Note on Ramanujan's arithmetical function $\tau(n)$. By Mr G. H. HARDY, Trinity College.

[*Received* 4 January, *read* 31 January 1927.]

1. In his remarkable memoir 'On certain arithmetical functions'* Ramanujan considers, among other functions of much interest, the function $\tau(n)$ defined by

$$f(x) = x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} = \Sigma\tau(n)x^n \quad \dots(1\cdot1).$$

This function is important in the theory of the representation of a number as a sum of 24 squares. In fact

$$(1+2x+2x^4+2x^9+\dots)^{24} = \Sigma r_{24}(n)x^n = \Sigma\delta_{24}(n)x^n + \Sigma e_{24}(n)x^n:$$

where $r_{24}(n)$ is the number of representations;

$$\tfrac{691}{16}\delta_{24}(n) = \sigma_{11}(n) - 2\sigma_{11}'(\tfrac{1}{2}n)\dagger,$$

where $\sigma_s(n)$ is the sum of the sth powers of the divisors of n, and $\sigma_s'(n)$ the sum of those of its odd divisors; and

$$\tfrac{691}{128}e_{24}(n) = (-1)^{n-1}259\tau(n) - 512\tau(\tfrac{1}{2}n)\dagger.$$

2. The associated Dirichlet's series

$$F(s) = \Sigma\frac{\tau(n)}{n^s} \quad \dots\dots(2\cdot1)$$

is convergent for sufficiently large positive s. Ramanujan arrived by conjecture at the very remarkable identity

$$F(s) = \Pi\left\{\frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}\right\} \quad \dots\dots(2\cdot2),$$

where the product extends over all primes p; and a proof of this formula has since been given by Mordell‡. It follows that, if we write $n = p_1^{a_1}p_2^{a_2}\dots p_r^{a_r}$ and

$$\cos\theta_p = \tfrac{1}{2}p^{-\frac{11}{2}}\tau(p) \quad \dots\dots(2\cdot3),$$

then

$$\tau(n) = n^{\frac{11}{2}}\frac{\sin(a_1+1)\theta_{p_1}}{\sin\theta_{p_1}}\dots\frac{\sin(a_r+1)\theta_{p_r}}{\sin\theta_{p_r}} \quad \dots(2\cdot4),$$

and $\tau(n)\tau(n') = \tau(nn')$ when n and n' are coprime.

* S. Ramanujan, 'On certain arithmetical functions', *Trans. Camb. Phil. Soc.*, 22 (1916), 159–184.

† Ramanujan, *l.c.*, 179, 184. A function with argument $\frac{1}{2}n$ is zero when n is odd.

‡ L. J. Mordell, 'On Mr Ramanujan's empirical expansions of modular functions', *Proc. Camb. Phil. Soc.*, 19 (1920), 117–124 (a paper communicated in 1917).

https://doi.org/10.1017/S0305004100011178 Published online by Cambridge University Press

3. The problem of determining the order of magnitude of $\tau(n)$ appears to be very difficult. Ramanujan proved, on the one hand that

$$\tau(n) = O(n^7)^* \qquad \text{......(3·1)},$$

and on the other that

$$\tau(n) = \Omega(n^5)^\dagger \qquad \text{......(3·2)}.$$

He also proved‡ that, if we assume the truth of (2·2), an assumption since justified by Mordell, and also of the inequality

$$\tau(p) \leqq 2p^{\frac{11}{2}} \qquad \text{......(3·3)},$$

then

$$\tau(n) \leqq n^{\frac{11}{2}} d(n) \qquad \text{......(3·4)}$$

and

$$\tau(n) \geqq n^{\frac{11}{2}} \qquad \text{......(3·5)}$$

for an infinity of values of n, so that (3·1) and (3·2) may be replaced by

$$\tau(n) = O(n^{\frac{11}{2}+\epsilon}) \qquad \text{......(3·6)},$$

for every positive ϵ, and

$$\tau(n) = \Omega(n^{\frac{11}{2}}) \qquad \text{......(3·7)}.$$

If then (3·3) were proved, the problem would be in essentials solved.

4. In this note I make three contributions to the problem. The first (which is all but trivial) is to show that (3·7) does not depend upon the unproved inequality (3·3) but only on the identities (2·2) and (2·4) established by Mordell, and is therefore certainly true. The second is to show that

$$\tau(n) = O(n^6) \qquad \text{......(4·1)},$$

which therefore replaces Ramanujan's (3·1); and the third is to show that, if

$$T(n) = \{\tau(1)\}^2 + \{\tau(2)\}^2 + \dots + \{\tau(n)\}^2 \qquad \text{......(4·2)},$$

then there are positive constants A and B such that

$$An^{12} < T(n) < Bn^{12} \qquad \text{......(4·3)}$$

for $n \geqq 1$. This shows that the average order of $\tau(n)$ is exactly $O(n^{\frac{11}{2}})$, and also gives an alternative proof of both (3·7) and (4·1), and indeed of more.

