J. Plasma Phys. (2021), *vol.* 87, 835870401
Published by Cambridge University Press
doi:10.1017/S0022377821000696

An energy and momentum conserving collisional bracket for the guiding-centre Vlasov–Maxwell–Landau model

Riccardo N. Iorio [1],† **and Eero Hirvijoki** [1]

[1]Department of Applied Physics, Aalto University, P.O. Box 11100, 00076 Aalto, Finland

(Received 18 March 2021; revised 16 June 2021; accepted 17 June 2021)

This paper proposes a metric bracket for representing Coulomb collisions in the so-called guiding-centre Vlasov–Maxwell–Landau model. The bracket is manufactured to preserve the same energy and momentum functionals as does the Vlasov–Maxwell part and to simultaneously satisfy a revised version of the H-theorem, where the equilibrium distributions with respect to collisional dynamics are identified as Maxwellians. This is achieved by exploiting the special projective nature of the Landau collision operator and the simple form of the system's momentum functional. A discussion regarding a possible extension of the results to electromagnetic drift-kinetic and gyrokinetic systems is included. We anticipate that energy conservation and entropy dissipation can always be manufactured whereas guaranteeing momentum conservation is a delicate matter yet to be resolved.

Key words: plasma dynamics, fusion plasma

1. Introduction

Since the 1980s much effort has been devoted to the study of magnetized plasmas to better comprehend the mechanisms that result in the confinement of particles and transport phenomena inside fusion devices. The complexity of the particles' trajectories in relation to the wide gap of time scales spanning from the electron cyclotron motion to the macroscopic phenomena has urged the development of perturbative time scale reduction techniques (see e.g. Grebogi, Kaufman & Littlejohn 1979; Cary & Kaufman 1981; Hatori & Washimi 1981; Kaufman & Holm 1984) to allow us to step over the computational limitations set by the gyromotion of particles (Littlejohn 1981, 1982, 1983, 1984).

When dealing with the description of a plasma through the Vlasov–Maxwell–Landau system, both gyrokinetic and guiding-centre theory are often applied to investigate solely the Vlasov–Maxwell part and the collision operator is neglected or heavily approximated, although the trend is starting to shift (see e.g. Bobylev & Nanbu 2000; Yoon & Chang 2014). While the complex character of the dissipationless dynamics may be addressed with systematic reduction techniques via the Lie-transform perturbation theory (see e.g. Hahm, Lee & Brizard 1988), following this approach for the dissipative part of the problem, the collisions, leads to difficult truncation problems in trying to ensure that the effects of

† Email address for correspondence: riccardo.iorio@aalto.fi

collisions do not violate, e.g. the laws of thermodynamics. Although gyrokinetic theory is quite well-established in the collisionless case (Brizard 2000; Sugama 2000; Burby *et al.* 2015*a*; Burby & Brizard 2019), dealing with collisions has proven to be puzzling when taking into account the non-uniformities of the background magnetic field.

After the initial work required to describe collisional test-particle guiding-centre dynamics (Brizard 2004; Hirvijoki *et al.* 2013), the first step in solving this matter was undertaken using electrostatic gyrokinetic theory (Burby, Brizard & Qin 2015*b*), where a collision operator compatible with the first and second law of thermodynamics as well as with conservation of the toroidal canonical momentum was found. Later, it was shown that the modern formulation of collisional electrostatic gyrokinetics exhibits a metriplectic structure (Hirvijoki & Burby 2020), which represents an extension of the Poisson bracket formalism of classical mechanics to dissipative systems that obey the laws of thermodynamics (see e.g. Kaufman & Morrison 1982; Grmela 1984*a*,*b*, 1985; Kaufman 1984; Morrison 1984*a*,*b*, 1986). Indeed, the metriplectic formulation has found applications in theoretical studies involving fluids or plasmas. Examples include magnetohydrodynamics (see e.g. Morrison & Greene 1980; Morrison 2009) and extended magnetohydrodynamics (see e.g. Materassi & Tassi 2012; Lingam 2015; Coquinot & Morrison 2020). In view of this, the possible existence of an asymptotic metriplectic reduction scheme that could generalize the already established reduction techniques of collisionless formulations as well as deal with the truncation problems that arise in the current construction of the collisional gyrokinetic theory was speculated.

The present work provides yet another indication of the possible existence of a metriplectic reduction theory and sheds further light on the issues of developing a collision operator for electromagnetic reduced plasma theories. We construct an energy and momentum conserving collisional bracket for the so-called guiding-centre Vlasov–Maxwell model (Brizard & Tronci 2016) and expand the discussion initiated by Hirvijoki & Burby (2020) on why extensions to drift-kinetic and gyrokinetic electromagnetic theories are so difficult. We start by briefly reviewing the concept of a collisional bracket in the context of particle phase space dynamics, i.e. with the Vlasov–Maxwell–Landau system. After this, a collisional bracket for the guiding-centre Vlasov–Maxwell system of Brizard & Tronci (2016) is given and conservation of energy and momentum is demonstrated together with an H-theorem. Finally, the successful treatment of electrostatic gyrokinetics (Hirvijoki & Burby 2020) is recalled to contextualize the difficulties encountered in trying to construct a momentum conserving bracket representative of collisions for electromagnetic drift-kinetic theory. Finding such a bracket would likely lead to better numerical algorithms (Hirvijoki 2021).

2. Collision operator as a metric bracket

The Vlasov–Maxwell–Landau model describes the evolution of distribution functions of species s and electromagnetic fields. In short, the model consists of the following equations:

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \frac{e_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \sum_{\bar{s}} C_{s\bar{s}}[f_s, f_{\bar{s}}], \tag{2.1}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j} = \nabla\times\boldsymbol{B}, \tag{2.2}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \tag{2.3}$$

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

$$\nabla \cdot \boldsymbol{E} = 4\pi\varrho, \tag{2.4}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.5}$$

where $\varrho(\boldsymbol{x}) = \sum_s e_s \int f_s \,\mathrm{d}\,\boldsymbol{v}$ is the charge density and $\boldsymbol{j}(\boldsymbol{x}) = \sum_s e_s \int \boldsymbol{v} f_s \,\mathrm{d}\,\boldsymbol{v}$ is the current density.

The Landau operator, which describes the effects arising from small-angle Coulomb collisions between the species s and $\bar{s}$, can be expressed as

$$C_{s\bar{s}}(f_s, f_{\bar{s}}) = -\sum_{\bar{s}} \frac{\nu_{s\bar{s}}}{m_s} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \delta(\boldsymbol{x} - \bar{\boldsymbol{x}}) f_s(z) f_{\bar{s}}(\bar{z}) \mathbb{Q}(\boldsymbol{v} - \bar{\boldsymbol{v}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S}, z, \bar{z}) \,\mathrm{d}\,\bar{z}, \tag{2.6}$$

where $\nu_{s\bar{s}} = 2\pi e_s e_{\bar{s}} \ln \Lambda$, the functional S is the entropy

$$\mathcal{S} = -\sum_s \int f_s \ln f_s \,\mathrm{d}\,z, \tag{2.7}$$

and the vector $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A}, z, \bar{z})$ is

$$\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A}, z, \bar{z}) = \frac{1}{m_s} \frac{\partial}{\partial \boldsymbol{v}} \frac{\delta \mathcal{A}}{\delta f_s}(z) - \frac{1}{m_{\bar{s}}} \frac{\partial}{\partial \bar{\boldsymbol{v}}} \frac{\delta \mathcal{A}}{\delta f_{\bar{s}}}(\bar{z}). \tag{2.8}$$

The coordinates $z = (\boldsymbol{x}, \boldsymbol{v})$ and $\bar{z} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{v}})$ refer to different phase-space locations and the functional derivative is identified via the Fréchet derivative:

$$\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \mathcal{A}[f_s + \epsilon \delta f_s] = \int \frac{\delta \mathcal{A}}{\delta f_s} \delta f_s \,\mathrm{d}\,z \equiv \delta \mathcal{A}[\delta f_s]. \tag{2.9}$$

