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An n-dimensional chemotaxis system with signal-dependent motility and generalized logistic source: global existence and asymptotic stabilization

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This paper deals with the global existence for a class of Keller–Segel model with signal-dependent motility and general logistic term under homogeneous Neumann boundary conditions in a higher-dimensional smoothly bounded domain, which can be written as

$$u_t = \Delta(\gamma(v)u) + \rho u - \mu u^l, \quad x \in \Omega,\ t > 0,$$
$$v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0.$$

It is shown that whenever $\rho \in \mathbb{R}$, $\mu > 0$ and

$$l > \max\left\{\frac{n+2}{2}, 2\right\},$$

then the considered system possesses a global classical solution for all sufficiently smooth initial data. Furthermore, the solution converges to the equilibrium

$$\left(\left(\frac{\rho_+}{\mu}\right)^{1/(l-1)}, \left(\frac{\rho_+}{\mu}\right)^{1/(l-1)}\right)$$

as $t \to \infty$ under some extra hypotheses, where $\rho_+ = \max\{\rho, 0\}$.

Keywords: Global existence; Boundedness; Large time behaviour; Signal-dependent motility

2010 Mathematics Subject Classification: 35A01; 35B45; 35K51; 92C17

1. Introduction and main results

Starting from the pioneering work of Keller and Segel [7] in 1970, the famous chemotaxis model

$$u_t = \nabla \cdot (D(u, v)\nabla u) - \nabla \cdot (\chi(u, v)\nabla v) + f(u),$$
$$v_t = \Delta v - v + u,$$

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has been extensively studied by lots of mathematicians from various aspects due to its important role not only in mathematics but also in biology and pharmacology. The model describes the chemotaxis process (from the Greek Chemo=chemical, taxis=arrangement), which may be defined as the influence of chemical substances on the movement of mobile species. For a broad overview over various types of chemotaxis processes, we refer the reader to the survey [4, 5, 15, 16] and the references therein.

The typical example of logistic source

$$f(u) = \rho u - \mu u^l,$$

where $\rho \in \mathbb{R}$, $\mu > 0$ and $l > 1$. In biological phenomena, the reproduction or death of the population plays an important role in its life. An interesting and challenging problem is to detect the generation of singularity of solutions, which has been proved for two- and higher-dimensional cases [27, 29, 33]. It is well-known that an appropriate logistic damping can prevent blow-up of solutions to the classical Keller–Segel system. The parabolic–elliptic Keller–Segel simplification (where v_t is replaced by 0) is considered in [22] and it is shown that if $l > 2$, then the system possesses a unique and uniformly bounded global classical solution. In [24], the existence of weak solutions is proved under more general conditions. For the parabolic–parabolic Keller–Segel system, in [26], it is shown that if $l = 2, \mu > 0$ is sufficiently large then the problem possesses a unique and uniformly bounded global-in-time classical solution. In [8], the global classical solution and large time behaviour are considered with $l = 2$. In [35], the author shows the uniform-in-time boundedness for the corresponding $2D$ Neumann initial-boundary value problem in a large class of cell kinetics including sub-logistic sources. Besides, many authors are interested in qualitative convergence of the solution [31, 32] and large time behaviour [3, 30] for such kind of model.

Moreover, due to the complexity of biological phenomenon and situation, many modified models have been constructed and considered by various authors.

Let us recall the chemotaxis system with signal-dependent motility

$$\begin{aligned}
&u_t = \Delta(\gamma(v)u), && x \in \Omega,\ t > 0,\\
&v_t = \Delta v - v + u, && x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x),\ v(x,0) = v_0(x), && x \in \Omega,
\end{aligned}$$

where both the cell diffusion rate and the chemotactic sensitivity depend nonlinearly on the signal concentration. Tao and Winkler [21] consider the model under the condition that the motility function γ satisfies

$$\gamma \in C^3([0, +\infty))$$

and

$$K_1 \leqslant \gamma(s) \leqslant K_2$$

for all $s > 0$ as well as

$$|\gamma'(s)| \leqslant K_3$$

for all $s > 0$ with certain positive constants K_1, K_2 and K_3 and they show the global existence of bounded solutions for such kind of model. In addition, Yoon and Kim [**36**] show the global existence of solutions for the model under the assumptions that γ is a power law case