* Ramanujan, *l.c.*, 168, 171.
† That is to say $\tau(n) \neq o(n^5)$. See Ramanujan, *l.c.*, 171.
‡ Ramanujan, *l.c.*, 175.

Proof of (3·7).

5. Ramanujan shows that (3·7) follows from (2·2) and (3·3). It is therefore only necessary to show that (3·7) is true if (3·3) is false for some p.

If (3·3) is false, $\cos\theta_p$ is real and θ_p complex, so that $\theta_p = k\pi + i\eta_p$, where k is an integer and η_p is real. We may suppose η_p positive, so that $e^{\eta_p} > 1$. Then, if $n = p^a$, we have

$$|\tau(n)| = n^{\frac{11}{2}} \left| \frac{\sin(a+1)\theta_p}{\sin\theta_p} \right| = n^{\frac{11}{2}} \left| \frac{\sinh(a+1)\eta_p}{\sinh\eta_p} \right|$$

$$> An^{\frac{11}{2}} e^{a\eta_p} = An^{\frac{11}{2}+\delta},$$

where A is a positive constant and $\delta = \eta_p/\log p > 0$, for all values of a; and this evidently completes the proof of (3·7).

Proof of (4·1).

6·1. The proof of (4·1) depends upon the following lemma, which is interesting in itself.

Lemma 1. *If $f(x)$ is the function* (1·1), *then*

$$f(x) = f(re^{i\theta}) = O\{(1-r)^{-6}\} \qquad \ldots\ldots(6·11)$$

uniformly in θ.

The proof of this lemma depends on the methods used by Littlewood and myself in one of our memoirs on Diophantine approximation*. Following the notations of that memoir, I write

$$x = q^2, \quad f(x) = \phi(q), \quad q = e^{\pi i\tau}, \quad \mathrm{T} = \frac{c+d\tau}{a+b\tau}, \quad Q = e^{\pi i\mathrm{T}},$$

where a, b, c, d are integers such that $ad - bc = 1$. Then the equation of transformation for $\phi(q)$ is

$$\phi(Q) = (a+b\tau)^{12}\phi(q).$$

Hence, if

$$\chi(q) = \mathfrak{R}\log\phi(q),$$

we have

$$\chi(q) = \chi(Q) - 12\log|a+b\tau|.$$

If we continue to follow both the ideas and the notation of the memoir referred to†, in which in particular

$$e^{-\pi y} = r, \quad \pi y = \log\frac{1}{r} \sim 1 - r, \quad |Q| = e^{-\pi\lambda},$$

* G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine approximation: II: The trigonometrical series associated with the elliptic theta-functions', *Acta Math.*, 37 (1914), 193–238.

† 226 *et seq.*

https://doi.org/10.1017/S0305004100011178 Published online by Cambridge University Press

we obtain

$$\chi(q) = \chi(Q) - 6 \log\left(\frac{1}{q'^2_{n+1}} + q_n^2 y^2\right) = \chi(Q) - 6\log y - 6\log\frac{1}{\lambda},$$

$$\chi(q) + 6\log y = -6\log(1/\lambda) + \mathfrak{R}\log\{Q^2(1-Q^2)^{24}(1-Q^4)^{24}\ldots\}$$

$$= -6\log\frac{1}{\lambda} - 2\pi\lambda + 24\,\Sigma\mathfrak{R}\log(1-Q^{2n}) < A - 6\log\frac{1}{\lambda} - 2\pi\lambda < A,$$

where the A's are constants*. But this is equivalent to

$$|f(x)| = |\phi(q)| < Ay^{-6} < A\left(\log\frac{1}{r}\right)^{-6} < A(1-r)^{-6},$$

which proves the lemma.

6·2. We can deduce (4·1) at once from (6·11). For

$$\tau(n) = \frac{1}{2\pi i}\int\frac{f(x)}{x^{n+1}}\,dx = O\{(1-r)^{-6}\} = O(n^6),$$

if the integration is effected round the circle of radius $r = 1 - 1/n$.

Proof of (4·3).

7·1. To prove (4·3) we require a further lemma.

Lemma 2. *If* $a_n \geqq 0$, $h > 0$, $k > 0$, $\alpha > 0$ *and*

$$h(1-r)^{-\alpha} < g(r) = \Sigma a_n r^n < k(1-r)^{-\alpha} \quad \ldots\ldots(7\cdot11)$$

for all values of r *less than and sufficiently near to* 1, *then there are positive constants* p *and* q *such that*

$$pn^\alpha < s_n = a_0 + a_1 + a_2 + \ldots + a_n < qn^\alpha \quad \ldots\ldots(7\cdot12)$$

for all sufficiently large values of n.

The second inequality (7·12) is immediate, since

$$s_n \leqq r^{-n}\sum_0^n a_\nu r^\nu \leqq r^{-n} g(r) \leqq 4g\left(1-\frac{1}{n}\right) < 4kn^\alpha$$

if $r = 1 - 1/n$ and n is sufficiently large. The first inequality is not quite so obvious.

