5
Uniform Rectifiability

The various rectifiability characterizations we have seen are qualitative. This is unavoidable if we start with qualitative size conditions of measure and densities. But if we start with quantitative size conditions like AD-regularity, then quantitative-equivalent uniform rectifiability conditions are in the core of the theory developed by David and Semmes in [146] and [147]. Most of the material of this chapter is from those books.

5.1 One-Dimensional Sets

Let us first have a quick look at one-dimensional sets. We say that $E \subset \mathbb{R}^n$ is *uniformly 1-rectifiable* if it is closed, AD-1-regular and contained in an AD-regular curve, that is, there is a curve Γ and a constant C such that $E \subset \Gamma$ and

$$\mathcal{H}^1(\Gamma \cap B(x,r)) \le Cr \text{ for all } x \in \mathbb{R}^n, r > 0.$$

The key to the related results for approximation by lines is in the travelling salesman-type conditions discussed in Section 3.4. Recall the definition of $\beta_E(x,r)$ from (3.1).

The following is a local version of Jones's travelling salesman Theorem 3.16:

Theorem 5.1 *If $E \subset \mathbb{R}^n$ is closed and AD-1-regular, then E is uniformly 1-rectifiable if and only if*

$$\int_0^R \int_{E \cap B(x,R)} \beta_E(y,r)^2 \, d\mathcal{H}^1 y \, r^{-1} dr \lesssim R \textit{ for all } x \in E, R > 0.$$

We also have a characterization by Menger curvature. Recall its definition from (3.2):

Theorem 5.2 *If $E \subset \mathbb{R}^2$ is closed and AD-1-regular, then E is uniformly 1-rectifiable if and only if*

$$\int_{E\cap B(a,R)}\int_{E\cap B(a,R)}\int_{E\cap B(a,R)} c(x,y,z)^2\, d\mathcal{H}^1 x\, d\mathcal{H}^1 y\, d\mathcal{H}^1 z \lesssim R$$

for all $a \in E, R > 0$.

That uniformly rectifiable sets satisfy this curvature condition was proved by Melnikov and Verdera in [335]. The other direction was proved in [327].

5.2 Lipschitz Maps and Approximation by Planes

Now $0 < m < n$ will be integers. In higher dimensions there is no simple definition of uniform rectifiability, but there are many interesting characterizations. Let us again take Lipschitz maps as the basis of the definition:

Definition 5.3 We say that $E \subset \mathbb{R}^n$ is *uniformly m-rectifiable* if it is closed, it is AD-m-regular and there are positive numbers M and θ such that for all $x \in E$ and $0 < R < d(E)$, there is a Lipschitz map $f\colon B^m(0,R) \to \mathbb{R}^n$ with $\mathrm{Lip}(f) \le M$ and

$$\mathcal{H}^m(E \cap B(x,R) \cap f(B^m(0,R))) \ge \theta R^m.$$

David and Semmes describe this property as E having *big pieces of Lipschitz images* of $\mathbb{R}^m$. There is a similar characterization by bi-Lipschitz images, but *big pieces of Lipschitz graphs* (graphs over m-planes) are strictly stronger by an unpublished Venetian blind construction of Hrycak, see [31] or [374]. However, iterating this, big pieces of big pieces of Lipschitz graphs are equivalent to uniform rectifiability, see [40]. This, with iteration, seems to be a general phenomenon, see Section 5.6 and [75].

It is clear that uniformly rectifiable sets are rectifiable, but not vice versa. Lipschitz graphs are basic examples of uniformly rectifiable sets.

Define the L^p versions of β's for any $1 \le p < \infty$ by

$$\beta_E^{m,p}(x,r) = \inf_{V \text{ affine } m\text{-plane}} \left(r^{-m} \int_{E\cap B(x,r)} \left(\frac{d(y,V)}{r}\right)^p d\mathcal{H}^m y \right)^{1/p}. \tag{5.1}$$

Set also $\beta_E^{m,\infty}(x,r) = \beta_E^m(x,r)$, recall (4.11).

