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ON TOPOLOGICAL APPROACHES TO THE JACOBIAN CONJECTURE IN $\mathbb{C}^N$

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Abstract We obtain a new theorem for the non-properness set S_f of a non-singular polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$. In particular, our result shows that if f is a counterexample to the Jacobian conjecture, then $S_f \cap Z \neq \emptyset$, for every hypersurface Z dominated by $\mathbb{C}^{n-1}$ on which some non-singular polynomial $h : \mathbb{C}^n \to \mathbb{C}$ is constant. Also, we present topological approaches to the Jacobian conjecture in $\mathbb{C}^n$. As applications, we extend bidimensional results of Rabier, Lê and Weber to higher dimensions.

Keywords: Jacobian conjecture; global injectivity; locally trivial fibrations; non-properness set

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1. Introduction and statement of the main results

This paper is strongly motivated by the arguments presented in the paper by Krasiński and Spodzieja [**18**].

Let $g = (g_1, g_2, \dots, g_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic mapping. We denote by $\operatorname{Jac}(g)(x)$ the Jacobian matrix of g at x. When $m = 1$, we denote this matrix by $\nabla g(x)$. When $m = n$, we denote by $\det \operatorname{Jac}(g)(x)$ the determinant of the Jacobian matrix of g at x. A point $y \in \mathbb{C}^m$ is a *regular value* of g if for each $x \in g^{-1}(y)$ the matrix $\operatorname{Jac}(g)(x)$ has maximum rank. We say that g is *non-singular* if its range contains only regular values. Let $J = (i_1, i_2, \dots, i_\ell)$, $i_1 < i_2 < \cdots < i_\ell$, be a sequence of integers in $\{1, 2, \dots, m\}$. We denote by G_J the mapping $G_J = (g_{i_1}, g_{i_2}, \dots, g_{i_\ell}) : \mathbb{C}^n \to \mathbb{C}^\ell$. When $J = (1, \dots, k-1, k+1, \dots m)$, we denote G_J by $G_{\widehat{k}}$.

A mapping $g : \mathbb{C}^n \to \mathbb{C}^m$ is said to be *proper at* $y \in \mathbb{C}^m$ if there exists a neighbourhood V of y such that $g^{-1}(\overline{V})$ is compact. The set of points at which g is not proper is denoted by S_g. We say that g is *proper* if S_g is the empty set $\emptyset$. When g is a polynomial mapping, the set S_g has been considered in many problems and applications; see, for instance, [12–14, 16, 19].

Recall that a mapping $\phi : X \to Y$, with $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ algebraic sets, is a *regular mapping* if ϕ is the restriction to X of a polynomial mapping defined in $\mathbb{C}^n$. We say that $\phi : X \to Y$ is a *biregular mapping* if ϕ and ϕ^{-1} are regular mappings, and in this case we say that X *is biregular to* Y.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a non-singular mapping, that is, $\det \operatorname{Jac} f(x)$ is a non-zero constant. Our main result is Theorem 1.2 below, which presents a new property of the non-properness set S_f of f. In order to state the theorem properly, we provide the following definition, motivated by [9].

Definition 1.1. Let $Z \subset \mathbb{C}^n$ be a non-singular hypersurface. We say that Z is dominated by $\mathbb{C}^{n-1}$ if there exists an onto proper regular mapping $\varphi : \mathbb{C}^{n-1} \to Z$.

Theorem 1.2. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a non-singular polynomial mapping. Then either S_f is the empty set, or $Z \cap S_f$ is not empty, for every non-singular hypersurface Z dominated by $\mathbb{C}^{n-1}$ on which some non-singular polynomial $h : \mathbb{C}^n \to \mathbb{C}$ is constant.*

We remark that sets Z with the properties of Theorem 1.2 include, for instance, graphs of polynomial functions $g : \mathbb{C}^{n-1} \to \mathbb{C}$.

