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A NON-PERIODIC INDEFINITE VARIATIONAL PROBLEM IN $\mathbb{R}^N$ WITH CRITICAL EXPONENT

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Abstract We consider the non-linear Schrödinger equation

$$-\Delta u + V(x)u = \mu f(u) + |u|^{2^*-2}u, \qquad (P_\mu)$$

in $\mathbb{R}^N$, $N \geq 3$, where V changes sign and $f(s)/s$, $s \neq 0$, is bounded, with V non-periodic in x. The existence of a solution is established employing spectral theory, a general linking theorem due to [12] and interaction between translated solutions of the problem at infinity with some qualitative properties of them.

Keywords: critical exponent; abstract linking theorem; indefinite problems; spectral theory

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1. Introduction

We investigate the existence of a non-trivial solution of the elliptic problem

$$-\Delta u + V(x)u = g(u), \quad u \in H^1(\mathbb{R}^N),\ u \neq 0 \qquad (P_\mu)$$

with $g(s) = \mu f(s) + |s|^{2^*-2}s$, $N \geq 3$, $\mu > 0$, under no periodicity condition on V that changes sign, and $s \mapsto f(s)/s$ is bounded.

This problem has been extensively studied considering several potentials V and non-linearities g. For the case where V and $g(s) = g(x, s)$ are periodic functions in the variable

x and g have a subcritical growth, we refer the reader to [7–9, 13, 16, 17] and references therein.

The case when g possesses critical growth and potential V changes sign, we refer the works of [2, 4, 14, 18], which are more closely related to this article. In all of them, the condition of periodicity is crucial in order to overcome the lack of compactness in $\mathbb{R}^N$.

On the other side, Maia e Soares in [12] considered the case when V is not periodic (namely, $V(x) \to V_\infty > 0$ as $|x| \to \infty$) and g is asymptotically linear at infinity, without any monotonicity condition on $s \mapsto g(s)/s$. The framework that the authors deal with the problem makes possible to apply the celebrated result due to Berestycki and Lions in [3] and ensures that the limit problem associated to (P_μ)

$$-\Delta u + V_\infty u = g(u), \quad u \in H^1(\mathbb{R}^N),\ u \neq 0 \qquad (P_{\mu,\ \infty})$$

has a non-trivial ground state solution $u_0 \in C^2(\mathbb{R}^N, \mathbb{R})$, which is positive, radially symmetric and decays exponentially, namely,

$$u_0(x) \leq \mathrm{e}^{-\sqrt{V_\infty}|x|} \quad \text{for all } x \in \mathbb{R}^N. \qquad (1.1)$$

As quoted by the authors in page 21, after some interactions between the energy functionals associated to (P_μ) and $(P_{\mu,\infty})$, property (1.1) of the solution u_0 was strongly needed in order to prove that the weak solution of (P_μ) is non-trivial (see Section 5 in [12], p. 19).

Our work complements all results cited above. Differently from them, the potential V is non-periodic and changes sign and the non-linearity $g(s) = \mu f(s) + |s|^{2^*-2}s$ possesses a critical growth, with $s \mapsto f(s)/s$ bounded. This scenario brings several difficulties.

The central idea of our approach is to apply the version of the Linking Theorem due to Maia and Soares in [12, Theorem 1.2] and, for that purpose, takes a positive ground state solution of $(P_{\mu,\infty})$ that has an exponential decay and makes some suitable interactions with problem (P_μ). Since our problem is critical, we cannot apply [3] directly as the authors do in [12]. Therefore, how to guarantee that $(P_{\mu,\infty})$ has some non-trivial ground state solution? And, then, is it possible to show that this solution has exponential decay as (1.1)?

In this paper, to prove that problem (P_μ) has a non-trivial solution, we first answer these two question, considering the elliptic problem

$$-\Delta u + V(x)u = \mu f(u) + |u|^{2^*-2}u, \quad u \in H^1(\mathbb{R}^N),\ u \neq 0 \qquad (P_\mu)$$

with $N \geq 3$, $\mu > 0$ and $u \in E := H^1(\mathbb{R}^N)$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a potential satisfying the conditions:

- (V_1) $V \in L^\infty(\mathbb{R}^N)$;
- (V_2) $\lim_{|x| \to +\infty} V(x) = V_\infty > 0$;
- (V_3) $0 \notin \sigma(L)$ and $\inf \sigma(L) < 0$, where $\sigma(L)$ is the spectrum of the operator $L = -\Delta + V$.
- (V_4) $V(x) \leq V_\infty - Ce^{-\gamma_1 |x|^{\gamma_2}}$, with $\gamma_1 > 0$ and $\gamma_2 \in (0, 1)$.

The conditions that we consider on the non-linearity $f \in C(\mathbb{R}, \mathbb{R})$ are the following:

(f_1) $\lim_{s\to 0} \dfrac{f(s)}{s} = 0$ and $f(s) = 0$, for all $s \in (-\infty, 0]$;

(f_2) We have $f(s) = 0$ for $s \leq 0$ and $\dfrac{|f(s)|}{|s|} < m$ for all $s \neq 0$;

(f_3) If $F(s) := \int_0^s f(t)\,\mathrm{d}t$ and $Q(s) := \dfrac{1}{2} f(s)s - F(s)$, then for all $s \in \mathbb{R} \setminus \{0\}$,

$$F(s) \geq 0, \ Q(s) > 0 \quad \text{and} \quad \lim_{s\to\infty} Q(s) = +\infty.$$

An important consequence of assumptions (f_1) and (f_2) is that, given $\varepsilon > 0$ and $2 \leq p \leq 2^*$, there exists $C_\varepsilon > 0$ such that

$$|F(s)| \leq \varepsilon |s|^2 + C_\varepsilon |s|^p \quad \text{and} \quad |f(s)| \leq \varepsilon |s| + C_\varepsilon |s|^{p-1} \tag{1.2}$$

for all $s \in \mathbb{R}$.

One of our main results is the next theorem.

Theorem 1.1. *Suppose that assumptions* $(V_1)-(V_4)$ *and* $(f_1)-(f_3)$ *hold. Then, there exists* $\mu^* > 0$ *such that, if* $\mu \geq \mu^*$*, problem* (P_μ) *has a nontrivial and nonnegative solution* u_μ *in* $H^1(\mathbb{R}^N)$*. Moreover, the limit problem*

$$\begin{cases} -\Delta u + V_\infty u = \mu f(u) + |u|^{2^*-2}u & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_{\mu,\,\infty}$}$$

has a ground state solution u_0 *such that, if* $\nu \in \left(0, \sqrt{V_\infty}\right)$*, then there exists* $C = C(m, \nu) > 0$ *satisfying*

$$|u_0(x)| \leq C|u_0|_\infty \mathrm{e}^{-\nu|x|}, \quad \forall x \in \mathbb{R}^N.$$

We stress here that, in order to prove Theorem 1.1, we do not need any monotonicity hypothesis on function $s \mapsto f(s)/s$, as one can find in the literature about similar problems. One example of non-linearity that satisfies our assumptions but $f(s)/s$, $s \neq 0$, is not increasing is $f(s) = \dfrac{s^7 - 1,5s^5 + 2s^3}{1 + s^6}$ for $s \in \mathbb{R}$. Then f satisfies our hypotheses; however, $f(s)/s$, $s \neq 0$, is not increasing.

This paper is organized into seven sections as follows. In § 2, we focus on providing the appropriate variational setting for the problem. In § 3, we obtain the geometry of a version of the Linking Theorem, and as a result, we obtain an appropriate Cerami sequence. This sequence is proven to be bounded in § 4. In § 5, we present the first part of the proof of Theorem 1.1, and in § 6, Appendix A, we present the second part of the proof and we also discuss the limit problem $(P_{\mu,\infty})$ and its ground state solution in detail. Finally, in Appendix B, § 7, we present a technical result on significant convergences.

2. Variational setting

Let $E := H^1(\mathbb{R}^N)$. The energy functional $I : E \to \mathbb{R}$ associated with equation (P) is given by

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x,$$

with $u \in E$. It is well known from conditions (V_2) and (V_3) that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad u \in L^2(\mathbb{R}^N) \tag{2.1}$$

has a sequence of eigenvalues $\lambda_1 < \lambda_2 \le \cdots \le \lambda_k < 0$ (see [6], Theorem 30, p. 150). We denote by ϕ_i the eigenfunction corresponding to λ_i, $i \in \{1, 2, \dots, k\}$ in $H^1(\mathbb{R}^N)$. Setting

$$E^- := \mathrm{span}\{\phi_i,\ i = 1, 2, \dots, k\} \quad \text{and} \quad E^+ := \left(E^-\right)^\perp,$$

we see that $E = E^+ \oplus E^-$. According to Stuart in [15], Theorem 3.15, the essential spectrum of $-\Delta + V$ is the interval $[V_\infty, +\infty)$, and this implies that $\dim E^- < \infty$. Having made these considerations, every function $u \in E$ may be written as $u = u^+ + u^-$ uniquely, where $u^+ \in E^+$ and $u^- \in E^-$. Hence, by using the arguments in Lemma 1.2 of [5], we may introduce the new inner product $\langle \cdot, \cdot \rangle$ in E, namely,

$$\langle u, v\rangle = \begin{cases} \displaystyle\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\,\mathrm{d}x & \text{if } u, v \in E^+, \\ \displaystyle -\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\,\mathrm{d}x & \text{if } u, v \in E^-, \\ 0 & \text{if } u \in E^+ \text{ and } v \in E^-. \end{cases}$$

such that the corresponding norm $\|\cdot\|$ is equivalent to $\|\cdot\|_E$, the usual norm in $E = H^1(\mathbb{R}^N)$. In addition, the functional I may be written as

$$I_\mu(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \mu \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x$$

for every function $u = u^+ + u^- \in E$. We call the attention to the fact that, since $\lambda_i \neq 0$ for all $i \in \{1, 2, \dots, k\}$, it follows from (2.1) and the definition of ϕ_i that

$$\int_{\mathbb{R}^N} u^+(x)v^-(x)\,\mathrm{d}x = 0 \tag{2.2}$$

for all functions $u^+ \in E^+$ and $v^- \in E^-$.

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To deal with compactness issues, we will prove several auxiliary results concerning the limit problem associated to (P_μ), namely,

$$\begin{cases} -\Delta u + V_\infty u = \mu f(u) + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases} \qquad (P_{\mu,\,\infty})$$

Hereafter, let us denote by $J_\mu : H^1(\mathbb{R}^N) \to \mathbb{R}$ the associated functional given by

$$J_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty u^2\right) \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x.$$

Also, let us consider the level $d_\mu := \inf_{u\in\mathcal{N}_\mu} J_\mu(u)$, where

$$\mathcal{N}_\mu := \left\{u \in H^1(\mathbb{R}^N)/\{0\} : \; J'_\mu(u) = 0\right\}.$$

3. Geometry of the Linking Theorem

In this section, we are going to show that functional I_μ satisfies the geometry of the following version of the Linking Theorem.

