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PERIODIC NAVIER SOLUTIONS FOR THE PLATE EQUATION WITH NON-MONOTONE NONLINEARITIES: THE MULTIDIMENSIONAL CASE

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Abstract We discuss the solvability of the periodic Navier problem for the plate equation with forced vibrations $x_{tt}(t,y)+\Delta^2 x(t,y)+l(t,y,x(t,y))=0$ in higher dimensions with side lengths being irrational numbers and the nonlinearity being superlinear. We also derive a new dual variational method.

Keywords: periodic Navier problem; semilinear plate equation; dual variational method

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Secondary 35A15

1. Introduction

The aim of this paper is to look for solutions to the problem

$$\left.\begin{aligned} x_{tt}(t,y)+\Delta^2 x(t,y)+l(t,y,x(t,y))&=0, && t\in\mathbb{R},\ y\in(0,\pi)^n,\\ x(t,y)=\Delta x(t,y)&=0, && t\in\mathbb{R},\ y\in\partial(0,\pi)^n,\\ x(t+T,y)&=x(t,y), && t\in\mathbb{R},\ y\in(0,\pi)^n. \end{aligned}\right\} \tag{1.1}$$

This type of equation, on rectangular plates with partially Navier boundary conditions, was derived in [**18**] as a nonlinear model for the dynamic suspension bridge to display torsional oscillations. The nonlinearity in that paper is of the form $l(t,y,x)=h(y,x)-f(t,y)$, where h (superlinear) describes the behaviour of hangers and f is the forcing term. The well-posedness of an initial–boundary-value problem was shown as well as the qualitative behaviour of the solutions (see [**35**]). As was announced in [**18**], the next step of the authors is to investigate the oscillating behaviour of equation (1.1). Our aim in this paper is to study the periodic oscillation of (1.1). Initial papers studying second order partial differential equations (PDEs) looking for time-periodic solutions concern wave equations of linear and semilinear type, typically with $T=2\pi$. In the nonlinear case (nonlinear l), usually $l=\epsilon f$ with $|\epsilon|$ sufficiently small and $f(t,y,\cdot)$ being strongly monotone (see the survey [**29**] and also [**5**, **19**]). Usually, to prove existence results one uses a variant of the Lyapunov–Schmidt method together with the theory of monotone operators. When f is only monotone, similar methods and Schauder's fixed-point theorem were used in [**13**]. In [**31**] Rabinowitz used his saddle point theorem in critical-point theory together with a Galerkin argument to prove the existence of weak solutions for a nonlinearity l being of class C^1 and sublinear at infinity. This initiated a number of papers in the literature devoted to the use of various techniques of modern critical-point theory in the study of semilinear wave equations (see [**12**,**33**] and references therein). The strongly monotone and weakly monotone nonlinearities were considered in [**14**, **22**, **28**]. In all the cited papers the monotonicity assumption (strong or weak) is the key property for overcoming the lack of compactness in the infinite-dimensional kernel of the equation $x_{tt}(t,y) - x_{yy}(t,y) = 0$ (which has periodic Dirichlet solutions). Willem [**38**], Hofer [**22**] and Coron [**15**] considered the class of wave equations in which $l(t,x,u) = g(u) + h(t,x)$ and $g(u)$ satisfies suitable linear growth conditions. In [**15**], for the autonomous case in which $h \equiv 0$, Coron established the existence of non-trivial solutions for non-monotone nonlinearities. The case in which l is the difference of two convex non-autonomous functions was investigated in [**4**]: in particular, the nonlinearity $l \in C([0,\pi] \times \mathbb{R}^2, \mathbb{R})$ has the form $l(t,y,x) = \lambda g(t,y,x) + \mu h(t,y,x)$ with $\lambda, \mu \in \mathbb{R}$, g superlinear in x, h sublinear in x, and both g and h are 2π-periodic in t and non-decreasing in x. The solutions to the considered problems are obtained using variational methods. The special form of l allows them to control the levels of the weak limits of certain Palais–Smale sequences since the functional corresponding to the wave equation does not satisfy the Palais–Smale condition (see also the references in [**4**]). In [**7**] existence and regularity of solutions (with $l = \epsilon f$) was proved for a large class of non-monotone forcing terms $f(t,y,x)$ including, for example, $f(t,y,x) = \pm x^{2k} + x^{2k+1} + h(t,y)$, $f(t,y,x) = \pm x^{2k} + \tilde{f}(t,y,x)$ with $\tilde{f}_x(t,y,x) \geqslant \beta > 0$. The proof is based on a variational Lyapunov–Schmidt reduction, minimization arguments and *a priori* estimate methods. A different approach, using a combination of analysis and group-invariance arguments to problems like (1.1) in $\mathbb{R}^2$ and S^2, is described in [**21**, **23**, **24**].

It is interesting that arithmetical properties of the ratio $\alpha = T/\pi$ play an important role in the solvability of the periodic Navier problem (1.1) over $[0,T] \times (0,\pi)^n$. The main reason for this is that the nature of the spectrum of the corresponding linear problem

$$x_{tt}(t,y) + \Delta^2 x(t,y) + g(t,y) = 0 \tag{1.2}$$

depends in an essential way on the arithmetical nature of α. It was pointed out by Borel [**10**] (for the wave equation in one dimension) that there exist numbers α, satisfying some arithmetical conditions, such that the linear problem need not have a solution in the class of analytic functions if g is analytic. Later, Novak [**30**] proved even more (for the same problem); that is, there exist irrationals α and functions g in L^2 such that this equation does not have any generalized periodic Dirichlet solutions. References on these questions can be found in [**34**]. The papers that treat the nonlinear version of (1.1) consider in most cases only the one-dimensional space variable, i.e. $n = 1$, autonomous nonlinearities ($l = l(x)$ or some cases of $l = l(y,x)$) and in all cases only the irrational numbers with bounded partial quotients (see, for example, [**3**, **9**, **16**, **17**] and references therein). Kuksin [**25**] (see also [**26**]) and Wayne [**36**] (see also [**37**]) were able to find, extending in a suitable way Kolmogorov–Arnold–Moser (KAM) techniques, periodic solutions to some Hamiltonian PDEs in one spatial dimension. As usual in KAM-type results, the periods of such persistent solutions satisfy a strong irrationality condition, the classical Diophantine condition, so that these orbits exist only on energy levels belonging to some Cantor set of positive measure. The main limitation of this method is the fact that standard KAM techniques require the linear frequencies to be well separated (i.e. we require non-resonance between the linear frequencies). To overcome such a difficulty, a new method for proving the existence of small-amplitude periodic solutions, based on the Lyapunov–Schmidt reduction, was developed in [**17**]. Rather than attempting to make a series of canonical transformations that bring the Hamiltonian into some normal form, the solution is constructed directly. Making the assumption that a periodic solution exists, one writes this solution as a Fourier series and substitutes that series into the partial differential equation. In this way one is reduced to solving two equations: the so-called (P) equation, which is infinite dimensional, where small denominators appear, and the finite-dimensional (Q) equation, which corresponds to resonances. Due to the presence of small divisors, the (P) equation is solved by the Nash–Moser implicit function theorem. Later, this method was improved by Bourgain to show the persistence of periodic solutions in higher spatial dimensions [**11**]. The first results on the existence of small-amplitude periodic solutions for some completely resonant PDEs were given in [**27**], for the specific nonlinearity $l(x) = x^3$, and in [**1**] when $l(x) = x^3 +$ higher-order terms. The approach of [**1**] is still based on the Lyapunov–Schmidt reduction. The (P) equation is solved for the strongly irrational frequencies $\omega \in W_\gamma$, where $W_\gamma = \{\omega \in \mathbb{R} \mid |\omega k - j| \geqslant \gamma/k,\ k \neq j\}$, through the contraction mapping theorem. Next, the (Q) equation is solved by looking for non-degenerate critical points of a suitable functional and continuing them, by means of the implicit function theorem, into families of periodic solutions of the nonlinear equation. The case of higher space dimensions was investigated in [**3**]. In [**8**] the authors prove, assuming only that the nonlinearity l satisfies $l(0) = l'(0) = \cdots = l^{(p-1)}(0) = 0$, $l^{(p)}(0) = ap! \neq 0$ for some $p \in \mathbb{N}$, $p \geqslant 2$, the existence of a large number of small-amplitude periodic solutions of (1.1) with fixed period. The case of a more general elliptic operator was treated in [**2**, **3**].

The aim of this paper is to consider the $n \geqslant 2$ case with T being irrational numbers such that $\alpha = T/\pi$ has not necessarily bounded partial quotients in its continued fraction and non-autonomous nonlinearity l for the plate equation (1.1). To the best of our knowledge, the above problem with α having unbounded partial quotients is also considered for the first time (except in some special case in [**17**]).

We assume that

(T) $T = \pi\alpha$, $\alpha > 0$, is irrational and satisfies $|\alpha - p/|q|^2| \geqslant c|q|^{-4}$ for all $p \in \mathbb{N}$, $|q| = \sqrt{\sum_{i=1}^{n} q_i^2}$, $q_i \in \mathbb{N}$, $i = 1, \ldots, n$, with some constant $c > 0$.

We note that condition (T) is still a non-resonance condition of Diophantine type, which, in fact, is the classical-type non-resonance condition ($|q|^2 \in \mathbb{N}$). The same non-resonance condition is used by Bambusi [**2**] for the nonlinear plate equation in the n-dimensional cube (the periodic Navier problem). Note that all irrational numbers satisfying the classical non-resonance condition of Diophantine type satisfy (T) as well; for several properties of such numbers see, for example, [**32**].