The claim that a non-singular polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial automorphism is the very well-known *Jacobian conjecture*, which hitherto remains unsolved; see, for instance, [3, 8] for details. From Hadamard's well-known global inversion theorem and the main result of Cynk and Rusek [5], f is an automorphism if and only if it is non-singular and S_f is the empty set. So the Jacobian conjecture will be proved if one shows that S_f is the empty set for any non-singular polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$.

From Jelonek [12, 13], it follows that for non-singular polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^n$, the set S_f is either empty or a hypersurface. Therefore, if there exists a counter-example $f : \mathbb{C}^n \to \mathbb{C}^n$ to the Jacobian conjecture, then the non-properness set S_f is a hypersurface such that $S_f \cap Z \neq \emptyset$ for every non-singular hypersurface $Z \subset \mathbb{C}^n$ satisfying the assumptions of Theorem 1.2.

Theorem 1.2 as well as the reasoning to prove it lead us to discuss results related to the notion of fibrations. We now turn to explaining these results which will culminate in generalizations of known bidimensional results to higher dimensions. We recall that a continuous mapping $g : X \to Y$ between topological spaces X and Y is a *trivial fibration* if there exist a topological space $\mathcal{F}$ and a homeomorphism $\varphi \colon \mathcal{F} \times Y \to X$ such that $pr_2 = g \circ \varphi$ is the second projection on Y. We say further that g is a *locally trivial fibration at* $y \in Y$ if there exists an open neighbourhood U of y in Y such that $g|_{g^{-1}(U)} \colon g^{-1}(U) \to U$ is a trivial fibration. We denote by $B(g)$ the set of points of Y where g is not a locally trivial fibration. The set $B(g)$ is usually called the *bifurcation* (or *atypical*) set of g. If $B(g)$ is the empty set we simply say that g is a *locally trivial fibration*.

In the case where $n = 2$, as a consequence of Abhyankar and Moh's embedding theorem [1], Lê and Weber [20] presented the following result.

Theorem 1.3 (Lê and Weber [20]). *Let $f = (f_1, f_2) \colon \mathbb{C}^2 \to \mathbb{C}^2$ be a non-singular polynomial mapping. If $B(f_1) = \emptyset$, then f is an automorphism.*

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As a consequence, the Jacobian conjecture in $\mathbb{C}^2$ can be reformulated in the following geometrical-topological way. See also [**27**, p. 781].

Conjecture 1.4 (Lê and Weber [20]). *Let* $f_1 : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial function. If* $B(f_1) \neq \emptyset$*, then for any polynomial function* $f_2 : \mathbb{C}^2 \to \mathbb{C}$ *there exists* $x \in \mathbb{C}^2$ *such that* $\det \operatorname{Jac}(f_1, f_2)(x) = 0$.

Analytical conditions ensuring locally trivial fibrations are known in the literature. So, in view of Theorem 1.3, for example, the use of such conditions is expected to obtain particular cases of the Jacobian conjecture. In this context, Rabier [**25**] considered analytical conditions to define the set $\widetilde{K}_\infty(g)$ for holomorphic mappings $g : \mathbb{C}^n \to \mathbb{C}^m$ (see Definition 3.1). He then obtained the following result.

Theorem 1.5 (Rabier [25, Theorem 9.1]). *Let* $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial mapping.*

(a) *If* f *is an automorphism, then* f *is non-singular and* $\widetilde{K}_\infty(f_1) = \widetilde{K}_\infty(f_2) = \emptyset$.