Theorem 3.1. *[12, Theorem 1.2] Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose there exist real constants $R > \rho > 0$, $\alpha > \beta$ and there exists an $e \in \partial B_1 \cap X$ such that $I \in C^1(E, \mathbb{R})$ satisfies*

- *(I_1)* $\quad I|_{\partial B_\rho \cap X} \geq \alpha$;
- *(I_2) Setting $M := \left(\bar{B}_R \cap V\right) \oplus \{re : 0 \leq r \leq R\}$, there exists an $h_0 \in C(M, E)$ such that*
  - *(i)* $\sup_{w\in M} I\left(h_0(w)\right) < +\infty$,
  - *(ii)* $\sup_{w\in \partial M} I\left(h_0(w)\right) = \beta$,
  - *(iii)* $h_0(\partial M) \cap (\partial B_\rho \cap X) = \emptyset$,
  - *(iv) There exists a unique $u \in h_0(M) \cap (\partial B_\rho \cap X)$ such that*

$$\deg\left(h_0, \mathrm{int}(M), u\right) \neq 0.$$

Then I possess a Cerami sequence on a level $c \geq \alpha$, which can be characterized as

$$c := \inf_{h\in\Gamma} \max_{w\in M} I(h(w)),$$

where $\Gamma := \{h \in C(M, E) : h|_{\partial M} = h_0\}$.

To prove that functional I_μ has the geometry of Theorem 3.1, let $u_0 \in H^1(\mathbb{R}^N)$ be a non-trivial ground state solution of problem $(P_{\mu,\infty})$ given by Proposition 6.7 in Appendix A. By hypothesis (f_2), u_0 is nonnegative.

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Given $w \in E$ and $y \in \mathbb{R}^N$, to simplify the notation, we write $w^+(\cdot - y)$ (or $w^-(\cdot - y)$) referring to the projection in E^+ (respectively, in E^-) of the translated function $w(\cdot - y)$.

Proceeding as Claim 4.5 in [11], we may prove that $u_0^+(\cdot - y)$ is a non-trivial function just choosing $y \in \mathbb{R}^N$ with norm sufficiently large. Now, let us consider $R > 0$, any non-trivial function $e \in E^+$ with $\|e\| = 1$ and the sets

$$M = \{w = te + v^-;\ \|w\| \le R,\ t \ge 0,\ v^- \in E^-\}$$

and

$$M_0 = \partial M = \{w = te + v^-;\ v^- \in E^-, \|w\| = R,\ t \ge 0 \text{ or } \|w\| \le R,\ t = 0\}.$$

Defining

$$h_0(w) = h_0\left(tRe + v^-\right) := u_0^+\left(\frac{\cdot - y}{tL}\right) + |v^-|,\ \text{for } t \in (0,1]$$

and $h_0(v^-) = |v^-|$, where $u_0 \in E$ is the non-trivial solution to the limit problem (P_∞) found before and $L > 0$ to be chosen, we have the following lemmas. The first one proves item (I_1) from Theorem 3.1.

Lemma 3.2. *There exists $\rho > 0$ such that*

$$\inf_{w \in \partial B_\rho \cap E^+} I_\mu(w) > 0.$$

Proof. For $\rho > 0$, let $w^+ \in E^+$ with $\|w^+\| = \rho$. Then, from (1.2), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$I_\mu(w^+) \ge \frac{1}{2}\rho^2 - \mu\varepsilon C\rho^2 - \mu C_\varepsilon C\rho^{2^*} - \frac{1}{2^*}\rho^{2^*},$$

where we used Sobolev embeddings and the equivalence of the norms. It follows that, for $\varepsilon < \dfrac{1}{4\mu C}$,

$$I_\mu(w^+) \ge \frac{1}{4}\rho^2 - \left(\frac{1}{2^*} + \mu C_\varepsilon C\right)\rho^{2^*}.$$

Now, choosing $0 < \rho < \left(\dfrac{2^*}{4(1 + 2^*\mu C_\varepsilon C)}\right)^{\frac{1}{2^*-2}}$, the result follows. □

The next result shows that item (ii) from Theorem 3.1 holds, choosing $\beta = 0$. Before stating this result, we remember an important result from spectral theory that characterizes the functions that belong to E^- as follows: $u \in E^-$ if and only if

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$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\, \mathrm{d}x < 0.$$

Thus, if $u \in E^-$, then $|u| \in E^-$.

In the sequel, we will denote $\|w\|^2_{V\infty} = \displaystyle\int_{\mathbb{R}^N} (|\nabla w|^2 + V_\infty w^2)\, \mathrm{d}x$ for any $w \in H^1(\mathbb{R}^N)$.

Lemma 3.3. *There exist* $R > 0$ *sufficiently large, which does not depend on* μ*, such that, for all* $\mu > 0$*,*

$$\sup_{w\in M_0} I_\mu(h_0(w)) = 0.$$

Proof. Denoting by $\Upsilon^+(t)(x) = u_0^+\left(\frac{x-y}{tL}\right)$, let us separate this proof in three possible cases. If $w = tRe + v^-$ with $t \in [0,1]$ and $\|v^-\| = R$, we have

$$\begin{aligned} I_\mu\left(h_0(w)\right) &\le \frac{1}{2}\left[\left\|\Upsilon^+(t)\right\|^2 - R^2\right] \\ &\le \frac{1}{2}\left[\|\Upsilon(t)\|^2_{V\infty} - R^2\right] \\ &\le \frac{1}{2}\left[\max_{t\in[0,1]} \|\Upsilon(t)\|^2_{V\infty} - R^2\right] < 0 \end{aligned}$$

for $R > 0$ large enough. Second, if $w = v^- \in \bar{B}_R(0) \cap E^-$, one has

$$I_\mu(h_0(w)) = I_\mu\left(v^-\right) \le 0.$$

To finish the proof, if $w = Re + v^-$, with $\|v^-\| \le R$, then

$$\begin{aligned} I_\mu\left(h_0(w)\right) &= \frac{1}{2}\left[\left\|\Upsilon^+(1)\right\|^2 - \|v^-\|^2\right] - \mu\int_{\mathbb{R}^N} F\left(\Upsilon^+(1) + |v^-|\right)\, \mathrm{d}x \\ &\quad - \frac{1}{2^*}\int_{\mathbb{R}^N} \left|\Upsilon^+(1) + |v^-|\right|^{2^*}\, \mathrm{d}x \\ &\le J_\mu\left(\Upsilon^+(1)\right) + \mu\int_{\mathbb{R}^N} \left[F\left(\Upsilon^+(1)\right) - F\left(\Upsilon^+(1) + |v|\right)\right] \mathrm{d}x \\ &\quad + \frac{1}{2^*}\int_{\mathbb{R}^N} \left[\left|\Upsilon^+(1)\right|^{2^*} - \left|\Upsilon^+(1) + |v^-|\right|^{2^*}\right]\, \mathrm{d}x \\ &\le 0, \end{aligned}$$

where we used the facts that, for $L > 0$ large enough, it holds that $J_\mu(\Upsilon(1)) < 0$; we also have for $|y|$ sufficiently large that $\|\Upsilon^-(1)\|$ is small enough such that $J_\mu(\Upsilon(1)) < 0$ implies $J_\mu\left(\Upsilon^+(1)\right) \le 0$ and

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$$\int_{\mathbb{R}^N}\left[\left|\Upsilon^+(1)\right|^{2^*}-\left|\Upsilon^+(1)+|v^-|\right|^{2^*}\right]\mathrm{d}x\approx\int_{\mathbb{R}^N}\left[\left|\Upsilon(1)\right|^{2^*}-\left|\Upsilon(1)+|v^-|\right|^{2^*}\right]\mathrm{d}x.$$

Moreover, we applied the non-decreasing condition for function F since $f(s)\geq 0$ for all s and for function $s\mapsto s^{2^*}$, for $s>0$. □

Now, let us demonstrate that item (iii) from Theorem 3.1 also is valid.

Lemma 3.4. *It holds that* $h_0(\partial M)\cap(\partial B_\rho\cap E^+)=\emptyset$.

Proof. Observe that

$$h_0(\partial M)=\left(\partial B_R\cap E^-\right)\oplus\left\{\Upsilon^+(t):0\leq t\leq 1\right\}\cup\left(\bar{B}_R\cap E^-\right)\cup\left(\bar{B}_R\cap E^-\right)\oplus\left\{\Upsilon^+(1)\right\}$$

and that

$$\left(\partial B_R\cap E^-\right)\oplus\{\Upsilon^+(t):0\leq t\leq 1\}\cap E^+=\emptyset.$$

In addition, to guarantee that $\left(\bar{B}_R\cap E^-\right)\oplus\{\Upsilon^+(1)\}\cap E^+\cap\partial B_\rho=\emptyset$, it is enough to choose a sufficiently large $L,|y|>0$ such that $I_\mu(\Upsilon^+(1))\approx I_\mu(\Upsilon(1))\leq J_\mu(\Upsilon(1))<-1$. Therefore, by Lemma 3.2, necessarily

$$\left\|\Upsilon^+(1)\right\|^2>\rho^2, \tag{3.1}$$

where $\Upsilon^+(1)=u_0^+\left(\frac{x-y}{L}\right)$. Then, we conclude that

$$h_0(\partial M)\cap\left(\partial B_\rho\cap E^+\right)=h_0(\partial M)\cap\partial B_\rho\cap E^+=\left(\bar{B}_R\cap E^-\right)\cap\partial B_\rho\cap E^+=\emptyset.$$

The lemma follows. □

Finally, let us prove that item (iv) from Theorem 3.1 is true.

Lemma 3.5. *There exists a unique* $u\in h_0(M)\cap(\partial B_\rho\cap E^+)$ *such that*

$$\deg\left(h_0,\operatorname{int}(M),u\right)\neq 0.$$

Proof. Consider the function $\psi:[0,1]\to\mathbb{R}$, given by $\psi(t)=\|\Upsilon^+(t)\|$, is strictly increasing and hence injective. Moreover, ψ is continuous, $\psi(0)=0$, and from (3.1), we have $\psi(1)>\rho$. Thus, from the Intermediate Value Theorem, there exists some (unique, since ψ is injective) $t_0\in(0,1)$ such that $\psi\left(t_0\right)=\rho$. Hence,

$$h_0(M)\cap\left(\partial B_\rho\cap E^+\right)=\left\{\Upsilon^+(t):t\in[0,1]\right\}\cap\partial B_\rho=\left\{\Upsilon^+\left(t_0\right)\right\},$$

and there exists an unique $w=\Upsilon^+\left(t_0\right)\in h_0(M)\cap(\partial B_\rho\cap E^+)$. Since $Rte\mapsto h_0(Rte)=\Upsilon^+(t)$ is injective, there exists a unique $u_0=Rt_0e\in\operatorname{int}(M)$ such that $h_0\left(u_0\right)=\Upsilon^+\left(t_0\right)$. Therefore, $\deg\left(h_0,\operatorname{int}(M),w\right)\neq 0$, proving (iv). □

4. Boundedness of Cerami sequences

We say that a sequence $(u_n) \subset E$ is a Cerami sequence at level c for functional I_μ if

$$I_\mu(u_n) \to c \text{ in } \mathbb{R} \quad \text{and} \quad \|I'_\mu(u_n)\|_{E^*}(1+\|u_n\|) \to 0 \ \in \ \mathbb{R},$$

as $n \to +\infty$. Before we state the next result, we note that, if (v_n) is a bounded sequence in E, then (v_n) satisfies either

(i) vanishing: for all $r>0$, $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |v_n|^2 \, \mathrm{d}x = 0$

or

(ii) non-vanishing: there exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,r)} |v_n|^2 \, \mathrm{d}x > \eta.$$

Lemma 4.1. *Let $(u_n) \subset E$ be a Cerami sequence at level $c>0$. Then, (u_n) has a bounded subsequence.*

Proof. Suppose by contradiction that $1 \le \|u_n\| \to \infty$ as $n \to +\infty$. Consider

$$v_n = \frac{u_n}{\|u_n\|}$$

and note that $\|v_n\| = 1$. The sequence (v_n) is bounded; however, we will show that neither (i) or (ii) is true. First, notice that from hypothesis (f_3),

$$\begin{aligned} c + o_n(1) &= I_\mu(u_n) - \frac{1}{2} I'_\mu(u_n)u_n \\ &= \mu \int_{\mathbb{R}^N} \left(\frac{f(u_n)u_n}{2} - F(u_n) \right) \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_n|^{2^*} \, \mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_n|^{2^*} \, \mathrm{d}x, \end{aligned}$$

which shows the boundedness of the sequence $(|u_n|_{2^*})$. It will be used in the calculations that follows.

