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ON A PROBLEM OF RICHARD GUY

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Abstract

In the 1993 Western Number Theory Conference, Richard Guy proposed Problem 93:31, which asks for integers n representable by $(x+y+z)^3/xyz$, where x, y, z are integers, preferably with positive integer solutions. We show that the representation $n = (x+y+z)^3/xyz$ is impossible in positive integers x, y, z if $n = 4^k(a^2+b^2)$, where $k, a, b \in \mathbb{Z}^+$ are such that $k \geq 3$ and $2 \nmid a+b$.

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1. Introduction

Let n be an integer. The equation

$$n = \frac{(x+y+z)^3}{xyz} \tag{1.1}$$

has been studied by several authors. Guy [7] asked for integers n representable by (1.1), where $x, y, z \in \mathbb{Z}$, preferably with $x, y, z \in \mathbb{Z}^+$. Guy's question is still open and only partial results have been published. According to [7], Montgomery found 539 solutions to (1.1) with $1 \leq x \leq y \leq z \leq 46300$. Bremner and Guy [1] found several solutions to (1.1) when n is in the range $|n| \leq 200$. Brueggeman [3] found four families of solutions to (1.1) involving only positive integers. In a short note [6], Garaev sketched a proof that (1.1) does not have solutions in positive integers if n is of the form $n = 8k-1$, $16k-4$, $32k-16$, $64k$ or $2^{2m+1}(2k-11)+27$, where $k, m \in \mathbb{Z}^+$. Garaev's proof was based on his work [5] on the cubic Diophantine equation $x^3+y^3+z^3 = nxyz$. In this paper, we find another family of integers n for which (1.1) has no solutions in positive integers.

THEOREM 1.1. *Let k, a, b be positive with $k \geq 3$ and $2 \nmid a+b$. Then the equation*

$$(x+y+z)^3 = 4^k(a^2+b^2)xyz$$

does not have positive integer solutions.

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Garaev's method is classical and uses the quadratic reciprocity law. Our method is based on an idea of Stoll [9] and uses Hilbert symbols and elliptic curves. We briefly outline the main idea.

Assume that we want to show that a rational number X is positive. The key is to find a rational number $D < 0$ such that $(X, D)_p = 1$ for all prime numbers p, where $(X, D)_p$ denotes the Hilbert symbol. Then the product formula for the Hilbert symbol (see Serre [8, Theorem 3, page 23]) forces $(X, D)_\infty = 1$. Since $D < 0$, we must have $X > 0$. Our experience shows that when X is the x-coordinate of a rational point on an elliptic curve of the form

$$y^2 = f(x),$$

where f is a cubic polynomial with rational coefficients, D is usually a factor of the discriminant of $f(x)$. This idea can be applied to several problems. For the representation of positive integers n in the form $n = (x + y + z)(1/x + 1/y + 1/z)$ or $n = (x + y + z + w)(1/x + 1/y + 1/z + 1/w)$, where $x, y, z, w \in \mathbb{Z}^+$, see [2, 11]. For the representation of positive integers n in the form $n = x/y + dy/z + z/w + dw/x$, where $x, y, z, w, d \in \mathbb{Z}^+$, see [4, 10].

2. Preliminaries

Let p be a prime number. Let $\mathbb{Q}_p$ denote the p-adic completion of $\mathbb{Q}$ with respect to p and $\mathbb{Z}_p$ the ring of p-adic integers in $\mathbb{Q}_p$. Let $\mathbb{Q}_p^3 = \{(x, y, z) : x, y, z \in \mathbb{Q}_p\}$ and $\mathbb{Z}_p^2 = \{x^2 : x \in \mathbb{Z}_p\}$. For $w \in \mathbb{Q}_p^*$, denote by $v_p(w)$ the exponent of the highest power of p dividing w. For λ and μ in $\mathbb{Q}_p^*$, the Hilbert symbol $(\lambda, \mu)_p$ is defined by

$$(\lambda, \mu)_p = \begin{cases} 1 & \text{if } \lambda x^2 + \mu y^2 = z^2 \text{ has a solution } (x, y, z) \neq (0, 0, 0) \text{ in } \mathbb{Q}_p^3, \\ -1 & \text{otherwise.} \end{cases}$$

For $\lambda, \mu \in \mathbb{R}$, the symbol $(\lambda, \mu)_\infty$ is $+1$ if $\lambda > 0$ or $\mu > 0$ and -1 otherwise. We implicitly understand that $\mathbb{Q}$ is a subfield of both $\mathbb{Q}_p$ and $\mathbb{R}$.