The matrix $\mathbb{Q}(\boldsymbol{\xi})$ is the familiar scaled projection matrix:

$$\mathbb{Q}(\boldsymbol{\xi}) = \frac{1}{|\boldsymbol{\xi}|} \left(\mathbb{I} - \frac{\boldsymbol{\xi}\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right), \tag{2.10}$$

with $\mathbb{I}$ as the identity matrix, and the standard form of the collision operator is recovered after computing the functional derivative of the entropy,

$$\frac{\delta \mathcal{S}}{\delta f_s} = -(1 + \ln f_s), \tag{2.11}$$

and then evaluating the expression:

$$f_s(z) f_{\bar{s}}(\bar{z}) \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S}, z, \bar{z}) = \frac{f_s(z)}{m_{\bar{s}}} \frac{\partial f_{\bar{s}}}{\partial \bar{\boldsymbol{v}}} - \frac{f_{\bar{s}}(\bar{z})}{m_s} \frac{\partial f_s}{\partial \boldsymbol{v}}. \tag{2.12}$$

If we multiply the collision operator with an arbitrary function g_s and integrate over the velocity space, partial integration provides

$$\int g_s(z) C_{s\bar{s}}(f_s, f_{\bar{s}}) \,\mathrm{d}\,z = \sum_{\bar{s}} \frac{\nu_{s\bar{s}}}{m_s} \int \frac{\partial g_s(z)}{\partial \boldsymbol{v}} \cdot \int \delta(\boldsymbol{x} - \bar{\boldsymbol{x}}) f_s(z) f_{\bar{s}}(\bar{z}) \mathbb{Q}(\boldsymbol{v} - \bar{\boldsymbol{v}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S}, z, \bar{z}) \,\mathrm{d}\,\bar{z} \,\mathrm{d}\,z, \tag{2.13}$$

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

while after exchanging the species indices,

$$\int g_{\bar{s}}(z) C_{s\bar{s}}(f_s, f_{\bar{s}})\,\mathrm{d}\,z$$
$$= -\sum_s \frac{\nu_{s\bar{s}}}{m_{\bar{s}}} \int \frac{\partial g_{\bar{s}}(\bar{z})}{\partial \bar{\boldsymbol{v}}} \cdot \int \delta(\boldsymbol{x}-\bar{\boldsymbol{x}}) f_s(z) f_{\bar{s}}(\bar{z}) \mathbb{Q}(\boldsymbol{v}-\bar{\boldsymbol{v}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S}, z, \bar{z})\,\mathrm{d}\,\bar{z}\,\mathrm{d}\,z. \tag{2.14}$$

By summing over the different species we obtain the following expression:

$$\sum_s \int g_s(z) C_{s\bar{s}}(f_s, f_{\bar{s}})\,\mathrm{d}\,z$$
$$= \sum_{s\bar{s}} \frac{1}{2} \iint \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{G}, z, \bar{z}) \cdot \mathbb{W}_{s\bar{s}}(z, \bar{z}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S}, z, \bar{z})\,\mathrm{d}\,\bar{z}\,\mathrm{d}\,z, \tag{2.15}$$

where we used the functional $G = \int g_s(z) f_s(z)\,\mathrm{d}z$. This rather peculiar form enables a straightforward identification of a functional bracket,

$$(\mathcal{A}, \mathcal{B}) = \sum_{s,\bar{s}} \frac{1}{2} \iint \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A}, z, \bar{z}) \cdot \mathbb{W}_{s\bar{s}}(z, \bar{z}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{B}, z, \bar{z})\,\mathrm{d}\,\bar{z}\,\mathrm{d}\,z, \tag{2.16}$$

where the positive semidefinite matrix $\mathbb{W}_{s\bar{s}}(z, \bar{z})$ is

$$\mathbb{W}_{s\bar{s}}(z, \bar{z}) = \nu_{s\bar{s}} \delta(\boldsymbol{x}-\bar{\boldsymbol{x}}) f_s(z) f_{\bar{s}}(\bar{z}) \mathbb{Q}(\boldsymbol{v}-\bar{\boldsymbol{v}}). \tag{2.17}$$

In terms of the bracket (2.16), collisional evolution of arbitrary functionals can then be generalized to the functional differential equation

$$\left.\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right|_{\mathrm{coll}} = (\mathcal{A}, \mathcal{S}). \tag{2.18}$$

A detailed account with intermediate steps is provided, e.g. by Morrison (1986) and Kraus & Hirvijoki (2017), and it is straightforward to verify that the bracket has the kinetic energy, momentum and mass functionals

$$\mathcal{K} = \sum_s \int \frac{m_s}{2} |\boldsymbol{v}|^2 f_s\,\mathrm{d}\,z, \tag{2.19}$$

$$\boldsymbol{\mathcal{P}} = \sum_s \int m_s \boldsymbol{v} f_s\,\mathrm{d}\,z, \tag{2.20}$$

$$\mathcal{M} = \int m_s f_s\,\mathrm{d}\,z, \tag{2.21}$$

as invariants, and that the equilibrium state is a Maxwellian. This last step is accomplished by noting that the functional derivatives of the total kinetic energy $\mathcal{K}$, the momentum $\boldsymbol{\mathcal{P}}$

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

and the mass $\mathcal{M}$ satisfy the identities:

$$\frac{\delta \mathcal{K}}{\delta f_s} = \frac{m_s}{2}|\boldsymbol{v}|^2, \quad \frac{\delta \boldsymbol{\mathcal{P}}}{\delta f_s} = m_s \boldsymbol{v}, \quad \frac{\delta \mathcal{M}}{\delta f_s} = m_s, \tag{2.22a–c}$$

which in turns yield the conditions:

$$\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{M}, z, \bar{z}) = 0, \quad \delta(\boldsymbol{x} - \bar{\boldsymbol{x}})\boldsymbol{\Gamma}_{s\bar{s}}(\boldsymbol{\mathcal{P}}, z, \bar{z}) = 0, \quad \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{K}, z, \bar{z}) \cdot \mathbb{W}_{s\bar{s}}(z, \bar{z}) = 0. \tag{2.23a–c}$$

The equilibrium state satisfies an energy principle, which arises from the existence of invariants of the bracket being degenerate. Following the energy–Casimir principle (Holm *et al.* 1985; Morrison 1998; Kraus & Hirvijoki 2017), the equilibrium condition is obtained from

$$\left.\left(\frac{\delta \mathcal{S}}{\delta f_s} + \lambda_s \frac{\delta \mathcal{M}}{\delta f_s} + \lambda_{\boldsymbol{\mathcal{P}}} \cdot \frac{\delta \boldsymbol{\mathcal{P}}}{\delta f_s} + \lambda_{\mathcal{K}} \frac{\delta \mathcal{K}}{\delta f_s}\right)\right|_{f=f_{eq}} = 0, \quad \text{for all } s, \tag{2.24}$$

which leads to the condition for equilibrium distribution functions:

$$-(1 + \ln f_s) + \lambda_s m_s + \lambda_{\boldsymbol{\mathcal{P}}} \cdot m_s \boldsymbol{v} + \lambda_{\mathcal{K}} \frac{m_s}{2}|\boldsymbol{v}|^2 = 0, \quad \text{for all } s. \tag{2.25}$$

From the latter it is straightforward to recognize that the equilibrium distributions are identified as Maxwellians:

$$f_s(z) = C\,\mathrm{e}^{\lambda_s m_s + \lambda_{\boldsymbol{\mathcal{P}}} \cdot m_s \boldsymbol{v} + \lambda_{\mathcal{K}}(m_s/2)|\boldsymbol{v}|^2}, \tag{2.26}$$

having common temperature and flow velocity but possibly different densities for each species. The values of the coefficients $\lambda_{s,\boldsymbol{\mathcal{P}},\mathcal{K}}$ and the factor C are computed from the initial state.

The convenience of the bracket formulation is that it brings the collisional evolution on equal footing with the infinite-dimensional Hamiltonian formulation of the dissipationless Vlasov–Maxwell part (see Morrison 1980; Weinstein & Morrison 1981; Marsden & Weinstein 1982). The kinetic system as a whole can then be formulated in terms of the so-called metriplectic dynamics of arbitrary functionals (Kaufman & Morrison 1982; Grmela 1984*a*,*b*, 1985; Kaufman 1984; Morrison 1984*a*,*b*, 1986). In dealing with perturbative reduction techniques for the dissipation-free Hamiltonian part, we are often concerned with preserving the underlying mathematical structure in the resulting dynamically reduced theories. We should aim at the same rigor also when accounting for the collisional evolution.

