $\alpha, \beta \in \Omega^*$ and $\alpha \perp \beta$, and demonstrated that all the required relations in Equations (1-2)–(1-4) (and also in Equation (1-5)) involving such transpositions hold in G . We complete the definitions by constructing the remaining transpositions $(\alpha \ \beta)$ with $\alpha, \beta \in \Omega$. Fix a sequence (k_1, k_2, \dots, k_n) with each $k_i \in \{0, 1, 2\}$. As an induction hypothesis, assume that we have constructed all transpositions $(\alpha \ \beta)$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are incomparable addresses satisfying $|\alpha_i|, |\beta_i| \geq k_i$ for $1 \leq i \leq d$. (The ‘base case’ is $(k_1, k_2, \dots, k_n) = (2, 2, \dots, 2)$, for which our assumption follows from the steps just completed.) Select an index d with $k_d > 0$ and for any pair of incomparable addresses α and β such that $|\alpha_i|, |\beta_i| \geq k_i$ for $i \neq d$ and such that one, or possibly both, of α_d or β_d has length $k_d - 1$, define

$$(\alpha \ \beta) := (\alpha.0_d \ \beta.0_d)(\alpha.1_d \ \beta.1_d). \tag{3-6}$$

Both transpositions on the right-hand side exist by our assumption. Furthermore, the transpositions on the right-hand satisfy the relations in Equations (1-2) and (1-5) and commute. It follows that $(\alpha \ \beta) = (\beta \ \alpha)$ and $(\alpha \ \beta)^2 = 1$. Conjugacy relations are established similarly to earlier steps, namely by considering cases (A)–(D) of Lemma 3.1. The method is the same for each case. For example, consider a case (B) conjugacy relation; that is, one of the form

$$(\gamma\eta \ \beta)^{(\gamma \ \delta)} = (\delta\eta \ \beta),$$

where some entry here has its d th coordinate of length $k_d - 1$. We may assume the entry with this shorter coordinate is either γ (and possibly also $\gamma\eta$) or β . If the d th coordinate of γ has length $k_d - 1$ and that of η is empty, then we use Equation (3-6) for both $(\gamma\eta \ \beta)$ and $(\gamma \ \delta)$. Note then $\eta.x_d = x_d.\eta$ for $x \in \{0, 1\}$, which permits us to calculate the following conjugate:

$$\begin{aligned} (\gamma\eta \ \beta)^{(\gamma \ \delta)} &= ((\gamma\eta.0_d \ \beta.0_d)(\gamma\eta.1_d \ \beta.1_d))^{(\gamma.0_d \ \delta.0_d)(\gamma.1_d \ \delta.1_d)} \\ &= (\delta\eta.0_d \ \beta.0_d)(\delta\eta.1_d \ \beta.1_d) = (\delta\eta \ \beta), \end{aligned}$$

relying upon relations that hold by the inductive assumption. The last step is either one of these assumed relations or is the definition of $(\delta\eta \ \beta)$ if it is the case that the d th coordinate of β or δ has length $k_d - 1$. Alternatively, if the d th coordinate of γ has length $k_d - 1$ and that of η is nonempty, write $\eta = x_d.\hat{\eta}$ for some $x \in \{0, 1\}$ and some (possibly empty) $\hat{\eta} \in \Omega$. In this case, we use Equation (3-6) for the definition of $(\gamma \ \delta)$

and calculate

$$\begin{aligned}(\gamma\eta\ \beta)^{(\gamma\ \delta)} &= (\gamma.x_d.\hat{\eta}\ \beta)^{(\gamma.0_d\ \delta.0_d)(\gamma.1_d\ \delta.1_d)} \\ &= (\delta.x_d.\hat{\eta}\ \beta) = (\delta\eta\ \beta).\end{aligned}$$

Conjugacy relations in case (B), where the d th coordinate of β has length $k_d - 1$, and those in cases (A), (C) and (D) are established similarly. Finally, the split relations in Equation (1-4) involving the transpositions defined in Equation (3-6) are either simply that definition or are inherited from split relations for the terms on the right-hand side of that formula.

It now follows, using this step repeatedly, that we have constructed transpositions $(\alpha\ \beta)$, for $\alpha, \beta \in \Omega$ with $\alpha \perp \beta$, in the group G and verified all relations in Equations (1-2)–(1-4) involving these transpositions. Consequently, by Theorem 1.1, there is a homomorphism $\phi: nV \rightarrow G$ mapping each transposition in nV to the corresponding element that we have defined in G . Moreover, Relation R7 tells us that the generator c is in the image of ϕ and hence G is isomorphic to a quotient of nV . However, all Relations R1–R7 listed are satisfied by the corresponding elements of nV and so there is a homomorphism from G into nV with nontrivial image. The fact that nV is simple therefore yields $G \cong nV$, completing the proof of Theorem 1.2.

PROOF OF COROLLARY 1.3. The subgroup $H = \langle a, b \rangle \cong \text{Sym}(\Delta)$ of G can be generated by a cycle x of length 4^n and a transposition t that can be assumed disjoint from c (as described via Relation R7). Note that c has odd order. Therefore, c and t are powers of $y = ct$ and $\{x, y\}$ is a generating set for G . Applying Tietze transformations to produce a presentation on generators x and y introduces two additional relations. This establishes the corollary. \square

Acknowledgements

The author thanks Collin Bleak for introducing him to Brin's family of groups nV , suggesting this project for investigation and for various helpful comments. It should also be recorded that the fact that nV can be generated by two elements, as appears in the proof of Corollary 1.3, was originally observed in collaboration with Bleak. The author also thanks Prof. Brin for his comments on a draft of this article and a referee of an earlier version for their helpful suggestions.

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