Proceedings of the Royal Society of Edinburgh, **150**, 2175–2188, *2020*
DOI:10.1017/prm.2019.12

Hyers–Ulam stability for equations with differences and differential equations with time-dependent and periodic coefficients

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(MS Received 7 September 2018; accepted 29 January 2019)

Let q be a positive integer and let (a_n) and (b_n) be two given $\mathbb{C}$-valued and q-periodic sequences. First we prove that the linear recurrence in $\mathbb{C}$

$$x_{n+2} = a_n x_{n+1} + b_n x_n, \quad n \in \mathbb{Z}_+ \tag{0.1}$$

is Hyers–Ulam stable if and only if the spectrum of the monodromy matrix $T_q := A_{q-1} \cdots A_0$ (i.e. the set of all its eigenvalues) does not intersect the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$, i.e. T_q is hyperbolic. Here (and in as follows) we let

$$A_n = \begin{pmatrix} 0 & 1 \\ b_n & a_n \end{pmatrix} \quad n \in \mathbb{Z}_+. \tag{0.2}$$

Secondly we prove that the linear differential equation

$$x''(t) = a(t)x'(t) + b(t)x(t), \quad t \in \mathbb{R}, \tag{0.3}$$

(where $a(t)$ and $b(t)$ are $\mathbb{C}$-valued continuous and 1-periodic functions defined on $\mathbb{R}$) is Hyers–Ulam stable if and only if $P(1)$ is hyperbolic; here $P(t)$ denotes the solution of the first-order matrix 2-dimensional differential system

$$X'(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad X(0) = I_2, \tag{0.4}$$

where I_2 is the identity matrix of order 2 and

$$A(t) = \begin{pmatrix} 0 & 1 \\ b(t) & a(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{0.5}$$

Keywords: Difference and differential equations, Discrete dichotomy, Hyers–Ulam stability

2010 *Mathematics subject classification:* 12H20, 34D09, 39B82

1. Short history of the problem

Oscar Perron originally settled (in the context of finite dimensional spaces) the equivalence between the exponential dichotomy of the homogeneous system $x'(t) = A(t)x(t)$ and the conditional stability of the following inhomogeneous one

$$y'(t) = A(t)y(t) + f(t, y(t)); \tag{1.1}$$

see [**17**]. The idea of passing from evolution equations to difference equations and vice versa (and used in the proof of theorem 2.12 below) has a long history that goes back to D. Henry [**15**].

The general framework of the stability problem for functional equations (called Hyers–Ulam stability) arose in 1940, due to a certain question posed by S. M. Ulam which was enunciated during a lecture he delivered in the Club of Mathematics at the University of Wisconsin. Much attention is given in the literature to Hyers–Ulam stability for differential and difference equations in a single variable; see [**1**, **2**, **6**–**8**, **13**, **14**, **16**, **19**–**21**, **23**, **24**, **27**, **28**] and the references therein. In [**18**], the Hyers–Ulam stability for linear recurrence of order n with constant coefficients is discussed. The main theorem in the above mentioned paper concerns the roots of the algebraic equation associated to the recurrence. The theorem says that the recurrence is Hyers–Ulam stable if all roots of its associated algebraic equation have modulus different to one. Brzdęk, Popa and Xu studied in [**6**] the case of nonstability for difference equations of order m. The results of the present paper generalizes and (or) complements various earlier outcomes proved or discussed in the recent monograph [**9**] and in the papers [**5**, **8**, **25**].

Of course it is difficult to find the exact solutions of differential equations and of equations with differences, so as a result it is important to find approximative solutions for these equations that are close (in a certain sense) to the exact solutions. Fixed point theorems (see e.g. [**12**, **22**]) and the increase of computer computing power makes it (generally speaking) easier to consider approximative solutions. However, it is not easy to determine the measure in which the approximate solutions are close to an exact solution. As a result it is important to establish simple criteria to ensure that all approximative solutions are close to an exact one. The purpose of our Hyers–Ulam programme is to highlight simple testing criteria for this type of stability. This article is part of this programme that was initiated by the papers [**1**, **2**, **10**, **11**]. An important novelty of this article is that the distance between an approximative solution and an exact solution (in the case of differential equations) is understood in the sense of the closed graph norm (see (2.13)) assuming implicitly that the solutions are from a Sobolev type space. A natural (open) problem is if the result in the continuous case is preserved with the 'sup' norm instead of the 'closed graph norm'.