In order to give the reader an insight into what condition (T) means, let us recall some fundamental facts from number theory. Let $\alpha = [a_0, a_1, a_2, \dots]$ ($a_0, a_1, a_2, \dots$ integers) be the continued fraction decomposition of the real number α [**32**]. The integers $a_0, a_1, a_2, \dots$ are the partial quotients of α and the rationals $p_n/q_n = [a_0, a_1, a_2, \dots, a_n]$ with p_n, q_n relatively prime integers, called the convergence of α, are such that $p_n/q_n \to \alpha$ as $n \to \infty$. An irrational number α is badly approximated if there is a constant $c(\alpha)$ such that

$$|\alpha - p/q| > c(\alpha)/q^2 \tag{1.3}$$

for every rational p/q; such a constant $c(\alpha)$ must satisfy $0 < c(\alpha) < 1/\sqrt{5}$. α is badly approximated if and only if the partial quotients in its continued fraction expansion are bounded, i.e. $|a_n| \leqslant K(\alpha)$, $n = 0, 1, 2, \dots$. There is a continuum of many badly approximated numbers, and there exists a continuum of many numbers that are not badly approximated. The set of irrational numbers with bounded partial quotients coincides with the set of numbers of constant type, which are the numbers α such that $q\|q\alpha\| \geqslant 1/r$ for some real number $r \geqslant 1$ and all integers $q > 0$, where $\|b\|$ denotes the distance between the irrational number b and the closest integer. By a classical theorem of Lagrange, all real quadratic irrationals have bounded partial quotients. It follows from results of Borel [**10**] and Bernstein [**6**] that the set of all irrational numbers having bounded partial quotients is a dense uncountable and null subset of the real line. Examples of transcendental numbers having bounded partial quotients are given by

$$\sum_{i=0}^{\infty} \frac{1}{n^{2^i}}$$

for $n \geqslant 2$ an integer. An example of a transcendental number with unbounded partial quotient is given by

$$\sum_{n=1}^{\infty} n^{-2} = \pi^2/6.$$

For π^2 we have

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{\theta+\varepsilon}}, \quad \theta = 11.85078\cdots$$

for all $\varepsilon > 0$ and q sufficiently large.

From the above we infer that the set of α satisfying (T) is non-empty!

We shall study (1.1) by variational method but using only tools from convex analysis, as we shall assume that our nonlinearity l is of the special type

$$l = j_1 F_x^1 + j_2 F_x^2 + \cdots + j_{n-1} F_x^{n-1} - F_x^n - G, \tag{1.4}$$

where $F^i(t, y, x)$ (F_x^i are derivatives), $i = 1, \dots, n$, are convex functions with respect to the third variable and $j_1, \dots, j_{n-1}$ are numbers having values either -1 or $+1$. The main tools we apply are the following standard results in convex analysis for a convex lower semi-continuous function $h\colon X \to \mathbb{R}$ in a reflexive Banach space, namely,

$$h(w) + h^*(w^*) = \langle w, w^* \rangle_{X,X^*} \iff w^* \in \partial h(w),\ w \in X,\ w^* \in X^* \tag{1.5}$$

(∂h means subdifferential in the sense of convex analysis and it is equal to h_w if h is differentiable and h^* is the Fenchel conjugate of h) and the Fenchel–Young inequality

$$h(w) + h^*(w^*) \geqslant \langle w, w^* \rangle \quad \text{for all } w \in X,\ w^* \in X^*. \tag{1.6}$$

The functional we study is

$$J(x) = \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2 - L(t,y,x(t,y)))\,\mathrm{d}y\,\mathrm{d}t, \tag{1.7}$$

where $L_x = l$, $\Omega = (0,\pi)^n$, defined on $U^1 = H^1_{\mathrm{per}}((0,T); {}^0H^2(\Omega))$, where ${}^0H^2(\Omega)$ is defined in the next section. Our approach is quite different from the papers mentioned above. First we consider $L(t,y,\cdot)$ convex, then we consider $L(t,y,\cdot)$ as the difference of convex functions (but a more general case than in [**4**]), and finally we consider $L(t,y,\cdot)$ as a general finite combination of convex functions (see (1.4)). Moreover, we use a new variational approach. Our aim is to investigate (1.1) by studying critical points of the functional (1.7) using in an essential way the form of l and the tools from convex analysis only. To this effect we apply an approach that is based on ideas developed in [**20**] ($n = 1$). Our aim is to find a nonlinear subset $\hat{X}$ of U^1 and study modifications of (1.7) only on $\hat{X}$. Moreover, we give clear relations between the constant c, the type of α and the type of nonlinearity l. Examples illustrating the theory will be given.

2. Main results

We put $Q = (0,T) \times \Omega$ with $\Omega = (0,\pi)^n$ and $A = \Delta^\gamma$, $\gamma = 1$ or $\gamma = 2$ for the elliptic operator with the domain $D(A) = {}^0H^{2\gamma}(\Omega)$, where ${}^0H^{2\gamma}(\Omega)$ is a Sobolev space of functions

$$\biggl\{x \in H^{2\gamma}(\Omega)\colon \frac{\partial x}{\partial y_i^{2l}}(y_1,\dots,y_{i-1},0,y_{i+1},\dots,y_n) = \frac{\partial x}{\partial y_i^{2l}}(y_1,\dots,y_{i-1},\pi,y_{i+1},\dots,y_n) = 0, \quad (y_1,\dots,y_{i-1},y_{i+1},\dots,y_n) \in \Omega_i,\ l = 0,1,\ i = 1,\dots,n\biggr\},$$

where $\Omega_i = \{(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\colon (y_1,\dots,y_{i-1},y_i,y_{i+1},\dots,y_n) \in \Omega,\ y_i \in (0,\pi)\}$ (see [**34**]). By a solution of problem (1.1) we mean a function $x \in U = H^{2,0}_{\mathrm{per}}(\mathbb{R} \times \Omega)$ that satisfies (1.1) in a strong sense, where $H^{2,0}_{\mathrm{per}}$ is the usual Sobolev space of periodic functions with respect to the first variable with period T and such that for each $t \in \mathbb{R}$, $x(t,\cdot) \in {}^0H^4(\Omega)$.

Let $\mathcal{L} \subset \mathbb{Z}^n$ be the lattice of the integer vectors $k = (k_1, \dots, k_n)$ such that $k_i \geqslant 1$ for $i = 1, \dots, n$. Put $|k| = \sqrt{\sum_{i=1}^n k_i^2}$, $|k|^2 = k_1^2 + \dots + k_n^2$ and $\sum_{j,k} = \sum_{j=-\infty}^{+\infty} \sum_{k \in \mathcal{L}}$. Let $\mathcal{H}^2 = H^0_{\text{per}}(\mathbb{R}; H^4(\Omega))$ be the usual Sobolev space. The norm $\|\cdot\|_{\mathcal{H}^2}$ of $g \in \mathcal{H}^2$ we define as the square root of $\sum_{j,k} |k|^8 |g_{j,k}|^2$, i.e.

$$\|g\|_{\mathcal{H}^2} = \left(\sum_{j,k} |k|^8 |g_{j,k}|^2 \right)^{1/2},$$

where

$$g_{j,k} = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \int_0^T \int_\Omega g(t,y) \mathrm{e}^{-ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n \, \mathrm{d}y \, \mathrm{d}t. \tag{2.1}$$

To formulate our main results we need a modification of [**34**, Theorem 6.3.1] for the case of the higher-dimension periodic Navier boundary conditions (1.1).

Proposition 2.1. *Let $g \in \mathcal{H}^2$. Then there exists an $\bar{x} \in U$ that is a unique solution to*

$$\left.\begin{aligned} x_{tt}(t,y) + \Delta^2 x(t,y) &= g(t,y), \\ x(t,y) = \Delta x(t,y) &= 0, \quad t \in \mathbb{R},\ y \in \partial\Omega, \\ x(t+T,y) &= x(t,y), \quad t \in \mathbb{R},\ y \in \Omega, \end{aligned}\right\} \tag{2.2}$$

with

$$\bar{x}(t,y) = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \sum_{j,k} (-j^2 4\alpha^{-2} + |k|^4)^{-1} g_{j,k} \mathrm{e}^{ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n, \tag{2.3}$$

where $g_{j,k}$ is as in (2.1) and such that

$$\|\bar{x}\|_U \leqslant B \|g\|_{\mathcal{H}^2} \tag{2.4}$$

with $B^2 = ((2\alpha)^4 + 1)\alpha^2 c^{-2}$ independent of g and α and c are defined as in (T).

Corollary 2.2. *Let $g \in \mathcal{H}^2$ and let $\bar{x} \in U$ be a periodic Navier solution to (2.2). Then there exists an $\hat{x} \in U$ that is a unique solution to*

$$\begin{aligned} x_{tt}(t,y) &= -\Delta^2 \bar{x}(t,y) + g(t,y), \\ x(t,y) &= 0, \qquad t \in \mathbb{R},\ y \in \partial\Omega, \\ x(t+T,y) &= x(t,y), \quad t \in \mathbb{R},\ y \in \Omega, \end{aligned}$$

with

$$\hat{x}(t,y) = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \sum_{j,k} (-j^2 4\alpha^{-2} + |k|^4)^{-1} g_{j,k} \mathrm{e}^{ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n$$

and such that

$$\|\hat{x}\|_U \leqslant B \|g\|_{\mathcal{H}^2}$$

with the same B as in Proposition 2.1.

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Remark 2.3. Note that the constant B is determined by α and c. We note that below the constant B will always denote that occurring in (2.4).