3. Collisional bracket for the guiding-centre Vlasov–Maxell model

The guiding-centre Vlasov–Maxwell system described by Brizard & Tronci (2016) consists of the following equations:

$$\frac{\partial F_s}{\partial t} + \dot{\boldsymbol{X}}_s \cdot \nabla f_s + \dot{v}_{\parallel,s} \frac{\partial F_s}{\partial v_\parallel} = 0, \tag{3.1}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}(\boldsymbol{j}_{gc} + \nabla \times \boldsymbol{M}_{gc}) = \nabla \times \boldsymbol{B}, \tag{3.2}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{3.3}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \varrho_{gc}, \tag{3.4}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3.5}$$

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

The phase-space advection of individual guiding-centres is given by the equations of motion:

$$\dot{\boldsymbol{X}} = v_\parallel \frac{\boldsymbol{B}^*}{B_\parallel^*} + \boldsymbol{E}^* \times \frac{c\boldsymbol{b}}{B_\parallel^*}, \tag{3.6}$$

$$\dot{v}_\parallel = \frac{e\boldsymbol{E}^*}{m} \cdot \frac{\boldsymbol{B}^*}{B_\parallel^*}, \tag{3.7}$$

where the so-called effective magnetic and electric fields are $\boldsymbol{B}^* = \boldsymbol{B} + (mc/e)v_\parallel \nabla \times \boldsymbol{b}$ and $\boldsymbol{E}^* = \boldsymbol{E} - (\mu/e)\nabla B - (mv_\parallel/e)\partial_t \boldsymbol{b}$ with $B_\parallel^* = \boldsymbol{b} \cdot \boldsymbol{B}^*$, and the guiding-centre current and charge density are given by

$$\boldsymbol{j}_{\mathrm{gc}}(\boldsymbol{x}) = \sum_s e_s \int \delta(\boldsymbol{x} - \boldsymbol{X})\dot{\boldsymbol{X}}_s F_s \,\mathrm{d}\boldsymbol{Z}_s^{\mathrm{gc}}, \tag{3.8}$$

$$\varrho_{\mathrm{gc}}(\boldsymbol{x}) = \sum_s e_s \int \delta(\boldsymbol{x} - \boldsymbol{X}) F_s \,\mathrm{d}\boldsymbol{Z}_s^{\mathrm{gc}}. \tag{3.9}$$

The magnetization in the system is given by

$$\boldsymbol{M}_{\mathrm{gc}}(\boldsymbol{x}) = \sum_s \int \delta(\boldsymbol{x} - \boldsymbol{X}) \left(\frac{m_s v_\parallel}{B} \mathbf{1}_\perp \cdot \dot{\boldsymbol{X}}_s - \mu \boldsymbol{b} \right) F_s \,\mathrm{d}\boldsymbol{Z}_s^{\mathrm{gc}}, \tag{3.10}$$

where $\mathbf{1}_\perp = \mathbf{1} - \boldsymbol{b}\boldsymbol{b}$, and $d\boldsymbol{Z}_s^{\mathrm{gc}}$ is the phase-space volume-element including the phase-space Jacobian that is proportional to $B_\parallel^*$. For more details and derivation of the equations, see the paper by Brizard & Tronci (2016).

As demonstrated by Brizard & Tronci (2016), the guiding-centre Vlasov–Maxwell system has a variational structure and conserved quantities that can be identified via analysis of the system's Noether symmetries. The global invariants are the total energy and momentum functionals:

$$\mathcal{H}^{\mathrm{gc}}[F, \boldsymbol{E}, \boldsymbol{B}] = \sum_s \int K_s F_s \,\mathrm{d}\boldsymbol{Z}_s^{\mathrm{gc}} + \frac{1}{8\pi} \int (|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) \,\mathrm{d}\boldsymbol{x}, \tag{3.11}$$

$$\boldsymbol{\mathcal{P}}^{\mathrm{gc}}[F, \boldsymbol{E}, \boldsymbol{B}] = \sum_s \int m_s v_\parallel \boldsymbol{b} F_s \,\mathrm{d}\boldsymbol{Z}_s^{\mathrm{gc}} + \frac{1}{4\pi c} \int \boldsymbol{E} \times \boldsymbol{B} \,\mathrm{d}\boldsymbol{x}, \tag{3.12}$$

with K being the individual guiding-centre kinetic energy,

$$K = \tfrac{1}{2} m v_\parallel^2 + \mu B. \tag{3.13}$$

If we are to add a collision operator to the right-hand side of (3.1), the same way that there exists one in (2.1), we should make sure that the corresponding metric structure preserves the functionals (3.11) and (3.12). However, such a procedure is not trivial when dealing with dissipative phenomena. The perturbation theory compatible with preserving Hamiltonian structures primarily operates at the level of the Lagrangian and not the Poisson structure. While this arrangement guarantees that truncations introduced to the perturbed Lagrangian facilitate a Poisson structure that satisfies the Jacobi identity, it does not directly instruct us on how to transform general brackets and functional derivatives. If the perturbation theory would directly operate on the Poisson structure, it might indicate how one could deal with the metric structure. Unfortunately, the truncation problem in applying the Lie-transformation perturbation theory to the Poisson structure still persists (Brizard *et al.* 2016). The alternative way is to use the particle phase-space collisional bracket (2.16) as a guide and to appropriately modify parts of it while simultaneously juggling with the conserved quantities. It is this latter route that we have adopted here.

Essentially, we look for a bracket that has a structure similar to the particle bracket, namely

$$(\mathcal{A},\mathcal{B})^{gc}=\sum_{s,\bar{s}}\frac{1}{2}\iint \boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{A},\boldsymbol{Z},\bar{\boldsymbol{Z}})\boldsymbol{\cdot}\mathbb{W}^{gc}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})\boldsymbol{\cdot}\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{B},\boldsymbol{Z},\bar{\boldsymbol{Z}})\,\mathrm{d}\bar{\boldsymbol{Z}}^{gc}_{\bar{s}}\,\mathrm{d}\boldsymbol{Z}^{gc}_{s},\tag{3.14}$$

where $\mathbb{W}^{gc}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ is required to be positive semidefinite to guarantee entropy dissipation. If we can have meaningful expressions for $\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{A},\boldsymbol{Z},\bar{\boldsymbol{Z}})$ and $\mathbb{W}^{gc}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ which resemble the analogous particle phase-space expressions, and guarantee that (3.11) and (3.12) remain invariants of the bracket, then (3.14) will be representative of small-angle Coulomb collisions and the form of the appropriate collision operator can be derived from the bracket directly.

Luckily, we have identified a manner in which the bracket (3.14) can be constructed to be compatible with the conserved total energy (3.11) and momentum (3.12) of the guiding-centre Vlasov–Maxwell system. To see how this works out, we define first

$$\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{A},\boldsymbol{Z},\bar{\boldsymbol{Z}})=\left(\frac{\boldsymbol{b}}{m}\frac{\partial}{\partial v_\parallel}+\frac{\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0}{B}\frac{\partial}{\partial\mu}\right)\frac{\delta\mathcal{A}}{\delta F}\bigg|_{s,\boldsymbol{Z}}-\left(\frac{\boldsymbol{b}}{m}\frac{\partial}{\partial v_\parallel}+\frac{\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0}{B}\frac{\partial}{\partial\mu}\right)\frac{\delta\mathcal{A}}{\delta F}\bigg|_{\bar{s},\bar{\boldsymbol{Z}}},\tag{3.15}$$

where $\varOmega$ is the cyclotron frequency and $\boldsymbol{\rho}_0$ is the lowest-order expression for the gyroradius. In the above expression, the first part is to be evaluated at the position $\boldsymbol{Z}$ with respect to the species s parameters and the second part in a similar manner but at $\bar{\boldsymbol{Z}}$ and with respect to species $\bar{s}$. Now, a direct computation provides

$$\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\boldsymbol{\mathcal{P}}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}})=\boldsymbol{b}(\boldsymbol{X})\boldsymbol{b}(\boldsymbol{X})-\boldsymbol{b}(\bar{\boldsymbol{X}})\boldsymbol{b}(\bar{\boldsymbol{X}}).\tag{3.16}$$

This expression closely resembles that encountered in the particle phase-space case. Consequently, if $\mathbb{W}^{gc}_{s\bar{s}}$ were to be proportional to $\delta(\boldsymbol{X}-\bar{\boldsymbol{X}})$, the momentum functional $\boldsymbol{\mathcal{P}}^{gc}$ would be an invariant of the proposed metric bracket in the sense of $(\boldsymbol{\mathcal{P}}^{gc},\mathcal{A})=0$ with respect to arbitrary $\mathcal{A}$.