The present paper is organized as follows. The next section contains the necessary definitions and preliminary results for the paper to be self-contained. At the end of this section the main results are presented. In the third section we present the proofs of these results and in the last section two concrete examples illustrating theoretical results are considered.

2. Background and the statement of the result

By $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ we denote the sets of integers, reals and respectively complex numbers and $\mathbb{Z}_+$ is the set of all nonnegative integers. $\mathbb{C}^m$ (with m given positive integer) is the set of all vectors $v = (\xi_1, \ldots, \xi_m)^T$ with $\xi_j \in \mathbb{C}$ for every integer $1 \leqslant j \leqslant m$; here and in as follows^T denotes the transposition. The norm on $\mathbb{C}^m$ is the well-known Euclidean norm defined by $\|v\| := (|\xi_1|^2 + \cdots + |\xi_m|^2)^{1/2}$. $\mathbb{C}^{m\times n}$ (with m and n given positive integers) denotes the set of all m by n matrices with complex entries. In particular, $\mathbb{C}^{m\times m}$ becomes a Banach algebra when endowed with the (Euclidean) matrix norm defined by

$$\|M\| := \sup_{\|v\|\leqslant 1} \|Mv\|, \quad v \in \mathbb{C}^m, \quad M \in \mathbb{C}^{m\times m}.$$

As is usual the rows and columns of a matrix $M \in \mathbb{C}^{m\times n}$ are identified by vectors of the corresponding dimensions and in that case its norm is the vector norm. Let M be a m by n matrix. The entry of M located at the intersection between the i^{th} row and the j^{th} column of the matrix M (with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$) is denoted by $[M]_{ij}$.

We outline the Hyers–Ulam problem for nonautonomous difference linear system of order m driven by a family $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ of m by m complex matrices. Consider the system

$$x_{n+1} = B_n x_n, \quad n \in \mathbb{Z}_+. \tag{2.1}$$

Let ε be a positive real number. A $\mathbb{C}^m$-valued sequence $(y_n)_{n\in\mathbb{Z}_+}$ is called a ε-approximative solution for (2.1) if

$$\|y_{n+1} - B_n y_n\| \leqslant \varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{2.2}$$

The family $\mathcal{B}$ (or the discrete system (2.1)) is said to be Hyers–Ulam stable if there exists a nonnegative constant L such that, for every $\varepsilon > 0$ and every ε-approximative solution (γ_n) of (2.1) there exists an exact solution (θ_n) of (2.1) such that

$$\sup_{n\in\mathbb{Z}_+} \|\gamma_n - \theta_n\| \leqslant L\varepsilon. \tag{2.3}$$

In the following we assume that the sequence (B_n) is q-periodic for some positive integer q. Denote by $T_q := B_0 \cdots B_{q-1}$ the monodromy matrix associated with the family $\mathcal{B}$. Recall that T_q is called hyperbolic if its spectrum does not intersect the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and the following result was stated in [[**11**], theorem 2.1, proposition 2.2].

Theorem 2.1. *The system (2.1) is Hyers–Ulam stable if and only if the monodromy matrix is hyperbolic or (with the terminology in [**11**]) it possesses a discrete dichotomy.*

Consider the following linear recurrence in $\mathbb{C}$ attached to the $\mathbb{C}$-valued and q-periodic sequences (a_n) and (b_n)

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$$x_{n+2} = a_n x_{n+1} + b_n x_n, \quad n \in \mathbb{Z}_+ \tag{2.4}$$

Recall (see [11]) that a sequence $(w_n)_{n\in\mathbb{Z}_+}$ of complex numbers is called an *ε-approximative solution* for (2.4) if

$$|w_{n+2} - a_n w_{n+1} - b_n w_n| \leqslant \varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{2.5}$$

The recurrence (2.4) is Hyers–Ulam stable if there exists a positive constant L such that for every $\varepsilon > 0$ and every ε - approximative solution (w_n) of (2.4) there exists an exact solution (θ_n) of (2.4) such that

$$\sup_{n\in\mathbb{Z}_+} |w_n - \theta_n| \leqslant L\varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{2.6}$$

For every positive integer n set $U(n, n) = I_2$ and for every pair (n, k) of nonnegative integers (with $n \geqslant k$) set $U(n, k) = A_{n-1} \cdots A_k$ (where

$$A_j := \begin{pmatrix} 0 & 1 \\ b_j & a_j \end{pmatrix}, \quad (j \in \mathbb{Z}_+).$$