First, suppose that hypothesis (i) is satisfied for sequence (v_n). Since the sequence (u_n) is a Cerami sequence, we have

$$\begin{aligned} o_n(1) = I'_\mu(u_n)\frac{u_n^+}{\|u_n\|^2} &= \frac{1}{\|u_n\|} I'_\mu(u_n)v_n^+ \\ &= \|v_n^+\|^2 - \mu \int_{\mathbb{R}^N} \left(\frac{f(u_n)}{u_n} v_n v_n^+ \right) \mathrm{d}x \end{aligned} \tag{4.1}$$

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and

$$o_n(1) = I'_\mu(u_n)\frac{u_n^-}{\|u_n\|^2} = \frac{1}{\|u_n\|}I'_\mu(u_n)v_n^-$$
$$= -\|v_n^-\|^2 - \mu\int_{\mathbb{R}^N}\left(\frac{f(u_n)}{u_n}v_n v_n^-\right)\,\mathrm{d}x. \tag{4.2}$$

Subtracting equation (4.2) from (4.1), we have

$$o_n(1) = \|v_n^+\|^2 + \|v_n^-\|^2 - \mu\int_{\mathbb{R}^N}\left(\frac{f(u_n)}{u_n}v_n(v_n^+ - v_n^-)\right)\,\mathrm{d}x$$

$$= \|v_n\|^2 - \mu\int_{\mathbb{R}^N}\left(\frac{f(u_n)}{u_n}v_n(v_n^+ - v_n^-)\right)\,\mathrm{d}x$$

$$= 1 - \mu\int_{\mathbb{R}^N}\left(\frac{f(u_n)}{u_n}v_n(v_n^+ - v_n^-)\right)\,\mathrm{d}x.$$

Thus,

$$\mu\int_{\Omega_n^+}\left(\frac{f(u_n)}{u_n}v_n(v_n^+ - v_n^-)\right)\,\mathrm{d}x = \mu\int_{\mathbb{R}^N}\left(\frac{f(u_n)}{u_n}v_n(v_n^+ - v_n^-)\right)\,\mathrm{d}x \to 1, \tag{4.3}$$

provided $f(s) = 0$ if $s \le 0$, where we define $\Omega_n^+ = \{x \in \mathbb{R}^N;\ u_n(x) > 0\}$. By equivalence of the norms, there exists a constant $\nu_0 > 0$ such that

$$\|w\|^2 \ge \nu_0\|w\|^2_{L^2(\mathbb{R}^N)} \tag{4.4}$$

for any $w \in E$. Given $0 < \varepsilon < \frac{1}{2}\nu_0$, by hypothesis (f_1), there exists $\delta > 0$ such that

$$\mu\frac{|f(s)|}{|s|} \le \varepsilon \quad \text{for} \quad 0 \ne |s| < \delta.$$

For each $n \in \mathbb{N}$, consider the set

$$\tilde{\Omega}_n = \{x \in \mathbb{R}^N;\ 0 < u_n(x) < \delta\}.$$

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Thus, from (4.4) and by Hölder's inequality,

$$\begin{aligned}\mu\int_{\tilde\Omega_n}\left(\frac{f(u_n)}{u_n}v_n(v_n^+-v_n^-)\right)\,\mathrm{d}x &\le \varepsilon\int_{\tilde\Omega_n}|v_n||v_n^+-v_n^-|\,\mathrm{d}x\\ &\le \varepsilon\|v_n\|_{L^2(\mathbb{R}^N)}\left(\|v_n^+\|_{L^2(\mathbb{R}^N)}+\|v_n^-\|_{L^2(\mathbb{R}^N)}\right)\\ &\le 2\varepsilon\|v_n\|^2_{L^2(\mathbb{R}^N)}\\ &\le \frac{2\varepsilon}{\nu_0}\|v_n\|^2=\frac{2\varepsilon}{\nu_0}<1,\end{aligned}$$

From (4.3), we conclude that

$$\liminf_{n\to\infty}\int_{\Omega_n^+\setminus\tilde\Omega_n}\left(\frac{f(u_n)}{u_n}v_n(v_n^+-v_n^-)\right)\,\mathrm{d}x>0. \tag{4.5}$$

Since the function $\dfrac{|f(\cdot)|}{|\cdot|}$ is bounded by (f_2), by Hölder's inequality with exponent $\dfrac{p}{2}>1$, we obtain

$$\int_{\Omega_n^+\setminus\tilde\Omega_n}\left(\frac{f(u_n)}{u_n}v_n(v_n^+-v_n^-)\right)\,\mathrm{d}x\le C|\Omega_n^+\setminus\tilde\Omega_n|^{\frac{p-2}{p}}\|v_n\|^{\frac{2}{p}}_{L^p(\mathbb{R}^N)}. \tag{4.6}$$

Assumption (i) and Lions's Lemma ensure that $\|v_n\|_{L^p(\mathbb{R}^N)}\to 0$. Therefore, up to a subsequence, from (4.5), we obtain

$$|\Omega_n^+\setminus\tilde\Omega_n|\to\infty,\quad\text{as } n\to\infty. \tag{4.7}$$

Now we consider two disjoint subsets of $\Omega_n^+\setminus\tilde\Omega_n$. Hypothesis (f_3) implies that there exists $R>0$ such that, if $s>R$,

$$\frac{1}{2}f(s)s-F(s)>1.$$

Without loss of generality, we assume $0<\delta<R$. For each $n\in\mathbb{N}$, let

$$A_n:=\{x\in\mathbb{R}^N; u_n(x)>R\}$$

and thus, by (4.1),

$$c+o_n(1)\ge\mu\int_{A_n}\left(\frac{1}{2}f(u_n(x))u_n(x)-F(u_n(x))\right)\,\mathrm{d}x>\mu|A_n|,$$

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which implies that the sequence $(|A_n|)$ is bounded. Consider also

$$B_n := \{x \in \mathbb{R}^N; \delta \le u_n(x) \le R\}.$$

Since $B_n = (\Omega_n^+ \setminus \tilde{\Omega}_n) \setminus A_n$, we have

$$|\Omega_n^+ \setminus \tilde{\Omega}_n| = |A_n| + |B_n|.$$

It follows from (4.7) and the boundedness of the sequence $(|A_n|)$ that

$$|B_n| \to \infty. \tag{4.8}$$

Since the interval $[\delta, R]$ is compact and the functions f and F are continuous, we have by hypothesis (f_3) that $\overline{\delta} := \inf_{s\in[\delta,R]} \left(\frac{1}{2}f(s)s - F(s)\right) > 0$. Thus, from (4.8),

$$\mu \int_{\mathbb{R}^N} \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) \mathrm{d}x \ge \mu \int_{B_n} \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) \mathrm{d}x$$

$$\ge \mu\overline{\delta}|B_n| \to \infty.$$

We have a contradiction with the fact that

$$\mu \int_{\mathbb{R}^N} \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) \mathrm{d}x \le I_\mu(u_n) - \frac{1}{2}I'_\mu(u_n)u_n = c + o_n(1).$$

Therefore, (i) does not hold for sequence (v_n). Now, suppose that (ii) holds for sequence (v_n). By equivalence of the norms, there exist constants $C_1, C_2 > 0$ such that

$$\|w\| \le C_1\|w\|_{H^1(\mathbb{R}^N)} \le C_2\|w\|, \text{ for all functions } w \in E. \tag{4.9}$$

Let $(y_n) \subset \mathbb{R}^N$ be the sequence given by hypothesis (ii). Consider $\tilde{v}_n(x) = v_n(x + y_n)$ and $\tilde{u}_n(x) = u_n(x + y_n)$. Note that $(\tilde{v}_n)$ is bounded in E. In fact, from (4.9), it follows that

$$\|\tilde{v}_n\| \le C_1\|\tilde{v}_n\|_{H^1(\mathbb{R}^N)} = C_1\|v_n\|_{H^1(\mathbb{R}^N)} \le C_2\|v_n\| = C_2.$$

Thus, up to a subsequence,

$$\begin{cases} \tilde{v}_n \rightharpoonup \tilde{v} & \text{weakly in } E, \\ \tilde{v}_n \to \tilde{v} & \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N). \end{cases} \tag{4.10}$$

We note that $\tilde{v} \ne 0$, since by (ii) and (4.10),

$$\int_{B(0,r)} \tilde{v}^2 \,\mathrm{d}x = \limsup_{n\to\infty} \int_{B(0,r)} \tilde{v}_n^2 \,\mathrm{d}x = \limsup_{n\to\infty} \int_{B(y_n,r)} v_n^2 \,\mathrm{d}x > \eta > 0.$$

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By (4.9), $\|\tilde{u}_n\| \geq \dfrac{C_1}{C_2}\|u_n\|$, which goes to infinity as $n \to \infty$. It follows from (4.10) that

$$0 \neq |\tilde{v}(x)| = \lim_{n\to\infty} |\tilde{v}_n(x)| = \lim_{n\to\infty} \frac{|\tilde{u}_n(x)|}{\|\tilde{u}_n\|} \quad \text{a.e. in } \Omega,$$

with $|\Omega| > 0$ and $\Omega \subset B(0,r)$. Since $\|\tilde{u}_n\| \to \infty$, we have $|\tilde{u}_n(x)| \to \infty$ a.e. in Ω. Thus, Fatou's Lemma yields

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*}\, \mathrm{d}x &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*}\, \mathrm{d}x \\ &\geq \liminf_{n\to\infty} \int_{\Omega} |\tilde{u}_n|^{2^*}\, \mathrm{d}x \\ &\geq \int_{\Omega} \liminf_{n\to\infty} |\tilde{u}_n|^{2^*}\, \mathrm{d}x \\ &= +\infty.\end{aligned}$$

However, this contradicts the fact that $(|u_n|_{2^*})$ is bounded. This implies that hypothesis (ii) does not hold for sequence (v_n). We conclude that, up to a subsequence, (u_n) is bounded. □

In the sequel, since M is a closed and bounded subset of a finite dimensional space, we have that M is compact. Therefore, by the continuity of h_0 and I_μ, for each $\mu > 0$, there exist $t_\mu > 0$ and $v_\mu^- \in E^-$ such that

$$\max_{u\in M} I_\mu(h_0(u)) = I_\mu(h_0(t_\mu Re + v_\mu^-)) = I_\mu\left(u_0^+\left(\frac{\cdot - y}{t_\mu L}\right) + |v_\mu^-|\right),$$

with $t_\mu Re + v_\mu^- \in M$. Let us prove that in fact $t_\mu > 0$. Otherwise, if $t_\mu = 0$, then by the definition of I_μ, we have $I_\mu(|v_\mu^-|) \leq 0$. However, by the proof of Lemma 3.2, we may choose several small values of $t > 0$ such that $tRe \in M$ and $\left\|u_0^+\left(\dfrac{\cdot - y}{tL}\right)\right\| < \rho$, satisfying $I_\mu(h_0(tRe)) = I_\mu\left(u_0^+\left(\dfrac{\cdot - y}{tL}\right)\right) > 0$, contradicting the maximality of $I_\mu(|v_\mu^-|)$. Therefore, $t_\mu > 0$.