(b) *If* f *is non-singular and* $\widetilde{K}_\infty(f_1) = \emptyset$*, then* f *is an automorphism.*

Rabier also proved the inclusion $B(g) \subset \widetilde{K}_\infty(g)$ for holomorphic mappings $g : \mathbb{C}^n \to \mathbb{C}^m$. Example 3.2 below shows that for $n \geq 3$, a polynomial mapping $f : \mathbb{C}^3 \to \mathbb{C}^3$ can be an automorphism even if $\widetilde{K}_\infty(F_{\widehat{k}}) \neq \emptyset$ for each $k = 1, 2, 3$; see also Remark 3.3. On the other hand, if $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial automorphism, then f is clearly non-singular and $B(F_{\widehat{k}}) = \emptyset$ for each $k = 1, 2, \ldots, n$. Moreover, it is known that a locally trivial fibration $g : \mathbb{C}^n \to \mathbb{C}^m$ has simply connected fibres (see Proposition 2.1). So it turns out that the next two results generalize Theorems 1.3 and 1.5 in different ways to higher dimensions.

Theorem 1.6. *Let* $f = (f_1, f_2, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ *be a non-singular polynomial mapping. If the fibres of* $F_{\widehat{n}}$ *out of* $B(F_{\widehat{n}})$ *are simply connected, then* f *is an automorphism. In particular, if* $B(F_{\widehat{n}}) = \emptyset$ *then* f *is an automorphism.*

Theorem 1.7. *Let* $f = (f_1, f_2, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ *be a non-singular polynomial mapping. Assume that the connected components of the fibres of* $F_{\widehat{k}} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ *out of* $B(F_{\widehat{k}})$ *are simply connected, for all* $k \in \{2, \ldots, n\}$*. Then* f *is an automorphism.*

We point out that it is enough to test the simply-connectedness in the hypotheses of above theorems over any open set of $\mathbb{C}^{n-1}$. This is so because $B(F_{\widehat{k}})$ is always contained in a hypersurface of $\mathbb{C}^{n-1}$; see details in the proof of Theorem 2.3.

In § 2, we relate the sets S_f and $B(F_{\widehat{k}})$ in Theorem 2.3 to each other and apply this theorem in the proofs of Theorems 1.6 and 1.7.

Now, analogously to Conjecture 1.4, as a direct application of Theorem 1.6, we obtain the following equivalent statement of the Jacobian conjecture in $\mathbb{C}^n$.

Conjecture 1.8. *Let* $F_{\widehat{n}} = (f_1, f_2, \ldots, f_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ *be a polynomial mapping. If* $B(F_{\widehat{n}}) \neq \emptyset$*, then for any polynomial function* $f_n : \mathbb{C}^n \to \mathbb{C}$ *there exists* $x \in \mathbb{C}^n$ *such that* $\det \operatorname{Jac}(F_{\widehat{n}}, f_n)(x) = 0$.

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We remark that from [**17**, Theorem 1.1] we have complete descriptions for the set $B(F_{\widehat{n}})$.

Another application of Theorem 1.6 is a topological proof of the bijectivity of the non-singular mappings $I + H : \mathbb{C}^4 \to \mathbb{C}^4$, with I the identity and H a homogeneous polynomial of degree three, which appear in Hubbers's classification [**11**]; see Remark 2.4 for details.

We end the paper with a result in $\mathbb{C}^2$.

Proposition 1.9. *Let $f_1 : \mathbb{C}^2 \to \mathbb{C}$ be a non-singular polynomial function. The following statements are equivalent.*

(a) *The connected components of a fibre $f_1^{-1}(c)$ are simply connected.*

(b) *There exists a polynomial $f_2 : \mathbb{C}^2 \to \mathbb{C}$ such that the mapping $(f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism.*

Therefore the assumptions on a non-singular polynomial function $f_1 : \mathbb{C}^2 \to \mathbb{C}$ in Theorems 1.6, 1.7 and Proposition 1.9 are equivalent. Moreover, these assumptions are equivalent to $B(f_1) = \emptyset$. From the results of Krasiński and Spodzieja [**18**, Theorem 4.1], it also follows that the above conditions are equivalent to the Hamiltonian vector field of f_1 defined in $\mathbb{C}[x]$ to be onto $\mathbb{C}[x]$.

2. Proofs of the theorems

We begin by recalling the following well-known property on trivial fibrations; see, for instance, [**28**, 11.6].