We need some properties of Hilbert symbols (see [8, pages 19–26] for proofs). Let $\lambda, \mu, \sigma \in \mathbb{Q}_p^*$.

(i) We have

$$(\lambda, \mu^2)_p = 1,$$
$$(\lambda, \mu\sigma)_p = (\lambda, \mu)_p(\lambda, \sigma)_p,$$
$$(\lambda, \mu)_\infty \prod_{p \text{ prime}} (\lambda, \mu)_p = 1.$$

(ii) Let $\lambda = p^\alpha u$ and $\mu = p^\beta v$, where $\alpha = v_p(\lambda)$ and $\beta = v_p(\mu)$. Then

$$(\lambda, \mu)_p = (-1)^{(1/2)\alpha\beta(p-1)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \quad \text{if } p \neq 2,$$
$$(\lambda, \mu)_p = (-1)^{(1/4)(u-1)(v-1)+(1/8)\alpha(v^2-1)+(1/8)\beta(u^2-1)} \quad \text{if } p = 2,$$

where by $(\frac{u}{p})$ denotes the Legendre symbol.

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3. Proof of Theorem 1.1

Assume that there exist positive integers x, y, z such that

$$(x+y+z)^3 = nxyz, \quad n = 2^{2k}(a^2+b^2).$$

Let $x+y+z = n\gamma z$, where $\gamma \in \mathbb{Q}$. Then $xy = n^2\gamma^3 z^2$. Therefore,

$$\left(\frac{x-y}{z}\right)^2 = (n\gamma - 1)^2 - 4n^2\gamma^3.$$

Let $X = -\gamma$ and $Y = (x-y)/z$. Then

$$Y^2 = 4n^2X^3 + (nX+1)^2. \tag{3.1}$$

Since $nX + 1 = -(x+y)/z < 0$,

$$nX + 1 < 0. \tag{3.2}$$

We show that (3.2) is impossible via the following lemmas.

LEMMA 3.1. *In (3.2),* $(-n, nX+1)_2 = 1$.

PROOF. Write $n = 2^{2k}n_1$, where $2 \nmid n_1$. We consider five cases according to the value of $v_2(nX)$.

Case 1: $v_2(nX) \geq 3$. Then $nX + 1 \equiv 1 \pmod 8$, so that $nX + 1 \in \mathbb{Z}_2^2$. Therefore,

$$(-n, nX+1)_2 = 1.$$

Case 2: $v_2(nX) = 2$. Then $nX + 1 \equiv 1 \pmod 4$, so that

$$(-n, nX+1)_2 = (-2^{2k}n_1, nX+1)_2 = (-n_1, nX+1)_2 = (-1)^{(1/4)(-n_1-1)nX} = 1.$$

Case 3: $v_2(nX) = 1$. Then $v_2(X) = 1 - 2k$. Let $X = 2^{1-2k}X_1$, where $2 \nmid X_1$. From (3.1),

$$Y^2 = \frac{n_1^2X_1^3}{2^{2k-5}} + (2n_1X_1+1)^2.$$

Therefore,

$$2^{2k-5}Y^2 = n_1^2X_1^3 + 2^{2k-5}(2n_1X_1+1)^2. \tag{3.3}$$

Since $2k - 5 > 0$ and $2 \nmid n_1X_1^3$, it follows from (3.3) that $2k - 5 + 2v_2(Y) = 0$. But this is impossible since $2 \nmid 2k - 5$.

Case 4: $v_2(nX) = 0$. Then $v_2(X) = -2k$. Let $X = 2^{-2k}X_1$, where $2 \nmid X_1$. From (3.1),

$$Y^2 = \frac{n_1^2X_1^3}{2^{2k-2}} + (n_1X_1+1)^2.$$

Hence,

$$2^{2k-2}Y^2 = n_1^2X_1^3 + 2^{2k-2}(n_1X_1+1)^2. \tag{3.4}$$

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Since $2k-2>0$ and $2 \nmid n_1^2X_1^3$, it follows from (3.4) that $2k-2+2v_2(Y)=0$. Taking (3.4) modulo 8 gives $X_1 \equiv 1 \pmod 8$. Hence, $X_1 \in \mathbb{Z}_2^2$. Let $X_1=\delta^2$, where $\delta \in \mathbb{Z}_2$. Since $-n_1U^2+(n_1X_1+1)V^2=1$ has a solution $(U,V)=(\delta,1)$,