Lemma 4.2. *It holds that $c_\mu \to 0$ as $\mu \to +\infty$, where $c_\mu = \inf_{h\in\Gamma} \max_{w\in M} I_\mu(h(w))$, with $\Gamma := \{h \in C(M,E) : h|_{\partial M} = h_0\}$.*

Proof. First, remember that $t_\mu > 0$ is not equal to zero. We claim that $t_\mu Re + v_\mu^- \to 0$ as $\mu \to +\infty$. In fact, since $t_\mu Re + v_\mu^- \in M$ and M is a compact set, passing to a subsequence if necessary, we may suppose that $t_\mu Re + v_\mu^- \to w_0 := t_0 Re + v_0^- \in M$

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strongly as $\mu \to +\infty$. Let us show that $w_0 = 0$. Otherwise, suppose $w_0 \neq 0$ and note that, by the equivalence of the norms,

$$0 \leq c_\mu \leq \max_{u \in M} I_\mu(h_0(u)) = I_\mu(h_0(t_\mu Re + v_\mu^-)) \\ \leq \frac{C}{2} \left\| u_0 \left(\frac{\cdot - y}{t_\mu L} \right) \right\|^2_{H^1(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} F(h_0(t_\mu Re + v_\mu^-))\,\mathrm{d}x.$$

After a change of variables, since $0 \leq t_\mu \leq 1$, it is possible to find a constant $C > 0$, which does not depend on μ, such that

$$\frac{C}{2} \left\| u_0 \left(\frac{\cdot - y}{t_\mu L} \right) \right\|^2_{H^1(\mathbb{R}^N)} \leq C$$

for all $\mu > 0$. At this moment, supposing $t_0 > 0$, since u_0 is positive and $u_0^-(\cdot - y) \to 0$ strongly in E as $|y| \to \infty$, we choose $y \in \mathbb{R}^N$ with norm large enough to get $F(h_0(w_0)) = F(h_0(t_0 Re + v_0^-)) = F(u_0^+((\cdot - y)/t_0 L) + |v_0^-|) \approx F(u_0((\cdot - y)/t_0 L) + |v_0^-|) > 0$. Therefore, Fatou's lemma provides

$$0 < \int_{\mathbb{R}^N} F(h_0(w_0))\,\mathrm{d}x \leq \liminf_{\mu \to \infty} \int_{\mathbb{R}^N} F(h_0(t_\mu Re + v_\mu^-))\,\mathrm{d}x.$$

Then, we obtain for $\mu > 0$ sufficiently large that

$$0 \leq c_\mu \leq C - \mu \int_{\mathbb{R}^N} F(h_0(t_\mu Re + v_\mu^-))\,\mathrm{d}x < 0,$$

which is an absurd. This contradiction shows that $w_0 = 0$ and proves our claim.

It follows from the continuity of the norm and of the function h_0 (using also that $h_0(0) = 0$) that

$$0 \leq c_\mu \leq \max_{u \in M} I_\mu(u) = I_\mu(h_0(t_\mu Re + v_\mu^-)) \\ \leq \frac{1}{2} \|h_0(t_\mu Re + v_\mu^-)\|^2 \\ \to 0$$

as $\mu \to \infty$, and the result follows. □

5. A nontrivial solution for (P_μ)

We begin with a technical result.

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Lemma 5.1. *If $\mu_2 > \mu_1 \geq 0$, there exists $C > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,|rmdx \leq C\,\mathrm{e}^{-\mu_1|x_1-x_2|}.$$

Proof. Since

$$\begin{aligned}\mu_1\,|x_1-x_2| + (\mu_2-\mu_1)\,|x-x_2| &\leq \mu_1\,(|x-x_1|+|x-x_2|) + (\mu_2-\mu_1)\,|x-x_2| \\ &= \mu_1\,|x-x_1| + \mu_2\,|x-x_2|\,,\end{aligned}$$

we have

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x-x_1|}\mathrm{e}^{-\mu_2|x-x_2|}\,\mathrm{d}x \leq \int_{\mathbb{R}^N} \mathrm{e}^{-\mu_1|x_1-x_2|}\mathrm{e}^{-(\mu_2-\mu_1)|x-x_2|}\,\mathrm{d}x = C\,\mathrm{e}^{-\mu_1|x_1-x_2|}.$$

The proof follows. □

Lemma 5.2. *For every $\mu > 0$, it holds that $c_\mu < d_\mu$.*

Proof. For simplicity, C will denote a positive constant, not necessarily the same one. By the definitions of the functionals I_μ and J_μ and fixing $u_\mu = v_\mu^- + t_\mu Re$, we have

$$\begin{aligned}
I_\mu(h_0(u_\mu)) &= \frac{1}{2}\|\Upsilon^+(t_\mu)\|^2 - \frac{1}{2}\|v_\mu^-\|^2 - \mu\int_{\mathbb{R}^N} F(\Upsilon^+(t_\mu)+|v^-|)\,\mathrm{d}x \\
&\quad - \frac{1}{2^*}\int_{\mathbb{R}^N} |\Upsilon^+(t_\mu)+|v^-||^{2^*}\,\mathrm{d}x \\
&\leq \frac{1}{2}\|\Upsilon^+(t_\mu)\|^2_{V_\infty} - \int_{\mathbb{R}^N} F(\Upsilon^+(t_\mu))\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |\Upsilon^+(t_\mu)|^{2^*}\,\mathrm{d}x \\
&\quad - \frac{1}{2^*}\int_{\mathbb{R}^N} |\Upsilon^+(t_\mu)+|v^-||^{2^*}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \big(F(\Upsilon^+(t_\mu)) - F(\Upsilon^+(t_\mu)+|v^-|)\big)\,\mathrm{d}x + \frac{1}{2^*}\int_{\mathbb{R}^N} |\Upsilon^+(t_\mu)|^{2^*}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} (V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x \\
&\leq J_\mu(\Upsilon^+(t_\mu)) + \frac{1}{2}\int_{\mathbb{R}^N} (V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \big(F(\Upsilon^+(t_\mu)) - F(\Upsilon^+(t_\mu)+|v^-|)\big)\,\mathrm{d}x \\
&\quad + \frac{1}{2^*}\int_{\mathbb{R}^N} \Big(|\Upsilon^+(t_\mu)|^{2^*} - |\Upsilon^+(t_\mu)+|v^-||^{2^*}\Big)\,\mathrm{d}x \\
&\leq J_\mu(\Upsilon^+(t_\mu)) + \frac{1}{2}\int_{\mathbb{R}^N} (V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x \qquad (5.1)
\end{aligned}$$

provided $|y|$ is large enough and F and $s \mapsto s^{2^*}$, for $s > 0$, are non-decreasing.

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Let us estimate the integral $\frac{1}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x$. We begin remembering that

$$u_0\left(\frac{x-y}{t_\mu L}\right)=\Upsilon(t_\mu)(x)=\Upsilon^+(t_\mu)(x)+\Upsilon^-(t_\mu)(x)$$

for all $x,y\in\mathbb{R}^N$. Thus, replacing x by $x+y$, we get

$$u_0\left(\frac{x}{t_\mu L}\right)=\Upsilon(t_\mu)(x+y)=\Upsilon^+(t_\mu)(x+y)+\Upsilon^-(t_\mu)(x+y)$$

for all $x,y\in\mathbb{R}^N$. Since $\|\Upsilon^-(t)(x+y)\|_2=\|\Upsilon^-(t)\|_2\to 0$ as $|y|\to\infty$ uniformly on $t\in[0,1]$ (see (3.8) and (3.9) in [12] and note that we are considering for $t=0$, $\Upsilon^-(0)=0$), it yields the pointwise convergence

$$\Upsilon^+(t_\mu)(x+y)\to u_0\left(\frac{x}{t_\mu L}\right)\ \text{as}\ |y|\to\infty. \tag{5.2}$$

We have from assumption (V_4) that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x&=\frac{1}{2}\int_{\mathbb{R}^N}(V(x+y)-V_\infty)\,(\Upsilon^+(t_\mu))^2(x+y)\,\mathrm{d}x\\&\le -C\int_{\mathbb{R}^N}\mathrm{e}^{-\gamma_1|x+y|^{\gamma_2}}(\Upsilon^+(t_\mu))^2(x+y)\,\mathrm{d}x.\\&\le -C\,\mathrm{e}^{-C|y|^{\gamma_2}}\int_{\mathbb{R}^N}\mathrm{e}^{-C|x|^{\gamma_2}}(\Upsilon^+(t_\mu))^2(x+y)\,\mathrm{d}x,\end{aligned}\tag{5.3}$$

where we used that, for all $x,y\in\mathbb{R}^N$, $|t_\mu Lx+y|^{\gamma_2}\le Ct_\mu^{\gamma_2}L^{\gamma_2}|x|^{\gamma_2}+C|y|^{\gamma_2}\le C|x|^{\gamma_2}+C|y|^{\gamma_2}$ and the function $s\mapsto -e^{-\gamma_1 s}$, $s>0$, is increasing.

Now using (5.2) and the Lebesgue Theorem, we obtain as $|y|\to\infty$ that

$$\int_{\mathbb{R}^N}\mathrm{e}^{-C|x|^{\gamma_2}}(\Upsilon^+(t_\mu))^2(x+y)\,\mathrm{d}x\to\int_{\mathbb{R}^N}\mathrm{e}^{-C|x|^{\gamma_2}}\left(u_0\left(\frac{x}{t_\mu L}\right)\right)^2\mathrm{d}x:=\alpha_0>0,$$

with $\alpha_0>0$ does not depending on y. It follows from this and (5.3) that

$$\frac{1}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)\,(\Upsilon^+(t_\mu))^2\,\mathrm{d}x\le -C\,\mathrm{e}^{-C|y|^{\gamma_2}} \tag{5.4}$$

for $|y|$ large enough.

Therefore, from (5.1), we obtain

$$I_\mu(h_0(u_\mu))\le J_\mu(\Upsilon^+(t_\mu))-C\,\mathrm{e}^{-C|y|^{\gamma_2}}. \tag{5.5}$$

By Mean Value Theorem (choosing the points $a:=\Upsilon(t_\mu)$ and $h:=-\Upsilon^-(t_\mu)$ that implies $a+h=\Upsilon^+(t_\mu)$) and the growth of f, we have for $2<p<2^*$ that

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$$\begin{aligned}\int_{\mathbb{R}^N} \left(F(\Upsilon(t_\mu)) - F(\Upsilon^+(t_\mu))\right) \mathrm{d}x &\le \int_{\mathbb{R}^N} \left|f(\Upsilon(t_\mu) + r_t\Upsilon^-(t_\mu))\right| |\Upsilon^-(t_\mu)| \,\mathrm{d}x \\ &\le \varepsilon \int_{\mathbb{R}^N} |\Upsilon(t_\mu)||\Upsilon^-(t_\mu)| \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}^N} (\Upsilon^-(t_\mu))^2 \,\mathrm{d}x \\ &\quad + C_\varepsilon \int_{\mathbb{R}^N} |\Upsilon(t_\mu)|^{p-1}|\Upsilon^-(t_\mu)| \,\mathrm{d}x \\ &\quad + C_\varepsilon \int_{\mathbb{R}^N} |\Upsilon^-(t_\mu)|^p \,\mathrm{d}x,\end{aligned} \tag{5.6}$$

where $r_t(x) \in (0,1)$. Following the same arguments, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^N} \left((\Upsilon(t_\mu))^{2^*} - (\Upsilon^+(t_\mu))^{2^*}\right) \mathrm{d}x \le C \int_{\mathbb{R}^N} |\Upsilon(t_\mu)|^{2^*-1}|\Upsilon^-(t_\mu)| \,\mathrm{d}x \\ + C \int_{\mathbb{R}^N} |\Upsilon^-(t_\mu)|^{2^*} \,\mathrm{d}x.\end{aligned} \tag{5.7}$$