Let $y \in \mathbb{C}^2$ be a given vector and (F_n) be a $\mathbb{C}^2$-valued given sequence. The solution of the following discrete Cauchy Problem

$$\begin{matrix} x_{n+1} = y_n + F^1_{n+1} \\ y_{n+1} = b_n x_n + a_n y_n + F^2_{n+1} \end{matrix}, \quad n \in \mathbb{Z}_+ \quad y = (x_0, y_0)^T, \tag{2.7}$$

(where $F_n = (F^1_n, F^2_n) \in \mathbb{C}^2$ and $F_0 = (0, 0)^T$)) is denoted and given by

$$\Phi_n(y, 0, (F_k)) := U(n, 0)y + \sum_{k=1}^{n} U(n, k)F_k. \tag{2.8}$$

In particular when (f_n) is a $\mathbb{C}$-valued sequence and taking $F^1_n = 0$ and $F^2_n = f_n$ let

$$\phi_n = \phi_n(y, 0, (f_k)) := [\Phi_n(y, 0, ((0, f_k)^T))]_{11}, \quad n \in \mathbb{Z}_+.$$

It is easy to see that (ϕ_n) is the solution of the scalar discrete Cauchy Problem

$$z_{n+2} = a_n z_{n+1} + b_n z_n + f_n, \quad n \in \mathbb{Z}_+ \quad z_0 = x_0, z_1 = y_0$$

and $\phi_{n+1} = [\Phi_n(y, 0, ((0, f_k)^T))]_{21}$.

Proposition 2.2 [11]. *The following two statements are equivalent.*

(1) *The recurrence (2.4) is Hyers–Ulam stable.*

(2) *There exists a positive constant L such that for every $\varepsilon > 0$, every $\mathbb{C}$-valued sequence $(f_n)_{n\in\mathbb{Z}_+}$ with $\sup_{n\in\mathbb{Z}_+} |f_n| \leqslant \varepsilon$, and every $x \in \mathbb{C}^2$ there exists $x_0 \in \mathbb{C}^2$ such that*

$$\sup_{n\in\mathbb{Z}_+} \left| \left[U(n, 0)(x - x_0) + \sum_{k=1}^{n} U(n, k)F_k \right]_{11} \right| \leqslant L\varepsilon, \tag{2.9}$$

where $F_{k+1} = (0, f_k)^T$ and $F_0 = (0, 0)^T$.

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Our first result reads as follows:

THEOREM 2.3. *Let (a_n) be a $\mathbb{C}$-valued and (b_n) be a $\mathbb{C}$-valued and periodic sequences with period q (q being a given positive integer). The following three statements are equivalent:*

(i) *The linear recurrence (2.4) is Hyers–Ulam stable.*

(ii) *The discrete linear system*

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= b_n x_n + a_n y_n \end{aligned} \tag{2.10}$$

is Hyers–Ulam stable.

(iii) *The monodromy matrix*

$$T_q = \begin{pmatrix} 0 & 1 \\ b_{q-1} & a_{q-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ b_0 & a_0 \end{pmatrix} \tag{2.11}$$

is hyperbolic.

REMARK 2.4.

(1) The equivalence between (ii) and (iii) follows from theorem 2.1.

(2) The case $q = 1$ (the autonomous case) was considered in [**20**]. Also, the case $q = 2$ was stated (under a slightly different but equivalent form) in [**11**]. The latter result depends on a technical lemma ([**11**], lemma 3.1) whose proof covers eight pages there.

Let $a(\cdot)$ and $b(\cdot)$ be $\mathbb{C}$-valued continuous and 1-periodic functions defined on $\mathbb{R}$. Consider the linear differential equation of second order.

$$x''(t) = a(t)x'(t) + b(t)x(t), \quad t \in \mathbb{R}, \tag{2.12}$$

and let $\varepsilon > 0$ be given.

Let k be a positive integer and denote by $CB^k(\mathbb{R}, \mathbb{C})$ the space of all $\mathbb{C}$-valued functions f and the derivatives $f', \ldots, f^{(k)}$ exist and are bounded and $f^{(k)}$ is continuous on $\mathbb{R}$. As is well-known $CB^1(\mathbb{R}, \mathbb{C})$ is a Banach space when it is endowed with the Closed Graph Norm given by

$$\|f\|_1 := \|f\|_\infty + \|f'\|_\infty, \quad f \in CB^1(\mathbb{R}, \mathbb{C}); \tag{2.13}$$

here $\| \cdot \|_\infty$ denotes the uniform norm on the space $CB(\mathbb{R}, \mathbb{C})$ that consists by all continuous and bounded functions on $\mathbb{R}$.