Proposition 2.1. *If $g\colon X \to Y$ is a locally trivial fibration and Y is a contractible space, then g is a trivial fibration.*

The following proposition will be used in the sequel to prove our main results.

Proposition 2.2. *Let $Y \subset \mathbb{C}^n$ be a non-singular simply connected algebraic curve. Then any locally injective regular function $g : Y \to \mathbb{C}$ is biregular.*

Proof. In this proof we follow the reasoning of [**18**, p. 310]. From the Riemann mapping theorem there is a biholomorphism $\phi : Y \to \mathbb{C}$, which is biregular, because proper holomorphic mappings between algebraic curves are regular mappings; see, for instance, [**26**, Theorem 4]. The composite function $h = g \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is a polynomial locally invertible function, $h(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$. Therefore, $g = h \circ \phi$ is a biregular mapping, which completes the proof. □

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a non-singular polynomial mapping. Since f is a local homeomorphism, it follows that f is a dominant mapping. Then, from [**12**, **13**], the non-properness set S_f is either empty or an algebraic hypersurface. Then $\mathbb{C}^n \setminus S_f$ is a connected subset of $\mathbb{C}^n$ (see, for instance, [**14**, Lemma 8.1]) and therefore f is an analytic cover of geometric degree $\mu(f)$ on $\mathbb{C}^n \setminus S_f$. Thus $\#f^{-1}(y) = \mu(f)$ for any $y \in \mathbb{C}^n \setminus S_f$.

By using once more that f is a local homeomorphism, it follows that $\#f^{-1}(z) < \mu(f)$ for any $z \in S_f$, and hence

$$S_f = \{y \in \mathbb{C}^n \mid \#f^{-1}(y) \neq \mu(f)\}. \tag{1}$$

For any $1 \leq k \leq n$, we denote by $\pi_{\widehat{k}} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection $\pi_{\widehat{k}}(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. With the above notation we have the following theorem.

Theorem 2.3. *Let $f = (f_1, f_2, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a non-singular polynomial mapping and $k \in \{1, 2, \ldots, n\}$. Assume that the connected components of the fibres of $F_{\widehat{k}}$ out of $B(F_{\widehat{k}})$ are simply connected. Then*

(a) *$S_f \subset \pi_{\widehat{k}}^{-1}(Z)$ for any algebraic set $Z \subset \mathbb{C}^{n-1}$ such that $B(F_{\widehat{k}}) \subset Z$;*

(b) *$S_f = \pi_{\widehat{k}}^{-1}(\pi_{\widehat{k}}(S_f))$.*

Proof. As explained before, it follows by [**12**, **13**] that S_f is either empty or an algebraic hypersurface. If $S_f = \emptyset$, we have nothing to show in items (a) and (b). Therefore, we assume that S_f is an algebraic hypersurface.

Proof of (a), let $Z \subset \mathbb{C}^{n-1}$ be an algebraic set such that $B(F_{\widehat{k}}) \subset Z$. If $Z = \mathbb{C}^{n-1}$ there is nothing to prove. So assume $Z \neq \mathbb{C}^{n-1}$. It follows that $L := \mathbb{C}^{n-1} \setminus Z$ is an open connected set (see [**14**, Lemma 8.1]) such that $F_{\widehat{k}}|_{F_{\widehat{k}}^{-1}(L)} : F_{\widehat{k}}^{-1}(L) \to L$ is a locally trivial fibration, and hence there exists $d_k \in \mathbb{N}$ such that $F_{\widehat{k}}^{-1}(\widetilde{y})$ has d_k connected components for each $\widetilde{y} \in L$.