Now using the exponential decay of $\Upsilon(t_\mu)$ given by Proposition 6.10 in Appendix A, with $\nu > 0$ to be chosen, we get from Lemma 5.1 that

$$\begin{aligned}\int_{\mathbb{R}^N} |\Upsilon(t_\mu)|^{2^*-1}|\Upsilon^-(t_\mu)| \,\mathrm{d}x &\le C \int_{\mathbb{R}^N} \left(\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|x-y|} \mathrm{e}^{-\delta|x|}\right) \mathrm{d}x \\ &\le C\, \mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|},\end{aligned} \tag{5.8}$$

just choosing $\nu > 0$ small enough. By the same arguments, we also have

$$\begin{aligned}\int_{\mathbb{R}^N} |\Upsilon(t_\mu)|^{p-1}|\Upsilon^-(t_\mu)| \,\mathrm{d}x &\le C \int_{\mathbb{R}^N} \left(\mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|x-y|} \mathrm{e}^{-\delta|x|}\right) \mathrm{d}x \\ &\le C\, \mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|}\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |\Upsilon(t_\mu)||\Upsilon^-(t_\mu)| \mathrm{d}x &\le C \int_{\mathbb{R}^N} \left(\mathrm{e}^{-\frac{\nu}{t_\mu L}|x-y|} \mathrm{e}^{-\delta|x|}\right) \mathrm{d}x \\ &\le C\, \mathrm{e}^{-\frac{\nu}{t_\mu L}|y|}.\end{aligned} \tag{5.10}$$

Now applying (5.8)–(5.10) in (5.6) and (5.7), and defining $\|\Upsilon^-(t_\mu)\|_{V_\infty} := \beta_y$, one has

$$\int_{\mathbb{R}^N} \left(F(\Upsilon(t_\mu)) - F(\Upsilon^+(t_\mu))\right) \mathrm{d}x \le \varepsilon C\, \mathrm{e}^{-\frac{\nu}{t_\mu L}|y|} + \varepsilon\beta_y^2 + C_\varepsilon C\, \mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|} + C_\varepsilon C \beta_y^p \tag{5.11}$$

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and

$$\int_{\mathbb{R}^N} \left((\Upsilon(t_\mu))^{2^*} - (\Upsilon^+(t_\mu))^{2^*}\right) \mathrm{d}x \le C\,\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|} + C\beta_y^{2^*}. \tag{5.12}$$

Choosing $0 < \varepsilon < \frac{1}{2}$, inequalities (5.11) and (5.12) provide

$$\begin{aligned}
J_\mu(\Upsilon(t_\mu)) &= \frac{1}{2}\|\Upsilon^+(t_\mu)\|^2 - \mu\int_{\mathbb{R}^N} F(\Upsilon^+(t_\mu))\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N}(\Upsilon^+(t_\mu))^{2^*}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\beta_y^2 - \mu\int_{\mathbb{R}^N}\left(F(\Upsilon(t_\mu)) - F(\Upsilon^+(t_\mu))\right)\mathrm{d}x \\
&\quad - \frac{1}{2^*}\int_{\mathbb{R}^N}\left((\Upsilon(t_\mu))^{2^*} - (\Upsilon^+(t_\mu))^{2^*}\right)\mathrm{d}x \\
&\ge J_\mu(\Upsilon^+(t_\mu)) - \left[\varepsilon C\,\mathrm{e}^{-\frac{\nu}{t_\mu L}|y|} + \varepsilon\beta_y^2 + C_\varepsilon C\,\mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|} + C_\varepsilon C\beta_y^p\right] \\
&\quad - \left[C\,\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|} + C\beta_y^{2^*}\right] + \frac{1}{2}\beta_y^2 \\
&\ge J_\mu(\Upsilon^+(t_\mu)) + \left(\frac{1}{2} - \varepsilon\right)\beta_y^2 - \varepsilon C\,\mathrm{e}^{-\frac{\nu}{t_\mu L}|y|} - C_\varepsilon C\,\mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|} - C_\varepsilon C\beta_y^p \\
&\quad - C\,\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|} - C\beta_y^{2^*}
\end{aligned}$$

Since $\beta_y \to 0$, we have

$$\left(\frac{1}{2} - \varepsilon\right)\beta_y^2 - C_\varepsilon C\beta_y^p - C\beta_y^{2^*} \ge 0$$

taking $|y|$ sufficiently large. It follows that

$$J_\mu(\Upsilon(t_\mu)) \ge J_\mu(\Upsilon^+(t_\mu)) - \varepsilon C\,\mathrm{e}^{-\frac{\nu}{t_\mu L}|y|} - C_\varepsilon C\,\mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|} - C\,\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|}. \tag{5.13}$$

Thus, returning to (5.1) with (5.4) and (5.13), we get

$$\begin{aligned}
I_\mu(h_0(u_\mu)) &\le J_\mu(\Upsilon(t_\mu)) - C\,\mathrm{e}^{-C|y|^{\gamma_2}} + \varepsilon C\,\mathrm{e}^{-\frac{\nu}{t_\mu L}|y|} + C_\varepsilon C\,\mathrm{e}^{-(p-1)\frac{\nu}{t_\mu L}|y|} \\
&\quad + C\,\mathrm{e}^{-(2^*-1)\frac{\nu}{t_\mu L}|y|}
\end{aligned} \tag{5.14}$$

with $|y|$ large enough. Since the function $\mathrm{e}^{-C|y|^{\gamma_2}}$ decays more slowly than the other terms of exponential functions (because $0 < \gamma_2 < 1$), we conclude from (5.14) that

$$I_\mu(h_0(u_\mu)) < J_\mu(\Upsilon(t_\mu)).$$

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To finish the proof, we proceed as follows.

$$c_\mu \le \max_{u\in M} I_\mu(h_0(u)) = I_\mu(h_0(u_\mu)) < J_\mu(\Upsilon(t_\mu)) \le \max_{t\in[0,1]} J_\mu(\Upsilon(t)) = J_\mu(u_0) = d_\mu.$$

We used the fact that the maximum $\max_{t\in[0,1]} J_\mu(\Upsilon(t))$ is achieved at the unique point t_0, where $J'_\mu(\Upsilon(t_0))\Upsilon(t_0) = 0$. This point t_0 is unique once $J'_\mu(\Upsilon(t_0))\Upsilon(t_0) = 0$ is equivalent to

$$t_0^{N-2}L^{N-2}\|\nabla u_0\|_2^2 + t_0^N L^N \|u_0\|_2^2 = t_0^N L^N \int_{\mathbb{R}^N} f(u_0)u_0 \,\mathrm{d}x + t_0^N L^N \|u_0\|_{2^*}^{2^*},$$

which has only one solution in t_0. Since u_0 is a non-trivial solution of $(P_{\mu,\infty})$, we can infer that the value of t_0 is $t_0 = \dfrac{1}{L}$. The proof is complete. □

Lemma 5.3. *Consider $(u_n) \subset E$, a $(Ce)_c$ sequence for functional I_μ such that*

$$u_n \rightharpoonup 0 \ \textit{in}\ H^1(\mathbb{R}^N) \quad \textit{and} \quad c \in \left(0, \frac{1}{NC_1^{\frac{N}{2}}} S^{\frac{N}{2}}\right),$$

where C_1 is given by (4.9). Then, there exist a sequence $(y_n) \subset E$ and $\rho, \eta > 0$, satisfying

$$\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 \,\mathrm{d}x \ge \eta. \tag{5.15}$$

Proof. Suppose, by contradiction, that (5.15) does not hold. Then, by [10, Lemma 8.4], we have, for $p \in (2, 2^*)$, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p \,\mathrm{d}x = 0.$$

Consequently, by (1.2) and (f_3),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x = 0. \tag{5.16}$$

Since $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and $\dim E^- < \infty$, then $\|u_n^-\| \to 0$. Thus, using that $I'_\mu(u_n)u_n = o_n(1)$ and (u_n) is bounded in $H^1(\mathbb{R}^N)$, we obtain by (5.16)

$$l := \int_{\mathbb{R}^N} |u_n|^{2^*} \,\mathrm{d}x + o_n(1) = \|u_n^+\|^2 - \|u_n^-\|^2 + o_n(1) = \|u_n^+\|^2 + o_n(1).$$

Hence, by (5.16) and inequality above,

$$o_n(1) + c = I_\mu(u_n) = \frac{1}{N}\int_{\mathbb{R}^N} |u_n|^{2^*} \,\mathrm{d}x + o_n(1) = \frac{1}{N} l. \tag{5.17}$$

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This implies that $l = Nc > 0$. Now considering S the best constant of the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we can conclude, by (4.9), that

$$l^{\frac{2}{2^*}} S \leq \int\limits_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x \leq C_1 \|u_n^+\|^2 + o_n(1) = C_1 l + o_n(1).$$

Using that $l = Nc$ and inequality above, we obtain $c \geq \dfrac{1}{NC_1^{\frac{N}{2}}} S^{\frac{N}{2}}$, which contradicts our hypothesis. □

Proposition 5.4. *Let $(u_n) \subset E$ be a $(Ce)_c$ sequence for functional I_μ. If*

$$c \in \left(0, \min\left\{d_\mu, \frac{1}{C_1^{\frac{N}{2}} N} S^{\frac{N}{2}}\right\}\right),$$

then problem (P_μ) has a non-trivial solution u_μ.

Proof. Using Lemma 4, there is $u_\mu \in E$ such that $u_n \rightharpoonup u_\mu$ in E and $I'_\mu(u_\mu) = 0$. Suppose, by contradiction, that $u_\mu \equiv 0$. Then, by Lemma 5.3, there are $(y_n) \subset \mathbb{R}^N$ and $\rho,\ \eta > 0$ such that (5.15) holds. Note that since $\dim E^-$ is finite, $\|u_n^-\| \to 0$ and, consequently, for all $v \in E$,

$$\begin{aligned}
\int_{\mathbb{R}^N} (\nabla u_n \nabla v + V(x) u_n v)\,\mathrm{d}x &= \int_{\mathbb{R}^N} \left(\nabla u_n^+ \nabla v + V(x) u_n^+ v\right)\mathrm{d}x + o_n(1),\\
\int_{\mathbb{R}^N} f(u_n) v \,\mathrm{d}x &= \int_{\mathbb{R}^N} f(u_n^+) v\,\mathrm{d}x + o_n(1),\\
\int_{\mathbb{R}^N} F(u_n)\,\mathrm{d}x &= \int_{\mathbb{R}^N} F(u_n^+)\,\mathrm{d}x + o_n(1),\\
\int_{\mathbb{R}^N} |u_n|^{2^*-1} u_n v\,\mathrm{d}x &= \int_{\mathbb{R}^N} |u_n^+|^{2^*-1} u_n^+ v\,\mathrm{d}x + o_n(1),\\
\int_{\mathbb{R}^N} |u_n|^{2^*}\,\mathrm{d}x &= \int_{\mathbb{R}^N} |u_n^+|^{2^*}\,\mathrm{d}x + o_n(1),
\end{aligned}$$

which imply $I_\mu(u_n) = I_\mu(u_n^+) + o_n(1)$ and $I'_\mu(u_n^+) = o_n(1)$.