DEFINITION 2.5. A function $y(\cdot) \in CB^2(\mathbb{R}, \mathbb{C})$ is called:

(i) an ε approximative solution of (2.12) (with a given $\varepsilon > 0$) if one has

$$|y''(t) - a(t)y'(t) - b(t)y(t)| \leqslant \varepsilon, \quad \forall t \in \mathbb{R}$$

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and

(ii) an exact solution of (2.12) if

$$y''(t) = a(t)y'(t) + b(t)y(t), \quad t \in \mathbb{R}. \tag{2.14}$$

REMARK 2.6. Clearly the algebraic sum of an exact solution and an ε-approximative solution of (2.12) is also an ε-approximative solution of (2.12).

DEFINITION 2.7. We say that (2.12) is Hyers–Ulam stable if there exists a positive constant L such that for every $\varepsilon > 0$ and every ε-approximative solution $y(\cdot)$ of (2.12) there exists an exact solution $\phi(\cdot)$ of (2.12) and

$$\|y(\cdot) - \phi(\cdot)\|_1 \leqslant L\varepsilon.$$

Consider the family of matrices

$$A(t) = \begin{pmatrix} 0 & 1 \\ b(t) & a(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{2.15}$$

Clearly, the map $t \mapsto A(t) : \mathbb{R} \to \mathbb{C}^{2\times 2}$ is 1-periodic and continuous.

By following [**2**] we introduce:

DEFINITION 2.8. A $\mathbb{C}^2$-valued function $Y(\cdot)$ is an ε-approximative solution of the differential system

$$X'(t) = A(t)X(t), \quad t \in \mathbb{R} \tag{2.16}$$

(for a given positive ε) if

$$\|Y'(\cdot) - A(\cdot)Y(\cdot)\|_\infty \leqslant \varepsilon.$$

DEFINITION 2.9. The system (2.16) is Hyers–Ulam stable if there exists a positive constant L such that for every ε-approximative solution $Y(\cdot)$ there exists an exact solution $\Phi(\cdot)$ of (2.16) and $\|Y(\cdot) - \Phi(\cdot)\|_\infty \leqslant L\varepsilon$.

Let $P(\cdot)$ denote the solution of the first-order 2 by 2 differential matrix system

$$X'(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad X(0) = I_2, \tag{2.17}$$

where I_2 is the identity matrix of order 2. As is well-known $P(t)$ is an invertible matrix for every $t \in \mathbb{R}$. For every pair $(t,s) \in \mathbb{R} \times \mathbb{R}$ let $U(t,s) := P(t)P^{-1}(s)$. Thus the solution $\Psi(t) := \Psi(t, 0, y, F)$ of the Cauchy Problem

$$X'(t) = A(t)X(t) + F(t), \quad t \in \mathbb{R}, \quad X(0) = y$$

(with $y \in \mathbb{C}^2$ given vector and $F(\cdot)$ a given continuous $\mathbb{C}^2$-valued function) is

$$\Psi(t) = U(t,0)y + \int_0^t U(t,s)F(s)\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

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REMARK 2.10.

(1) Clearly, if $f(\cdot)$ is a $\mathbb{C}$-valued continuous function defined on $\mathbb{R}$ and $\|f\|_\infty < \varepsilon$ then any solution of the differential equation

$$x''(t) - a(t)x'(t) - b(t)x(t) = f(t), \quad t \in \mathbb{R} \tag{2.18}$$

is an ε-approximative solution of (2.12).

(2) Conversely, every ε-approximative solution of (2.12) comes in this way. Indeed, let $y(\cdot)$ be an ε-approximative solution of (2.12) and set $f(t) := y''(t) - a(t)y'(t) - b(t)y(t)$. Then $\|f\|_\infty \leqslant \varepsilon$ and $y(\cdot)$ is the solution of (2.18) with the initial conditions $x(0) = y(0)$ and $x'(0) = y'(0)$.