Now let $y \in \pi_{\widehat{k}}^{-1}(L)$ and $V_1 \cup \cdots \cup V_{d_k}$ be the decomposition of $F_{\widehat{k}}^{-1}(\pi_{\widehat{k}}(y))$ into its connected components. From Proposition 2.2, it follows that $f_k|_{V_j} : V_j \to \mathbb{C}$ is a biregular function for each $j = 1, \ldots, d_k$. This shows that

$$\#f^{-1}(y) = d_k, \quad \forall y \in \pi_{\widehat{k}}^{-1}(L). \tag{2}$$

Since S_f is an algebraic hypersurface and $\pi_{\widehat{k}}^{-1}(L)$ is open, it follows that $\pi_{\widehat{k}}^{-1}(L) \setminus S_f \neq \emptyset$, which by (1) and (2) gives that $d_k = \mu(f)$. Therefore it follows that $\pi_{\widehat{k}}^{-1}(L) \subset \mathbb{C}^n \setminus S_f$, proving (a).

Proof of (b). We know that $B(F_{\widehat{k}})$ is contained in an algebraic hypersurface $Z \subset \mathbb{C}^{n-1}$; see, for instance, the main result of [**15**] or [**29**, Corollaire 5.1]. Let $Z_1 \cup \ldots \cup Z_l$ be the decomposition of Z into its irreducible components. It follows that $\pi_{\widehat{k}}^{-1}(Z_1) \cup \ldots \cup \pi_{\widehat{k}}^{-1}(Z_l)$ is the decomposition of $\pi_{\widehat{k}}^{-1}(Z)$ into its irreducible components. By statement (a), $S_f \subset \pi_{\widehat{k}}^{-1}(Z)$, and since S_f and $\pi_{\widehat{k}}^{-1}(Z)$ are algebraic hypersurfaces, it follows that there are indices $i_1, \ldots, i_j \in \{1, 2, \ldots, l\}$ such that

$$S_f = \pi_{\widehat{k}}^{-1}(Z_{i_1}) \cup \cdots \cup \pi_{\widehat{k}}^{-1}(Z_{i_j}) = \pi_{\widehat{k}}^{-1}(Z_{i_1} \cup \cdots \cup Z_{i_j}).$$

Since $\pi_{\widehat{k}}$ is onto, it follows that $\pi_{\widehat{k}}^{-1}(\pi_{\widehat{k}}(S_f)) = S_f$, proving statement (b). □

We can also give the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $Z \cap S_f = \emptyset$, with $Z \subset \mathbb{C}^n$ a non-singular hypersurface dominated by $\mathbb{C}^{n-1}$. Let $h : \mathbb{C}^n \to \mathbb{C}$ be the non-singular polynomial which is constant on Z. We may suppose that $Z = h^{-1}(0)$ and that h is irreducible.

Since Z is dominated by $\mathbb{C}^{n-1}$, it follows by the main result of [**9**] that Z is simply connected.

Let $V := f^{-1}(Z) = g^{-1}(0)$, where $g := h \circ f$. From the assumption $Z \cap S_f = \emptyset$ and (1) it follows that the restricted mapping $f|_V : V \to Z$ is a cover mapping with degree $\mu(f)$. Since Z is simply connected, V has $\mu(f)$ connected components, say $V = V_1 \cup \cdots \cup V_{\mu(f)}$, and $f|_{V_i}$ are biholomorphisms for each $i = 1, \ldots, \mu(f)$. Since $f|_{V_i} : V_i \to Z \subset \mathbb{C}^n$ are proper regular mappings, it thus follows that $f|_{V_i}^{-1} : Z \to V_i$ are regular mappings. This is so because if p, q are mappings of algebraic sets, with p holomorphic, q regular and proper, and $q \circ p$ is regular then so is p; see [**26**, Theorem 3(ii)]. Hence $f|_{V_i}$ are biregular mappings for $i = 1, \ldots, \mu(f)$.

The connected components of V are the irreducible components of V. Let q_j be an irreducible polynomial such that $V_j = q_j^{-1}(0)$, $j = 1, \ldots, \mu(f)$. Since g is non-singular, it follows that $g = \gamma q_1 \cdots q_{\mu(f)}$, for a suitable $\gamma \in \mathbb{C}^*$.