Next we perform a similar direct computation with respect to the total energy functional. This provides

$$\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}})=(v_\parallel\boldsymbol{b}+\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0)|_{s,\boldsymbol{Z}}-(v_\parallel\boldsymbol{b}+\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0)|_{\bar{s},\bar{\boldsymbol{Z}}},\tag{3.17}$$

and is analogous to the expression one finds in the particle phase-space case: $v_\parallel\boldsymbol{b}+\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0$ is the familiar lowest-order expression for the particle velocity in the guiding-centre coordinates including parallel streaming and Larmor rotation around the magnetic field line but neglecting drifts across the field lines. Consequently, if we choose $\mathbb{W}^{gc}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ so that the vector $\boldsymbol{\varGamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}})$ will belong to its null space, then the bracket will have the total energy functional an as invariant in the sense of $(\mathcal{H}^{gc},\mathcal{A})^{gc}=0$ with respect to arbitrary $\mathcal{A}$. A natural choice is to look for a solution that closely resembles the particle

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

phase-space case, and this happens to be

$$\mathbb{W}^{\mathrm{gc}}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}}) = \nu_{s\bar{s}}\delta(\boldsymbol{X}-\bar{\boldsymbol{X}})F_s(\boldsymbol{Z})F_{\bar{s}}(\bar{\boldsymbol{Z}})\mathbb{Q}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}})). \tag{3.18}$$

The choices of (3.15) and (3.18) together with the general expression for the bracket (3.14) now guarantee that if the collisional evolution of functionals were to be given by

$$\left.\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\right|_{\mathrm{coll}} = (\mathcal{A},\mathcal{S})^{\mathrm{gc}}, \tag{3.19}$$

and driven by the entropy functional

$$\mathcal{S} = -\sum_s \int F_s \ln F_s \,\mathrm{d}\boldsymbol{Z}^{\mathrm{gc}}_s, \tag{3.20}$$

both the energy and momentum conservation would be satisfied and the entropy dissipation would be guaranteed.

The energy–Casimir principle in this case provides the equilibrium condition:

$$\left.\left(\frac{\delta\mathcal{S}}{\delta F_s} + \boldsymbol{\lambda}_{\boldsymbol{\mathcal{P}}^{\mathrm{gc}}}\boldsymbol{\cdot}\frac{\delta}{\delta F_s}\boldsymbol{\mathcal{P}}^{\mathrm{gc}}[F,\boldsymbol{E},\boldsymbol{B}] + \lambda_{\mathcal{H}^{\mathrm{gc}}}\frac{\delta}{\delta F_s}\mathcal{H}^{\mathrm{gc}}[F,\boldsymbol{E},\boldsymbol{B}]\right)\right|_{F=F_{\mathrm{eq}}} = 0, \quad \text{for all } s. \tag{3.21}$$

which by means of the identities

$$\frac{\delta\mathcal{H}^{\mathrm{gc}}[F,\boldsymbol{E},\boldsymbol{B}]}{\delta F_s} = K_s, \qquad \frac{\delta\boldsymbol{\mathcal{P}}^{\mathrm{gc}}[F,\boldsymbol{E},\boldsymbol{B}]}{\delta F_s} = m_s v_\parallel \boldsymbol{b}, \tag{3.22a,b}$$

leads to the condition for equilibrium distribution functions

$$-(1+\ln F_s) + \boldsymbol{\lambda}_{\boldsymbol{\mathcal{P}}^{\mathrm{gc}}}\boldsymbol{\cdot} m_s v_\parallel \boldsymbol{b} + \lambda_{\mathcal{H}^{\mathrm{gc}}}K_s = 0, \quad \text{for all } s. \tag{3.23}$$

The latter shows that the equilibrium distributions with respect to collisional dynamics are identified as an exponentiation of the guiding-centre kinetic energy function divided by a common temperature, with a possible common drift in the parallel direction of the equilibrium magnetic field.

Finally, we point out that the choice of the vector $\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{A},\boldsymbol{Z},\bar{\boldsymbol{Z}})$ in (3.15) is not arbitrary. It closely resembles that used by Hirvijoki & Burby (2020): from the guiding-centre Poisson bracket

$$\begin{aligned}\{F,G\}^{\mathrm{gc}}_{s,\boldsymbol{Z}} = &\frac{e}{mc}\left(\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\theta}\right)\\ &+\frac{\boldsymbol{B}^*}{mB^*_\parallel}\boldsymbol{\cdot}\left(\nabla^*F\frac{\partial G}{\partial v_\parallel} - \nabla^*G\frac{\partial F}{\partial v_\parallel}\right)\\ &-\frac{c\boldsymbol{b}}{mB^*_\parallel}\boldsymbol{\cdot}\nabla^*F\times\nabla^*G,\end{aligned} \tag{3.24}$$

using $\partial\boldsymbol{\rho}_0/\partial\theta = \boldsymbol{b}\times\boldsymbol{\rho}_0$, $\varOmega = eB/mc$ and further assuming a locally homogeneous background plasma, provides

$$\left\{\boldsymbol{x}+\boldsymbol{\rho}_0, \frac{\delta\mathcal{A}}{\delta F}\right\}^{\mathrm{gc}}_{s,\boldsymbol{Z}} \approx \left.\left(\frac{\boldsymbol{b}}{m}\frac{\partial}{\partial v_\parallel} + \frac{\varOmega\boldsymbol{b}\times\boldsymbol{\rho}_0}{B}\frac{\partial}{\partial\mu}\right)\frac{\delta\mathcal{A}}{\delta F}\right|_{s,\boldsymbol{Z}}. \tag{3.25}$$

Despite this resemblance, if one were to construct the metric bracket with the exact guiding-centre Poisson bracket as done by Hirvijoki & Burby (2020), the momentum

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

functional $\mathcal{P}^{gc}[F,\boldsymbol{E},\boldsymbol{B}]$ in (3.12) would not remain an invariant of the bracket. This highlights the non-triviality of finding a collisional bracket with the correct functional form and invariants.

4. Explicit expressions for the gyroaverages

While the structure of the collisional bracket (3.14) might appear somewhat intimidating, parts of it can be handled analytically. The gyroangle dependency in the bracket materializes in the expressions:

$$\boldsymbol{b}|_{X,\bar{X}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot\boldsymbol{b}|_{X,\bar{X}}, \tag{4.1}$$

$$\boldsymbol{b}|_{X,\bar{X}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot(\boldsymbol{b}\times\boldsymbol{\rho}_0/|\boldsymbol{\rho}_0|)|_{Z,\bar{Z}}, \tag{4.2}$$

$$(\boldsymbol{b}\times\boldsymbol{\rho}_0/|\boldsymbol{\rho}_0|)|_{Z,\bar{Z}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot(\boldsymbol{b}\times\boldsymbol{\rho}_0/|\boldsymbol{\rho}_0|)|_{Z,\bar{Z}}, \tag{4.3}$$

where the syntax $|_X$, $|_{\bar{X}}$, $|_Z$ and $|_{\bar{Z}}$ refer to the different possible combinations of positions with which the Landau operator can be evaluated. The presence of the localizing $\delta(\boldsymbol{X}-\bar{\boldsymbol{X}})$ in the bracket, however, simplifies the above equations by forcing the guiding-centre positions of the particles $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ to the same configuration space position. In that specific case, we can evaluate

$$\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}})|_{X=\bar{X}}=(v_\parallel-\bar{v}_\parallel)\boldsymbol{b}+v_{\perp,s}(\boldsymbol{X},\mu)\boldsymbol{k}(\boldsymbol{X},\theta)-\bar{v}_{\perp,\bar{s}}(\boldsymbol{X},\bar{\mu})\boldsymbol{k}(\boldsymbol{X},\bar{\theta}), \tag{4.4}$$

with $v_{\perp,s}(\boldsymbol{X},\mu)=\sqrt{2\mu B(\boldsymbol{X})/m_s}$ and the unit vector $\boldsymbol{k}=\boldsymbol{b}\times\boldsymbol{\rho}_0/|\boldsymbol{\rho}_0|$, which is perpendicular to the magnetic field.