PROPOSITION 2.11. *The differential equation (2.12) is Hyers–Ulam stable (in the sense of Definition 2.7) provided there exists a positive constant L such that for every $\varepsilon > 0$, every $\mathbb{C}$-valued continuous function f (with $\|f\|_\infty \leqslant \varepsilon$), and every $x \in \mathbb{C}^2$ there exists $x_0 \in \mathbb{C}^2$ and*

$$\left\| \left[U(\cdot, 0)(x - x_0) + \int_0^{\cdot} U(\cdot, s)F(s)\, \mathrm{d}s \right] \right\|_\infty \leqslant \frac{L\varepsilon}{2}, \tag{2.19}$$

where $F(s) = (0, f(s))^T$.

Proof. Is enough to see that if $\phi(\cdot)$ is a solution of

$$y''(t) - a(t)y'(t) - b(t)y(t) = f(t)$$

with the initial condition

$$(\phi(0), \phi'(0)) = (x - x_0)^T,$$

then

$$U(t, 0)(x - x_0) + \int_0^t U(t, s)F(s)\, \mathrm{d}s = (\phi(t), \phi'(t))^T, \quad t \in \mathbb{R}. \tag{2.20}$$

□

THEOREM 2.12. *Let $a(\cdot)$ and $b(\cdot)$ be $\mathbb{C}$-valued continuous and 1-periodic functions defined on $\mathbb{R}$. The following three statements are equivalent:*

(i) *The differential equation (2.12) is Hyers–Ulam stable.*

(ii) *The linear differential system*

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= b(t)x(t) + a(t)y(t), \quad t \in \mathbb{R} \end{aligned} \tag{2.21}$$

is Hyers–Ulam stable.

(iii) *The monodromy matrix $P(1)$ is hyperbolic.*

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3. Proofs

Proof of theorem 2.3. We already noticed that, via theorem 2.1, the statements (ii) and (iii) are equivalent. □

Proof of **(i) ⇒ (iii)**. We argue by contradiction. Assume that the complex number λ_1 (with $|\lambda_1| = 1$) is an eigenvalue of T_q. Under such an assumption we show that for every $\varepsilon > 0$ and every initial condition $y = (z_0, z_1)^T$ there exists a sequence (f_n) with $\|(f_n)\|_\infty \leqslant \varepsilon$ such that the sequence (ϕ_n) is unbounded, and that contradicts (2.9) for $y := x - x_0$. Noticing that

$$\Phi_n := \Phi_n(y, 0, ((0, f_k)^T)) = (\phi_n, \phi_{n+1})^T$$

and thus (for our purpose) is enough to prove that $([\Phi_n]_{11})$ (or $([\Phi_n]_{21})$) is unbounded. We break the proof into four cases. □

Case 1. Let $\sigma(T_q) = \{\lambda_1, \lambda_2\} \subset \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $[T_q]_{12} \neq 0$.

Let $\varepsilon > 0$, $u_0 = (0, (\varepsilon/2[T_q]_{12}))^T \in \mathbb{C}^2$. Notice that $\|u_0\| \leqslant \varepsilon$. Set

$$F_{nq} := \lambda_1^n u_0, (n \in \mathbb{Z}_+) \quad \text{and} \quad F_k = 0 \quad \text{when } k \text{ is not a multiple of } q. \tag{3.1}$$

From the Spectral Decomposition Theorem, (see e.g. [**1**, lemma 4.5]), there are 2 by 2 matrices B and C such that

$$T_q^n = \lambda_1^n B + \lambda_2^n C \quad n \in \mathbb{Z}_+.$$

Now, taking into account (2.8) one has

$$\begin{aligned} E_{\lambda_1}\Phi_{nq}(x - x_0, 0, (F_k)) &= \lambda_1^n B(x - x_0) + \sum_{k=1}^{n} E_{\lambda_1} U(nq, kq)\lambda_1^k u_0 \\ &= \lambda_1^n B(x - x_0) + \sum_{k=1}^{n} E_{\lambda_1} T^{n-k} \lambda_1^k u_0 \\ &= \lambda_1^n [B(x - x_0) + nBu_0], \quad n \in \mathbb{Z}_+, \end{aligned} \tag{3.2}$$

where

$$B = \frac{1}{\lambda_1 - \lambda_2}(T_q - \lambda_2 I_2) \tag{3.3}$$

and E_{λ_1} is the elementary (Riesz) projection; see for example ([**11**] pages 911–912) for further details.

Hence,

$$|[nBu_0]_{11}| = n\frac{\varepsilon}{2|\lambda_1 - \lambda_2|} \to \infty \text{ as } n \to \infty$$

and thus the sequence $(|[E_{\lambda_1}\Phi_{nq}]_{11}|)_n$ is unbounded and a contradiction arises.