Let us prove that $\lim\limits_{n\to+\infty} |y_n| \to +\infty$ occurs. In fact, if there exists $\overline{R} > 0$ such that $B_\rho(y_n) \subset B_{\overline{R}}(0) \subset \mathbb{R}^N$, for all $n \in \mathbb{N}$, then, since (u_n) converges strongly to 0 in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we see that

$$\limsup_{n\to+\infty} \int_{B_\rho(y_n)} u_n^2\,\mathrm{d}x \leq \limsup_{n\to+\infty} \int_{B_{\overline{R}}(0)} u_n^2\,\mathrm{d}x = 0,$$

contradicting (5.15).

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Now define $w_n(x) := u_n^+(x+y_n)$, for all $x \in \mathbb{R}^N$. Therefore, (w_n) is bounded in E, and there exists $w_n \rightharpoonup w_\mu \in E$. Using the arguments above, we have

$$I_\mu(u_n^+) = J_\mu(w_n) + o_n(1) \quad \text{and} \quad J'_\mu(w_\mu) = 0. \tag{5.18}$$

We claim that $w_\mu \neq 0$. In fact, by (5.15),

$$\begin{aligned} \eta \leq \limsup_{n\to+\infty} \int_{B_\rho(y_n)} |u_n|^2 \,\mathrm{d}x &= \limsup_{n\to+\infty} \int_{B_\rho(y_n)} |u_n^+|^2 \,\mathrm{d}x + o_n(1) \\ &= \limsup_{n\to+\infty} \int_{B_\rho(0)} |w_n|^2 \,\mathrm{d}x + o_n(1) \\ &= \int_{B_\rho(0)} |w_\mu|^2 \,\mathrm{d}x. \end{aligned}$$

Hence,

$$\begin{aligned} d_\mu \leq J_\mu(w_\mu) &= J_\mu(w_\mu) - \frac{1}{2}J'_\mu(w_\mu)w_\mu \\ &\leq \liminf_{n\to\infty}\left[I_\mu(w_n) - \frac{1}{2}I'_\mu(w_n)w_n\right] \\ &= \liminf_{n\to\infty}\left[I_\mu(u_n) - \frac{1}{2}I'_\mu(u_n)u_n\right] \\ &\leq I_\mu(u_n) + o_n(1) \\ &= c_\mu. \end{aligned}$$

Therefore, u_μ may not be trivial, and the proof of the lemma is complete. □

At this moment, the first part of the proof of Theorem 1.1 may be presented. In § 2, we proved that I_μ satisfies all conditions in Theorem 3.1, what implies the existence of a Cerami sequence (u_n) at level $c_\mu > 0$, where $c_\mu = \inf_{h\in\Gamma}\max_{w\in M} I_\mu(h_0(w))$ with $\Gamma := \{h \in C(M,E) : h|_{\partial M} = h_0\}$. This sequence is bounded by Lemma 4 and then (u_n) converges weakly to a solution u_μ of (P_μ). To show that u_μ is non-trivial, we have from Lemma 5.2 that $c_\mu < d_\mu$ and, from Lemma 4.2, we choose $\mu > 0$ large enough to obtain $c_\mu < \dfrac{1}{C_1^{\frac{N}{2}}N}S^{\frac{N}{2}}$ and apply Proposition 5.4 to get a non-trivial solution u_μ of problem (P_μ), which is non-negative because of hypothesis (f_2), as we wished to prove.

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6. Appendix A: One ground state solution to the Limit Problem $(P_{\mu,\infty})$ and some of its properties

To prove the existence of ground state solution to the limit problem $(P_{\mu,\infty})$, we consider

$$\begin{cases} -\Delta u + V_\infty u = \mu f(u) + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$P_{\mu,\,\infty}$}$$

Hereafter, let us denote by $J_\mu : H^1(\mathbb{R}^N) \to \mathbb{R}$ the associated functional given by

$$J_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty u^2\right) \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x.$$

In this section, we consider $H^1(\mathbb{R}^N)$ endowed with the following norm

$$\|u\|_{H^1}^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty |u|^2\right) \mathrm{d}x.$$

Notice that weak solutions of problem $(P_{\mu,\infty})$ in $H^1(\mathbb{R}^N)$ are critical points of functional $J_\mu \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Let us show that functional J_μ has a mountain pass geometry.

Proposition 6.1. *The following statements hold.*

(i) There exist $\alpha,\ \rho > 0$ *such that*

$$J_\mu(u) \geq \alpha, \text{ for all } u \in H^1(\mathbb{R}^N), \text{ with } \|u\|_{H^1} = \rho.$$

(ii) For all $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, *we have*

$$\limsup_{t\to+\infty} J_\mu(tu) \leq -\infty.$$

Proof. Using (1.2) and Sobolev embeddings, we obtain

$$\begin{aligned} J_\mu(u) &\geq \frac{1}{2}\|u\|_{H^1}^2 - \mu\left(\frac{\epsilon}{2}\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x\right) - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_{H^1}^2 - \frac{\mu\epsilon}{2}C\|u\|_{H^1}^2 - \mu C C_\epsilon \|u\|_{H^1}^p - C\|u\|_{H^1}^{2^*} \\ &= \|u\|_{H^1}^2\left[\left(\frac{1}{2} - \frac{\mu\epsilon}{2}C\right) - \mu C C_\epsilon \|u\|_{H^1}^{p-2} - C\|u\|_{H^1}^{2^*-2}\right] \end{aligned}$$

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Choosing $\epsilon \in \left(0, \dfrac{1}{\mu C}\right)$ and taking $\|u\|_{H^1}$ small enough, we can determine positive numbers α and ρ such that

$$J_\mu(u) \geq \alpha, \text{ for all } u \in H^1(\mathbb{R}^N), \quad \text{with } \|u\|_{H^1} = \rho.$$

To prove the second item, let us consider $u \neq 0$ and $t > 0$. Then, by (f_3),

$$\begin{aligned} J_\mu(tu) &= \frac{t^2}{2}\|u\|_{H^1}^2 - \mu \int_{\mathbb{R}^N} F(tu)\,\mathrm{d}x - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &< t^2\left[\frac{1}{2}\|u\|_{H^1}^2 - \frac{t^{2^*-2}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x\right]. \end{aligned}$$

Letting $t \to +\infty$, we obtain

$$\limsup_{t\to+\infty} J_\mu(tu) \leq t^2\left[\frac{1}{2}\|u\|_{H^1}^2 - \frac{t^{2^*-2}}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x\right] \to -\infty.$$

□

We say that a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ is a Palais-Smale sequence at level b_μ for the functional J_μ if

$$J_\mu(u_n) \to b_\mu \quad \text{and} \quad \|J_\mu'(u_n)\| \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N),$$

as $n \to \infty$, where

$$b_\mu := \inf_{\eta\in\Gamma} \max_{t\in[0,1]} J_\mu(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], H^1(\mathbb{R}^N)) : \eta(0) = 0,\ J_\mu(\eta(1)) < 0\}.$$

Notice that Proposition 6.1 implies the existence of a Palais-Smale sequence at level b_μ for the functional J_μ. Using this Palais-Smale sequence, we show the existence of non-trivial critical point for J_μ, but we need to show some technical results. First, let $S > 0$ be the best constant to the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Lemma 6.2. *Consider $(u_n) \subset H^1(\mathbb{R}^N)$ a Palais-Smale sequence at level b_μ for the functional J_μ such that*

$$u_n \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^N) \quad \text{and} \quad b_\mu \in \left(0, \frac{1}{N}S^{\frac{N}{2}}\right).$$

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Then, there exist a sequence $(y_n) \subset \mathbb{R}^N$ *and* $\rho, \eta > 0$ *such that*

$$\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 \,\mathrm{d}x \geq \eta. \tag{6.1}$$

Proof. One proceeds exactly as in Proposition 5.4. □

Lemma 6.3. *If* $\mu \to +\infty$, *then* $b_\mu \to 0$.

Proof. Consider a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\|\varphi\|_{H^1}^2 = 2$. Then, by Lemma 6.1, there exists $t_\mu > 0$ such that

$$J_\mu(t_\mu \varphi) = \max_{t\geq 0} J_\mu(t\varphi).$$

We are going to show that, up to subsequence, $t_\mu \to 0$ when $\mu \to +\infty$. First, using the characterization of b_μ and (f_3), we have

$$\begin{aligned} 0 < b_\mu &\leq t_\mu^2 - \mu \int_{\mathbb{R}^N} F(t_\mu\varphi)\,\mathrm{d}x - \frac{t_\mu^{2^*}}{2^*} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,\mathrm{d}x \\ &\leq t_\mu^2 \left(\frac{1}{2} - \frac{t_\mu^{2^*-2}}{2^*} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,\mathrm{d}x \right), \end{aligned}$$

and this implies that $(t_\mu) \subset \mathbb{R}$ is bounded. Hence, up to a subsequence, (t_μ) converges to some $t_0 \geq 0$. To prove that $t_0 = 0$, let us suppose, by contradiction, that $t_0 > 0$. Then, by (f_3),

$$\begin{aligned} 0 < b_\mu &\leq t_\mu^2 - \mu \int_{\mathbb{R}^N} F(t_\mu\varphi)\,\mathrm{d}x - \frac{t_\mu^{2^*}}{2^*} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,\mathrm{d}x \\ &\leq t_0^2 - \frac{\mu}{2} \int_{\mathbb{R}^N} F(t_0\varphi)\,\mathrm{d}x - \frac{t_0^{2^*}}{2^*} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,\mathrm{d}x + o_\mu(1), \end{aligned}$$

which implies that

$$0 \leq \limsup_{\mu\to+\infty} b_\mu \leq -\infty.$$

Therefore, up to a subsequence, (t_μ) converges to 0. Hence, by (f_3) one more time,

$$0 < b_\mu \leq t_\mu^2 - \mu \int_{\mathbb{R}^N} F(t_\mu\varphi)\,\mathrm{d}x - \frac{t_\mu^{2^*}}{2^*} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,\mathrm{d}x \leq t_\mu^2,$$

for all $\mu > 0$. This proves the lemma. □

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Proposition 6.4. *There exists $\mu^* > 0$ such that the limit problem $(P_{\mu,\infty})$ has a nontrivial solution.*

Proof. By Proposition 6.1, we get a Palais-Smale sequence $(u_n) \subset H^1(\mathbb{R}^N)$ at level $b_\mu > 0$. Using the proof of Lemma 4 with a slight modification, we can show that the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Then, there exists $u_\infty \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u_0 \in H^1(\mathbb{R}^N)$.

If $u_0 \neq 0$ then, using a density argument, we have a nontrivial solution of $(P_{\mu,\infty})$. On the other hand, if $u_0 \equiv 0$, we can find $\mu^* > 0$ such that, by Lemma 6.3,

$$0 < b_\mu < \frac{1}{N} S^{\frac{N}{2}}, \quad \forall \ \mu \geq \mu^*.$$

Then, by Lemma 6.2, there exist a sequence $(y_n) \subset \mathbb{R}^N$ and $\rho, \eta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_\rho(y_n)} |u_n|^2 \, \mathrm{d}x \geq \eta. \tag{6.2}$$

Setting $w_n := u_n(x + y_n)$ and, since the problem is invariant under translations, we have that $(w_n) \subset \mathbb{R}^N$ is a bounded Palais-Smale sequence. Hence, there exists $w_0 \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $w_n \rightharpoonup w_0 \in H^1(\mathbb{R}^N)$ and, by (6.2) and Sobolev embeddings,

$$\limsup_{n\to\infty} \int_{B_\rho(0)} |w_0|^2 \, \mathrm{d}x = \limsup_{n\to\infty} \int_{B_\rho(0)} |w_n|^2 \, \mathrm{d}x \geq \eta,$$

which implies that $w_0 \neq 0$ and, using a density argument one more time, we have a non-trivial solution of $(P_{\mu,\infty})$. □

At this moment, we will concentrate in showing that the limit problem has a ground state solution. For this, let us consider

$$\mathcal{N}_\mu := \left\{ \ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \ J'_\mu(u) = 0 \right\}.$$

Before stating the results, observe that Lemmas 5.4 and 6.2 are still valid when we consider a sequence $(u_n) \subset \mathcal{N}_\mu$ instead of a $(Ce)_c$ sequence for functional J_μ.