This means that we can evaluate the double gyroaveages analytically. Defining the operator $\langle\cdot\rangle=1/(4\pi^2)\int\cdot\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta}$ and the parameter

$$s(\boldsymbol{X},\mu,\bar{\mu})=\frac{2v_{\perp,s}(\boldsymbol{X},\mu)\bar{v}_{\perp,\bar{s}}(\boldsymbol{X},\bar{\mu})}{(v_\parallel-\bar{v}_\parallel)^2+v^2_{\perp,s}(\boldsymbol{X},\mu)+\bar{v}^2_{\perp,\bar{s}}(\boldsymbol{X},\bar{\mu})}, \tag{4.5}$$

we use the scalar identity $\boldsymbol{k}(\boldsymbol{X},\theta)\cdot\boldsymbol{k}(\boldsymbol{X},\bar{\theta})=\cos(\theta-\bar{\theta})$ together with the expression for $\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}})|_{X=\bar{X}}$ and the double gyroaverages of (4.1), (4.2), and (4.3) become

$$\langle\mathbb{Q}^{\boldsymbol{bb}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\frac{1}{\pi}\left(\frac{s}{2v_\perp\bar{v}_\perp}\right)^{3/2}((v_\perp^2+\bar{v}_\perp^2)\mathcal{I}_2(s)-2v_\perp\bar{v}_\perp\mathcal{I}_3(s)), \tag{4.6}$$

$$\langle\mathbb{Q}^{\boldsymbol{bk}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\frac{1}{\pi}\left(\frac{s}{2v_\perp\bar{v}_\perp}\right)^{3/2}(v_\parallel-\bar{v}_\parallel)(\bar{v}_\perp\mathcal{I}_3(s)-v_\perp\mathcal{I}_2(s)), \tag{4.7}$$

$$\langle\mathbb{Q}^{\boldsymbol{bk}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\langle\mathbb{Q}^{\boldsymbol{kb}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}, \tag{4.8}$$

$$\langle\mathbb{Q}^{\boldsymbol{b\bar{k}}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\frac{1}{\pi}\left(\frac{s}{2v_\perp\bar{v}_\perp}\right)^{3/2}(v_\parallel-\bar{v}_\parallel)(\bar{v}_\perp\mathcal{I}_2(s)-v_\perp\mathcal{I}_3(s)), \tag{4.9}$$

$$\langle\mathbb{Q}^{\boldsymbol{b\bar{k}}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\langle\mathbb{Q}^{\boldsymbol{\bar{k}b}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}, \tag{4.10}$$

$$\langle\mathbb{Q}^{\boldsymbol{kk}}(\boldsymbol{\Gamma}^{gc}_{s\bar{s}}(\mathcal{H}^{gc},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}=\frac{1}{\pi}\left(\frac{s}{2v_\perp\bar{v}_\perp}\right)^{3/2}\left((v_\parallel-\bar{v}_\parallel)^2\mathcal{I}_2(s)+\bar{v}_\perp^2\mathcal{I}_1(s)\right), \tag{4.11}$$

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

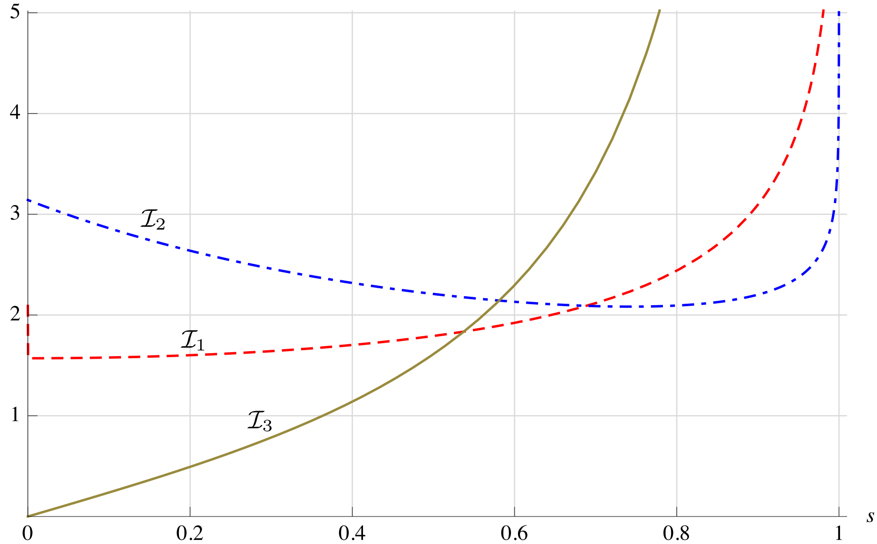

FIGURE 1. The functions $\mathcal{I}_1(s)$, $\mathcal{I}_2(s)$ and $\mathcal{I}_3(s)$.

$$\langle \mathbb{Q}^{\bar{k}\bar{k}}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}} = \frac{1}{\pi}\left(\frac{s}{2v_\perp \bar{v}_\perp}\right)^{3/2}\left((v_\parallel - \bar{v}_\parallel)^2\mathcal{I}_2(s) + v_\perp^2\mathcal{I}_1(s)\right), \tag{4.12}$$

$$\langle \mathbb{Q}^{k\bar{k}}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}} = \langle \mathbb{Q}^{\bar{k}k}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}}, \tag{4.13}$$

$$\langle \mathbb{Q}^{kk}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{X=\bar{X}} = \frac{1}{\pi}\left(\frac{s}{2v_\perp \bar{v}_\perp}\right)^{3/2}\left(v_\perp \bar{v}_\perp\mathcal{I}_1(s) + (v_\parallel - \bar{v}_\parallel)^2\mathcal{I}_3(s)\right). \tag{4.14}$$

The notation with the superscripts refers to the double contraction of the matrix $\mathbb{Q}$ with respect to $\boldsymbol{b}$ and $\boldsymbol{k}$ etc. The functions $\mathcal{I}_i(s)$ are defined in terms of the complete elliptic integrals of the first and second kind:

$$\mathcal{I}_1(s) = \frac{4}{s^2\sqrt{1-s}}\left(K\left[\frac{2s}{s-1}\right] - (1-s)E\left[\frac{2s}{s-1}\right]\right), \tag{4.15}$$

$$\mathcal{I}_2(s) = \frac{2}{(1+s)\sqrt{1-s}}E\left[\frac{2s}{s-1}\right], \tag{4.16}$$

$$\mathcal{I}_3(s) = \frac{2}{s(1+s)\sqrt{1-s}}\left(E\left[\frac{2s}{s-1}\right] - (1+s)K\left[\frac{2s}{s-1}\right]\right). \tag{4.17}$$

The functional form of (4.15)–(4.17), is such that all these expressions share a common singularity at the point $s = 1$ (see figure 1). The singularity results from the condition:

$$(v_\parallel - \bar{v}_\parallel)^2 + (v_\perp - \bar{v}_\perp)^2 = 0, \tag{4.18}$$

which corresponds to the same singularity that appears in the projection matrix (2.10). From a physical standpoint, the singularity at $s = 1$ correspond to the colliding particles residing at the same point in the velocity space. Instructions on performing the double gyroaveraging are provided in Appendix A.

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

5. Discussion

The collisional bracket for the guiding-centre Vlasov–Maxwell–Landau system that we have presented and analysed in the previous sections is the first one of its kind for any temporally reduced electromagnetic kinetic plasma model. As there nevertheless exists a collisional bracket for the electrostatic gyrokinetic model (Hirvijoki & Burby 2020), one can only wonder why have we not encountered analogous brackets or energetically-consistent collision operators for other reduced electromagnetic kinetic plasma theories, namely the electromagnetic drift-kinetic and gyrokinetic models. What are we missing?