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Case 2. Let $\sigma(T_q) = \{\lambda_1, \lambda_2\} \subset \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $[T_q]_{12} = 0$.

2.1. Let

$$T_q = \begin{pmatrix} \lambda_2 & 0 \\ * & \lambda_1 \end{pmatrix}.$$

Thus (via (3.3)) one has

$$B = \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix}.$$

Set (F_k) as in (3.1) with $u_0 := (0, (\varepsilon/2))^T$. Then $\|u_0\| \leqslant \varepsilon$ and

$$|[nBu_0]_{21}| = n\frac{\varepsilon}{2} \to \infty.$$

2.2.

$$\text{Let } T_q = \begin{pmatrix} \lambda_1 & 0 \\ * & \lambda_2 \end{pmatrix} \text{ and one has } B = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}.$$

Set

$$F_k = \begin{cases} \lambda_1^n u_0, & \text{if } k = nq - 1 \\ 0, & \text{if } k + 1 \text{ is not a multiple of } q \end{cases} \tag{3.4}$$

where $u_0 := (0, (\varepsilon/2))^T$. One has

$$\begin{aligned} E_{\lambda_1}\Phi_{nq}(x - x_0, 0, (F_k)) &= \lambda_1^n B(x - x_0) + \sum_{k=1}^{n} E_{\lambda_1} U(nq, kq - 1)\lambda_1^k u_0 \\ &= \lambda_1^n B(x - x_0) + \sum_{k=1}^{n} E_{\lambda_1} T^{n-k} U(q, q - 1)\lambda_1^k u_0 \\ &= \lambda_1^n [B(x - x_0) + nBA_{q-1}u_0] \end{aligned} \tag{3.5}$$

and

$$BA_{q-1} = \begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}$$

and this yields

$$|[nBA_{q-1}u_0]_{11}| = n\frac{\varepsilon}{2} \to \infty, \ (\text{ as } n \to \infty).$$

Case 3. Let $\sigma(T_q) = \{\lambda_1\}$ and $[T_q]_{12} \neq 0$.

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In this case the Spectral Decomposition Theorem yields

$$T_q^n = \lambda_1^n(nB + I_2) \quad (n \in \mathbb{Z}_+), \quad \text{with } B := \frac{1}{\lambda_1}(T_q - \lambda_1 I_2). \tag{3.6}$$

Set again (F_n) as in (3.1) with $u_0 := (0, (\varepsilon/2))^T$. Thus $\|u_0\| \leqslant \varepsilon$ and

$$\lambda_1^{-n}\Phi_{nq}(x - x_0, 0, (F_k)) = (nB + I_2)(x - x_0) + \left[nI_2 + \frac{n(n-1)}{2}B\right]u_0$$

and this yields

$$[\lambda_1^{-n}\Phi_{nq}]_{11} = [(nB + I_2)(x - x_0)]_{11} + \frac{n(n-1)}{2}[B]_{12}\frac{\varepsilon}{2}$$

that is unbounded since $[B]_{12} = [T_q]_{12} \neq 0$.

Case 4. Let $\sigma(T_q) = \{\lambda_1\}$ and $[T_q]_{12} = 0$.

Set $G_{nq-1} = \lambda_1^n u_0$ (whenever n is a positive integer) and $G_k = 0$ elsewhere (i.e. when $k+1$ is not a multiple of q). Here u_0 is the same as in the third case. Now $\Phi_{nq} = \Phi_{nq}(x - x_0, 0, (G_k))$ is given by

$$\begin{aligned}
\Phi_{nq} &= \lambda_1^n(nB + I_2)(x - x_0) + \sum_{k=1}^{n} U(nq, kq)U(kq, kq-1)G_{kq-1} \\
&= \lambda_1^n(nB + I_2)(x - x_0) + \sum_{k=1}^{n} T^{n-k}A_{q-1}\lambda_1^k u_0 \\
&= \lambda_1^n\left[(nB + I_2)(x - x_0) + \sum_{k=1}^{n}[(n-k)B + I_2]A_{q-1}u_0\right] \\
&= \lambda_1^n\left[(nB + I_2)(x - x_0) + \left(\frac{n(n-1)}{2}B + nI_2\right)A_{q-1}u_0\right].
\end{aligned} \tag{3.7}$$

Since

$$B = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \quad (\text{for some } a \in \mathbb{C}),$$

one has

$$\lambda_1^{-n}[\Phi_{nq}]_{11} = [x - x_0]_{11} + [nA_{q-1}u_0]_{11} = [x - x_0]_{11} + n\frac{\varepsilon}{2}$$

that is unbounded.