$$\gamma(s) = \frac{c_0}{s^k}, \quad c_0 > 0,\ k > 0 \tag{1.1}$$

and the motility function γ decreases as the density of the chemical substance increases, i.e.,

$$\gamma'(s) < 0$$

for all $s > 0$. In [**6**], the authors show the global existence of solutions for the model

$$\begin{aligned}
&u_t = \Delta(\gamma(v)u) + \mu u(1-u), && x \in \Omega,\ t > 0,\\
&v_t = \Delta v + u - v, && x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega, t > 0,\\
&u(x,0) = u_0(x),\ v(x,0) = v_0(x), && x \in \Omega,
\end{aligned}$$

in the two-dimensional domain with some assumptions that $\mu > 0$ and the motility function γ satisfies

$$\gamma \in C^3([0,+\infty)), \quad \gamma > 0,\ \gamma' < 0 \ \text{ on } [0,+\infty) \tag{1.2}$$

and

$$\lim_{s\to\infty} \frac{\gamma'(s)}{\gamma(s)} \text{ exists.} \tag{1.3}$$

To extend this result to the higher dimensions, Wang and Wang [**23**] show the global existence for such kind of the system using the method of approximation and Liu and Xu [**9**] get the global existence and the large time behaviour for such kind of model. For similar results on related systems involving superquadratic degradation terms, we refer the reader to the papers [**10–13**].

In the paper, we consider the following signal-dependent motility model with general logistic term:

$$\begin{aligned}
&u_t = \Delta(\gamma(v)u) + \rho u - \mu u^l, && x \in \Omega,\ t > 0,\\
&v_t = \Delta v - v + u, && x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x),\ v(x,0) = v_0(x), && x \in \Omega,
\end{aligned} \tag{1.4}$$

where we assume the following:

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- $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ and ν is an unit outer normal vector of $\partial\Omega$.

- u and v denote the density of the bacteria and the chemoattractant, respectively. The initial data satisfies

$$\begin{aligned} u_0 \in C^0(\overline{\Omega}),\, v_0 \in W^{1,\infty}(\Omega), \\ u_0 \geqslant 0 \,\text{and}\, v_0 \geqslant C_0 > 0, \end{aligned} \tag{1.5}$$

  where C_0 is a positive constant.

- $\rho \in \mathbb{R}$, $\mu > 0$ and $l > \max\{(n+2)/2, 2\}$ are constants.

- we suppose that the motility function γ satisfies

$$\begin{aligned} &\gamma \in C^3((0,+\infty)), \quad \gamma > 0, \gamma' < 0 \text{ on } (0,+\infty) \text{ and} \\ &\frac{\gamma'}{\gamma} \text{ is bounded on } [\varepsilon,+\infty) \text{ for any } \varepsilon > 0. \end{aligned} \tag{1.6}$$

  There are many functions which satisfy the above conditions. For instance

$$\gamma(v) = \frac{a}{(1+bv)^m}, \quad \gamma(v) = 1 - \frac{v}{\sqrt{1+v^2}},\ \gamma(v) = v^{-a},$$

  where a, b, m are positive constants. In addition, we suppose a stronger assumption that the motility function γ satisfies

$$\gamma \in C^3([0,+\infty)), \quad \gamma > 0, \gamma' < 0 \text{ on } [0,+\infty) \text{ and } \frac{\gamma'}{\gamma} \text{ is bounded on } [0,+\infty), \tag{1.7}$$

  when we study the large time behaviour of the system (1.4). Note that the assumption (1.6) is weaker than the assumption (1.7) which means the assumption (1.7) implies the assumption (1.6). But the inverse is not ture. The main difference between the assumption (1.6) and (1.7) is that (1.6) contains a class of functions such as the precise power-type example v^{-a} which is of singular behaviour at $v = 0$.

Main results

The goal of this paper is to study the global boundedness and large time behaviour of the chemotaxis system (1.4). Our main results read as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and the motility function γ satisfies the condition (1.6). $\rho \in \mathbb{R}, \mu > 0$ and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants. If the initial value (u_0, v_0) satisfies (1.5), then there exists a pair (u, v) of nonnegative functions

$$(u,v) \in \left[C^0(\overline{\Omega} \times [0,+\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,+\infty))\right]^2,$$

which solves (1.4) in the classical sense.