We *claim that* $q_j|_{V_i} : V_i \to \mathbb{C}$ *is constant for each* $i \neq j$ *in* $\{1, \ldots, \mu(f)\}$. Indeed, if this is not true for some i and j and $\psi : \mathbb{C}^{n-1} \to Z$ is the onto proper regular mapping from the assumption, the non-constant polynomial $q_j \circ f|_{V_i}^{-1} \circ \psi : \mathbb{C}^{n-1} \to \mathbb{C}$ has a zero, and so $V_i \cap V_j \neq \emptyset$. This contradiction proves the claim.

So, since g is non-singular, it follows from the Nullstellensatz that $q_j = \beta_{ij} q_i + \alpha_{ij}$, for polynomials β_{ij} and constants α_{ij}. Thus for $i \neq j$, we have $q_i = \beta_{ji}\beta_{ij} q_i + \beta_{ji}\alpha_{ij} + \alpha_{ji}$, and total degree considerations show that $\beta_{ij} \in \mathbb{C}^*$. Therefore, $g = P(q_1)$, with P a suitable polynomial of degree $\mu(f)$. Since $\nabla g = P'(q_1)\nabla q_1$ is nowhere zero in $\mathbb{C}^n$, it follows that the degree of P must be 1, that is, $\mu(f) = 1$. Then f is injective and hence it is an automorphism from [**5**]. Therefore $S_f = \emptyset$, which ends the proof. □

Pinchuk [**24**] presented a non-singular polynomial mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not invertible, thus providing a counterexample to the *real Jacobian conjecture*. In this example, we have $S_f \cap Z_c = \emptyset$, for any line $Z_c := \{(c, y) \mid y \in \mathbb{R}\}$ and $c < -1$; see, for instance, [**4**]. Therefore, Theorem 1.2 does not hold for non-singular polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^n$.

We now provide the proof of Theorem 1.6.

Proof of Theorem 1.6. Since $B(F_{\widehat{n}})$ is contained in a hypersurface $Z \subset \mathbb{C}^{n-1}$ ([**15**] or [**29**, Corollaire 5.1]), it follows from statement (a) of Theorem 2.3 that $S_f \subset \pi_{\widehat{n}}^{-1}(Z)$. Let $y \notin \pi_{\widehat{n}}^{-1}(Z)$. From (1) and [**5**], it is enough to prove that $\# f^{-1}(y) = 1$.

From the hypothesis, the fibre $F_{\widehat{n}}^{-1}(\pi_{\widehat{n}}(y))$ is simply connected. It thus follows by Proposition 2.2 that f_n is injective in this fibre. So $\# f^{-1}(y) = 1$, and we are done. □

In the following remark we present an application of Theorem 1.6.

Remark 2.4. An important result on the Jacobian conjecture given by Bass, Connel and Wright in [**3**] is that the Jacobian conjecture in all dimensions follows if one proves that for all $n \geq 2$, non-singular polynomial mappings of the form $f = I + H : \mathbb{C}^n \to \mathbb{C}^n$, where I is the identity mapping and H is a homogeneous polynomial of degree three, are injective. In [**11**], Hubbers classified the non-singular polynomial mappings $I + H$, for

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$n = 4$, up to linear conjugations, obtaining eight families of mappings. Then he proved the bijectivity of each family by applying a criterion described by van den Essen in [**7**], which is based on the calculation of the Gröbner basis of an ideal defined from the components of f. Here we give a new and topological proof of the bijectivity of each mapping in Hubbers's classification, using Theorem 1.6. Indeed, with the enumeration of [**11**, Theorem 2.7] or [**8**, Theorem 7.1.2], it is straightforward to check the simply-connectedness of the fibres of:

$$F_{\widehat{4}} = (f_1, f_2, f_3)\colon \mathbb{C}^4 \to \mathbb{C}^3 \quad \text{for families 1, 2, 7 and 8,}$$
$$F_{\widehat{2}} = (f_1, f_3, f_4)\colon \mathbb{C}^4 \to \mathbb{C}^3 \quad \text{for family 3,}$$
$$F_{\widehat{3}} = (f_1, f_2, f_4)\colon \mathbb{C}^4 \to \mathbb{C}^3 \quad \text{for families 4, 5 and 6.}$$

Thus each family is an automorphism by Theorem 1.6.