According to [146], Jones and Fang have produced three-dimensional Lipschitz graphs which show that Theorem 5.1 does not hold for $\beta_E^{3,\infty}$. But we have the following:

Theorem 5.4 *Let $1 \le p \le \infty$, if $m = 1$, and $1 \le p < 2m/(m-2)$, if $m > 1$. If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E is uniformly m-rectifiable if and only if*

$$\int_0^R \int_{E\cap B(x,R)} \beta_E^{m,p}(y,r)^2 \, d\mathcal{H}^m y \, r^{-1} dr \lesssim R^m \text{ for all } x \in E, R > 0. \tag{5.2}$$

In addition to being natural as the L^p versions of the uniform approximation of sets, the β's have counterparts in L^p differentiation of functions due to a result of Dorronsoro in [181].

The validity of (5.2) is often stated as E satisfying the *geometric lemma*. One also says that $\beta_E^{m,p}(x,r)^2 \, d\mathcal{H}^m x \, r^{-1} dr$ is a Carleson measure on $E\times(0,\infty)$:

Definition 5.5 Let $E \subset \mathbb{R}^n$. A Borel measure λ on $E \times (0,\infty)$ is a *Carleson measure* if

$$\lambda(B(x,R)\times(0,R)) \lesssim R^m \text{ for all } x \in E, R > 0.$$

A set $A \subset E \times (0,\infty)$ is a *Carleson set* if $\chi_A(x,r)\, d\mathcal{H}^m x \, r^{-1} dr$ is a Carleson measure.

This terminology comes from Carleson's solution of the complex analysis corona problem in 1962 and the methods he introduced. Several other Carleson measure and set conditions characterizing uniform rectifiability can be found in [147]. Below we shall see some of them and some more recent ones.

Condition (5.2) guarantees that E is well approximable by planes at most scales, so it is a relative of the existence of approximate tangent planes of rectifiable sets. Next we shall give different conditions in that spirit.

Definition 5.6 Let $E \subset \mathbb{R}^n$ be AD-m-regular. We say that E satisfies *weak geometric lemma* if for every $\varepsilon > 0$,

$$\{(x,r) \in E \times (0,\infty) \colon \beta_E^m(x,r) > \varepsilon\}$$

is a Carleson set, that is,

$$\int_0^R \mathcal{H}^m \left(\{y \in E \cap B(x,R) \colon \beta_E^m(y,r) > \varepsilon\}\right) r^{-1} dr \lesssim R^m \text{ for all } x \in E, R > 0.$$

The weak geometric lemma does not imply even ordinary rectifiability, recall Section 4.7. But it is useful in combination with some other conditions. As in the case of rectifiability and tangent measures, the corresponding bilateral approximation does the job. Recall the bilateral β from (4.12). The *bilateral weak geometric lemma* characterizes uniform rectifiability:

Theorem 5.7 *If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E is uniformly m-rectifiable if and only if for every $\varepsilon > 0$,*

$$\{(x,r) \in E \times (0,\infty)\colon b\beta_E^m(x,r) > \varepsilon\}$$

is a Carleson set.

A weaker condition where approximation is allowed with unions of planes already implies uniform rectifiability. To state this, define $ub\beta_E(x,r)$ as in (4.12), but V is replaced by a union of m-planes. Then, see [147], Proposition II.3.18,

Theorem 5.8 *If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E is uniformly m-rectifiable if and only if for every $\varepsilon > 0$,*

$$\{(x,r) \in E \times (0,\infty)\colon ub\beta_E(x,r) > \varepsilon\}$$

is a Carleson set.

I have stated this a bit more exotic characterization since it will be useful in Chapter 10.

There is also a local symmetry characterization. Recall a similar condition in Lemma 4.12 and its role in the proof of Theorem 4.10.

Theorem 5.9 *If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E is uniformly m-rectifiable if and only if for every $\varepsilon > 0$,*

$$\{(x,r) \in E \times (0,\infty)\colon \exists y,z \in E \cap B(x,r) \text{ such that } d(2y-z,E) > \varepsilon r\}$$

is a Carleson set.

The local symmetry condition follows immediately from the bilateral approximation in Theorem 5.7; the converse requires more work.