Dropping all the extra indices and species labels for clarity, we know that the particle velocity can be represented in the reduced coordinates to a reasonable accuracy as $\boldsymbol{v} = \dot{\boldsymbol{X}} + \dot{\boldsymbol{\rho}}_0$, where $\dot{\boldsymbol{X}}$ captures the parallel streaming and the slower drifts while $\dot{\boldsymbol{\rho}}_0$ captures the fast Larmor rotation. In an electrostatic theory, it then makes sense to write the particle velocity as

$$\boldsymbol{v} = \{\boldsymbol{X} + \boldsymbol{\rho}_0, H\}, \tag{5.1}$$

with H the single drift-centre or gyrocenter Hamiltonian. Further, we know that the velocity-space derivative of a particle phase-space function f can be expressed in terms of the non-canonical particle phase-space Poisson bracket. This admits the transformation of the velocity derivatives in the Landau operator to the reduced coordinates in a manner which circumvents much of the difficulties in the underlying asymptotic transformation:

$$\frac{1}{m}\frac{\partial f}{\partial \boldsymbol{v}} \mapsto \{\boldsymbol{X} + \boldsymbol{\rho}_0, F\}. \tag{5.2}$$

Here F is the distribution function and $\{\cdot,\cdot\}$ is the Poisson bracket in the reduced coordinates. For details, see the paper by Brizard (2004). Combining this information with how the single-particle velocity can be expressed in the electrostatic case, it then seems logical to express the vector $\boldsymbol{\Gamma}(\mathcal{A})$ in the collisional bracket as

$$\boldsymbol{\Gamma}(\mathcal{A}) = \left\{\boldsymbol{X} + \boldsymbol{\rho}_0, \frac{\delta\mathcal{A}}{\delta F}\right\}_z - \left\{\boldsymbol{X} + \boldsymbol{\rho}_0, \frac{\delta\mathcal{A}}{\delta F}\right\}_{\bar{z}}. \tag{5.3}$$

Because the Hamiltonian functional in the electrostatic case satisfies $\delta\mathcal{H}/\delta F = H$, an energy-conserving collisional bracket for electrostatic theories can then be found simply by choosing $\mathbb{Q}(\boldsymbol{\Gamma}(\mathcal{H}))$ in the matrix $\mathbb{W}$. Furthermore, if the background magnetic vector potential $\boldsymbol{A}_0$ present in the unperturbed guiding-centre one-form is axially symmetric and the one-form itself gyrogauge independent, then

$$\{\boldsymbol{X} + \boldsymbol{\rho}_0, p_\phi\} = \hat{\boldsymbol{z}} \times (\boldsymbol{X} + \boldsymbol{\rho}_0), \tag{5.4}$$

where p_ϕ is the single guiding-centre canonical toroidal momentum. As the canonical toroidal momentum functional in the electrostatic case also satisfies $\delta\mathcal{P}_\phi/\delta F = p_\phi$, the bracket will satisfy canonical toroidal momentum conservation as long as the localizing delta-function in the matrix $\mathbb{W}$ is chosen with respect to the difference in the particle positions $\boldsymbol{X} + \boldsymbol{\rho}_0$ and $\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}}_0$ (see the papers by Burby *et al.* 2015*b* and Hirvijoki & Burby 2020 for details).

In perturbed electromagnetic theories, such as the drift-kinetic and gyrokinetic ones, things are different. The single drift-centre or gyrocentre Lagrangian is presented as

$$L = \vartheta_\alpha \dot{z}^\alpha - K(\boldsymbol{E}_1, \boldsymbol{B}_1) + (e/c)A_{1,i}\dot{X}^i - e\varphi_1, \tag{5.5}$$

with ϑ_α the six time-independent components of the unperturbed guiding-centre one-form, K is the kinetic energy function, and $(\boldsymbol{A}_1, \varphi_1)$ and $(\boldsymbol{B}_1, \boldsymbol{E}_1)$ are the perturbed

https://doi.org/10.1017/S0022377821000696 Published online by Cambridge University Press

electromagnetic potentials and fields, respectively. In this case, the particle velocity in reduced coordinates becomes

$$\boldsymbol{v} = \{\boldsymbol{X} + \boldsymbol{\rho}_0, K\} - \{\boldsymbol{X} + \boldsymbol{\rho}_0, \boldsymbol{X}\} \cdot e\boldsymbol{E}_1. \tag{5.6}$$

An analogous expression was used also by Brizard & Chandre (2020) in a different context. Also note that now the Poisson bracket contains contributions from the time-dependent perturbation $\boldsymbol{B}_1$ as well. Formally, the electrostatic case is recovered by simply setting the perturbations $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ to zero, moving the electrostatic part of the electric field inside the bracket, and by combining K and φ_1 into $H = K(\boldsymbol{E}_1) + e\varphi_1$.

Imagine now that we were to try a similar route as in the electrostatic case. We shall use the drift-kinetic system documented by Hirvijoki *et al.* (2020) as an example. In this case, the canonical toroidal momentum and energy functionals conserved by the collisionless dynamics can be expressed in the following form:

$$\mathcal{P}_\phi[F, \boldsymbol{E}_1, \boldsymbol{B}_1] = \int \left(p_\phi - \frac{\partial K}{\partial \boldsymbol{E}_1} \times \frac{\boldsymbol{B}_1}{c} \cdot \boldsymbol{e}_\phi \right) F \, \mathrm{d}\boldsymbol{Z} + \frac{1}{4\pi} \int \boldsymbol{E}_1 \times \frac{\boldsymbol{B}_1}{c} \cdot \boldsymbol{e}_\phi \, \mathrm{d}\boldsymbol{x}, \tag{5.7}$$

$$\mathcal{H}[F, \boldsymbol{E}_1, \boldsymbol{B}_1] = \int \left(K - \frac{\partial K}{\partial \boldsymbol{E}_1} \cdot \boldsymbol{E}_1 \right) F \, \mathrm{d}\boldsymbol{Z} + \frac{1}{8\pi} \int \left(|\boldsymbol{E}_1|^2 + |\boldsymbol{B}_0 + \boldsymbol{B}_1|^2 \right) \mathrm{d}\boldsymbol{x}, \tag{5.8}$$

From where, we infer

$$\frac{\delta \mathcal{P}_\phi}{\delta F} = p_\phi - \frac{\partial K}{\partial \boldsymbol{E}_1} \times \frac{\boldsymbol{B}_1}{c} \cdot \boldsymbol{e}_\phi, \tag{5.9}$$

$$\frac{\delta \mathcal{H}}{\delta F} = K - \frac{\partial K}{\partial \boldsymbol{E}_1} \cdot \boldsymbol{E}_1. \tag{5.10}$$

Most importantly, this means that

$$\frac{\delta \mathcal{P}_\phi}{\delta F} \neq p_\phi, \tag{5.11}$$

$$\left\{ \boldsymbol{X} + \boldsymbol{\rho}_0, \frac{\delta \mathcal{H}}{\delta F} \right\} \neq \boldsymbol{v}. \tag{5.12}$$

There does not appear to be an immediate, simple way to modify (5.3) so that one would recover $\boldsymbol{\Gamma}(\mathcal{H}) = \boldsymbol{v} - \bar{\boldsymbol{v}}$, where $\boldsymbol{v}$ is given by (5.6), and $\boldsymbol{\Gamma}(\mathcal{P}_\phi) = \{\boldsymbol{X} + \boldsymbol{\rho}_0, p_\phi\}$, which guarantee the conservation laws in the electrostatic case.

Were we to, e.g. add some operator to the expression (5.3), we would need to find a way to cancel the inequalities above while acting only on the functional derivatives,

$$\frac{\delta \mathcal{P}_\phi}{\delta \boldsymbol{E}_1} = \frac{\boldsymbol{B}_1 \times \boldsymbol{e}_\phi}{4\pi} - \int \delta(\boldsymbol{x} - \boldsymbol{X}) \frac{\partial^2 K}{\partial \boldsymbol{E}_1 \partial \boldsymbol{E}_1} \times \boldsymbol{B}_1 \cdot \boldsymbol{e}_\phi F \, \mathrm{d}\boldsymbol{Z}, \tag{5.13}$$

$$\frac{\delta \mathcal{P}_\phi}{\delta \boldsymbol{B}_1} = -\frac{\boldsymbol{D} \times \boldsymbol{e}_\phi}{4\pi} - \int \delta(\boldsymbol{x} - \boldsymbol{X}) \frac{\partial^2 K}{\partial \boldsymbol{B}_1 \partial \boldsymbol{E}_1} \times \boldsymbol{B}_1 \cdot \boldsymbol{e}_\phi F \, \mathrm{d}\boldsymbol{Z}, \tag{5.14}$$

$$\frac{\delta \mathcal{H}}{\delta \boldsymbol{E}_1} = \frac{\boldsymbol{D}}{4\pi} - \int \delta(\boldsymbol{x} - \boldsymbol{X}) \frac{\partial^2 K}{\partial \boldsymbol{E}_1 \partial \boldsymbol{E}_1} \cdot \boldsymbol{E}_1 F \, \mathrm{d}\boldsymbol{Z}, \tag{5.15}$$

$$\frac{\delta \mathcal{H}}{\delta \boldsymbol{B}_1} = \frac{\boldsymbol{B}_0 + \boldsymbol{B}_1}{4\pi} - \int \delta(\boldsymbol{x} - \boldsymbol{X}) \frac{\partial^2 K}{\partial \boldsymbol{B}_1 \partial \boldsymbol{E}_1} \cdot \boldsymbol{E}_1 F \, \mathrm{d}\boldsymbol{Z}, \tag{5.16}$$

where the electric displacement field is $\boldsymbol{D}_1(\boldsymbol{x}) = \boldsymbol{E}_1(\boldsymbol{x}) - \int \delta(\boldsymbol{x} - \boldsymbol{X}) \partial K / \partial \boldsymbol{E}_1 F \, \mathrm{d}\boldsymbol{Z}$. It is, however, difficult to imagine how the expression $-(\partial K / \partial \boldsymbol{E}_1) \times (\boldsymbol{B}_1 / c) \cdot \boldsymbol{e}_\phi$, which appears to be preventing momentum conservation, could be expressed in terms of some operator acting on $\delta\mathcal{P}_\phi/\delta\boldsymbol{E}_1$ and $\delta\mathcal{P}_\phi/\delta\boldsymbol{B}_1$. The operator would also need to be linear in its action to preserve the bilinearity of the total bracket. Even if such an operator could be found, it is difficult to imagine it to simultaneously address the other inequality when acting on $\mathcal{H}$ instead of $\mathcal{P}_\phi$.