We can now also present the proof of Theorem 1.7.

Proof of Theorem 1.7. By applying statement (b) of Theorem 2.3 for each k, it follows that there is a set $B \subset \mathbb{C}$ such that $S_f = B \times \mathbb{C}^{n-1}$. Since S_f is either empty or a hypersurface, it then follows that there exists $z \in \mathbb{C}$ such that the affine hyperplane $Z = \{z\} \times \mathbb{C}^{n-1}$ is disjoint of S_f. The result thus follows by Theorem 1.2. □

It is well known that analytic and geometric conditions can be used to estimate $B(F_{\widehat{k}})$; see, for instance, [**6**, **15**, **17**, **19**, **25**]. Thus, we may use these conditions to ensure the topological hypothesis related to $B(F_{\widehat{k}})$ in Theorems 1.6, 1.7 and 2.3.

Remark 2.5. Splitting a complex mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ into real and imaginary parts, we obtain an associated real mapping $f^{\mathbb{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. In [**21**, Corollary 2] it was proved that *the (complex) Jacobian conjecture is equivalent to the simply-connectedness of all connected components of fibres of* $f^{\mathbb{R}}_{i_1 \ldots i_{2n-2}}$, *for all combinations* $(i_1 < \ldots < i_{2n-2})$ *of* $\{1, \ldots, 2n\}$. Note that this requires this topological condition to be verified for $2n^2 - n$ mappings from $\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$ for mixing real and imaginary parts. Our Theorem 1.7 improves this equivalence by proving that it is enough to check the same topological condition just for $n - 1$ mappings instead of the $2n^2 - n$ cases of [**21**].

3. On the Rabier condition

In this section we recall the definition of the set $\widetilde{K}_\infty(g)$, for holomorphic mappings $g : \mathbb{C}^n \to \mathbb{C}^m$. We also present the example of a polynomial automorphism f in $\mathbb{C}^3$ such that $\widetilde{K}_\infty(F_{\widehat{k}}) \neq \emptyset$ for $k = 1, 2, 3$, as mentioned in the introduction. We end this section with a discussion of our contributions related to already known results.

Definition 3.1 (Rabier [25]). Let $g\colon \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping, with $n \geq m$. We set

$$\widetilde{K}_\infty(g) := \{t \in \mathbb{C}^m \mid \exists \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n, \lim_{j\to\infty} \|x_j\| = \infty, \lim_{j\to\infty} g(x_j) = t \text{ and } \lim_{j\to\infty} \nu(\mathrm{Jac}(g)(x_j)) = 0\}, \tag{3}$$

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where $\nu(A) := \inf_{\|\varphi\|=1} \|A^*(\varphi)\|$, for a linear mapping $A\colon \mathbb{C}^n \to \mathbb{C}^m$ and its adjoint $A^*\colon (\mathbb{C}^m)^* \to (\mathbb{C}^n)^*$. We say that g satisfies the *Rabier condition* if $\widetilde{K}_\infty(g) = \emptyset$.

For $g : \mathbb{C}^n \to \mathbb{C}$, we have $\nu(\nabla g(x)) = \|\nabla g(x)\|$, and if g is non-singular, Definition 3.1 recovers the classical *Palais–Smale condition*.

We observe that different functions instead of ν produce the same set $\widetilde{K}_\infty(g)$; see, for instance,[**15**, **19**]. Other conditions related to $\widetilde{K}_\infty(g)$ can be found, for instance, in [**6**, **19**].