Instead of approximating sets by planes, one can approximate the Hausdorff measure on E with Lebesgue measures on planes. It is more natural to state this for measures. So we define an AD-m-regular measure μ to be uniformly m-rectifiable if its support is uniformly m-rectifiable. For a ball $B(x,r)$ and $\mu,\nu \in \mathcal{M}(B(x,r))$ define

$$F_{x,r}(\mu,\nu) = \sup\left\{\left|\int f\,d\mu - \int f\,d\nu\right|\colon \operatorname{spt} f \subset B(x,r), \operatorname{Lip}(f) \le 1\right\}. \quad (5.3)$$

Then $F_{x,r}$ is a metric that metrizes weak convergence, see, for example, [321, p. 195]. Tolsa introduced in [413] the following α coefficients for a measure μ:

$$\alpha_\mu^m(x,r) = r^{-m-1}\inf\left\{F_{x,r}(\mu, c\mathcal{H}^m \llcorner V)\colon c \ge 0, V \text{ an affine } m\text{-plane}\right\}. \quad (5.4)$$

He proved the following:

Theorem 5.10 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is AD-m-regular, then μ is uniformly m-rectifiable if and only if*

$$\int_0^R \int_{B(x,R)} \alpha_\mu^m(x,r)^2 \, d\mu x \, r^{-1} dr \lesssim R^m \text{ for all } x \in \mathbb{R}^n, R > 0. \tag{5.5}$$

It is easy to show that the α^m numbers dominate the $\beta^{m,1}$ numbers, so one direction follows from Theorem 5.4. The other direction uses corona decompositions, see Section 5.5.

In [414] Tolsa proved an $L^p, 1 \leq p \leq 2$, version of this; the case $p = 1$ is essentially the above. Then the α coefficients are defined using the mass transport, Wasserstein, distance

$$W_p(\mu, \nu) = \inf_\pi \left(\int |x - y|^p \, d\pi(x, y) \right)^{1/p},$$

where the infimum is taken over all probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$ whose marginals are μ and ν. Dabrowski [124,126] proved the similar characterization of (non-uniform) rectifiability.

In [32] Azzam and Dabrowski gave a characterization of the L^p norms $||f||_{L^p(\mu)}$ for uniformly rectifiable measures μ in terms of the α's.

5.3 Density Ratios

There is also an analogue of Preiss's Theorem 4.11 'existence of density is equivalent to rectifiability':

Theorem 5.11 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is AD-m-regular, then μ is uniformly m-rectifiable if and only if the complement in* $\operatorname{spt}\mu \times (0, \infty)$ *of the set*

$$\{(x, r) \in \operatorname{spt}\mu \times (0, \infty) \colon \text{there exists } \delta(x, r) > 0 \text{ such that}$$
$$|\mu(B(y, t)) - \delta(x, r)t^m| < \varepsilon r^m \text{ for all } y \in \operatorname{spt}\mu \cap B(x, r), 0 < t \leq r\}$$

is a Carleson set for all $\varepsilon > 0$.

The 'if' part is the more difficult one. The proof for that is based on uniform measures. It can be shown that the condition of the theorem implies that μ can be approximated at most locations and scales by m-uniform measures. For $m = 1, 2, n - 1$ the m-uniform measures are either flat or conical as was discussed after Preiss's Theorem 4.11. Using this, David and Semmes completed the proof of Theorem 5.11 for $m = 1, 2, n - 1$ in [147]. They had proven the 'only if' direction already [146]. For the remaining dimensions no such concrete information about uniform measures is available. However, Preiss's paper contains enough information (uniform measures are 'flat at infinity') so that Tolsa could finish the proof in [416].

Of course, Theorem 5.11 implies that uniform measures are uniformly rectifiable. Tolsa showed more in the same paper: the uniform measures have the 'big pieces of Lipschitz graphs' property.