Assumption M. *Let $F^1, F^2, \dots, F^n$ be functions of the variables (t, y, x) and let a function G of the variable (t, y) be given. $F^1, F^2, \dots, F^n$ are measurable with respect to (t, y) in $[0, T] \times \Omega$ for all x in $\mathbb{R}$, are continuously differentiable and convex with respect to x in $\mathbb{R}$, and $(t, y) \to F^i(t, y, 0)$ are integrable on $(0, T) \times \Omega$ and satisfy*

$$F^i(t, y, x) \geqslant a_i(t, y)x + b_i(t, y) \tag{2.5}$$

for some $a_i, b_i \in L^2([0, T] \times \Omega)$, $i = 1, \dots, n$, for all $(t, y) \in [0, T] \times \Omega$, $x \in \mathbb{R}$, $G(\cdot, \cdot) \in \mathcal{H}^2$. Let $j_1, \dots, j_{n-1}$ be a sequence of numbers having values either -1 or $+1$. Assume that our original nonlinearity (see (1.1)) has the form

$$l = -j_1 F_x^1 - j_2 F_x^2 - \cdots - j_{n-1} F_x^{n-1} - F_x^n - G. \tag{2.6}$$

There exist constants E_{n-1}, $\mathcal{F}$ such that $l_{n-1}(x), F_x^n(x) \in \mathcal{H}^2$ $(H_x(h) = H_x(\cdot, \cdot, h(\cdot, \cdot)))$ for x from

$$X_{l_n} = \{x \in U\colon \|x\|_U \leqslant B(E_{n-1} + \mathcal{F})\} \tag{2.7}$$

and $\|l_{n-1}(x)\|_{\mathcal{H}^2}$, $\|F_x^n(x)\|_{\mathcal{H}^2} + \|G(\cdot, \cdot)\|_{\mathcal{H}^2}$ are bounded in X_{l_n} by E_{n-1}, $\mathcal{F}$, respectively, where

$$l_{n-1} = -j_1 F_x^1 - j_2 F_x^2 - \cdots - j_{n-1} F_x^{n-1}.$$

Moreover, assume that $F_x^n(x) \in L^2$ for all $x \in U$, that

$$\int_0^T \int_\Omega F_x^n(t, y, x(t, y))\, \mathrm{d}y\, \mathrm{d}t \to \pm\infty \quad \text{when} \quad \int_0^T \int_\Omega x(t, y)\, \mathrm{d}y\, \mathrm{d}t \to \pm\infty,$$

and an argument minimum of

$$\min_{x \in H(X_{l_n})} \int_0^T \int_\Omega F^n(t, y, x(t, y))\, \mathrm{d}y\, \mathrm{d}t \tag{2.8}$$

is the same as that of

$$\min_{x \in H(X_{l_n})} \left(\int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t, y)|^2 + \tfrac{1}{2}|x_t(t, y)|^2)\, \mathrm{d}y\, \mathrm{d}t + \int_0^T \int_\Omega \sum_{i=1}^{n-1} j_i F^i(t, y, x(t, y))\, \mathrm{d}y\, \mathrm{d}t \right), \tag{2.9}$$

where $H(X_{l_n})$ is defined below.

Put $\bar{F}_x^n = F_x^n + G$, $\bar{F}^n = F^n + xG$. Define in X_{l_n} the map $X_{l_n} \ni x \to H(x) = v$, where v is a solution of the periodic Navier problem for

$$v_{tt}(t, y) + \Delta^2 v(t, y) = -l_{n-1}(t, y, x(t, y)) + \bar{F}_x^n(t, y, x(t, y)) \quad \text{almost everywhere on } (0, T) \times \Omega.$$

Comments on Assumption M. Assumption (2.5) is standard in convex analysis when one wants to apply its tools. Most continuous convex functions that appear in practice satisfy it. The aim is to consider a nonlinearity of type (2.6) and to use tools from convex analysis to treat nonlinearities of equations that are not derivatives of convex functions, for example, x^4. Consider, for example, $x^4 = (x^7 + x^5 + x^4) - (x^7 + x^5)$: it is a difference of the derivatives of two convex functions $(\frac{1}{8}x^8 + \frac{1}{6}x^6 + \frac{1}{5}x^5)$ and $(\frac{1}{8}x^8 + \frac{1}{6}x^6)$. The strongest assumption is the boundedness of $\|l_{n-1}(x)\|_{\mathcal{H}^2}$, $\|F_x^n(x)\|_{\mathcal{H}^2} + \|G(\cdot,\cdot)\|_{\mathcal{H}^2}$ in X_{l_n} by E_{n-1}, $\mathcal{F}$. The form of (2.7) suggests that l is linear, at least for large x. Thus, if l is superlinear, we have to consider it in a small neighbourhood of zero and for a small α (small period T). The last assumption concerning the argument of the minimum is a technical assumption and it relates to the use of tools of convex analysis. In order to check it we define the nonlinearity of the equation with the help of $n-2$ terms of the type F_x^j and calculate the minimum of (2.9) but with $n-2$ terms with x_0 as an argument minimum of (2.9), and then add the term $-\delta(x(t,y) - x_0(t,y))^2$ as the $(n-1)$th term in (2.9) and put $F_x^n(x) = \delta(x(t,y) - x_0(t,y))^2$ with sufficiently small $\delta > 0$. Examples are given in the following sections.

We now formulate the main theorem of the paper.

Theorem 2.4 (main theorem). *Under Assumption M there exists an $\bar{x} \in H(X_{l_n})$ such that $J(\bar{x}) = \inf_{x\in H(X_{l_n})} J(x)$ and $\bar{x}$ is a solution to (1.1).*

We now formulate a theorem that gives us additional information on solutions to (1.1), which is important in classical mechanics. This theorem is new for problem (1.1).

Theorem 2.5. *Let $\bar{x}$ be such that $J^{l_n}(\bar{x}) = \inf_{x\in H(X_{l_n})} J^{l_n}(x)$. Then there exists $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)$ such that for almost every (a.e.) $(t,y) \in (0,T)\times\Omega$,*

$$\left.\begin{aligned} \bar{p}(t,y) &= \bar{x}_t(t,y), \\ \bar{q}(t,y) &= \Delta\bar{x}(t,y), \\ \bar{p}_t(t,y) + \Delta\bar{q}(t,y) + l(t,y,\bar{x}(t,y)) &= 0 \end{aligned}\right\} \tag{2.10}$$

and

$$J^{l_n}(\bar{x}) = J_D^{l_n}(\bar{p},\bar{q},\bar{z}_1,\dots,\bar{z}_{n-2},\bar{z}_n),$$

where

$$\bar{z}_i = j_i F_x^i(t,y,\bar{x}(t,y)), \quad i = 1,\dots,n-1, \tag{2.11}$$

$$\begin{aligned} J^{l_n}(\bar{x}) = &\int_0^T \int_\Omega (-\tfrac{1}{2}\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega (j_1F^1(t,y,\bar{x}(t,y)) + j_2F^2(t,y,\bar{x}(t,y)) + \dots + \bar{F}^n(t,y,\bar{x}(t,y)))\,\mathrm{d}y\,\mathrm{d}t, \end{aligned}$$

$$
\begin{aligned}
J_D^{l_n}&(\bar{p}, \bar{q}, \bar{z}_1, \ldots, \bar{z}_{n-2}, \bar{z}_n) \\
&= \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2 \, \mathrm{d}y \, \mathrm{d}t - \frac{1}{2}\int_0^T \int_\Omega |\bar{p}(t,y)|^2 \, \mathrm{d}y \, \mathrm{d}t \\
&\quad - j_1 \int_0^T \int_\Omega F^{1*}(t, y, \bar{z}_1(t,y)) \, \mathrm{d}y \, \mathrm{d}t \\
&\quad \vdots \\
&\quad - j_{n-2} \int_0^T \int_\Omega F^{n-2*}(t, y, \bar{z}_{n-2}(t,y)) \, \mathrm{d}y \, \mathrm{d}t \\
&\quad - \int_0^T \int_\Omega F^{n-1*}(t, y, -(\bar{p}_t(t,y) + \Delta\bar{q}(t,y) - \bar{z}_1(t,y) \\
&\qquad\qquad\qquad - \cdots - \bar{z}_{n-2}(t,y) - \bar{z}_n(t,y))) \, \mathrm{d}y \, \mathrm{d}t,
\end{aligned} \tag{2.12}
$$

F^{i*}, $\bar{F}^{n*}$ are the Fenchel conjugates of F^i, $\bar{F}^n$, $i = 1, \ldots, n-1$, with respect to the third variable. Moreover, $\bar{x} \in H(X_{l_n})$.

The proofs of the theorems are given in §§ 3 and 4. They consist of several steps. First we prove Proposition 2.1. Next we prove Theorem 2.4 (the main theorem). We consider first the nonlinearity l consisting only of one function $j_1 F_x^1$, then we consider the case for the difference of two functions $j_1 F_x^1 - F_x^2$, and then by induction the general case.

3. Proof of Proposition 2.1

We use ideas based on [**34**, Theorem 6.3.1]. We know that $x \in L^2((0,T); L^2(\Omega))$ belongs to U if and only if

$$\sum_{j,k} (|k|^8 + |j|^4)|x_{j,k}|^2 < \infty, \tag{3.1}$$

where

$$x_{j,k} = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \int_0^T \int_\Omega x(t,y) \mathrm{e}^{-ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n \, \mathrm{d}y \, \mathrm{d}t.$$

Hence,

$$x(t,y) = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \sum_{j,k} x_{j,k} \mathrm{e}^{ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n. \tag{3.2}$$

The square root of (3.1) defines a norm in U. Similarly, for $g \in \mathcal{H}^2 \subset L^2((0,T); L^2(\Omega))$ we have

$$g(t,y) = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \sum_{j,k} g_{j,k} \mathrm{e}^{ij(2\pi/T)t} \sin k_1 y_1 \cdots \sin k_n y_n \tag{3.3}$$

with

$$\sum_{j,k} |k|^8 |g_{j,k}|^2 < \infty.$$

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Substituting (3.2) and (3.3) in (2.2) gives

$$(-j^2 4\alpha^{-2} + |k|^4)x_{j,k} = g_{j,k}, \quad j \in \mathbb{Z},\ k \in \mathcal{L}. \tag{3.4}$$

By Assumption (T) we can write a solution $\bar{x}$ of the problem (2.2) in the form (2.3). This function belongs to U since

$$\sum_{j,k} (|k|^8 + |j|^4)(-j^2 4\alpha^{-2} + |k|^4)^{-2}|g_{j,k}|^2 \leqslant B^2\|g\|^2_{\mathcal{H}^2} \tag{3.5}$$

with B some constant independent of g. This inequality is a direct consequence of the relation

$$\sup\{(|k|^8 + |j|^4)(-j^2 4\alpha^{-2} + |k|^4)^{-2}(1 + |j|)^0|k|^{-8};\ (j,k) \in \mathbb{Z} \times \mathcal{L}\} < \infty.$$