$$(-n_1,\, n_1X_1+1)_2 = 1.$$

Therefore,

$$(-n,\, 1+nX)_2 = (-2^{2k}n_1,\, n_1X_1+1)_2 = (-n_1,\, n_1X_1+1)_2 = 1.$$

Case 5: $v_2(nX)<0$. Then $v_2(X)<-2k$. Let $X=2^{-t}X_1$, where $2 \nmid X_1$, $t \in \mathbb{Z}$ and $t>2k$. From (3.1),

$$Y^2 = \frac{n_1^2X_1^3}{2^{3t-4k-2}} + (2^{2k-t}n_1X_1+1)^2.$$

Hence,

$$2^{3t-4k-2}Y^2 = n_1^2X_1^3 + 2^{t-2}(n_1X_1+2^{t-2k})^2. \tag{3.5}$$

Since $t>2k>2$ and $2 \nmid n_1^2X_1^3$, it follows from (3.5) that $3t-4k-2+2v_2(Y)=0$. Therefore, $2 \mid t$. Since $t>2k \geq 6$, we also have $t-2>4$. Taking (3.5) modulo 8 gives $X_1 \equiv 1 \pmod 8$. Since $2 \mid 2k-t$,

$$\begin{aligned}(-n,\, nX+1)_2 &= (-2^{2k}n_1,\, 2^{2k-t}(n_1X_1+2^{t-2k}))_2 \\ &= (-n_1,\, n_1X_1+2^{t-2k})_2 = (-1)^{(1/4)(-n_1-1)(n_1X_1+2^{t-2k}-1)}.\end{aligned} \tag{3.6}$$

Here, $4 \mid 2^{t-2k}$, because $2 \mid t$, and $t>2k$ and $X_1 \equiv 1 \pmod 8$. Therefore,

$$\begin{aligned}(-n_1-1)(n_1X_1+2^{t-2k}-1) &\equiv -(n_1+1)(n_1+2^{t-2k}-1) \pmod 8 \\ &\equiv -(n_1+1)(n_1-1) \equiv 0 \pmod 8.\end{aligned}$$

Therefore,

$$(-1)^{(1/4)(-n_1-1)(n_1X_1+2^{t-2k}-1)} = 1. \tag{3.7}$$

So, from (3.6) and (3.7) again, $(-n,\, nX+1)_2 = 1$. □

LEMMA 3.2. *Let p be an odd prime. In (3.1),* $(-n,\, nX+1)_p = 1$.

PROOF. We consider three cases according to the value of $v_p(nX)$.

Case 1: $v_p(nX) \geq 1$. Then $nX+1 \equiv 1 \pmod p$, so that $nX+1 \in \mathbb{Z}_p^2$. Hence,

$$(-n,\, nX+1)_p = 1.$$

Case 2: $v_p(nX)=0$. Then $v_p(X)=-v_p(n)$. Let $n=p^sn_1$ and $X=p^{-s}X_1$, where $s \geq 0$, $p \nmid n_1$ and $p \nmid X_1$.

Case 2.1: $s=0$. First suppose that $p \nmid (n_1X_1+1)$. Since both $-n_1$ and n_1X_1+1 are units in $\mathbb{Z}_p$, we have $(-n_1,\, n_1X_1+1)_p = 1$. Therefore,

$$(-n,\, nX+1)_p = (-n_1,\, n_1X_1+1)_p = 1.$$

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On the other hand, if $p \mid (n_1X_1 + 1)$, then, from (3.1),

$$Y^2 = 4n_1^2X_1^3 + (n_1X_1 + 1)^2.$$

Therefore, X_1 is a square modulo p and so $X_1 \in \mathbb{Z}_p^2$. Let $X_1 = \omega^2$, where $\omega \in \mathbb{Z}_p$ and $p \nmid \omega$. Then

$$-n_1 \equiv \frac{1}{X_1} \equiv \omega^{-2} \pmod{p},$$

so that $-n_1 \in \mathbb{Z}_p^2$ and

$$(-n, nX + 1)_p = (-n_1, n_1X_1 + 1)_p = 1.$$

Case 2.2: $s > 0$. From (3.1),

$$Y^2 = \frac{4n_1^2X_1^3}{p^s} + (n_1X_1 + 1)^2,$$

so that

$$p^sY^2 = 4n_1^2X_1^3 + p^s(n_1X_1 + 1)^2. \tag{3.8}$$

Since $p \nmid 4n_1^2X_1^3$ and $s > 0$, it follows from (3.8) that $s + 2v_p(Y) = 0$. Therefore, $2 \mid s$.