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REMARK 1.1. The assumption (1.6) on the motility function γ in theorem 1.1 is weaker than (1.1) in [**36**] and (1.2), (1.3) in [**6**]. On one hand, the power law case (1.1) satisfies the assumption (1.6), i.e., the motility function can be singular at $v = 0$. On the other hand, the condition that

$$\lim_{v\to\infty} \frac{\gamma'(v)}{\gamma(v)}$$

exists in [**6**] can imply the assumption (1.6). But the vise is not true.

REMARK 1.2. If we replace the assumption (1.6) with (1.7), i.e., we exclude the singular at $v = 0$, then we can also get the global existence of the solution for such kind of model by the same method. Moreover, the solution of (1.4) is bounded in $\Omega \times (0, +\infty)$; namely, there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \leqslant C$$

for all $t > 0$.

Next, under the stronger assumption 1.7 on the motility function, we will give the large time behaviour of the solution for system (1.4) when $\rho \leqslant 0$.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and the motility function γ satisfies the condition* (1.7)*. $\rho \leqslant 0$, $\mu > 0$ and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants. Then the solution of (1.4) *satisfies*

$$\|u(\cdot, t)\|_{L^\infty} \to 0 \tag{1.8}$$

and

$$\|v(\cdot, t)\|_{L^\infty} \to 0$$

as $t \to +\infty$.

Finally, under the stronger assumption (1.7) on the motility function and an extra condition on

$$K := \sup_{0\leqslant s<+\infty} \frac{|\gamma'(s)|^2}{\gamma(s)},$$

we also can show the large time behaviour of the solution for the system (1.4) when $\rho > 0$.

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THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and the motility function γ satisfies the condition (1.7). $\rho > 0$, $\mu > 0$ and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants. Assume

$$K := \sup_{0 \leqslant v < +\infty} \frac{|\gamma'(v)|^2}{\gamma(v)} < 16\mu\left(\frac{\rho}{\mu}\right)^{(l-3)/(l-1)}.$$

Then the solution of (1.4) satisfies

$$\left\|u(\cdot,t) - \left(\frac{\rho}{\mu}\right)^{1/(l-1)}\right\|_{L^\infty} \to 0 \tag{1.9}$$

and

$$\left\|v(\cdot,t) - \left(\frac{\rho}{\mu}\right)^{1/(l-1)}\right\|_{L^\infty} \to 0$$

as $t \to +\infty$.

Plan of the paper

This paper is arranged as follows. Section (2) is devoted to the local existence of solutions and extensibility of the chemotaxis system with signal-dependent motility and generalized logistic source. In addition, we show some important estimates of u and v. With the above paving, we can prove the global classical solution to the system (1.4) in § (3). Finally, in §§ (4) and (5) we show the large time behaviour of the solution for system (1.4) under the condition $\rho \leqslant 0$ and $\rho > 0$ respectively.

2. Preliminaries: local existence and some inequalities

Firstly, we give the existence of local solutions of (1.4). The proof, refer to [**17**], is based on the Schauder fixed point theorem. Alternatively, the existence of local solutions could also have been obtained by applying the abstract theory [**2**]. Here we omit the proof of the following lemma due to the standard method.

LEMMA 2.1 Local existence. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and the motility function γ satisfies the condition (1.6). $\rho \in \mathbb{R}$, $\mu > 0$ and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants. If the initial data satisfies the condition (1.5), then there exist $T_{\max} \in (0, \infty]$ and a pair (u, v) of nonnegative functions

$$(u, v) \in \left[C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right]^2,$$

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which solves (1.4) *in the classical sense in* $\Omega \times (0, T_{\max})$. *Moreover, we have*

$$\text{either } T_{\max} = \infty \text{ or } \limsup_{t \nearrow T_{\max}} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}) = \infty.$$

Now, we prove some basic properties of solutions to the system (1.4).

2.1. Boundedness of u

We have the following basic boundedness information on the solution u.