To prove it, let us put

$$\begin{aligned}
\Sigma_1 &= \{(j,k) \in \mathbb{Z} \times \mathcal{L};\ \alpha^{-1}|j| < |k|^2\},\\
\Sigma_2 &= \{(j,k) \in \mathbb{Z} \times \mathcal{L};\ |k| \leqslant \alpha^{-1}|j| \leqslant 2|k|^2\},\\
\Sigma_3 &= \{(j,k) \in \mathbb{Z} \times \mathcal{L};\ 2|k|^2 < \alpha^{-1}|j|\}.
\end{aligned}$$

We confine ourselves to estimates on the set Σ_2 (the other cases are similar) and again apply assumption (T):

$$\begin{aligned}
&(|k|^8 + |j|^4)(-j^2 4\alpha^{-2} + |k|^4)^{-2}|k|^{-8}\\
&\qquad \leqslant ((2\alpha)^4 + 1)|k|^8\alpha^4|k|^{-8}\left(\alpha + \frac{2|j|}{|k|^2}\right)^{-2}\left(\alpha - \frac{2|j|}{|k|^2}\right)^{-2}|k|^{-8}\\
&\qquad \leqslant ((2\alpha)^4 + 1)\alpha^{4-2}c^{-2}|k|^{8-8}\\
&\qquad \leqslant ((2\alpha)^4 + 1)\alpha^2 c^{-2}\\
&\qquad < \infty.
\end{aligned}$$

Hence, we also get estimate (2.4) with $B^2 = ((2\alpha)^4 + 1)\alpha^2 c^{-2}$.

3.1. Proof of Corollary 2.2

We follow the reasoning in the proof of Proposition 2.1. It is enough to observe, as $\bar{x}_{j,k}$ satisfies (3.4), that $\hat{x}_{j,k} = (-j^2 4\alpha^{-2} + |k|^4)^{-1}g_{j,k}$.

4. Proof of existence of solutions and their regularity for problem (1.1)

4.1. Simple case: the function $l = F_x$

First consider another equation

$$\left.\begin{aligned}
x_{tt}(t,y) + \Delta^2 x(t,y) + F_x(t,y,x(t,y)) &= 0,\\
x(t,y) = \Delta x(t,y) = 0, \quad t \in \mathbb{R},\ y &\in \partial\Omega,\\
x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y &\in \Omega,
\end{aligned}\right\} \tag{4.1}$$

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and corresponding to it the functional

$$J^F(x) = \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2 - F(t,y,x(t,y)))\,\mathrm{d}y\,\mathrm{d}t \tag{4.2}$$

defined on U^1. For that problem we assume the following hypotheses.

(G1) $F(t,y,x)$ is measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$, is continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y)\in(0,T)\times\Omega$, and $(t,y)\to F(t,y,0)$ is integrable on $(0,T)\times\Omega$, $F_x(t,y,x) = F_x^1(t,y,x) + F^2(t,y)$, $(t,y,x)\in(0,T)\times\Omega\times\mathbb{R}$, $F^2(\cdot,\cdot)\in\mathcal{H}^2$.

(G2) There exists a constant $E>0$ such that for $x\in X_F$,

$$\|F_x^1(x)\|_{\mathcal{H}^2} \leqslant E, \tag{4.3}$$

where

$$X_F = \{x\in U\colon \|x\|_U \leqslant B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^2})\}.$$

(G3) $F(t,y,x)\geqslant a(t,y)x + b(t,y)$ for some $a,b\in L^2((0,T)\times\Omega)$ for all $(t,y)\in(0,T)\times\Omega$, $x\in\mathbb{R}$.

Remark 4.1. Note that, except for convexity, the restrictions for F_x^1 are not strong, and they are natural.

Remark 4.2. The convexity assumption of $F(t,y,\cdot)$ is strong. For example, x^5 is non-convex. To overcome that problem (at least partly) we study in a later section the case in which $l = j_1F_x^1 + j_2F_x^2 + \cdots + F_x^n + G$, where F^i are convex and the j_i take values in $\{-1,+1\}$. Then $x^5 = x^6 + x^5 + x^2 - x^6 - x^2$ is equal to the difference of two convex functions $x^6 + x^5 + x^2$ and $x^6 + x^2$.

Exploiting the definition of the set X_F and Proposition 2.1, we prove the following lemma.

Lemma 4.3. *Let $x\in X_F$ and let v be a solution of the periodic Navier problem for*

$$v_{tt}(t,y) + \Delta^2 v(t,y) = -F_x(t,y,x(t,y)) \quad \textit{almost everywhere on } (0,T)\times\Omega. \tag{4.4}$$

Then

$$\|v\|_U \leqslant B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^2}).$$

Proof. Fix an arbitrary $x\in X_F$; thus, $F_x(x)\in\mathcal{H}^2$. Hence, by Proposition 2.1 there exists a unique solution $v\in U$ of the periodic Dirichlet problem for (4.4) satisfying

$$\|v\|_U \leqslant B\|F_x(x)\|_{\mathcal{H}^2}.$$

Next we get the following estimate:

$$B\|F_x(x)\|_{\mathcal{H}^2} \leqslant B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^2}).$$

Hence, we obtain

$$\|v\|_U \leqslant B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^2}).$$

□

Define in X_F the map $X_F \ni x \to H(x) = v$, where v is a solution of the periodic Navier problem (4.4). From the above lemma we see that $H(X_F) \subset X_F$. Put

$$X^F = \{\hat{x} \in U\colon \hat{x}_{tt}(t,y) = -\Delta^2 v(t,y) - F_x(t,y,v(t,y)),\ \text{where } v \in H(X_F)\}.$$

By Lemma 4.3 and Corollary 2.2 we see that $X^F \subset X_F$.

Remark 4.4. Let us observe that if $\bar{x}$ is a solution to (4.1), then, by the above lemma, it has to belong to X^F. Moreover, by Lemma 4.3, X^F is bounded in U. If $\{x_n\}$ is weakly convergent in U, then corresponding to it, by (4.4), $\{v_n\}$ is also weakly convergent—this is a consequence of the fact that a weakly convergent sequence in U is pointwise convergent. Therefore, X^F is weakly compact in U.

Next define the set X^{Fd}: an element $(p,q) \in H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)$ belongs to X^{Fd} provided that there exist $\hat{x} \in H(X_F)$, $x \in X^F$ such that for a.e. $(t,y) \in (0,T)\times\Omega$,

$$p(t,y) = x_t(t,y) \quad \text{and} \quad p_t(t,y) = -\Delta q(t,y) - F_x(t,y,\hat{x}(t,y)) \quad \text{with } q(t,y) = \Delta\hat{x}(t,y).$$

By Lemma 4.3 and Corollary 2.2 the set X^{Fd} is non-empty.

The dual functional to (4.2) is usually taken as

$$\begin{aligned} J_D^F(p,q) = \int_0^T \int_\Omega F^*(t,y,-(p_t(t,y) + \Delta q(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\ + \frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t, \end{aligned} \tag{4.5}$$

where F^* is the Fenchel conjugate of F with respect to the third variable and $J_D^F\colon H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega) \to \mathbb{R}$.

We will look at relationships between the functionals J^F and J_D^F on the sets X^F and X^{Fd}, respectively, using the variational principle at extreme points. It relates the critical values of both functionals and provides the necessary conditions that must be satisfied by the solution to problem (4.1).

Now we state the simple result of the paper, which is an existence theorem for a particular case of problem (4.1).

Theorem 4.5. *There exists an $\bar{x} \in H(X_F)$ such that*

$$\inf_{x\in H(X_F)} J^F(x) = J^F(\bar{x}).$$

Moreover, there exists $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)$ such that

$$J_D^F(\bar{p},\bar{q}) = J^F(\bar{x}) \tag{4.6}$$

and the following system holds:

$$\left.\begin{aligned} \bar{x}_t(t,y) &= \bar{p}(t,y), \\ \Delta\bar{x}(t,y) &= \bar{q}(t,y), \\ \bar{p}_t(t,y) + \Delta\bar{q}(t,y) &= -F_x(t,y,\bar{x}(t,y)). \end{aligned}\right\} \tag{4.7}$$

This result is new. However it has a strong assumption, the convexity of $F(t,y,\cdot)$, on the nonlinearity F_x. First we illustrate, by way of an example, a case of the above theorem.

Example 4.6. Assume that $n=4$ and choose T such that an α, which it defines (according to (T)), ensures that $B \leqslant \frac{1}{3}$. Let us consider $F^1(t,y,x)=\frac{1}{6}x^6+\frac{1}{5}x^5+\frac{1}{2}x^2+1$. Of course, $F^1(t,y,\cdot)$ is convex. Let $F^2(\cdot,\cdot)$ be any function in $(0,T)\times\Omega$ belonging to $\mathcal{H}^2$. We would like to stress that this B does not depend on the nonlinearity F^1 (see Remark 2.3). Note that $\|F^1_x(x)\|_{\mathcal{H}^2} \leqslant \|x\|^5_U + \|x\|^4_U + \|x\|_U$. Thus, take $E=1$, and then assumptions (G1)–(G3) are satisfied, so by the above theorem there exists an $\bar{x} \in X^F$ that is a solution to (4.1).

Remark 4.7. We can consider also the case in which $F(t,y,x)=x^6+x^5+x^2$. Then we take $F^1(t,y,x)=\frac{1}{6}x^6+\frac{1}{5}x^5+\frac{1}{2}x^2+x$ and $F^2(t,y)=-1$. Thus, the theorem in this case asserts that there exists a non-trivial solution to (4.1).

4.1.1. *The auxiliary results*

By (G1)–(G3), the definition of X_F, and the mean-value theorem, we obtain the following lemma.