Lemma 6.5. *Consider $\mu^* > 0$ given by Proposition 6.4. If $\mu \geq \mu^*$, then the following proprieties hold.*

(i) $\mathcal{N}_\mu \neq \emptyset$.

(ii) *There exists $\rho_\mu > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^{2^*} \, \mathrm{d}x \geq \rho_\mu > 0, \quad \forall \ u \in \mathcal{N}_\mu.$$

(iii) *If $u \in \mathcal{N}_\mu$, then* $J_\mu(u) \geq \dfrac{1}{N} \displaystyle\int_{\mathbb{R}^N} |u|^{2^*} \, \mathrm{d}x \geq \frac{\rho_\mu}{N} > 0.$

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Proof. We deduce immediately item (i) from the existence of a non-trivial solution obtained by Proposition 6.4. To prove the second item, we are going to use (1.2) to ensure that

$$|f(s)| \leq \epsilon|s| + C_\epsilon|s|^{2^*-1}, \quad \forall\, s \in \mathbb{R}.$$

Then, considering $u \in \mathcal{N}_\mu$ and using Sobolev embeddings,

$$\begin{aligned} \|u\|_{H^1}^2 &= \mu \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &\leq \epsilon\mu \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x + (\mu C_\epsilon + 1)\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &\leq \epsilon\mu\|u\|_{H^1}^2 + C_{\epsilon,\mu}\|u\|_{H^1}^{2^*}. \end{aligned} \tag{6.3}$$

Choosing $\epsilon \in \left(0, \dfrac{1}{\mu}\right)$, then there exists $K_\mu > 0$ such that

$$\|u\|_{H^1}^2 \geq K_\mu, \quad \forall\, u \in \mathcal{N}_\mu. \tag{6.4}$$

Using again (6.3) and (6.4), we can find $\rho_\mu > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \geq \rho_\mu > 0, \quad \forall\, u \in \mathcal{N}_\mu.$$

Let us show item (iii). Using (f_3) and item (ii), we obtain, for $u \in \mathcal{N}_\mu$,

$$\begin{aligned} J_\mu(u) &= J_\mu(u) - \frac{1}{2}J_\mu(u)u \\ &= \mu \int_{\mathbb{R}^N} \left(\frac{f(u)u}{2} - F(u)\right)\,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \\ &\geq \frac{\rho_\mu}{N} > 0 \end{aligned}$$

□

The above Lemma ensures that

$$0 < d_\mu := \inf_{u \in \mathcal{N}_\mu} J_\mu(u). \tag{6.5}$$

The next result establishes the existence of a non-trivial ground state solution w_∞ to problem $(P_{\mu,\infty})$.

Lemma 6.6. *If $\mathcal{N}_\mu \neq \emptyset$, let $(u_n) \subset \mathcal{N}_\mu$ be a minimizing sequence for J_μ. Then (u_n) is bounded in $H^1(\mathbb{R}^N)$. Moreover, there exists $\mu^* > 0$ such that the infimum of J_μ on $\mathcal{N}_\mu$ is attained for all $\mu > \mu^*$.*

Proof. The existence of a minimizing sequence $(u_n) \subset \mathcal{N}_\mu$ is assured by Lemma 6.5. Using the same arguments as in Lemma 4.1, we show that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Hence, by usual arguments and Sobolev embeddings, there exists $w_\infty \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{cases} \nabla u_n(x) \to \nabla w_\infty(x) \quad a.e \text{ in } \mathbb{R}^N. \\ u_n \to w_\infty \text{ in } L^s_{\text{loc}}(\mathbb{R}^N) \quad \text{for any } 1 \le s < 2^*, \\ u_n(x) \to w_\infty(x) \quad \text{for a.e } x \in \mathbb{R}^N. \end{cases} \tag{6.6}$$

The pointwise convergence of the gradient in (6.6) is guaranteed by Lemma 7.1 in the Appendix B.

By density arguments, $J'_\mu(w_\infty)(w_\infty) = 0$. Observe that, if $w_\infty \neq 0$, then $w_\infty \in \mathcal{N}_\mu$, and hence, by Fatou's Lemma and (f_3),

$$\begin{aligned} d_\mu &\le J_\mu(w_\infty) = J_\mu(w_\infty) - \frac{1}{2}J'_\mu(w_\infty)w_\infty \\ &= \mu \int_{\mathbb{R}^N} \left(\frac{f(w_\infty)w_\infty}{2} - F(w_\infty) \right) \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |w_\infty|^{2^*} \mathrm{d}x \\ &\le \mu \int_{\mathbb{R}^N} \left(\frac{f(u_n)u_n}{2} - F(u_n) \right) \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u_n|^{2^*} \mathrm{d}x + o_n(1) \\ &\le J_\mu(u_n) - \frac{1}{2}J'_\mu(u_n)u_n + o_n(1) = J_\mu(u_n) + o_n(1) = d_\mu + o_n(1). \end{aligned}$$

Then, we conclude that (u_n) converges strongly to w_∞ in $H^1(\mathbb{R}^N)$ and hence $J_\mu(w_\infty) = d_\mu$.

On the other hand, if $w_\infty \equiv 0$, we define $w_n(x) := u_n(x + y_n)$. Then, arguing as we did in Lemmas 5.3, 5.4 and 6.3, there exists $\mu^* > 0$ such that if $\mu > \mu^*$, then there is $\widetilde{w_{\mu,\infty}} \in \mathcal{N}_{\mu,\infty}$ satisfying, up to a subsequence,

$$\begin{cases} \nabla w_n(x) \to \nabla \widetilde{w_\infty}(x) \quad a.e \text{ in } \mathbb{R}^N, \\ w_n \to \widetilde{w_\infty} \text{ in } L^s_{\text{loc}}(\mathbb{R}^N) \quad \text{for any } 2 \le s < 2^*, \\ w_n(x) \to \widetilde{w_\infty}(x) \quad \text{a.e } x \in \mathbb{R}^N, \\ J_\mu(u_n) = J_\mu(w_n). \end{cases} \tag{6.7}$$

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Then, by Fatou's Lemma and (f_3),

$$\begin{aligned} d_\mu \leq J_\mu(\widetilde{w_{\mu,\infty}}) &= J_\mu(\widetilde{w_{\mu,\infty}}) - \frac{1}{2}J'_\mu(\widetilde{w_{\mu,\infty}})\widetilde{w_{\mu,\infty}} \\ &\leq \liminf_{n\to\infty}\left[J_\mu(w_n) - \frac{1}{2}J'_\mu(w_n)w_n\right] \\ &= \liminf_{n\to\infty}\left[J_\mu(u_n) - \frac{1}{2}J'_\mu(u_n)u_n\right] \\ &\leq J_\mu(u_n) + o_n(1) \\ &= d_\mu, \end{aligned}$$

which completes the proof. □

Proposition 6.7. *There exists $\mu^* > 0$ such that problem $(P_{\mu,\infty})$ has a ground state solution $u_0 \in H^1(\mathbb{R}^N)$ for all $\mu > \mu^*$.*

Proof. It follows directly of Lemmas 6.5 and 6.6. □

Without loss of generality, we may suppose that $u_0 \geq 0$ in $\mathbb{R}^N$. To see this, it is enough to truncate the functional J_μ, considering $(\max\{w, 0\})^{2^*}$ in place of $|w|^{2^*}$ and using hypothesis (f_2).

The next lemma is an important consequence of the standard regularity arguments. It will be used to apply the Divergence Theorem in the sequel.

Lemma 6.8. *It holds that $u_0 \in H^2(\mathbb{R}^N)$.*

We also may suppose that $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$. Indeed, it is enough to solve problem $(P_{\mu,\infty})$ in $H^1_{\text{rad}}(\mathbb{R}^N)$ and use the Symmetric Criticality Principle. This implies the following lemma whose proof is an immediate consequence of Strauss inequality.

Lemma 6.9. *There exists $C = C(N) > 0$ such that, for all $x \neq 0$,*

$$|u_0(x)| \leq C|u_0|_\infty \frac{1}{|x|^{\frac{N-1}{2}}}.$$

The next result is the most important concerning the qualitative properties of u_0. It will guarantee that the solution u_0 has an appropriate exponential decay, which was used before to relate two important levels in order to obtain a strong convergence of a Cerami sequence. Recall that $\dfrac{|f(s)|}{|s|} < m$ for all $s \in \mathbb{R}$ from hypothesis (f_2).

Proposition 6.10. *If $\nu \in \left(0, \sqrt{V_\infty}\right)$ and $\mu \geq \mu^*$, then there exists $C = C(m, \nu) > 0$ such that*

$$|u_0(x)| \leq C\|u_0\|_\infty e^{-\nu|x|}, \quad \forall x \in \mathbb{R}^N.$$

Proof. For $x \neq 0$, we have that

$$\Delta\left(\mathrm{e}^{-\nu|x|}\right) = \left(\nu^2 - \frac{N-1}{|x|}\nu\right)\mathrm{e}^{-\nu|x|}. \tag{6.8}$$

Define $C := \mathrm{e}^{\nu R} > 0$, where $R > 0$ to be chosen, and

$$w(x) := u_0(x) - C|u_0|_\infty \mathrm{e}^{-\nu|x|}.$$

By the definition of C, we get $w(x) \leq 0$ for $|x| \leq R$. Let us prove that this inequality still holds for $|x| > R$. For this end, consider the set

$$\Omega = \{x \in \mathbb{R}^N;\ w_+(x) > 0\},$$

where $w_+(x) = \max\{w(x), 0\}$. Suppose, by contradiction, that $\Omega \neq \emptyset$. Since $w_+ \in H^1(\mathbb{R}^N)$, it follows that Ω is a Lebesgue measurable set, which satisfies

$$\Omega \subset \mathbb{R}^N \setminus \overline{B_R(0)} := D(R).$$

Therefore, by the Divergence Theorem (note that $w_+ = 0$ on $\partial D(R)$), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_+|^2 \,\mathrm{d}x &= \int_{D(R)} \nabla w_+ \cdot \nabla w_+ \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla w \cdot \nabla w_+ \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla w_+ \,\mathrm{d}x - C|u|_\infty \int_{\mathbb{R}^N} \nabla(e^{-\nu|x|}) \cdot \nabla w_+ \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla w_+ \,\mathrm{d}x + C|u|_\infty \int_{\mathbb{R}^N} \Delta(e^{-\nu|x|}) w_+ \,\mathrm{d}x.
\end{aligned}$$

Using the definition of w and that u_0 is a solution of $(P_{\beta,\infty})$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_+|^2 \,\mathrm{d}x &= \int_\Omega \left(\mu f(u_0) - V_\infty u_0 + (u_0)^{2^*-1} + C\|u_0\|_\infty \Delta(\mathrm{e}^{-\nu|x|})\right) w \,\mathrm{d}x \\
&= \int_\Omega \left[\left(\mu \frac{f(u_0)}{u_0} - V_\infty + (u_0)^{2^*-2}\right) u_0 \right. \\
&\qquad \left. + C|u_0|_\infty \left(\nu^2 - \frac{N-1}{|x|}\nu\right)\mathrm{e}^{-\nu|x|}\right] w \,\mathrm{d}x \\
&\leq \int_\Omega \left[\left(\mu\varepsilon + C_\varepsilon (u_0)^{p-2} - V_\infty + (u_0)^{2^*-2}\right) u_0 \right. \\
&\qquad \left. + C|u_0|_\infty \left(\nu^2 - \frac{N-1}{|x|}\nu\right)\mathrm{e}^{-\nu|x|}\right] w \,\mathrm{d}x,
\end{aligned}$$