LEMMA 2.2. *If* $\rho \in \mathbb{R}$, $\mu > 0$ *and* $l > 1$ *are constants, then there exists* $C > 0$ *such that*

$$\int_\Omega u \leqslant C \tag{2.1}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation in (1.4) over Ω and using the condition $\partial u/\partial \nu = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = \rho \int_\Omega u - \mu \int_\Omega u^l \tag{2.2}$$

for all $t \in (0, T_{\max})$. In addition, based on Hölder's inequality, we conclude the fact

$$\int_\Omega u^l \geqslant \frac{1}{|\Omega|^{l-1}} \left(\int_\Omega u\right)^l$$

for all $t \in (0, T_{\max})$ which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \leqslant \rho_+ \int_\Omega u - \frac{\mu}{|\Omega|^{l-1}} \left(\int_\Omega u\right)^l$$

for all $t \in (0, T_{\max})$. Solving this ODI and noticing the positivity of u, we get (2.1). □

LEMMA 2.3. *If* $\rho \in \mathbb{R}$, $\mu > 0$ *and* $l > 1$ *are constants, then for each* $a > 0$, *there exists* $C > 0$ *such that*

$$\int_0^t e^{-a(t-s)} \int_\Omega u^l \leqslant C \tag{2.3}$$

for all $t \in (0, T_{\max})$.

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Proof. According to lemma 2.2, there exists $C_1 > 0$ such that

$$\int_\Omega u \leqslant C_1$$

for all $t \in (0, T_{\max})$. Multiplying (2.2) by e^{as} and integrating in time $s \in (0,t)$, we have

$$\begin{aligned}\mu \int_0^t e^{as} \int_\Omega u^l &= -\int_0^t e^{as} \frac{\mathrm{d}}{\mathrm{d}s} \int_\Omega u + \rho \int_0^t e^{as} \int_\Omega u \\ &\leqslant \int_\Omega u_0 + (a + \rho_+) \int_0^t e^{as} \int_\Omega u \\ &\leqslant C_1 \left(1 + \frac{a + \rho_+}{a}\right) e^{at}\end{aligned}$$

for all $t \in (0, T_{\max})$ which implies (2.3). Hence, we finish the proof of the lemma. □

2.2. Boundedness of v

As a preparation for deriving the global existence of solutions, the following important estimates for the solution v [28] are essential.

LEMMA 2.4. *If $T_{\max} < +\infty$, then there exists a lower bound $\underline{v} > 0$ such that*

$$\inf_{x \in \Omega} v(x,t) \geqslant \underline{v}$$

for all $t \in (0, T_{\max})$.

Proof. By the comparison principle with the positivity of u, we know from the second equation of (1.4) that

$$v(x,t) \geqslant e^{-t} \inf_{y \in \Omega} v_0(y)$$

for all $(x,t) \in \Omega \times (0, T_{\max})$ which implies

$$v(x,t) \geqslant e^{-T_{\max}} \inf_{y \in \Omega} v_0(y)$$

for all $(x,t) \in \Omega \times (0, T_{\max})$. □

LEMMA 2.5. *Assume $\lambda > 0$ is the first nonzero eigenvalue of $-\Delta$ in $\Omega \subset \mathbb{R}^n$ with the Neumann boundary condition and $q > 2$. If $u \in C^0(\overline{\Omega} \times (0, T_{\max}))$ satisfies*

$$\sup_{t \in (0, T_{\max})} \int_0^t e^{-(\lambda+1)(t-s)} \|u(s)\|_{L^q}^q \,\mathrm{d}s < +\infty, \tag{2.4}$$

then for each

$$r \in \begin{cases} [1, \frac{nq}{n+2-q}), & q \leqslant n+2, \\ [1, \infty], & q > n+2, \end{cases}$$

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and the solution $v \in L^\infty((0,T_{\max});W^{1,r}(\Omega))$ *which satisfies*

$$\begin{aligned} &v_t = \Delta v - v + u, && x \in \Omega,\ t \in (0,\infty), \\ &\frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega,\ t \in (0,\infty), \\ &v(0) = v_0(x), && x \in \Omega. \end{aligned}$$

then there exists a constant $C > 0$ *such that*

$$\|v(\cdot,t)\|_{W^{1,r}} \leqslant C$$

for all $t \in (0,T_{\max})$.