Lemma 4.8. *There exist constants M_1, M_2 such that*

$$M_1 \leqslant \int_0^T \int_\Omega F(t,y,x(t,y))\,\mathrm{d}y\,\mathrm{d}t \leqslant M_2$$

for all $x \in X_F$.

Lemma 4.9. *The functional J attains its infimum on $H(X_F)$, i.e.* $\inf_{x\in H(X_F)} J^F(x) = J^F(\bar{x})$*, where $\bar{x} \in H(X_F)$.*

Proof. By definition of the set X_F and Lemma 4.8, we see that the functional J^F is bounded on X_F. We denote by $\{x^j\}$ a minimizing sequence for J^F in $H(X_F)$. This sequence has a subsequence that we denote again by $\{x^j\}$ converging weakly in U and strongly in U^1, and hence also strongly in $L^2((0,T)\times\Omega;\mathbb{R})$, to a certain element $\bar{x}\in U$. Moreover, $\{x^j\}$ is also convergent almost everywhere. Thus, by construction of the set $H(X_F)$ and Remark 4.4, we observe that $\bar{x}\in H(X_F)$. Hence,

$$\liminf_{j\to\infty} J^F(x^j) \geqslant J^F(\bar{x}).$$

Thus,

$$\inf_{x\in H(X_F)} J^F(x) = J^F(\bar{x}).$$

□

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4.1.2. *Variational principle*

As the set X_F is a ball in U it is clear that we cannot directly apply any known theorem to derive the necessary conditions.

Theorem 4.10. *Let* $\inf_{x\in H(X_F)} J^F(x) = J^F(\bar{x})$. *Then there exists* $(\bar{p},\bar{q}) \in H^1((0,T)\times \Omega)\times H^2((0,T)\times\Omega)$ *such that for a.e.* $(t,y)\in(0,T)\times\Omega$,

$$\bar{p}(t,y) = \bar{x}_t(t,y), \tag{4.8}$$

$$\bar{q}(t,y) = \Delta\bar{x}(t,y), \tag{4.9}$$

$$\bar{p}_t(t,y) + \Delta\bar{q}(t,y) + F_x(t,y,\bar{x}(t,y)) = 0 \tag{4.10}$$

and

$$J^F(\bar{x}) = J_D^F(\bar{p},\bar{q}).$$

Moreover, by Remark 4.4, $\bar{x}\in H(X_F)$.

Proof. Let $\bar{x}\in H(X_F)$ be such that $J^F(\bar{x}) = \inf_{x\in H(X_F)} J^F(x)$. This means that there exists an $\hat{x}\in H(X_F)$ such that

$$\hat{p}(t,y) = \hat{x}_t(t,y) \tag{4.11}$$

and

$$\hat{p}_t(t,y) = -\Delta\hat{q}(t,y) - F_x(t,y,\bar{x}(t,y)), \tag{4.12}$$

for a.e. $(t,y)\in(0,T)\times\Omega$, where $\hat{q}$ is given by

$$\hat{q}(t,y) = \Delta\bar{x}(t,y). \tag{4.13}$$

By the definitions of J^F, J_D^F, relations (4.12), (4.13) (cf. (1.5)) and the Fenchel–Young inequality (see (1.6)), it follows that

$$\begin{aligned}
J^F(\bar{x}) &= \int_0^T\int_\Omega(-\tfrac{1}{2}|\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2 - F(t,y,\bar{x}(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\
&\geqslant \int_0^T\int_\Omega\langle\bar{x}_t(t,y),\hat{p}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega|\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^T\int_\Omega\langle\Delta\bar{x}(t,y),\hat{q}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega|\hat{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^T\int_\Omega\langle\bar{x}(t,y),\hat{p}_t(t,y)+\Delta\hat{q}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega F(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_0^T\int_\Omega F^*(t,y,-(\hat{p}_t(t,y)+\Delta\hat{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^T\int_\Omega|\hat{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega|\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

Therefore, we get that

$$J^F(\bar{x}) \geqslant J_D^F(\hat{p},\hat{q}).$$

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Next observe that (again applying (1.5), but now to (4.11), and the Fenchel–Young inequality (1.6))

$$\begin{aligned}\inf_{x\in H(X_F)} J^F(x) &= J^F(\bar{x})\\ &\leqslant J^F(\hat{x})\\ &= \int_0^T\int_\Omega (\tfrac{1}{2}|\hat{x}_t(t,y)|^2 - \langle \hat{x}_t(t,y), \hat{p}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t\\ &\quad - \int_0^T\int_\Omega (\tfrac{1}{2}|\Delta\hat{x}(t,y)|^2 - \langle \Delta\hat{x}(t,y), \hat{q}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t\\ &\quad - \int_0^T\int_\Omega (F(t,y,\hat{x}(t,y)) + \langle \hat{x}(t,y), \hat{p}_t(t,y) + \Delta\hat{q}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t\\ &\leqslant \frac{1}{2}\Big(-\int_0^T\int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega |\hat{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t\Big)\\ &\quad + \int_0^T\int_\Omega F^*(t,y,-(\hat{p}_t(t,y)+\Delta\hat{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t\\ &= J_D^F(\hat{p},\hat{q}),\end{aligned}$$

and so

$$J^F(\bar{x}) \leqslant J_D^F(\hat{p},\hat{q}).$$

Thus, we have the equality $J^F(\bar{x}) = J_D^F(\hat{p},\hat{q})$, which implies that

$$\begin{aligned}&\int_0^T\int_\Omega F^*(t,y,-(\hat{p}_t(t,y)+\Delta\hat{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega F(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad + \frac{1}{2}\int_0^T\int_\Omega |\hat{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |\Delta\bar{x}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad\qquad = \frac{1}{2}\int_0^T\int_\Omega |\bar{x}_t(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

This together with $\hat{p}_t(\cdot) + \Delta\hat{q}(\cdot) = -F_x(\cdot,\bar{x}(\cdot))$ gives

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_\Omega |\bar{x}_t(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad - \int_0^T\int_\Omega \langle \bar{x}_t(t,y), \hat{p}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |\hat{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad\qquad + \frac{1}{2}\int_0^T\int_\Omega |\Delta\bar{x}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega (\langle \Delta\bar{x}(t,y), \hat{q}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t = 0.\end{aligned}$$

Hence, by standard convexity arguments we obtain the equality

$$\hat{p}(t,y) = \bar{x}_t(t,y).$$

Thus, taking into account (4.12) we infer that

$$\bar{x}_{tt}(t,y) + \Delta^2\bar{x}(t,y) = -F_x(t,y,\bar{x}(t,y)),$$

and therefore there exist $\bar{p} = \bar{x}_t$ and $\bar{q} = \Delta\bar{x}$, i.e. (4.8)–(4.10) are satisfied and so the assertions of the theorem are satisfied. Moreover, $\bar{x} \in H(X_F)$. □

4.2. Simple case: the function $l = -F_x$

A similar theorem to Theorem 4.5 is true for the problem

$$\left.\begin{aligned} x_{tt}(t,y) + \Delta^2 x(t,y) - F_x(t,y,x(t,y)) = 0,\\ x(t,y) = \Delta x(t,y) = 0, \quad t \in \mathbb{R},\ y \in \Omega,\\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in \Omega, \end{aligned}\right\} \tag{4.14}$$

with the corresponding functional

$$J^{F-}(x) = \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2 + F(t,y,x(t,y)))\,\mathrm{d}y\,\mathrm{d}t \tag{4.15}$$

defined on U^1 with the same hypotheses (G1)–(G3) on F and the set X_F. Counterparts of Lemmas 4.3–4.9 are still valid as the sign of F does not change their proofs. But of course, now $H(x) = v$ is defined by

$$v_{tt}(t,y) + \Delta^2 v(t,y) = F_x(t,y,x(t,y)) \quad \text{almost everywhere on } (0,T)\times\Omega$$

and

$$X^{-F} = \{\hat{x} \in U : \hat{x}_{tt}(t,y) = -\Delta^2 v(t,y) + F_x(t,y,v(t,y)), \text{ where } v \in H(X_F)\}.$$

Hence, we get for (4.14) the following theorem.

Theorem 4.11. *Let* $\inf_{x\in H(X_F)} J^{F-}(x) = J^{F-}(\bar{x})$. *Then there exists* $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega) \times H^2((0,T)\times\Omega)$ *such that for a.e.* $(t,y) \in (0,T)\times\Omega$,

$$\left.\begin{aligned} \bar{p}(t,y) = \bar{x}_t(t,y),\\ \bar{q}(t,y) = \Delta\bar{x}(t,y),\\ \bar{p}_t(t,y) + \Delta\bar{q}(t,y) - F_x(t,y,\bar{x}(t,y)) = 0 \end{aligned}\right\} \tag{4.16}$$

and

$$J^{F-}(\bar{x}) = J_D^{F-}(\bar{p},\bar{q}),$$

where

$$J_D^{F-}(\bar{p},\bar{q}) = -\int_0^T\int_\Omega F^*(t,y,-(\bar{p}_t(t,y) + \Delta\bar{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega |\bar{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t.$$

Moreover, $\bar{x} \in H(X_F)$.

To prove the above theorem, in the next section we also need an extension of Corollary 2.2 to a nonlinear case.

Proposition 4.12. *Let F be as in Theorem 4.11 and let $\bar{x} \in U$ be such that $\inf_{x\in H(X_F)} J^{F-}(x) = J^{F-}(\bar{x})$. There then exists an $\hat{x} \in U$ that is a solution to*

$$\begin{gathered} x_{tt}(t,y) - F_x(t,y,x(t,y)) = -\Delta^2 \bar{x}(t,y), \\ x(t,y) = 0, \qquad t \in \mathbb{R},\ y \in \partial\Omega, \\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in \Omega, \end{gathered} \tag{4.17}$$

and such that

$$\|\hat{x}\|_U \leqslant B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^2}). \tag{4.18}$$

First note that the functional corresponding to (4.17) has the form

$$J^{F-}(x) = \int_0^T \int_\Omega (\tfrac{1}{2}|x_t(t,y)|^2 + F(t,y,x(t,y)) - \tfrac{1}{2}|\Delta\bar{x}(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t$$

and by the assumption on F we see that it is strictly convex, lower semi-continuous and Gateux differentiable in U. Thus, it attains its minimum $\hat{x}$ in the set $H(X_F)$ and $\hat{x}$ satisfies (4.17) as well as estimate (4.18).