First suppose that $p \nmid n_1X_1 + 1$. Since both $-n_1$ and $n_1X_1 + 1$ are units in $\mathbb{Z}_p$, we have $(-n_1, n_1X_1 + 1)_p = 1$. Therefore,

$$(-n, nX + 1)_p = (-p^sn_1, n_1X_1 + 1)_p = (-n_1, n_1X_1 + 1)_p = 1.$$

On the other hand, if $p \mid n_1X_1 + 1$, then, from (3.8), X_1 is a square modulo p. Therefore, $X_1 \in \mathbb{Z}_p^2$. Let $X_1 = \zeta^2$, where $\zeta \in \mathbb{Z}_p$ and $p \nmid \zeta$. Then

$$-n_1 \equiv \frac{1}{X_1} \equiv \zeta^{-2} \pmod{p},$$

so that $-n_1 \in \mathbb{Z}_p^2$. Therefore,

$$(-n, nX + 1)_p = (-p^sn_1, n_1X_1 + 1)_p = (-n_1, n_1X_1 + 1)_p = 1.$$

Case 3: $v_p(nX) < 0$. Let $n = p^sn_1$ and $X = p^{-t}X_1$, where $t > s \geq 0$, $p \nmid n_1$ and $p \nmid X_1$. From (3.1),

$$Y^2 = \frac{4n_1^2X_1^3}{p^{3t-2s}} + \left(\frac{n_1X_1}{p^{t-s}} + 1\right)^2.$$

Therefore,

$$p^{3t-2s}Y^2 = 4n_1^2X_1^3 + p^t(n_1X_1 + p^{t-s})^2. \tag{3.9}$$

Since $p \nmid 4n_1^2X_1^3$ and $t > 0$, it follows from (3.9) that $3t - 2s + 2v_2(Y) = 0$. Thus, $2 \mid t$.

Case 3.1: $2 \mid s$. Since $-n_1$ and $n_1X_1 + p^{t-s}$ are units in $\mathbb{Z}_p$,

$$(-n_1, n_1X_1 + p^{t-s})_p = 1.$$

Therefore,

$$(-n, nX+1)_p = (-p^s n_1, p^{s-t}(n_1X_1 + p^{t-s}))_p = (-n_1, n_1X_1 + p^{t-s})_p = 1.$$

Case 3.2: $2 \nmid s$. Then $p^s \parallel n = 4^k(a^2+b^2)$. We first show that $p \equiv 1 \pmod 4$. Indeed, if $p \equiv 3 \pmod 4$, then a^2+b^2 can only be divisible by an even power of p, contrary to the assumption that s is odd. Therefore, $p \equiv 1 \pmod 4$.

Taking (3.9) modulo p shows that X_1 is a square modulo p. Therefore, $X_1 \in \mathbb{Z}_p^2$. Since $2 \nmid s$, $2 \nmid s-t$, $t-s > 0$ and $(-n_1, n_1X_1 + p^{t-s})_p = 1$,

$$\begin{aligned}(-n, nX+1)_p &= (-p^s n_1, p^{s-t}(n_1X_1 + p^{t-s}))_p = (-pn_1, p(n_1X_1 + p^{t-s}))_p \\ &= (p,p)_p \cdot (p, n_1X_1 + p^{t-s})_p \cdot (-n_1, p)_p \cdot (-n_1, n_1X_1 + p^{t-s})_p \\ &= (-1)^{(p-1)/2} \cdot \left(\frac{n_1X_1 + p^{t-s}}{p}\right) \cdot \left(\frac{-n_1}{p}\right) \\ &= \left(\frac{n_1X_1}{p}\right)\left(\frac{-n_1}{p}\right) = \left(\frac{n_1}{p}\right)\left(\frac{-n_1}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{n_1}{p}\right)^2 = 1,\end{aligned}$$

because $p \equiv 1 \pmod 4$ and X_1 is a square modulo p. □

LEMMA 3.3. *We have* $(-n, nX+1)_\infty = 1$.

PROOF. By Lemmas 3.1 and 3.2, $(-n, nX+1)_p = 1$ for all prime numbers p. Since

$$(-n, nX+1)_\infty \cdot \prod_{p \text{ prime}} (-n, nX+1)_p = 1,$$

it follows that $(-n, nX+1)_\infty = 1$. □

The consequence of Lemma 3.3 is that the equation $-n\alpha^2 + (nX+1)\beta^2 = 1$ has real solutions. Hence, $nX+1 > 0$, contradicting $nX+1 < 0$ from (3.2).

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