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where we used (1.2). Now, choosing $\varepsilon > 0$ such that $\mu\varepsilon + \nu^2 - V_\infty < 0$, which implies in particular that $\mu\varepsilon - V_\infty < 0$, we obtain from Lemma 6.9 that

$$\int_{\mathbb{R}^N} |\nabla w_+|^2 \, \mathrm{d}x \leq \int_\Omega \left[\left(\mu\varepsilon - V_\infty + \frac{c_1}{|x|^{\frac{(N-1)(p-2)}{2}}} + \frac{c_2}{|x|^{\frac{(N-1)(2^*-2)}{2}}} \right) u_0 \right. \\ \left. + C|u_0|_\infty \left(\nu^2 - \frac{N-1}{|x|}\nu \right) e^{-\nu|x|} \right] w \, \mathrm{d}x$$

for some constants $c_1, c_2 > 0$ that do not depend on $R > 0$. We choose $R > 0$ sufficiently large such that, for $|x| > R$,

$$\mu\varepsilon - V_\infty + \frac{c_1}{|x|^{\frac{(N-1)(p-2)}{2}}} + \frac{c_2}{|x|^{\frac{(N-1)(2^*-2)}{2}}} < 0.$$

Noting that $u_0(x) > C|u_0|_\infty \mathrm{e}^{-\nu|x|}$ in Ω, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_+|^2 \, \mathrm{d}x &\leq \int_\Omega \left[\left(\mu\varepsilon - V_\infty + \frac{c_1}{|x|^{\frac{(N-1)(p-2)}{2}}} + \frac{c_2}{|x|^{\frac{(N-1)(2^*-2)}{2}}} \right) C|u_0|_\infty \mathrm{e}^{-\nu|x|} \right. \\ &\quad \left. + \left(\nu^2 - \frac{N-1}{|x|}\nu \right) C|u_0|_\infty \mathrm{e}^{-\nu|x|} \right] w \, \mathrm{d}x. \\ &\leq \int_\Omega \left[\mu\varepsilon - V_\infty + \nu^2 + \frac{c_1}{|x|^{\frac{(N-1)(p-2)}{2}}} + \frac{c_2}{|x|^{\frac{(N-1)(2^*-2)}{2}}} \right] \\ &\quad \times C|u_0|_\infty \mathrm{e}^{-\nu|x|} w \, \mathrm{d}x. \end{aligned}$$

If necessary, we take $R > 0$ even large so that

$$\mu\varepsilon - V_\infty + \nu^2 + \frac{c_1}{|x|^{\frac{(N-1)(p-2)}{2}}} + \frac{c_2}{|x|^{\frac{(N-1)(2^*-2)}{2}}} < 0,$$

what is possible in view of $\mu\varepsilon - V_\infty + \nu^2 < 0$. Then, we arrive at

$$0 \leq \int_{\mathbb{R}^N} |\nabla w_+|^2 \, \mathrm{d}x < 0,$$

an absurd. Thus, $\Omega = \emptyset$ and the proposition follows. □

All this section proves the second and final part of Theorem 1.1.

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7. Appendix B: A technical result

Lemma 7.1. *Let $(u_n) \subset \mathcal{N}_\mu$ be a sequence satisfying $u_n \rightharpoonup w_\infty$ in $H^1(\mathbb{R}^N)$ for some $w_\infty \in H^1(\mathbb{R}^N)$. Then, passing to a subsequence, $\nabla u_n \to \nabla w_\infty$ strongly in $[L^2_{\rm loc}(\mathbb{R}^N)]^N$ and $\nabla u_n(x) \to \nabla w_\infty(x)$ almost everywhere $x \in \mathbb{R}^N$.*

Proof. We will adapt some ideas found in [1]. Since $u_n \rightharpoonup w_\infty$ in $H^1(\mathbb{R}^N)$, then, up to a subsequence,

$$\begin{cases} u_n \to w_\infty \text{ in } L^s_{\rm loc}(\mathbb{R}^N) \quad \text{for any } 1 \le s < 2^*, \\ \\ u_n(x) \to w_\infty(x) \quad \text{for a.e } x \in \mathbb{R}^N. \end{cases} \tag{7.1}$$

Given any $R>0$, let $\psi \in C_0^\infty(\mathbb{R}^N,[0,1])$ be such that $\psi \equiv 1$ in $B_R(0)$ and $\psi \equiv 0$ in $\mathbb{R}^N \backslash B_{2R}(0)$. Then, by Cauchy–Schwarz inequality and the fact that $\int_{\mathbb{R}^N} \psi \nabla w_\infty \nabla (u_n - w_\infty)\, {\rm d}x = o_n(1)$ (because of the weak convergence $u_n \rightharpoonup w_\infty$), we have the following facts:

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla w_\infty \nabla ((u_n - w_\infty)\psi)\, {\rm d}x &= \int_{\mathbb{R}^N} \psi \nabla w_\infty \nabla (u_n - w_\infty)\, |rm dx \\ &\quad + \int_{\mathbb{R}^N} (u_n - w_\infty) \nabla w_\infty \nabla \psi \, {\rm d}x \\ &= o_n(1), \end{aligned}$$

$$\int_{\mathbb{R}^N} (u_n - w_\infty) \nabla (u_n - w_\infty) \nabla \psi \, {\rm d}x = o_n(1),$$

$$\int_{\mathbb{R}^N} V_\infty u_n (u_n - w_\infty) \psi \, {\rm d}x = \int_{B_{2R}(0)} V_\infty u_n (u_n - w_\infty) \psi \, {\rm d}x = o_n(1)$$

and

$$\int_{\mathbb{R}^N} f(u_n)(w_\infty - u_n)\psi \, {\rm d}x = \int_{B_{2R}(0)} f(u_n)(w_\infty - u_n)\psi \, {\rm d}x = o_n(1),$$

where in the last convergence, we used the growth of *f*. Moreover, let $2^*-1 < s < 2^*$ and consider $r = \dfrac{s}{2^*-1} > 1$. Then, $r' := \dfrac{r}{r-1}$ satisfies $r' = \dfrac{s}{s-2^*+1} < 2^*$. By Hölder inequality with exponents r and r', we get from (7.1) and from the boundedness of (u_n) in $L^s(\mathbb{R}^N)$ that

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$$\begin{aligned}\left|\int_{\mathbb{R}^N}|u_n|^{2^*-1}(u_n-w_\infty)\psi\,\mathrm{d}x\right| &= \left|\int_{B_{2R}(0)}|u_n|^{2^*-1}(u_n-w_\infty)\psi\,\mathrm{d}x\right|\\ &\leq \left[\int_{B_{2R}(0)}|u_n|^{r(2^*-1)}\,\mathrm{d}x\right]^{1/r}\\ &\quad\times\left[\int_{B_{2R}(0)}|u_n-w_\infty|^{r'}|\psi|^{r'}\,\mathrm{d}x\right]^{1/r'}\\ &\leq \left[\int_{B_{2R}(0)}|u_n|^{s}\,\mathrm{d}x\right]^{1/r}\left[\int_{B_{2R}(0)}|u_n-w_\infty|^{r'}\,\mathrm{d}x\right]^{1/r'}\\ &= o_n(1).\end{aligned}$$

Therefore, remembering that $u_n\in\mathcal{N}_\mu$, one obtains

$$\begin{aligned}o_n(1) &= J'_\mu(u_n)u_n\psi-J'_\mu(u_n)w_\infty\psi-\int_{\mathbb{R}^N}\nabla w_\infty\nabla\left((u_n-w_\infty)\psi\right)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\nabla u_n\nabla\left((u_n-w_\infty)\psi\right)\,\mathrm{d}x+\int_{\mathbb{R}^N}V_\infty u_n(u_n-w_\infty)\psi\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}f(u_n)(w_\infty-u_n)\psi\,\mathrm{d}x+\int_{\mathbb{R}^N}|u_n|^{2^*-1}(w_\infty-u_n)\psi\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N}\nabla w_\infty\nabla\left((u_n-w_\infty)\psi\right)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\nabla(u_n-w_\infty)\nabla\left((u_n-w_\infty)\psi\right)\,\mathrm{d}x+o_n(1)\\ &= \int_{\mathbb{R}^N}\psi|\nabla(u_n-w_\infty)|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}(u_n-w_\infty)\nabla(u_n-w_\infty)\nabla\psi\,\mathrm{d}x+o_n(1)\\ &\geq \int_{B_R(0)}|\nabla(u_n-w_\infty)|^2\,\mathrm{d}x+o_n(1).\end{aligned}$$

Since $R>0$ is arbitrary, this implies that, passing to a subsequence, $\nabla u_n\to\nabla w_\infty$ in $[L^2_{\text{loc}}(\mathbb{R}^N)]^N$.

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To complete the proof of the lemma, let us show that $\nabla u_n(x) \to \nabla w_\infty(x)$ a.e. $x \in \mathbb{R}^N$ by using a diagonal process. Since $\nabla u_n \to \nabla w_\infty$ in $[L^2(B_1(0))]^N$, there exists an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that the subsequence $(\nabla u_n(x))_{n\in\mathbb{N}_1}$ converges to $\nabla w_\infty(x)$ a.e. $x \in B_1(0)$. Since $(\nabla u_n)_{n\in\mathbb{N}_1}$ converges to ∇w_∞ in $[L^2(B_2(0))]^N$, we obtain a subsequence $(\nabla u_n)_{n\in\mathbb{N}_2}$ with $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $(\nabla u_n(x))_{n\in\mathbb{N}_2}$ converges to $\nabla w_\infty(x)$ a.e. $x \in B_2(0)$. Proceeding in this way, we find infinite subsets of indexes $\mathbb{N}_{k+1} \subset \mathbb{N}_k \subset \mathbb{N}$ such that $(\nabla u_n(x))_{n\in\mathbb{N}_k}$ converges to $\nabla w_\infty(x)$ a.e. $x \in B_k(0)$. Consider $\mathbb{N}^* = \{n_1^*, n_2^*, \ldots, n_k^*, \ldots\} \subset \mathbb{N}$ with n_k^* being the kth element of $\mathbb{N}_k$. Therefore, $(\nabla u_n(x))_{n\in\mathbb{N}^*}$ is, from its kth element, a subsequence of $(\nabla u_n(x))_{n\in\mathbb{N}_k}$ and hence converges to $\nabla w_\infty(x)$ a.e. $x \in B_k(0)$. For each $k \in \mathbb{N}$, there exists $Z_k \subset B_k(0)$ of zero measure such that $(\nabla u_n(x))_{n\in\mathbb{N}_k}$ converges to $\nabla w_\infty(x)$ for all $x \in B_k(0)\backslash Z_k$. Take $Z := \cup_{m=1}^\infty Z_m$. Then, Z has zero measure and, for all $x \in \mathbb{R}^N\backslash Z$, we have $x \in B_k(0)\backslash Z_k$ for some $k \in \mathbb{N}$ and $(\nabla u_n(x))_{n\in\mathbb{N}^*}$ converges to $\nabla w_\infty(x)$. This shows that, up to a subsequence, $\nabla u_n(x) \to \nabla w_\infty(x)$ a.e. $x \in \mathbb{R}^N$ and completes the proof. □

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