Proof of Theorem 4.11. Let $\bar{x} \in H(X_F)$ be such that $J^{-F}(\bar{x}) = \inf_{x\in H(X_F)} J^{-F}(x)$. Let us define an $\hat{x} \in X^{-F}$ such that

$$\hat{p}(t,y) = \hat{x}_t(t,y) \tag{4.19}$$

and

$$\hat{p}_t(t,y) = -\Delta^2 \bar{q}(t,y) + F_x(t,y,\hat{x}(t,y)), \tag{4.20}$$

for a.e. $(t,y) \in (0,T)\times\Omega$, where $\hat{q}$ is given by

$$\bar{q}(t,y) = \Delta\bar{x}(t,y). \tag{4.21}$$

From the above Proposition 4.12 we see that such an $\hat{x}$ exists. By the definitions of J^{-F} and J_D^{-F} it follows that

$$\begin{aligned} J^{-F}(\bar{x}) &= \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2 + F(t,y,\bar{x}(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\ &\geqslant \int_0^T \int_\Omega \langle \bar{x}_t(t,y), \hat{p}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \langle \Delta\bar{x}(t,y), \bar{q}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \langle \bar{x}(t,y), \hat{p}_t(t,y) + \Delta\hat{q}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega F(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega F^*(t,y,(\hat{p}_t(t,y) + \Delta\bar{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t. \end{aligned}$$

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Therefore, we get that

$$J^{-F}(\bar{x}) \geqslant J_D^{-F}(\hat{p},\bar{q}).$$

Next observe that

$$\begin{aligned}
\inf_{x\in H(X_F)} J^{-F}(x) &= J^{-F}(\bar{x}) \\
&\leqslant J^{-F}(\hat{x}) \\
&= \int_0^T \int_\Omega (\tfrac{1}{2}|\hat{x}_t(t,y)|^2 - \langle \hat{x}_t(t,y), \hat{p}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega (\tfrac{1}{2}|\Delta\hat{x}(t,y)|^2 - \langle \Delta\hat{x}(t,y), \bar{q}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega (F(t,y,\hat{x}(t,y)) + \langle \hat{x}(t,y), \hat{p}_t(t,y) + \Delta\bar{q}(t,y)\rangle)\,\mathrm{d}y\,\mathrm{d}t \\
&\leqslant \frac{1}{2}\biggl(-\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t\biggr) \\
&\quad + \int_0^T \int_\Omega F^*(t,y,(\hat{p}_t(t,y) + \Delta\bar{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\
&= J_D^{-F}(\hat{p},\bar{q}),
\end{aligned}$$

and so

$$J^{-F}(\bar{x}) \leqslant J_D^{-F}(\hat{p},\bar{q}).$$

Thus, we have the equality $J^{-F}(\bar{x}) = J_D^{-F}(\hat{p},\bar{q})$, which implies that

$$\begin{aligned}
&\int_0^T \int_\Omega F^*(t,y,(\hat{p}_t(t,y) + \Delta\bar{q}(t,y)))\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega F(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega |\Delta\bar{x}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad\qquad = \frac{1}{2}\int_0^T \int_\Omega |\bar{x}_t(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

This together with $\hat{p}_t(\cdot) + \Delta\bar{q}(\cdot) = F_x(\cdot,\hat{x}(\cdot))$ gives

$$\frac{1}{2}\int_0^T \int_\Omega |\bar{x}_t(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \int_0^T \int_\Omega \langle \bar{x}_t(t,y), \hat{p}(t,y)\rangle\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Hence, by standard convexity arguments we obtain the equality

$$\hat{p}(t,y) = \bar{x}_t(t,y).$$

Thus, taking into account (4.20) we infer that

$$\bar{x}_{tt}(t,y) + \Delta^2\bar{x}(t,y) - F_x(t,y,\bar{x}(t,y)) = 0,$$

and therefore the assertions of the theorem are satisfied. □

4.3. The case of the nonlinearity $F - G$

We now consider the more complicated problem

$$\left.\begin{aligned} x_{tt}(t,y) + \Delta^2 x(t,y) - G_x(t,y,x(t,y)) - j_1 F_x(t,y,x(t,y)) = 0, \\ x(t,y) = \Delta x(t,y) = 0, \quad t \in \mathbb{R},\ y \in \partial\Omega, \\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in \Omega, \end{aligned}\right\} \tag{4.22}$$

with the corresponding functional

$$J^{FG}(x) = \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2 + G(t,y,x(t,y)) + j_1 F(t,y,x(t,y)))\,\mathrm{d}y\,\mathrm{d}t$$

defined in U^1, where $j_1 \in \{-1,1\}$. The case in which L ($l = L_x$) is a difference of two convex non-autonomous functions, for the wave equation, was investigated in [**4**]; there the nonlinearity $l \in C([0,\pi] \times \mathbb{R}^2, \mathbb{R})$ had the form $l(t,y,x) = \lambda g(t,y,x) + \mu h(t,y,x)$ with $\lambda, \mu \in \mathbb{R}$, g superlinear in x, h sublinear in x, and both g and h were 2π-periodic in t and non-decreasing in x.

For problem (4.22) we assume that the following hypotheses hold.

(GG1) $F(t,y,x)$ and $G(t,y,x)$ are measurable with respect to (t,y) in $(0,T) \times \Omega$ for all x in $\mathbb{R}$, are continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y) \in (0,T) \times \Omega$, and $(t,y) \to F(t,y,0) - G(t,y,0)$ is integrable on $(0,T) \times \Omega$.

(GG2) There exist constants E, $\mathcal{G}$ such that for $x \in X_{FG} = \{x \in U\colon \|x\|_U \leqslant B(E+\mathcal{G})\}$, $G_x(x) \in \mathcal{H}^2$, $F_x(x) \in \mathcal{H}^2$ and $\|F_x(x)\|_{\mathcal{H}^2}$, $\|G_x(x)\|_{\mathcal{H}^2}$ are bounded in X_{FG} by E, $\mathcal{G}$, respectively. Moreover, assume that $G_x(x) \in L^2$ for all $x \in U$, that

$$\int_0^T \int_\Omega G_x(t,y,x(t,y))\,\mathrm{d}y\,\mathrm{d}t \to \pm\infty \quad \text{when} \quad \int_0^T \int_\Omega x(t,y)\,\mathrm{d}y\,\mathrm{d}t \to \pm\infty,$$

and that an argument minimum of

$$\min_{x \in H(X_{FG})} \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2 + j_1 F(t,y,x(t,y)))\,\mathrm{d}y\,\mathrm{d}t$$

is the same as that of

$$\min_{x \in H(X_{FG})} \int_0^T \int_\Omega G(t,y,x(t,y))\,\mathrm{d}y\,\mathrm{d}t,$$

where $H(X_{FG})$ is defined in the lemma below.

(GG3) $F(t,y,x)$ and $G(t,y,x)$ satisfy (G3).

We have, analogously to the simple case, the following lemma.

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Lemma 4.13. *Let $x \in X_{FG}$ and let v be a solution of the periodic Navier problem for*

$$v_{tt}(t,y) + \Delta^2 v(t,y) = j_1 F_x(t,y,x(t,y)) + G_x(t,y,x(t,y)) \quad \text{almost everywhere on } (0,T)\times\Omega. \tag{4.23}$$

Then

$$\|v\|_U \leqslant B(E+\mathcal{G}).$$

Define in X_{FG} the map $X_{FG} \ni x \to H(x) = v$, where v is a solution of the periodic Navier problem (4.23). Next, define the set X^d_{FG}: an element $(\hat{p},\hat{q},z) \in H^1((0,T)\times\Omega) \times H^2((0,T)\times\Omega) \times H^1((0,T)\times\Omega)$ belongs to X^d_{FG} provided that there exist $x \in X_{FG}$, $\hat{x} \in H(X_{FG})$ such that for a.e. $(t,y) \in (0,T)\times\Omega$,

$$\hat{p}(t,y) = \hat{x}_t(t,y) \quad \text{and} \quad \hat{p}_t(t,y) + \Delta\hat{q}(t,y) = j_1 F_x(t,y,x(t,y)) + z(t,y)$$

with

$$\hat{q}(t,y) = \Delta\hat{x}(t,y) \quad \text{and} \quad z(t,y) = G_x(t,y,x(t,y)).$$

By Lemma 4.13, the set X^d_{FG} is non-empty.

The dual functional to J^{FG} is then taken as

$$\begin{aligned} J^{FG}_D(p,q,z) = &-j_1\int_0^T\int_\Omega F^*(t,y,-(p_t(t,y)+\Delta q(t,y)-z(t,y)))\,\mathrm{d}y\,\mathrm{d}t \\ &-\int_0^T\int_\Omega G^*(t,y,z(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &+\frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t, \end{aligned} \tag{4.24}$$

where F^*, G^* are the Fenchel conjugates of F, G with respect to the third variable and

$$J^{FG}_D\colon H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)\times H^1((0,T)\times\Omega) \to \mathbb{R}.$$

Similarly to the case of the functional J^F, we have the following result.

Lemma 4.14. *The functional J^{FG} attains its minimum on $H(X_{FG})$, that is,*

$$\inf_{x\in H(X_{FG})} J^{FG}(x) = J^{FG}(\overline{x}),$$

where $\bar{x} \in H(X_{FG})$.

