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A CHARACTERISATION OF CLOSED SUBALGEBRAS OF $\mathcal{B}(H)$

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We show that the class of Banach algebras A isomorphic with norm-closed (non-self-adjoint) subalgebras of $\mathcal{B}(H)$ is characterized by the condition that the norms of polynomials in A be dominated by the norms of the same polynomials in $\mathcal{B}(H)$.

Definition 1. If H is a Hilbert space, $\mathcal{B}(H)$ denotes the Banach algebra of all bounded operators on H. A Banach algebra which is bi-continuously isomorphic with a closed subalgebra of $\mathcal{B}(H)$, for some Hilbert space H, will be called an *R-algebra*, and an *IR-algebra* if the isomorphism is isometric. If X is a compact Hausdorff space, then $C(X)$ denotes the Banach algebra of all continuous complex-valued functions on X, with the sup norm. A *uniform algebra* is a closed subalgebra of some $C(X)$. A *Q-algebra* is a Banach algebra A which is bicontinuously isomorphic with the quotient of a uniform algebra by a closed ideal, and A is an *IQ-algebra* if the isomorphism is isometric.

Definition 2. Unless otherwise qualified, the word "polynomial" will mean a polynomial in several non-commuting variables, without constant term. If $p(X_1, \ldots, X_n)$ is such a polynomial, A a Banach algebra and $\delta > 0$, then we define

$$\|p\|_{A,\delta} = \sup \{\|p(x_1, \ldots, x_n)\| : x_i \in A, \|x_i\| \leq \delta \ (1 \leq i \leq n)\}.$$

We have separate notations for two important special cases: $\|p\|_\infty$ for $\|p\|_{C,1}$, where C denotes the complex numbers, and $\|p\|_\eta$ for $\|p\|_{\mathcal{B}(H),1}$, where H is, say, a separable Hilbert space; (any other infinite-dimensional Hilbert space H produces the same norm).

We recall the main results about Q-algebras and their relation to R-algebras.

Theorem (Craw; see (2)). *A commutative Banach algebra A is a Q-algebra if and only if there exist $M, \delta > 0$ such that $\|p\|_{A,\delta} \leq M\|p\|_\infty$ for all polynomials p. Further, A is IQ if and only if this condition holds with $M = \delta = 1$.*

Theorem (Cole; see (2)). *Every Q-algebra is an R-algebra.*

Theorem (Varopoulos (4)). *Not every commutative R-algebra is Q-algebra.*

Our theorem is a sort of non-commutative analogue of Craw's result, though, by Varopoulos' theorem, it reduces to something different in the commutative case (see Corollary).

Theorem. *A Banach algebra A is an R-algebra if and only if there exist* $M, \delta > 0$ *such that* $\|p\|_{A,\delta} \leq M\|p\|_\eta$ *for all polynomials p. Further, A is IR if and only if this condition holds with* $M = \delta = 1$.

Proof. That every R-algebra satisfies the stated condition, with $M = \delta = 1$ for an IR-algebra, is clear. The proof of the converse parallels that of Craw's theorem, using $\mathcal{B}(H)$ instead of C. Thus, we let $\Lambda = \{a \in A : \|a\| \leq \delta\}$, $\Delta = \{z \in \mathcal{B}(H) : \|z\| \leq 1\}$, and X the Cartesian product Δ^Λ. Let $B(X, \mathcal{B}(H))$ denote the C*-algebra of all bounded functions $\phi : X \to \mathcal{B}(H)$, with the sup norm: $\|\phi\| = \sup\{\|\phi(x)\| : x \in X\}$. For each $a \in \Lambda$, we define $\zeta_a \in B(X, \mathcal{B}(H))$ by $\zeta_a(x) = x(a)$ $(x \in X)$. Let U_0 be the subalgebra of $B(X, B(H))$ generated by $\{\zeta_a : a \in \Lambda\}$, and let U be the closure of U_0. Let π be the homomorphism of U_0 onto A defined by

$$\pi(p(\zeta_{a_1}, \ldots, \zeta_{a_n})) = p(a_1, \ldots, a_n)$$

for all polynomials $p(X_1, \ldots, X_n)$ and all n-tuples $(a_1, \ldots, a_n)$ of distinct elements of Λ. Since $\|\zeta_a\| = 1$ $(a \in \Lambda)$, the given condition $\|p\|_{A,\delta} \leq M\|p\|_\eta$ ensures that π is continuous, with norm at most M. Therefore π extends to a homomorphism of U onto A. Thus A is bicontinuously isomorphic with a quotient of the closed subalgebra of U of the C*-algebra $B(X, \mathcal{B}(H))$.

The remainder of the proof is a non-commutative analogue of Cole's theorem, due to Bernard.

Theorem (Bernard (1)). *Let* Γ *be a C*-algebra with identity. Let U be a closed subalgebra of* Γ *containing the identity, and let I be a closed ideal of U. Then U/I is an IR-algebra.*

The provisos concerning the identity may clearly be dropped, by adjoining the identity to U if it is not already in U. Applying Bernard's theorem to our situation shows that A is an R-algebra.

If $M = \delta = 1$, then the isomorphism induced by π is an isometry, and the isometric nature of Bernard's theorem completes the proof that A is IR.

For the commutative version, we define $\|p\|_\eta$ for a polynomial $p(X_1, \ldots, X_n)$ in commuting variables by

$$\|p\|_\eta = \sup \|p(T_1, \ldots, T_n)\|$$

the supremum being taken over all n-tuples $(T_1, \ldots, T_n)$ of commuting contractions on a separable Hilbert space.

Corollary 1. *A commutative Banach algebra A is an R-algebra if and only if there exist M, $\delta > 0$ such that $\|p\|_{A,\delta} \leq \|p\|_\eta$ for all polynomials $p(X_1, \ldots, X_n)$ in commuting variables $X_1, \ldots, X_n$ and without constant term. Further, commutative IR-algebras are characterised by this condition with $M = \delta = 1$.*

The main force of the theorem is that there is *some* condition on the norms of polynomials which characterises R-algebras. Of course, the function $\|\cdot\|_\eta$ is not easily calculated, and there is a need for more usable conditions. However, the fact that there is a condition of this form is of some help. By methods similar to those used by Davie ((3) pp. 38–39) to construct Arens regular, non-Q algebras, we may prove:

Corollary 2. *There exist commutative, Arens regular, non-R algebras.*

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