Finally, we note that because the electric perturbation $\boldsymbol{E}_1$ in the kinetic energy function K often appears at second order, one could imagine using $\delta\mathcal{H}/\delta F = K - \partial K/\partial\boldsymbol{E}_1 \cdot \boldsymbol{E}_1$ as an approximation of K. Further, if the $\boldsymbol{E}\times\boldsymbol{B}$ velocity could be neglected in collisions, then using the $\boldsymbol{\Gamma}(\mathcal{A})$ of (5.3) together with $\mathbb{Q}(\boldsymbol{\Gamma}(\mathcal{H}))$ for the matrix $\mathbb{W}$ would guarantee energy conservation and entropy dissipation, even if momentum conservation would not be achieved. In general, though, resolving these issues rigorously calls for a more systematic approach to performing asymptotic dynamical reduction of metric brackets.

6. Summary

In this paper, we have presented a metric bracket to account for Coulomb collisions in the so-called guiding-centre Vlasov–Maxwell–Landau model. The bracket has been shown to preserve the system energy and momentum functionals, and to satisfy an H-theorem. We have also discussed in detail the issues that arise if the same concept is to be applied to electromagnetic drift-kinetic or gyrokinetic theories: while we are, in principle, able to manufacture an energy conserving and entropy dissipating bracket, we have not found a way to guarantee momentum conservation, regardless of whether energy is conserved or not.

Based on our findings, we conclude that a systematic tool for performing asymptotic dynamical reduction of collisional process, or more precisely of metric brackets, is necessary. Although Lie-transform perturbation theory is an established tool to handle asymptotic dynamical reduction of dissipation-free dynamics, no similar compatible theory exists yet to handle structure-preserving dissipative dynamics.

Acknowledgements

R.I. and E.H. thank Dr T. Kiviniemi for useful comments regarding the manuscript.

Editor Emanuele Tassi thanks the referees for their advice in evaluating this article.

Funding

This work has been supported by the Academy of Finland grant nos. 316088 and 315278.

Declaration of interests

The authors report no conflict of interest.

Appendix A. Computation of the gyroaverages

The computation of the gyroaverages involves contracting the Landau tensor with the unit vectors $\boldsymbol{b}$ and $\boldsymbol{k} = \boldsymbol{b}\times(\boldsymbol{\rho}_0/|\boldsymbol{\rho}_0|)$ evaluated at the positions $|_X$, $|_{\bar{X}}$, $|_Z$ and $|_{\bar{Z}}$. In principle the following combinations are needed:

$$\left.\begin{aligned}&\boldsymbol{b}|_{X,\bar{X}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot\boldsymbol{b}|_{X,\bar{X}},\\&\boldsymbol{b}|_{X,\bar{X}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot\boldsymbol{k}|_{Z,\bar{Z}},\\&\boldsymbol{k}|_{Z,\bar{Z}}\cdot\mathbb{Q}(\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}}))\cdot\boldsymbol{k}|_{Z,\bar{Z}},\end{aligned}\right\}\tag{A1}$$

but in practice the condition $\boldsymbol{X} = \bar{\boldsymbol{X}}$ can be applied, credited to the presence of the localizing delta function in the bracket. Then double integration on the angular variables is performed with the operator $\langle\cdot\rangle = 1/(4\pi^2)\int \cdot\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta}$.

For the computation, one needs expressions involving the vector $\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})$. The norm of the vector can be expressed as

$$|\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|_{\boldsymbol{X}=\bar{\boldsymbol{X}}} = \left(\frac{2v_{\perp,s}\bar{v}_{\perp,\bar{s}}}{s}\right)^{1/2}\sqrt{1 - s\cos(\theta - \bar{\theta})}, \tag{A2}$$

with $v_{\perp,s} = v_{\perp,s}(\boldsymbol{X}, \mu)$, $\bar{v}_{\perp,\bar{s}} = \bar{v}_{\perp,\bar{s}}(\boldsymbol{X}, \bar{\mu})$ and $s = s(\boldsymbol{X}, \mu, \bar{\mu})$ the parameters defined in (4.5). As the scaled projection matrix $\mathbb{Q}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))$ takes the form

$$\mathbb{Q}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})) = \frac{1}{|\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|}\left[\mathbb{I} - \frac{\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})}{|\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|^2}\right], \tag{A3}$$

also the following dyad is needed:

$$\begin{aligned}
&\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|_{\boldsymbol{X}=\bar{\boldsymbol{X}}}\\
&\quad = (v_{\parallel} - \bar{v}_{\parallel})^2\boldsymbol{bb} + v_{\perp,s}^2\boldsymbol{kk} - v_{\perp,s}\bar{v}_{\perp,\bar{s}}(\boldsymbol{k}\bar{\boldsymbol{k}} + \bar{\boldsymbol{k}}\boldsymbol{k}) + \bar{v}_{\perp,\bar{s}}^2\bar{\boldsymbol{k}}\bar{\boldsymbol{k}}\\
&\qquad + (v_{\parallel} - \bar{v}_{\parallel})[v_{\perp,s}(\boldsymbol{bk} + \boldsymbol{kb}) - \bar{v}_{\perp,\bar{s}}(\boldsymbol{b}\bar{\boldsymbol{k}} + \bar{\boldsymbol{k}}\boldsymbol{b})].
\end{aligned} \tag{A4}$$

It is also useful to note the identity $\boldsymbol{k}\cdot\bar{\boldsymbol{k}} = \cos(\theta - \bar{\theta})$.

As an example, suppose we wish to evaluate the average related to the contraction of the Landau tensor with the unit vector $\boldsymbol{b}$, namely $\langle\mathbb{Q}^{\boldsymbol{bb}}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{\boldsymbol{X}=\bar{\boldsymbol{X}}}$. The contraction with $\boldsymbol{b}$ provides

$$\boldsymbol{b}\cdot\mathbb{Q}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\cdot\boldsymbol{b}|_{\boldsymbol{X}=\bar{\boldsymbol{X}}} = \frac{|\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|^2_{\boldsymbol{X}=\bar{\boldsymbol{X}}} - (v_{\parallel} - \bar{v}_{\parallel})^2}{|\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}})|^3_{\boldsymbol{X}=\bar{\boldsymbol{X}}}}, \tag{A5}$$

substituting the relevant expressions, one identifies two integrals

$$\left.\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{[1 - s\cos(\bar{\theta} - \theta)]^{3/2}} &= \frac{8\pi}{(s+1)\sqrt{1-s}}E\left(\frac{2s}{s-1}\right),\\
\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(\bar{\theta} - \theta)\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{[1 - s\cos(\bar{\theta} - \theta)]^{3/2}} &= 8\pi\frac{\sqrt{1-s}}{s(s^2-1)}\left[(s+1)K\left(\frac{2s}{s-1}\right) - E\left(\frac{2s}{s-1}\right)\right],
\end{aligned}\right\} \tag{A6}$$

after which the result from § 4 is recovered

$$\langle\mathbb{Q}^{\boldsymbol{bb}}(\boldsymbol{\varGamma}^{\mathrm{gc}}_{s\bar{s}}(\mathcal{H}^{\mathrm{gc}}, \boldsymbol{Z}, \bar{\boldsymbol{Z}}))\rangle|_{\boldsymbol{X}=\bar{\boldsymbol{X}}} = \frac{1}{\pi}\left(\frac{s}{2v_{\perp}\bar{v}_{\perp}}\right)^{3/2}((v_{\perp}^2 + \bar{v}_{\perp}^2)\mathcal{I}_2(s) - 2v_{\perp}\bar{v}_{\perp}\mathcal{I}_3(s)). \tag{A7}$$

The rest of the gyroaverages are computed in exactly the same manner by contracting the appropriate unit vectors with the projection matrix and performing the angular integrals.