Theorem 4.15. *Let $J^{FG}(\bar{x}) = \inf_{x\in H(X_{FG})} J^{FG}(x)$. Then there exists $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)$ such that for a.e. $(t,y)\in(0,T)\times\Omega$,*

$$\left.\begin{aligned} \bar{p}(t,y) &= \bar{x}_t(t,y), \\ \bar{q}(t,y) &= \Delta\bar{x}(t,y), \\ \bar{p}_t(t,y) + \Delta\bar{q}(t,y) - G_x(t,y,\bar{x}(t,y)) - j_1F_x(t,y,\bar{x}(t,y)) &= 0 \end{aligned}\right\} \tag{4.25}$$

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and

$$J^{FG}(\bar{x}) = J_D^{FG}(\bar{p}, \bar{q}, \bar{z}),$$

where $\bar{z}(t, y) = G_x(t, y, \bar{x}(t, y))$. *Moreover,* $\bar{x} \in H(X_{FG})$.

Example 4.16. We show how to use the above theorem to solve the non-convex superlinear problem, assuming that $n = 4$ and with T such that α (defined by (T)) is such that $B \leqslant \frac{1}{3}$ and

$$\left.\begin{aligned} x_{tt} + \Delta^2 x + x^4 = 0,& \\ x(t, y) = \Delta x(t, y) = 0, \quad t \in \mathbb{R},\ y \in \partial\Omega,& \\ x(t + T, y) = x(t, y), \quad t \in \mathbb{R},\ y \in \Omega.& \end{aligned}\right\} \tag{4.26}$$

Let us put $\frac{1}{5}x^5 = \frac{1}{6}x^6 + \frac{1}{5}x^5 + \frac{1}{2}x^2 - \frac{1}{6}x^6 - \frac{1}{6}x^2$. Then assume that

$$F(t, y, x) = \tfrac{1}{6}x^6 + \tfrac{1}{5}x^5 + \tfrac{1}{2}x^2 \quad \text{and} \quad G(t, y, x) = \tfrac{1}{6}x^6 + \tfrac{1}{6}x^2.$$

$F(t, y, \cdot)$ and $G(t, y, \cdot)$ are convex. Note that $\|G_x(x)\|_{\mathcal{H}^2} \leqslant \|x\|_U^5 + \|x\|_U$. Thus, take, for example, $E = 1$ and $\mathcal{G} = 1$. Then assumptions (GG1)–(GG3) are satisfied, so by the above theorem there exists an $\bar{x} \in H(X_{FG})$ that is a solution to (4.26).

Proof of Theorem 4.15. Let us fix $j_1 = 1$, the $j_1 = -1$ case is similar. Let $\bar{x} \in H(X_{FG})$ be such that $J^{FG}(\bar{x}) = \inf_{x \in H(X_{FG})} J^{FG}(x)$. Let us choose $\hat{x} \in U$ (by the assumption on G_x, such an $\hat{x}$ exists) such that

$$J^{FG}(\bar{x}) = J^F(\bar{x}) + \int_0^T \int_\Omega G_x(t, y, \hat{x}(t, y))\bar{x}(t, y)\,\mathrm{d}y\,\mathrm{d}t - \int_0^T \int_\Omega G^*(t, y, G_x(\hat{x}(t, y)))\,\mathrm{d}y\,\mathrm{d}t.$$

By Theorem 4.10, putting

$$\hat{z}(t, y) = G_x(t, y, \hat{x}(t, y))$$

we have that

$$J^F(\bar{x}) + \int_0^T \int_\Omega G_x(t, y, \hat{x}(t, y))\bar{x}(t, y)\,\mathrm{d}y\,\mathrm{d}t = J_D^{F+}(\bar{p}, \bar{q}, \hat{z}), \tag{4.27}$$

where in that case

$$\begin{aligned} J_D^{F+}(\bar{p}, \bar{q}, \hat{z}) = \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t, y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_\Omega |\bar{p}(t, y)|^2\,\mathrm{d}y\,\mathrm{d}t& \\ + \int_0^T \int_\Omega F^*(t, y, -(\bar{p}_t(t, y) + \Delta\bar{q}(t, y) - \hat{z}(t, y)))\,\mathrm{d}y\,\mathrm{d}t& \end{aligned}$$

and

$$\begin{aligned} \bar{p}(t, y) &= \bar{x}_t(t, y), \\ \bar{q}(t, y) &= \Delta\bar{x}(t, y), \\ \bar{p}_t(t, y) + \Delta\bar{q}(t, y) - \hat{z}(t, y) &= -F_x(t, y, \bar{x}(t, y)). \end{aligned} \tag{4.28}$$

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Thus, by (4.27) we have the equality

$$J^{FG}(\bar{x}) + \int_0^T \int_\Omega G^*(t,y,G_x(\hat{x}(t,y)))\,\mathrm{d}y\,\mathrm{d}t = J_D^{F+}(\bar{p},\bar{q},\hat{z}).$$

From the above equalities we furthermore infer that

$$G_x(t,y,\hat{x}(t,y)) = G_x(t,y,\bar{x}(t,y)).$$

Putting this into (4.28) we obtain the assertions of the theorem. □

4.4. The more general case: proof of the main theorem

Let us consider now a sequence of convex (with respect to the third variable) functions $F^1, F^2, \dots, F^n$ of the variables (t,y,x) and a function G of the variable (t,y). Let $j_1,\dots,j_{n-1}$ be a sequence of numbers having values either -1 or $+1$. Let us assume that our original nonlinearity (see (1.1)) has the form

$$l = -j_1F_x^1 - j_2F_x^2 - \dots - j_{n-1}F_x^{n-1} - F_x^n - G.$$

Let Assumption M hold. To prove existence of a solution to (1.1) with a nonlinearity l we use an induction argument. To this effect, let us put $l_{n-1} = -j_1F_x^1 - j_2F_x^2 - \dots - j_{n-2}F_x^{n-2} - F_x^{n-1}$ and consider the problem

$$\left.\begin{aligned} x_{tt}(t,y) + \Delta^2 x(t,y) + l_{n-1}(t,y,x(t,y)) - G(t,y) = 0, &\quad t\in\mathbb{R},\ y\in\Omega,\\ x(t,y) = \Delta x(t,y) = 0, &\quad t\in\mathbb{R},\ y\in\partial\Omega,\\ x(t+T,y) = x(t,y), &\quad t\in\mathbb{R},\ y\in\Omega. \end{aligned}\right\} \tag{4.29}$$

For this problem we assume that the following hypotheses hold.

($\mathrm{G}_{n-1}1$) $F^1, F^2, \dots, F^{n-1}$ are measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$, are continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y)\in(0,T)\times\Omega$, and $(t,y)\to F^i(t,y,0)$ are integrable on $(0,T)\times\Omega$, $n=1,\dots,n-1$.

($\mathrm{G}_{n-1}2$) There exist some constants E_{n-1}, $\mathcal{G}$ such that, for

$$x \in X_{l_{n-1}} = \{x\in U\colon \|x\|_U \leqslant B(E_{n-1}+\mathcal{G})\},$$

$l_{n-1}(x)\in\mathcal{H}^2$, $G(\cdot)\in\mathcal{H}^2$ and $\|l_{n-1}(x)\|_{\mathcal{H}^2}$ and $\|G(\cdot)\|_{\mathcal{H}^2}$ are bounded in $X_{l_{n-1}}$ by E_{n-1} and $\mathcal{G}$, respectively.

($\mathrm{G}_{n-1}3$) $F^1, F^2, \dots, F^{n-1}$ satisfy (G3).

Similarly to Lemma 4.3, we have the following lemma.

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Lemma 4.17. *Let $x \in X_{l_{n-1}}$ and let v be a solution of the periodic Navier problem for*

$$v_{tt}(t,y) + \Delta^2 v(t,y) = -l_{n-1}(t,y,x(t,y)) + G(t,y) \quad \text{almost everywhere on } (0,T)\times\Omega, \tag{4.30}$$
$$v(t,y) = \Delta v(t,y) = 0, \quad t\in\mathbb{R},\ y\in\Omega,$$
$$v(t+T,y) = v(t,y), \quad t\in\mathbb{R},\ y\in\Omega.$$

Then

$$\|v\|_U \leqslant B(E_{n-1} + \mathcal{G}).$$

Define in $X_{l_{n-1}}$ the map $X_{l_{n-1}} \ni x \to H(x) = v$, where v is a solution of the periodic Navier problem (4.30).

The induction hypothesis is stated as follows.

(IH) Under hypotheses ($\mathrm{G}_{n-1}1$), ($\mathrm{G}_{n-1}2$) and ($\mathrm{G}_{n-1}3$), problem (4.29) has a solution in $H(X_{l_{n-1}})$, i.e. we assume that the following theorem holds.

Theorem 4.18. *Let $\bar{x}$ be such that $J^{l_{n-1}}(\bar{x}) = \inf_{x\in H(X_{l_{n-1}})} J^{l_{n-1}}(x)$. Then there exists $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)$ such that for a.e. $(t,y)\in(0,T)\times\Omega$,*

$$\bar{p}(t,y) = \bar{x}_t(t,y),$$
$$\bar{q}(t,y) = \Delta\bar{x}(t,y),$$
$$\bar{p}_t(t,y) + \Delta\bar{q}(t,y) + l_{n-1}(t,y,\bar{x}(t,y)) - G(t,y) = 0$$

and

$$J^{l_{n-1}}(\bar{x}) = J_D^{l_{n-1}}(\bar{p},\bar{q},\bar{z}_1,\dots,\bar{z}_{n-2}),$$

where

$$\bar{z}_i = j_i F_x^i(t,y,\bar{x}(t,y)), \quad i=1,\dots,n-2,$$
$$J^{l_{n-1}}(\bar{x}) = \int_0^T\int_\Omega (-\tfrac{1}{2}\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \left(\sum_{i=1}^{n-2} j_i F^i(t,y,\bar{x}(t,y)\right) + \bar{F}^{n-1}(t,y,\bar{x}(t,y)))\,\mathrm{d}y\,\mathrm{d}t,$$
$$\begin{aligned} J_D^{l_{n-1}}(\bar{p},\bar{q},\bar{z}_1,\dots,\bar{z}_{n-2}) &= \frac{1}{2}\int_0^T\int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega |\bar{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega \sum_{i=1}^{n-2} j_i F^{i*}(t,y,\bar{z}_i(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega \bar{F}^{n-1*}(t,y,-(\bar{p}_t(t,y) + \Delta\bar{q}(t,y) - \bar{z}_1(t,y) \\ &\qquad - \cdots - \bar{z}_{n-2}(t,y)))\,\mathrm{d}y\,\mathrm{d}t, \end{aligned}$$

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F^{i*} are the Fenchel conjugates of F^i, $i = 1, \dots, n-1$, with respect to third variable, and $\bar{F}^{n-1} = F^{n-1} + xG$. Moreover, $\bar{x} \in H(X_{l_{n-1}})$.