Proof of **(ii)** $\Rightarrow$ **(i)**. Is enough to see that for every $\varepsilon > 0$, every $\mathbb{C}$-valued sequence (f_n) with $\|(f_n)\|_\infty \leqslant \varepsilon$ and every vector $y \in \mathbb{C}^2$ one has

$$\|\Phi_n(y, 0, ((0, f_k)^T))\|_\infty \geqslant \|[\Phi_n(y, 0, ((0, f_k)^T))]_{11}\|_\infty. \qquad \square$$

Proof of theorem 2.12. From [**2**, theorem 1.3] it follows that (ii) and (iii) are equivalent. □

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Proof of **(i)** $\Rightarrow$ **(iii)**. We argue by contradiction. Assume that the spectrum of $T := P(1)$ intersects the unit circle. Let $\mathrm{e}^{\mathrm{i}\mu}$ with $\mu \in \mathbb{R}$ be an eigenvalue of T. In view of proposition 2.11 it is enough to prove that for some $\varepsilon > 0$ and some continuous function f (with $\|f\|_\infty \leqslant \varepsilon$) and all $(x - x_0) \in \mathbb{C}^2$, the map

$$t \mapsto X(t) = X(t, 0, x - x_0, F) := U(t, 0)(x - x_0) + \int_0^t U(t, s)F(s)\,\mathrm{d}s, \tag{3.8}$$

(where $F(s) = (0, f(s))^T$) is unbounded on $\mathbb{R}_+$. □

Let $\varepsilon > 0$ be given and let $u_0 \in \mathbb{C}^2$ with $[u_0]_{11} = 0$, $[u_0]_{21} \neq 0$ and $\|u_0\| \leqslant \varepsilon$. Let G be the 1-periodic, $\mathbb{C}^2$-valued map that is defined on the interval $[0, 1]$ by

$$G(s) := \sin(\pi s)\frac{P(s)}{M}u_0, \quad s \in [0, 1] \tag{3.9}$$

(where $M = \sup_{t\in[0,1]} \|U(t, 0)\|$) and set $F(s) := \mathrm{e}^{\mathrm{i}\mu s}G(s)$. Thus, via (3.8), successively one has

$$\begin{aligned} X(n) &= U(n, 0)(x - x_0) + \sum_{k=0}^{n-1} \int_k^{k+1} U(n, s)F(s)\,\mathrm{d}s \\ &= U(n, 0)(x - x_0) + \sum_{k=0}^{n-1} \int_0^1 U(n, k + r)\,\mathrm{e}^{\mathrm{i}\mu(k+r)}G(r)\,\mathrm{d}r \\ &= U(n, 0)(x - x_0) + \sum_{k=0}^{n-1} \int_0^1 T^{n-k-1}\,\mathrm{e}^{\mathrm{i}\mu k}U(1, r)\,\mathrm{e}^{\mathrm{i}\mu r}G(r)\,\mathrm{d}r \\ &= T^n(x - x_0) + \frac{L_\mu}{M}\sum_{k=0}^{n-1} T^{n-k}\,\mathrm{e}^{\mathrm{i}\mu k}u_0, \end{aligned} \tag{3.10}$$

where $L_\mu := \int_0^1 \mathrm{e}^{\mathrm{i}\mu r} \sin \pi r\,\mathrm{d}r$. Using the well-known Euler formula

$$\left(\sin \pi t = \frac{1}{2i}(\mathrm{e}^{\mathrm{i}\pi t} - \mathrm{e}^{-\mathrm{i}\pi t}), \quad i \in \mathbb{C}, i^2 = -1\right)$$

it is easy to see (we omit the details) that L_μ is a nonzero complex number for all $\mu \in \mathbb{R}$.

Now, whenever $[T]_{21} \neq 0$ (as in the proof of theorem 2.3) we can choose a suitable vector u_0 such that the sequence $(X(n))$ is unbounded and a contradiction arises.

When $[T]_{21} = 0$, let G_1 be the 1-periodic, $\mathbb{C}^2$-valued map that is defined on the interval $[0, 1]$ by

$$G_1(s) := \sin(\pi s)\frac{P(s)A_0}{M_1}u_0, \quad s \in [0, 1] \tag{3.11}$$

(where $M_1 = \sup_{t\in[0,1]} \|U(t, 0)A_0\|$) and set $F_1(s) := \mathrm{e}^{\mathrm{i}\mu s}G_1(s)$.