Proof. By the variation-of-constants formula, we get

$$v = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(s)\,\mathrm{d}s \tag{2.5}$$

for all $t \in (0,T_{\max})$. By the well-known smoothing estimates for the Neumann Laplace semigroup[25] and Hölder's inequality, there exist $C_1, C_2, \lambda > 0$ such that

$$\|\nabla e^{t\Delta}\varphi\|_{L^r} \leqslant C_1\left(1 + t^{-(1/2)-(n/2)\left(\frac{1}{l}-\frac{1}{r}\right)_+}\right)e^{-\lambda t}\|\varphi\|_{L^l}$$

for all $\varphi \in L^l(\Omega)$ and all $t > 0$ and

$$\|\nabla e^{t\Delta}\varphi\|_{L^r} \leqslant C_2\|\nabla\varphi\|_{L^\infty}$$

for all $\varphi \in L^\infty(\Omega)$ and all $t > 0$. Let q' satisfies

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Therefore, using (2.5) and Hölder's inequality, we conclude

$$\begin{aligned} \|\nabla v\|_{L^r} &\leqslant C_2\|\nabla v_0\|_{L^\infty} + \int_0^t e^{-(t-s)}\|\nabla e^{(t-s)\Delta}u(s)\|_{L^r}\,\mathrm{d}s \\ &\leqslant C_2\|\nabla v_0\|_{L^\infty} + C_1\int_0^t e^{-(\lambda+1)(t-s)} \\ &\quad\times\left(1 + (t-s)^{-(1/2)-(n/2)(1/q-(1/r))_+}\right)\|u(s)\|_{L^q}\,\mathrm{d}s \\ &\leqslant C_2\|\nabla v_0\|_{L^\infty} + C_1\left(\int_0^t e^{-(\lambda+1)(t-s)}\|u(s)\|_{L^q}^q\,\mathrm{d}s\right)^{1/q} \\ &\quad\left(\int_0^t e^{-(\lambda+1)(t-s)}\left(1 + (t-s)^{-(1/2)-(n/2)(1/q-(1/r))_+}\right)^{q'}\,\mathrm{d}s\right)^{1/q'} \end{aligned} \tag{2.6}$$

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for all $t \in (0, T_{\max})$. For $q \leqslant n+2$, we have

$$\begin{aligned} & -\left[\frac{1}{2}+\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)_{+}\right] q' \\ & > -\left[\frac{1}{2}+\frac{n}{2}\left(\frac{1}{q}-\frac{n+2-q}{nq}\right)\right] \frac{q}{q-1} \\ & = -1. \end{aligned} \tag{2.7}$$

For $q > n+2$, we have

$$\begin{aligned} & -\left[\frac{1}{2}+\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)_{+}\right] q' \\ & \geqslant -\left[\frac{1}{2}+\frac{n}{2q}\right] \frac{q}{q-1} \\ & > -1. \end{aligned} \tag{2.8}$$

Simple calculus shows

$$\begin{aligned} & \int_0^t e^{-(\lambda+1)(t-s)} \left(1+(t-s)^{-(1/2)-(n/2)(1/q-(1/r))_+}\right)^{q'} \mathrm{d}s \\ & = \frac{1}{\lambda+1} \int_0^{(\lambda+1)t} e^{-y} \left(1+\left(\frac{1}{\lambda+1} y\right)^{-(1/2)-(n/2)(1/q-(1/r))_+}\right)^{q'} \mathrm{d}y \\ & \leqslant \frac{2^{q'}}{\lambda+1} \int_0^{+\infty} e^{-y} \left(1+\left(\frac{1}{\lambda+1} y\right)^{-\left[1/2+n/2(1/q-(1/r))_+\right]q'}\right) \mathrm{d}y \end{aligned}$$

for all $t \in (0, T_{\max})$. By (2.7) and (2.8), there exists $C_3 > 0$ such that

$$\int_0^t e^{-(\lambda+1)(t-s)} \left(1+(t-s)^{-(1/2)-(n/2)(1/q-(1/r))_+}\right)^{q'} \mathrm{d}s \leqslant C_3 \tag{2.9}$$

for all $t \in (0, T_{\max})$. Combining (2.4), (2.6) and (2.9), we can easily get the desired conclusion. □

Lemma 2.6. *If $l > n+2$, then there exists $C > 0$ such that*

$$\|v(\cdot, t)\|_{W^{1,\infty}} \leqslant C$$

for all $t \in (0, T_{\max})$. If $\max\{(n+2)/2, 2\} < l \leqslant n+2$, then for any sufficient small $\varepsilon > 0$, there exists $C > 0$ such that

$$\|v(\cdot, t)\|_{W^{1,\frac{nl}{n+2-l}-\varepsilon}} \leqslant C$$

for all $t \in (0, T_{\max})$.