The next example, taken from the class presented in [**23**], shows that for $f : \mathbb{C}^n \to \mathbb{C}^n$, $n \geq 3$, a version of Theorem 1.5 (a) does not hold if we use the Rabier condition on the mappings $F_{\widehat{k}} : \mathbb{C}^n \to \mathbb{C}^{n-1}$.

Example 3.2 (see [23]). Let $f = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by

$$f_1(x, y, z) = x + yh(x, y, z), \quad f_2(x, y, z) = y, \quad f_3(x, y, z) = h(x, y, z).$$

where $h(x, y, z) = z - 3x^5y + 2x^7y^2$. We have that $\det \operatorname{Jac}(f) \equiv 1$, and that f is an automorphism whose inverse is

$$f^{-1}(p, q, r) = (p - qr, q, r + 3q(p - qr)^5 - 2q^2(p - qr)^7).$$

We also have that $F_{\widehat{3}} = (f_1, f_2) : \mathbb{C}^3 \to \mathbb{C}^2$, $F_{\widehat{2}} = (f_1, f_3) : \mathbb{C}^3 \to \mathbb{C}^2$ and $F_{\widehat{1}} = (f_1, f_3) : \mathbb{C}^3 \to \mathbb{C}^2$ do not satisfy the Rabier condition; see Definition 3.1. In fact, to prove that $\widetilde{K}_\infty(F_{\widehat{3}}) \neq \emptyset$, we may use the sequence $\lambda(n) = (n, 1/n^2, 0)$. For $F_{\widehat{2}}$ and $F_{\widehat{1}}$, we may use the sequences $\gamma(n) = (1/n, n^2, 1/n^3)$ and $\delta(n) = (n, 1/n^2, n^3)$, respectively.

Remark 3.3. The proof from [**20**] of Theorem 1.3 depends on Abhyankar and Moh's result. On the other hand, the proof of Theorem 1.5 (b) presented in [**25**] depends on a formula by Adjamagbo and van den Essen [**2**, Corollary 1.4]. It is known that a non-singular polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ satisfies the Rabier condition (i.e. $\widetilde{K}_\infty(f) = \emptyset$) if and only if f is a locally trivial fibration; see [**10**, **22**, **27**]. Thus, the proof of Theorem 1.6 also gives different proofs of Theorems 1.3 and 1.5.

4. The bidimensional case

Proof of Proposition 1.9. Clearly (b) implies (a).

Assume (a). Without loss of generality we assume that $c = 0$ and write $f_1^{-1}(0) = V_1 \cup \cdots \cup V_d$ for the decomposition of $f_1^{-1}(0)$ into its connected components. Each V_j is an irreducible component of $f_1^{-1}(0)$, and so $V_j = q_j^{-1}(0)$, where q_j is an irreducible polynomial. From Proposition 2.2, each connected component of $f_1^{-1}(c)$ is biregular to $\mathbb{C}$. Since f_1 is non-singular, it follows that $f_1 = \gamma q_1 \cdots q_d$ with $\gamma \in \mathbb{C}$. As in the proof of Theorem 1.2, we conclude that $d = 1$, and so $f_1^{-1}(0)$ is biregular to $\mathbb{C}$. Now, from [**1**], it follows that there exists an automorphism $h : \mathbb{C}^2 \to \mathbb{C}^2$ such that $g_1(x_1, x_2) = f_1 \circ h(x_1, x_2) = x_1$. Let $g_2(x_1, x_2) = x_2$ and define $f_2(x_1, x_2) = g_2 \circ h^{-1}(x_1, x_2)$. Then (f_1, f_2) is an automorphism. □

As we said in the introduction section, the assumptions in Theorems 1.6 and 1.7 on a non-singular polynomial function $f_1 : \mathbb{C}^2 \to \mathbb{C}$ are equivalent. An open question is whether

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a non-singular polynomial mapping $F_{\widehat{k}} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ whose connected components of the fibres are simply connected necessarily has connected fibres.

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