We write $r = e^{-y}$, so that $1 - r \sim y$. Then plainly

$$G(y) = \Sigma s_n e^{-ny} > \tfrac{1}{2}hy^{-\alpha-1} \quad \ldots\ldots(7\cdot13)$$

if y is sufficiently small. We choose c so that

$$c > 2\alpha, \quad 8qc^\alpha e^{-c} < h \quad \ldots\ldots(7\cdot14),$$

* The function $-6\log(1/\lambda) - 2\pi\lambda$ becomes negatively infinite when $\lambda \to 0$ or $\lambda \to \infty$, and has a maximum when $\pi\lambda = 3$. Here $\lambda > \frac{1}{2}$.

and take $y = c/\nu$, where ν is a large integer; and we write

$$G(y) = \sum_{1}^{\nu} s_n e^{-ny} + \sum_{\nu+1}^{\infty} s_n e^{-ny} = G_1(y) + G_2(y) \quad \ldots(7\cdot15).$$

Then

$$G_2(y) < q \sum_{\nu+1}^{\infty} n^a e^{-ny} < q \int_{\nu}^{\infty} u^a e^{-uy}\, du = q y^{-a-1} \int_{c}^{\infty} w^a e^{-w}\, dw,$$

if ν is sufficiently large*. But

$$\chi(c) = \int_c^\infty w^a e^{-w}\, dw = c^a e^{-c} + a \int_c^\infty w^{a-1} e^{-w}\, dw < c^a e^{-c} + \tfrac{1}{2}\chi(c),$$

since $c > 2a$; and so

$$G_2(y) < 2qc^a e^{-c} y^{-a-1} < \tfrac{1}{4} h y^{-a-1} \quad \ldots\ldots(7\cdot16),$$

by (7·14). From (7·13)—(7·16) we deduce

$$G_1(y) = G(y) - G_2(y) > \tfrac{1}{4} h y^{-a-1} \quad \ldots\ldots(7\cdot17).$$

But

$$G_1(y) = \sum_{n \leqq \nu} s_n e^{-ny} \leqq \frac{s_\nu}{1 - e^{-y}} \leqq \frac{2s_\nu}{y} \quad \ldots\ldots(7\cdot18),$$

if y is small enough. Comparing (7·17) and (7·18) we see that

$$s_\nu > \tfrac{1}{8} h y^{-a} = \tfrac{1}{8} h c^{-a} \nu^a$$

for sufficiently large ν, which proves the first inequality (7·11).

7·2. The second inequality (4·3) is an immediate deduction from Lemmas 1 and 2. For

$$g(r) = \Sigma \{\tau(n)\}^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta$$
$$= O\{(1-r)^{-12}\} = O(n^{12})$$

when $r = 1 - 1/n$, by Lemma 1; and $\mathrm{T}(n) < Bn^{12}$ by Lemma 2. This inequality plainly includes (4·1), since $\mathrm{T}(n) \geqq \{\tau(n)\}^2$.

To prove the first inequality (4·3) we use Jacobi's identity†

$$\psi(x) = \{(1-x)(1-x^2)(1-x^3)\ldots\}^3 = 1 - 3x + 5x^3 - 7x^6 + \ldots,$$

where the indices are the triangular numbers.

Since $f = x\psi^8$, we have

$$g(r) = \frac{r^2}{24} \int_0^{2\pi} |\psi(re^{i\theta})|^{16}\, d\theta > A \left(\int_0^{2\pi} |\psi(re^{i\theta})|^2\, d\theta \right)^8$$
$$> A(1 + 3^2 r^2 + 5^2 r^6 + 7^2 r^{12} + \ldots)^8$$

* The maximum of $u^a e^{-uy}$ occurs for $u = a/y$, outside the range of integration because $c > a$.

† *Fundamenta nova*, § 66.

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when $r \to 1$. But if $r = e^{-\delta}$, we have

$$\Sigma (2n+1)^2 r^{\frac{1}{2}n(n+1)} \sim 4 \Sigma n^2 r^{n^2} \sim 4 \int_0^\infty t^2 e^{-\delta t^2} dt$$

$$\sim A\delta^{-\frac{3}{2}} \sim A(1-r)^{-\frac{3}{2}},$$

and so

$$g(r) > A(1-r)^{-12};$$

and therefore, by Lemma 2, $T(n) > An^{12}$.

Since

$$T(n) \leqq n \operatorname*{Max}_{\nu \leqq n} \{\tau(\nu)\}^2,$$

this gives an alternative proof of (3·7).

[Added 9 Feb. 1927.] The note was written nearly 10 years ago, my interest in the matter reviving recently as a result of editorial work in connection with the forthcoming edition of Ramanujan's works. The index 6 of (4·1) is the $\frac{1}{4}r$ which occurs in the recent work of Landau, Petersson, and Walfisz concerning the number of representations of n by r squares, and the associated lattice-point problem; and (4·1) itself must be included implicitly in their general results. See H. Petersson, 'Über die Anzahl der Gitterpunkte in mehrdimensionalen Ellipsoiden', *Hamburg Math. Abhandlungen*, 5 (1926), 116–150; and the memoirs of Landau and Walfisz there referred to.