Now consider our problem (see (1.1)).

Using hypothesis (IH), we have the following lemma.

Lemma 4.19. *Let $x \in X_{l_n}$ and let v be a solution of the periodic Navier problem for*

$$v_{tt}(t,y) + \Delta^2 v(t,y) = -l_{n-1}(t,y,x(t,y)) + \bar{F}^n_x(t,y,x(t,y)) \quad \text{almost everywhere on } (0,T) \times \Omega. \tag{4.31}$$

$$v(t,y) = \Delta v(t,y) = 0, \quad t \in \mathbb{R},\ y \in \partial\Omega,$$
$$v(t+T,y) = v(t,y), \quad t \in \mathbb{R},\ y \in \Omega.$$

Then

$$\|v\|_U \leqslant B(E_{n-1} + \mathcal{F}).$$

Example 4.20. Following the same argument as in the previous examples (with the same n, T small), we obtain that l is a polynomial with respect to the variable x with suitable small positive coefficients (functions of the variables (t,y) belonging to $L^\infty((0,T) \times \Omega)$) that satisfies ($\mathrm{G}_{n-1}1$)–($\mathrm{G}_{n-1}3$) for some chosen constants E_{n-1} and $\mathcal{F}$.

Define in X_{l_n} the map $X_{l_n} \ni x \to H(x) = v$, where v is a solution of the periodic Navier problem (4.31). Define a functional corresponding to the problem (see (1.1)) with l defined by (2.6):

$$J^{l_n}(x) = \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta x(t,y)|^2 + \tfrac{1}{2}|x_t(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \sum_{i=1}^{n-1} j_i F^i(t,y,x(t,y))\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \bar{F}^n(t,y,x(t,y))\,\mathrm{d}y\,\mathrm{d}t. \tag{4.32}$$

Next, define the set $X^d_{l_n}$: an element $(p,q,z_1,\dots,z_{n-1}) \in H^1((0,T) \times \Omega) \times H^2((0,T) \times \Omega) \times \dots \times H^1((0,T) \times \Omega)$ belongs to $X^d_{l_n}$ provided that there exist $\hat{x} \in X_{l_n}$, $x \in H(X_{l_n})$ such that for a.e. $(t,y) \in (0,T) \times \Omega$,

$$p_t(t,y) + \Delta q(t,y) = z_1(t,y) + \dots + z_{n-1}(t,y) + \bar{F}^n_x(t,y,\hat{x}(t,y))$$

and

$$p(t,y) = x_t(t,y) \quad \text{with } q(t,y) = \Delta x(t,y),$$
$$z_1(t,y) = j_1 F^1_x(t,y,\hat{x}(t,y)), \ \dots,\ z_{n-1}(t,y) = j_{n-1} F^{n-1}_x(t,y,\hat{x}(t,y)).$$

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The dual functional to (4.32) is then taken as

$$\begin{aligned} J_D^{l_n}&(p,q,z_1,\dots,z_{n-2},z_n)\\ &= -j_{n-1}\int_0^T\int_\Omega F^{n-1*}(t,y,-(p_t(t,y)+\Delta q(t,y)-z_1(t,y)\\ &\qquad\qquad -\cdots - z_{n-2}(t,y)-z_n(t,y)))\,\mathrm{d}y\,\mathrm{d}t\\ &\quad + j_1\int_0^T\int_\Omega F^{1*}(t,y,z_1(t,y))\,\mathrm{d}y\,\mathrm{d}t\\ &\quad \vdots \qquad\qquad (4.33)\\ &\quad + j_{n-2}\int_0^T\int_\Omega F^{n-2*}(t,y,z_{n-2}(t,y))\,\mathrm{d}y\,\mathrm{d}t\\ &\quad + \frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t, \qquad (4.34)\end{aligned}$$

where F^{i*} are the Fenchel conjugates of F^i with respect to the third variable, $i = 1,\dots,n-1$, and

$$J_D^{l_n}\colon H^1((0,T)\times\Omega)\times H^2((0,T)\times\Omega)\times\cdots\times H^1((0,T)\times\Omega)\to\mathbb{R}.$$

Similarly to the case of the functional J^{FG}, we have the following result.

Lemma 4.21. *The functional J^{l_n} attains its infimum on $H(X_{l_n})$, that is,*

$$\inf_{x\in H(X_{l_n})} J^{l_n}(x) = J^{l_n}(\bar{x}),$$

where $\bar{x}\in H(X_{l_n})$.

We are now in position to prove Theorem 2.5.

Proof of the main theorem. Let us fix $j_{n-1}=1$, the $j_{n-1}=-1$ case is similar. Let $\bar{x}\in H(X_{l_n})$ be such that $J^{l_n}(\bar{x}) = \inf_{x\in H(X_{l_n})} J^{l_n}(x)$. Let us choose $\hat{x}\in U$ (by the assumption on F_x^n, such an $\hat{x}$ exists) such that

$$\begin{aligned} J^{l_n}(\bar{x}) &= \int_0^T\int_\Omega (-\tfrac{1}{2}\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \sum_{i=1}^{n-1} j_i F^i(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t\\ &\quad + \int_0^T\int_\Omega F_x^n(t,y,\hat{x}(t,y))\bar{x}(t,y)\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega F^{n*}t,y,F_x^n(\hat{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

By Theorem 4.18, putting

$$\hat{z}_n(t,y) = F_x^n(t,y,\hat{x}(t,y))$$

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we have that

$$\begin{aligned} J^{l_{n-1}}(\bar{x}) &= \int_0^T \int_\Omega (-\tfrac{1}{2}|\Delta\bar{x}(t,y)|^2 + \tfrac{1}{2}|\bar{x}_t(t,y)|^2)\,\mathrm{d}y\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \sum_{i=1}^{n-2} j_i F^i(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t - \int_0^T \int_\Omega F^{n-1}(t,y,\bar{x}(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega F_x^n(t,y,\hat{x}(t,y))\bar{x}(t,y)\,\mathrm{d}y\,\mathrm{d}t \\ &= J_D^{l_{n-1}}(\bar{p},\bar{q},\bar{z}_1,\dots,\bar{z}_{n-2},\hat{z}_n), \end{aligned} \tag{4.35}$$

where in that case

$$\begin{aligned} &J_D^{l_{n-1}}(\bar{p},\bar{q},\bar{z}_1,\dots,\bar{z}_{n-2},\hat{z}_n) \\ &\quad = \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\int_0^T \int_\Omega |\bar{p}(t,y)|^2\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad - j_1 \int_0^T \int_\Omega F^{1*}(t,y,\bar{z}_1(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad \vdots \\ &\qquad - j_{n-2} \int_0^T \int_\Omega F^{n-2*}(t,y,\bar{z}_{n-2}(t,y))\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad + \int_0^T \int_\Omega F^{n-1*}(t,y,-(\bar{p}_t(t,y) + \Delta\bar{q}(t,y) - \bar{z}_1(t,y) \\ &\qquad\qquad - \cdots - \bar{z}_{n-2}(t,y) - \hat{z}_n(t,y)))\,\mathrm{d}y\,\mathrm{d}t \end{aligned}$$

and

$$\begin{gathered} \bar{p}(t,y) = \bar{x}_t(t,y), \\ \bar{q}(t,y) = \Delta\bar{x}(t,y), \\ \bar{z}_i(t,y) = j_i F_x^i(t,y,\bar{x}(t,y)), \quad i = 1,\dots,n-2, \\ \bar{p}_t(t,y) + \Delta\bar{q}(t,y) - \bar{z}_1(t,y) - \cdots - \bar{z}_{n-2}(t,y) - \hat{z}_n(t,y) = -F_x^{n-1}(t,y,\bar{x}(t,y)). \end{gathered} \tag{4.36}$$

From (4.35) and the above equalities we furthermore obtain that

$$F_x^n(t,y,\hat{x}(t,y)) = F_x^n(t,y,\bar{x}(t,y)).$$

Putting this into (4.36) we obtain the assertions of the theorem. □

5. Conclusions

The nonlinear terms in problems of type (1.1) are usually monotone functions or sublinear at infinity (see the survey [**29**]). In [**4**] Bartsch *et al*. considered a nonlinear term that is the difference of two monotone functions. In this paper we extend the theory to a

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nonlinearity that is a finite linear combination of monotone functions. The open question arises as to whether the nonlinearity l can be of the form $l = f + g$, where f is a monotone function and g is only continuous. It was pointed out in [**10**] that arithmetical properties of the ratio $\alpha = T/\pi$ play an important role in solvability of the periodic Navier problem (1.1) (see also an interesting discussion on that problem in [**34**]). There are only a few papers that treat the problem in the case in which T is an irrational number such that $\alpha = T/\pi$ has not necessarily bounded partial quotients in its continued fraction with a nonlinear l. The case in which the spatial dimension $n \geqslant 2$ is investigated only by a few authors; an interesting discussion on current methods is contained in [**3**, § 4]. We have proved, for $n \geqslant 2$, that if α satisfies assumption (T), then with l a finite linear combination of monotone functions, problem (1.1) has a solution in $H^{2,0}_{\mathrm{per}}(\mathbb{R} \times \Omega)$, i.e. a strong solution. Moreover, we proved that this solution satisfies a variational and duality principle.

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