Chousionis, Garnett, Le and Tolsa [99] characterized uniform rectifiability with the differences of density ratios, as in Theorem 4.15:

Theorem 5.12 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be AD-m-regular. Then μ is uniformly m-rectifiable if and only if for all $x \in \operatorname{spt}\mu, R > 0$,*

$$\int_0^R \int_{B(x,R)} \left| \frac{\mu(B(y,r))}{r^m} - \frac{\mu(B(y,2r))}{(2r)^m} \right|^2 d\mu y\, r^{-1} dr \lesssim R^m.$$

Azzam and Hyde [37] proved a sufficient condition for uniform rectifiability in terms of density ratios involving Hausdorff content.

5.4 Projections

There is no satisfactory characterization of uniform rectifiability in terms of projections. Tao [406] proved a quantitative multiscale version of Besicovitch's projection Theorem 3.13 quantizing Besicovitch's proof. However, this does not seem to relate to David–Semmes uniform rectifiability. But we have the following theorem due to Orponen [374]:

Theorem 5.13 *If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E has big pieces of Lipschitz graphs if and only if there is $\theta > 0$ such that for every $x \in E$ and $0 < r < d(E)$ there is $V \in G(n,m)$ for which*

$$\mathcal{H}^m(P_W(E \cap B(x,r)) \geq \theta r^m \text{ for } W \in B(V,\theta). \tag{5.6}$$

It is easy to see that big pieces of Lipschitz graphs imply (5.6): if G is a Lipschitz graph over V, then it is also a Lipschitz graph over W when W is sufficiently close to V. David and Semmes showed earlier that one projection satisfying (5.6) is enough if E in addition satisfies the weak geometric lemma, recall Definition 5.6. Obviously one projection alone does not imply even rectifiability – think about the four corners Cantor set of Example 3.6. Orponen showed that (5.6) implies the weak geometric lemma. His complicated argument involves, among many other things, ingredients from the classical Besicovitch–Federer argument for the proof of Theorem 4.17. The example of Hrycak mentioned in Section 5.2 shows that uniformly rectifiable sets need not satisfy (5.6).

For a related result of Martikainen and Orponen, see [311]. Dabrowski and Villa [129] proved an analyst's travelling salesman theorem for sets satisfying (5.6).

Chang, Dabrowski, Orponen and Villa [89] proved a quantitative result in the plane, saying roughly that if the average length of the projections of E is nearly maximal, as compared to the diameter of E, then a large part of E can be covered with a Lipschitz graph.

5.5 Basic Tools

Many of the proofs use generalized dyadic cubes constructed by David in [135], and another construction was given by Christ in [111]. The standard dyadic cubes usually are not good enough, and they are replaced by a family Δ of Borel subsets of an AD-m-regular set E. In case E is unbounded, which is not essential, Δ splits into subfamilies Δ_j, $j \in \mathbb{Z}$. Each Δ_j is a disjoint partition of E, $\mathcal{H}^m(Q) \sim 2^{jm}$ for $Q \in \Delta_j$, and if $Q \in \Delta_j, Q' \in \Delta_k$, $j \leq k$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$. This is very much as for the standard dyadic cubes, but there is a fourth more special 'small boundary' property: there is $\alpha > 0$ such that for $Q \in \Delta_j$ and for all $0 < \tau < 1$,

$$\mathcal{H}^m(\{x \in Q\colon d(x, E \setminus Q) \leq \tau 2^j\}) + \mathcal{H}^m(\{x \in E \setminus Q\colon d(x, Q) \leq \tau 2^j\}) \lesssim \tau^{\alpha} 2^{jm}.$$

This is more rarely used, but it is useful in particular in connection with singular integrals.

Often the proofs use stopping time arguments in the spirit we discussed in Section 4.6. This leads to the general concept of corona decompositions.

In the *corona decomposition*, the family Δ as above is decomposed into a good subfamily $\mathcal{G}$ and a bad subfamily $\mathcal{B}$. Further, $\mathcal{G}$ is decomposed into stopping time families $\mathcal{S}$. Each of them has a unique maximal top cube $Q(\mathcal{S})$ which contains all other cubes of $\mathcal{S}$. In addition, if $Q \in \mathcal{S}$ and $Q \subset Q' \subset Q(\mathcal{S})$, then $Q' \in \mathcal{S}$, and either all children of Q belong to $\mathcal{S}$ or none of them do. Then we say that E admits a corona decomposition if for all positive numbers η and θ such corona decomposition can be found with the following two properties.