REFERENCES

BOBYLEV, A.V. & NANBU, K. 2000 Theory of collision algorithms for gases and plasmas based on the boltzmann equation and the Landau-Fokker-Planck equation. *Phys. Rev.* E **61** (4), 4576.

BRIZARD, A.J. 2000 Variational principle for nonlinear gyrokinetic Vlasov–Maxwell equations. *Phys. Plasmas* **7** (12), 4816–4822.

BRIZARD, A.J. 2004 A guiding-center Fokker–Planck collision operator for nonuniform magnetic fields. *Phys. Plasmas* **11** (9), 4429–4438.

BRIZARD, A.J. & CHANDRE, C. 2020 Hamiltonian formulations for perturbed dissipationless plasma equations. *Phys. Plasmas* **27** (12), 122111.

BRIZARD, A.J., MORRISON, P.J., BURBY, J.W., DE GUILLEBON, L. & VITTOT, M. 2016 Lifting of the Vlasov–Maxwell bracket by lie-transform method. *J. Plasma Phys*. **82** (6).

BRIZARD, A.J. & TRONCI, C. 2016 Variational formulations of guiding-center Vlasov–Maxwell theory. *Phys. Plasmas* **23** (6), 062107.

BURBY, J.W. & BRIZARD, A.J. 2019 Gauge-free electromagnetic gyrokinetic theory. *Phys. Lett.* A **383** (18), 2172–2175.

BURBY, J.W., BRIZARD, A.J., MORRISON, P.J. & QIN, H. 2015*a* Hamiltonian gyrokinetic Vlasov–Maxwell system. *Phys. Lett.* A **379** (36), 2073–2077.

BURBY, J.W., BRIZARD, A.J. & QIN, H. 2015*b* Energetically consistent collisional gyrokinetics. *Phys. Plasmas* **22** (10), 100707.

CARY, J.R. & KAUFMAN, A.N. 1981 Ponderomotive effects in collisionless plasma: a lie transform approach. *Phys. Fluids* **24** (7), 1238–1250.

COQUINOT, B. & MORRISON, P.J. 2020 A general metriplectic framework with application to dissipative extended magnetohydrodynamics. *J. Plasma Phys*. **86** (3).

GREBOGI, C., KAUFMAN, A.N. & LITTLEJOHN, R.G. 1979 Hamiltonian theory of ponderomotive effects of an electromagnetic wave in a nonuniform magnetic field. *Phys. Rev. Lett*. **43** (22), 1668.

GRMELA, M. 1984*a* Bracket formulation of dissipative fluid mechanics equations. *Phys. Lett.* A **102** (8), 355–358.

GRMELA, M. 1984*b* Particle and bracket formulations of kinetic equations. In *Fluids and Plasmas: Geometry and Dynamics*, Contemporary Mathematics, pp. 125–132. American Mathematical Society.

GRMELA, M. 1985 Bracket formulation of dissipative time evolution equations. *Phys. Lett.* A **111** (1-2), 36–40.

HAHM, T.S., LEE, W.W. & BRIZARD, A. 1988 Nonlinear gyrokinetic theory for finite-beta plasmas. *Phys. Fluids* **31** (7), 1940–1948.

HATORI, T. & WASHIMI, H. 1981 Covariant form of the ponderomotive potentials in a magnetized plasma. *Phys. Rev. Lett*. **46** (4), 240.

HIRVIJOKI, E. 2021 Structure-preserving marker-particle discretizations of coulomb collisions for particle-in-cell codes. *Plasma Phys. Control. Fusion* **63** (4), 044003.

HIRVIJOKI, E., BRIZARD, A., SNICKER, A. & KURKI-SUONIO, T. 2013 Monte Carlo implementation of a guiding-center Fokker-Planck kinetic equation. *Phys. Plasmas* **20** (9), 092505.

HIRVIJOKI, E. & BURBY, J.W. 2020 Collisional gyrokinetics teases the existence of metriplectic reduction. *Phys. Plasmas* **27** (8), 082307.

HIRVIJOKI, E., BURBY, J.W., PFEFFERLÉ, D. & BRIZARD, A.J. 2020 Energy and momentum conservation in the Euler–Poincaré formulation of local Vlasov–Maxwell-type systems. *J. Phys. A: Math. Theor*. **53** (23), 235204.

HOLM, D.D., MARSDEN, J.E., RATIU, T. & WEINSTEIN, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep*. **123** (1-2), 1–116.

KAUFMAN, A.N. 1984 Dissipative Hamiltonian systems: a unifying principle. *Phys. Lett.* A **100** (8), 419–422.

KAUFMAN, A.N. & HOLM, D.D. 1984 The lie-transformed vlasov action principle: relativistically covariant wave propagation and self-consistent ponderomotive effects. *Phys. Lett.* A **105** (6), 277–279.

KAUFMAN, A.N. & MORRISON, P.J. 1982 Algebraic structure of the plasma quasilinear equations. *Phys. Lett.* A **88** (8), 405–406.

KRAUS, M. & HIRVIJOKI, E. 2017 Metriplectic integrators for the landau collision operator. *Phys. Plasmas* **24** (10), 102311.

LINGAM, M. 2015 Dissipative effects in magnetohydrodynamical models with intrinsic magnetization. *Commun. Nonlinear Sci. Numer. Simul*. **28** (1-3), 223–231.

LITTLEJOHN, R.G. 1981 Hamiltonian formulation of guiding center motion. *Phys. Fluids* **24**, 1730.

LITTLEJOHN, R.G. 1982 Hamiltonian perturbation theory in noncanonical coordinates. *J. Math. Phys*. **23**, 742.

LITTLEJOHN, R.G. 1983 Variational principles of guiding centre motion. *J. Plasma Phys*. **29** (1), 111–125.

LITTLEJOHN, R.G. 1984 Geometry and guiding center motion. In *Fluids and Plasmas: Geometry and Dynamics* (ed. J. E. Marsden), Contemporary Mathematics, vol. 28, pp. 151–167. American Mathematical Society.

MARSDEN, J.E. & WEINSTEIN, A. 1982 The Hamiltonian structure of the Maxwell-Vlasov equations. *Phys.* D *Nonlinear Phenom*. **4** (3), 394–406.

MATERASSI, M. & TASSI, E. 2012 Metriplectic framework for dissipative magneto-hydrodynamics. *Phys.* D*: Nonlinear Phenom*. **241** (6), 729–734.

MORRISON, P.J. 1980 The Maxwell-Vlasov equations as a continuous Hamiltonian system. *Phys. Lett.* A **80** (5–6), 383–386.

MORRISON, P.J. 1984*a* Bracket formulation for irreversible classical fields. *Phys. Lett.* A **100** (8), 423–427.

MORRISON, P.J. 1984*b* Some observations regarding brackets and dissipation. *Tech. Rep.*. Center for Pure and Applied Mathematics Report PAM–228, University of California, Berkeley.

MORRISON, P.J. 1986 A paradigm for joined Hamiltonian and dissipative systems. *Phys.* D*: Nonlinear Phenom*. **18**, 410–419.

MORRISON, P.J. 1998 Hamiltonian description of the ideal fluid. *Rev. Mod. Phys*. **70** (2), 467.

MORRISON, P.J. 2009 On Hamiltonian and action principle formulations of plasma dynamics. In *AIP Conference Proceedings*, vol. 1188, pp. 329–344. American Institute of Physics.

MORRISON, P.J. & GREENE, J.M. 1980 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett*. **45** (10), 790.

SUGAMA, H. 2000 Gyrokinetic field theory. *Phys. Plasmas* **7** (2), 466–480.

WEINSTEIN, A. & MORRISON, P.J. 1981 Comments on: the Maxwell-Vlasov equations as a continuous Hamiltonian system. *Phys. Lett.* A **86** (4), 235–236.

YOON, E.S. & CHANG, C.S. 2014 A Fokker-Planck-Landau collision equation solver on two-dimensional velocity grid and its application to particle-in-cell simulation. *Phys. Plasmas* **21** (3), 032503.