Proof. Combining lemma 2.3 and lemma 2.5, we can easily draw the conclusion of the lemma. □

2.3. L^p estimate of u

Now, we derive the L^p boundedness of u. Combining the hypothesis (1.6), lemma 2.2, lemma 2.4, lemma 2.6 and some important inequalities, we achieve the boundedness of u in $L^\infty((0,T_{\max});L^p(\Omega))$ for arbitrary $p \geqslant 2$.

LEMMA 2.7. *If $\rho \in \mathbb{R}$, $\mu > 0$ and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants, $T_{\max} < +\infty$ and the hypothesis (1.6) holds, then for any $p \geqslant 2$, there exists $C > 0$ such that

$$\int_\Omega u^p \leqslant C$$

for all $t \in (0, T_{\max})$.

Proof. Using u^{p-1} with $p \geqslant 2$ as a test function for the first equation in (1.4), integrating the resulting equation by parts and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + (p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + \mu\int_\Omega u^{p+l-1} \\
&= -(p-1)\int_\Omega \gamma'(v)u^{p-1}\nabla u\cdot\nabla v + \rho\int_\Omega u^p \\
&\quad \leqslant \frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^p|\nabla v|^2 + \rho\int_\Omega u^p
\end{aligned}$$

for all $t \in (0, T_{\max})$. In view of Young's inequality, we get that there exists $C_1 > 0$ such that

$$(\rho+1)\int_\Omega u^p \leqslant \frac{\mu}{2}\int_\Omega u^{p+l-1} + C_1$$

for all $t \in (0, T_{\max})$. Hence, we can easily get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \frac{p(p-1)}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 + p\int_\Omega u^p + \frac{\mu p}{2}\int_\Omega u^{p+l-1} \\
&\quad \leqslant \frac{p(p-1)}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^p|\nabla v|^2 + C_1 p
\end{aligned} \tag{2.10}$$

for all $t \in (0, T_{\max})$ where v is positive in $\overline{\Omega} \times [0, T_{\max})$.

Now, we estimate the first term of the right-hand side of above inequality (2.10). In order to use lemma 2.6, we need to compute it in two cases.

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The case $l > n+2$. In view of the hypothesis 1.6 and lemma 2.4, we can find a constant $C_2 > 0$ such that

$$\frac{|\gamma'(v)|}{\gamma(v)} \leqslant C_2 \tag{2.11}$$

for all $t \in (0, T_{\max})$. Here we use Young's inequality, the hypothesis 1.6, lemma 2.4, lemma 2.6 and (2.11) to find $C_3 > 0$ such that

$$\begin{aligned} \frac{p(p-1)}{2} \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^p |\nabla v|^2 &\leqslant \frac{p(p-1)}{2} C_2^2 \gamma(\underline{v}) \|\nabla v\|_{L^\infty}^2 \int_\Omega u^p \\ &\leqslant \frac{\mu p}{4} \int_\Omega u^{p+l-1} + C_3 \end{aligned}$$

for all $t \in (0, T_{\max})$ which combined with (2.10) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p \int_\Omega u^p + \frac{p(p-1)}{2} \int_\Omega \gamma(v) u^{p-2} |\nabla u|^2 + \frac{\mu p}{4} \int_\Omega u^{p+l-1} \leqslant C_1 p + C_3$$

for all $t \in (0, T_{\max})$ and thereby concludes the proof.