The bad cubes and the maximal top cubes satisfy the Carleson packing condition for every $Q \in \Delta$:

$$\sum_{Q' \subset Q, Q' \in \mathcal{B}} \mathcal{H}^m(Q') + \sum_{\mathcal{S}\colon\, Q(\mathcal{S}) \subset Q} \mathcal{H}^m(Q(\mathcal{S})) \lesssim \mathcal{H}^m(Q).$$

For every stopping time family $\mathcal{S}$, there exists a Lipschitz graph $\Gamma(\mathcal{S})$ such that $\mathrm{Lip}(\Gamma(\mathcal{S})) \leq \eta$ and $d(x, \Gamma(\mathcal{S})) \leq \theta d(Q)$ whenever $x \in 2Q$ and $Q \in \mathcal{S}$.

David and Semmes proved in [146]

Theorem 5.14 *If $E \subset \mathbb{R}^n$ is closed and AD-m-regular, then E is uniformly rectifiable if and only if it admits a corona decomposition.*

This often is a useful link for going from one characterizing property to another. For instance, in [146] David and Semmes proved that the geometric lemma implies corona decomposition which implies big pieces of Lipschitz images.

5.6 Parabolic Rectifiability

Hofmann, Lewis and Nyström introduced parabolic uniform rectifiability in [236] and [237]. Later there were many related papers, but now I only comment on the recent work of Bortz, Hoffman, Hofmann, Luna Garcia and Nyström in [75] and [76], see the discussions and references there for other developments. Although there are deep results, this theory is not yet very fully developed and mainly the codimension one case has been considered.

In the following balls, Hausdorff measures and AD-regularity are defined with the parabolic metric d in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$:

$$d((x,s),(y,t)) = |x-y| + \sqrt{|s-t|},\ (x,s),(y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

Notice that the Hausdorff dimension of $\mathbb{R}^{n+1}$ is then $n+2$ and the codimension one Hausdorff measure is $\mathcal{H}^{n+1}$. Define the parabolic β by

$$p\beta_E(x,r) = \inf_V \left(r^{-n-1} \int_{E\cap B(x,r)} \left(\frac{d(y,V)}{r} \right)^2 d\mathcal{H}^{n+1}y \right)^{1/2}, \tag{5.7}$$

where the infimum is taken over all vertical hyperplanes V, that is, the planes $V = W \times \mathbb{R}$, where W is an affine $(n-1)$-plane in $\mathbb{R}^n$. Using only vertical hyperplanes is natural since they and the 'horizontal' plane $H = \{(x,t) : t = 0\}$ are the only linear hyperplanes invariant under the dilations $\delta_r(x,t) = (rx, r^2t)$. Notice that $d(\delta_r(x,t), \delta_r(x',t')) = rd((x,t),(x',t'))$.

Definition 5.15 A closed set $E \subset \mathbb{R}^{n+1}$ is *parabolic uniformly rectifiable* if it is AD-$(n+1)$-regular and

$$\int_0^R \int_{E\cap B(x,R)} p\beta_E(x,r)^2\, d\mathcal{H}^{n+1}x\, r^{-1}dr \lesssim R^{n+1} \text{ for all } x \in E, R > 0. \tag{5.8}$$

For David–Semmes uniformly rectifiable sets, Lipschitz graphs are basic examples. But not here. There are examples of parabolic Lipschitz graphs that are not uniformly rectifiable. Basic parabolic uniformly rectifiable sets are *regular Lipschitz graphs*, that is, graphs over vertical planes of regular Lipschitz functions g. Here $g\colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ is a regular Lipschitz function if it is Lipschitz in the parabolic metric d and the half derivative with respect to t belongs to BMO, with BMO defined using parabolic balls. The half derivative can be defined by

$$D^t_{1/2} g(x,t) = \int_{\mathbb{R}} \frac{g(x,s) - g(x,t)}{|s-t|^{3/2}}\, ds.$$

See [178] for several characterizations of what it means that $D^t_{1/2} g \in BMO$.