In this case, (as in (3.10)) $X(n)$ (corresponding to F_1) is given by

$$X(n, 0, x - x_0, F_1) = T^n(x - x_0) + \frac{L_\mu}{M} \sum_{k=0}^{n-1} T^{n-k} A_0 \, \mathrm{e}^{\mathrm{i}\mu k} u_0$$

and it is unbounded (we omit the details).

Proof of **(ii)** $\Rightarrow$ **(i)**. Since the system (2.21) is Hyers–Ulam stable, for some $L > 0$ and every $\varepsilon > 0$, every $\mathbb{C}$-valued functions f and g with

$$\max\{\|f\|_\infty, \|g\|_\infty\} \leqslant \frac{\varepsilon}{2}$$

and every $x \in \mathbb{C}^2$ there exists $x_0 \in \mathbb{C}^2$ such that

$$\left\| \left[U(\cdot, 0)(x - x_0) + \int_0^{\cdot} U(\cdot, s) G(s) \, \mathrm{d}s \right] \right\|_\infty \leqslant \frac{L\varepsilon}{2}, \tag{3.12}$$

where $G(s) = (g(s), f(s))^T$, [**2**, remark 2.4]. □

In particular, for $g = 0$ and taking into account that

$$U(t, 0)(x - x_0) + \int_0^t U(t, s) F(s) \, \mathrm{d}s = (\phi(t), \phi'(t))^T, \quad t \in \mathbb{R} \tag{3.13}$$

(where $F(s) = (0, f(s))^T$ and $\phi(\cdot)$ is the solution of (2.18) with the initial condition $(\phi(0), \phi'(0)) = (x - x_0)^T)$, (3.12) and (3.13) yield

$$\|\phi\|_1 = \|\phi\|_\infty + \|\phi'\|_\infty \leqslant L\varepsilon$$

and the assertion follows.

4. Examples

Example 4.1. The linear recurrence of order 2

$$x_{n+2} = \sin \frac{2n\pi}{3} x_{n+1} + \cos \frac{2n\pi}{3} x_n, \quad n \in \mathbb{Z} \tag{4.1}$$

is Hyers–Ulam stable.

Indeed, with the above notation one has

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

and the monodromy matrix associated to (4.1) is

$$T_3 = A_2 A_1 A_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{5}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

The characteristic equation of T_3 is $\lambda^2 - (3\sqrt{3}/4)\lambda - (1/4) = 0$ and the absolute value of its solutions is different to 1.

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EXAMPLE 4.2. The differential equation of order 2

$$x''(t) = \exp(2\pi t)x'(t), \quad t \in \mathbb{R} \tag{4.2}$$

is not Hyers–Ulam stable.

Indeed, with the above notation, for $t \in \mathbb{R}$, one has

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & \exp(2\pi \mathrm{i} t) \end{pmatrix}$$

and the map $t \mapsto A(t) : \mathbb{R} \to \mathbb{C}^{2\times 2}$ is continuous and 1-periodic. For the Cauchy Problem (2.17) we associate the recurrence

$$X_{n+1}(t) = I_2 + \int_0^t A(s)X_n(s)\,\mathrm{d}s, \quad X_0 = I_2,$$

whose solution is given by

$$X_n(t) = I_2 + \int_0^t A(t_n)dt_n + \cdots + \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} A(t_n)\cdots A(t_1)\,\mathrm{d}t_1\cdots \mathrm{d}t_n.$$

As is well-known, for each $t \in \mathbb{R}$, $X_n(t) \to P(t)$ as $n \to \infty$ (in the matrix norm) and the convergence is uniformly on compact intervals of $\mathbb{R}$.

A simple calculation (that is left as an exercise) shows, for each positive integer n, one has $[X_n(1)]_{11} = 1$ and $[X_n(1)]_{21} = 0$.

Thus $[P(1)]_{11} = 1$ and $[P(1)]_{21} = 0$ and that shows 1 is an eigenvalue of $P(1)$, i.e. $P(1)$ is not hyperbolic. Now we apply theorem 2.12 to get the assertion.

REMARK 4.3. We believe that the methods used in the proof of the results in this article could be adapted to obtain similar results in the case of higher order recurrences in the operator framework as in [**3**, **26**].

Declaration

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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