The case $\max\{(n+2)/2, 2\} < l \leqslant n+2$. Using inequality $|X-Y|^2 \geqslant 1/2|X|^2 - |Y|^2$ for $X, Y \in \mathbb{R}^n$, we get

$$\begin{aligned} \int_\Omega \gamma(v) u^{p-2} |\nabla u|^2 &= \frac{4}{p^2} \int_\Omega \left| \gamma^{1/2}(v) \nabla u^{p/2} \right|^2 \\ &= \frac{4}{p^2} \int_\Omega \left| \nabla \left(\gamma^{1/2}(v) u^{p/2} \right) - \frac{\gamma'(v)}{2\gamma^{1/2}(v)} u^{p/2} \nabla v \right|^2 \\ &\geqslant \frac{2}{p^2} \int_\Omega \left| \nabla \left(\gamma^{1/2}(v) u^{p/2} \right) \right|^2 - \frac{1}{p^2} \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^p |\nabla v|^2 \end{aligned}$$

for all $t \in (0, T_{\max})$ which combining with (2.10) implies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p \int_\Omega u^p + \frac{p-1}{p} \int_\Omega \left| \nabla \left(\gamma^{1/2}(v) u^{p/2} \right) \right|^2 + \frac{\mu p}{2} \int_\Omega u^{p+l-1} \\ &\quad \leqslant \frac{(p-1)(p^2+1)}{2p} \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^p |\nabla v|^2 + C_1 p \end{aligned} \tag{2.12}$$

for all $t \in (0, T_{\max})$. Noticing the condition $\max\{(n+2)/2, 2\} < l \leqslant n+2$ which implies $nl/(n+2-l) > \max\{2, n\}$, we can choose ε sufficiently small such that $nl/(n+2-l) - \varepsilon > \max\{2, n\}$. By lemma 2.6, there exists $C_4 > 0$ such that

$$\left(\int_\Omega |\nabla v|^{nl/(n+2-l)-\varepsilon} \right)^{\frac{1}{\frac{nl}{n+2-l}-\varepsilon}} \leqslant C_4 \tag{2.13}$$

for all $t \in (0, T_{\max})$. Let

$$a = \frac{nl - \varepsilon(n+2-l)}{nl - (\varepsilon+2)(n+2-l)}.$$

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Owing to Hölder's inequality, (2.11) and (2.13), we get

$$\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^p |\nabla v|^2$$
$$\leqslant C_2^2 \left(\int_\Omega \gamma^a(v) u^{ap}\right)^{1/a} \left(\int_\Omega |\nabla v|^{nl/(n+2-l)-\varepsilon}\right)^{\frac{2}{\frac{nl}{n+2-l}-\varepsilon}}$$
$$\leqslant C_2^2 C_4^2 \left\| \gamma^{1/2}(v) u^{p/2} \right\|_{L^{2a}}^2 \tag{2.14}$$

for all $t \in (0, T_{\max})$. Using Gagliardo–Nirenberg inequality thus entails the existence of $C_5 > 0$ such that

$$\left\|\gamma^{1/2}(v)u^{p/2}\right\|_{L^{2a}}^2 \leqslant C_5 \left(\left\| \nabla \left(\gamma^{1/2}(v) u^{p/2}\right)\right\|_{L^2}^{((a-1)n)/a} \times \|\gamma^{1/2}(v)u^{p/2}\|_{L^2}^{(2a-(a-1)n)/a} + \|\gamma^{1/2}(v)u^{p/2}\|_{L^{2/p}}^2 \right) \tag{2.15}$$

for all $t \in (0, T_{\max})$. Combining (2.14) with (2.15) by means of Young's inequality provides $C_6 > 0$ such that

$$\frac{(p-1)(p^2+1)}{2p} \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} u^p |\nabla v|^2$$
$$\leqslant \frac{(p-1)(p^2+1)}{2p} C_2^2 C_4^2 \left\|\gamma^{1/2}(v)u^{p/2}\right\|_{L^{2a}}^2$$
$$\leqslant \frac{(p-1)(p^2+1)}{2p} C_2^2 C_4^2 C_5 \left(\left\| \nabla \left(\gamma^{1/2}(v) u^{p/2}\right)\right\|_{L^2}^{(a-1)n/a} \times \|\gamma^{1/2}(v)u^{p/2}\|_{L^2}^{2a-(a-1)n/a} + \|\gamma^{1/2}(v)u^{p/2}\|_{L^{2/p}}^2 \right)$$
$$\leqslant \frac{p-1}{2p} \int_\Omega \left|\nabla \left(\gamma^{1/2}(v)u^{p/2}\right)\right|^2 + \frac{\mu p}{4} \int_\Omega u^{p+l-1} + C_6$$

for all $t \in (0, T_{\max})$ which combining with (2.12) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + p \int_\Omega u^p + \frac{p-1}{2p} \int_\Omega \left|\nabla \left(\gamma^{1/2}(v)u^{p/2}\right)\right|^2 + \frac{\mu p}{4} \int_\Omega u^{p+l-1} \leqslant C_1 p + C_6$$

for all $t \in (0, T_{\max})$ and thereby concludes the proof. □

3. Global existence

In this section, we aim to show the unique global-in-time solution of the system (1.4) by using lemma 2.1 and lemma 2.7. Our main method is the standard Alikakos–Moser iteration. The details are as follows.