In [75] the authors introduced a very general setting for corona decompositions in metric spaces. They proved for any family $\mathcal{E}$ of AD-m-regular sets with bounded constants that if an AD-m-regular set E admits a corona decomposition with $\mathcal{E}$ (that is, $\mathcal{E}$ replaces Lipschitz graphs in the definition of Section 5.5), then E has big pieces of big pieces of $\mathcal{E}$. Combining this with [76], they have

Theorem 5.16 *If $E \subset \mathbb{R}^n \times \mathbb{R}$ is closed and AD-$(n+1)$-regular, then the following are equivalent:*

(1) *E is parabolic uniformly rectifiable.*
(2) *E admits a corona decomposition with regular Lipschitz graphs.*
(3) *E has big pieces of big pieces of regular Lipschitz graphs.*

The study of parabolic rectifiability is motivated by a desire to develop the boundary behaviour theory for the heat equation $\Delta_x u(x,t) = \partial_t u(x,t)$ in the same vein as has been done for the Laplace equation, see Chapter 11. In particular, the preference for regular Lipschitz graphs over ordinary Lipschitz graphs in the parabolic metric comes from heat equation examples.

What should parabolic (non-uniform) rectifiability mean? In light of the above, a natural definition would seem to be the one based on covering with regular Lipschitz graphs. This was suggested in [317], but it has not yet been developed. It would probably be the right notion from an analysis (heat equation) point of view. But we could also ask for a notion that would correspond to the almost everywhere existence of approximate tangent planes and to tangent measures, and this turns out to be different. This question has been studied in [325]. Recall the remarks after Definitions 4.4 and 4.6 according to which approximate tangent planes and tangent measures can be defined in the parabolic setting as there.

Let now m be any integer with $1 \leq m \leq n+1$. The linear subspaces of $\mathbb{R}^{n+1}$ of the parabolic Hausdorff dimension m that are invariant under the parabolic dilations δ_r are again the vertical hyperplanes when $m = n+1$, the lines through 0 in $H = \{(x,t)\colon t = 0\}$ when $m = 1$ and the linear m-dimensional subspaces of

H and linear vertical $(m-1)$-dimensional subspaces of $\mathbb{R}^n \times \mathbb{R}$ when $2 \le m \le n$. Let us denote by $P(n,m)$ the family of such linear subspaces. The following theorem was proved in [325]. All the concepts are parabolic.

Theorem 5.17 *Let $E \subset \mathbb{R}^n \times \mathbb{R}$ be $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$. Then the following are equivalent:*

(1) *For every $\varepsilon > 0$, there are Lipschitz graphs G_i over $V_i \in P(n,m)$ with Lipschitz constants less than ε such that $\mathcal{H}^m(E \setminus \cup_i G_i) = 0$.*
(2) *There are C^1 graphs G_i over $V_i \in P(n,m)$ such that $\mathcal{H}^m(E \setminus \cup_i G_i) = 0$.*
(3) *E has an approximate tangent m-plane $V \in P(n,m)$ at $\mathcal{H}^m$ almost all of its points.*
(4) *For $\mathcal{H}^m$ almost all $a \in E$ there is an m-flat measure $\lambda_a = \mathcal{H}^m \llcorner V_a, V_a \in P(n,m)$ such that $\mathrm{Tan}(\mathcal{H}^m \llcorner E, a) = \{c\lambda_a : 0 < c < \infty\}$.*
(5) *For $\mathcal{H}^m$ almost all $a \in E$ $\mathcal{H}^m \llcorner E$ has a unique tangent measure at a.*

The proof has many ingredients similar to the Euclidean case. That (5) implies (4) follows from Theorem 4.7. Here C^1 is defined in a parabolic sense. One candidate for parabolic rectifiability is given by these conditions.

For $2 \le m \le n$, the planes in $P(n,m)$ have linear dimension m or $m-1$, but they can be treated simultaneously.

We need Lipschitz graphs with arbitrarily small Lipschitz constants because Lipschitz graphs themselves need not satisfy the other conditions, as shown in [325]. There one also finds an example which satisfies the conditions of Theorem 5.17 but intersects every regular Lipschitz graph in measure zero. So these two possible classes of rectifiable sets are different.