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The proof of theorem 1.1. Suppose that $T_{\max} < +\infty$, then thanks to lemma 2.7, we get that for any $p > n$, there exists $C_1 > 0$ such that

$$\int_\Omega u^p \leqslant C_1$$

for all $t \in (0, T_{\max})$. Thus using well-known theorem to the second equation of (1.4), we can easily conclude that there exists $C_2 > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,\infty}} \leqslant C_2 \tag{3.1}$$

for all $t \in (0, T_{\max})$. Employing a standard Alikakos–Moser iteration [**1**], we infer that there exists $C_3 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty} \leqslant C_3 \tag{3.2}$$

for all $t \in (0, T_{\max})$. For a statement precisely covering the present situation we refer to [**18**]. Hence, the theorem 1.1 is proved due to (3.1), (3.2) and the extensibility criterion lemma 2.1. □

4. Large time behaviour: the case $\rho \leqslant 0$

Based on [**20**, **34**], the large time behaviour of the solution in the case $\rho \leqslant 0$ can be obtained by the quantitative decay estimate with respect to the norm in L^1.

LEMMA 4.1. *If $\rho \leqslant 0$, $\mu > 0$ are constants and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\},$$

then there exists $C > 0$ such that

$$\int_\Omega u \leqslant C\,(1+t)^{-(1/l-1)}$$

and

$$\int_\Omega v \leqslant C\,(1+t)^{-(1/l-1)}$$

for all $t \in (0, +\infty)$.

Proof. Integrating the first equation in (1.4) over Ω, using Hölder's inequality and solving a simple ODE inequality, we have

$$\begin{aligned}\int_\Omega u &\leqslant \left(\left(\int_\Omega u_0\right)^{-(l-1)} + \frac{\mu(l-1)}{|\Omega|^{l-1}}t\right)^{-(1/l-1)}\\ &\leqslant \max\left\{\int_\Omega u_0, |\Omega|\mu^{-(1/l-1)}(l-1)^{-(1/l-1)}\right\}(1+t)^{-(1/l-1)}\\ &:= C_1\,(1+t)^{-(1/l-1)}\end{aligned}$$

for all $t \in (0, +\infty)$ which yields the L^1 norm of u.

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Next, integrating the second equation of (1.4) over Ω, we see that

$$y(t) := \int_\Omega v$$

satisfies

$$\begin{aligned} y'(t) &= -y(t) + \int_\Omega u \\ &\leqslant -y(t) + C_1 (1+t)^{-(1/l-1)} \end{aligned}$$

for all $t \in (0, +\infty)$. Let

$$M := \max\left\{ \left(1 + \frac{2}{l-1}\right)^1 /(l-1) \int_\Omega v_0, 2C_1 \left(1 + \frac{2}{l-1}\right)^1 /(l-1) \right\}$$

and

$$\bar{y}(t) = M \left(1 + \frac{2}{l-1} + t\right)^{-(1/l-1)}$$

for all $t \in (0, +\infty)$. We infer that

$$\bar{y}(0) = M \left(1 + \frac{2}{l-1}\right)^{-(1/l-1)} \geqslant \int_\Omega v_0 = y(0)$$

and

$$\begin{aligned} &\bar{y}'(t) + \bar{y}(t) - C_1 (1+t)^{-(1/l-1)} \\ &= -\frac{M}{l-1} \left(1 + \frac{2}{l-1} + t\right)^{-(1/l-1)-1} \\ &\quad + M \left(1 + \frac{2}{l-1} + t\right)^{-(1/l-1)} - C_1 (1+t)^{-(1/l-1)} \\ &= M \left(1 + \frac{2}{l-1} + t\right)^{-(1/l-1)-1} \left(-\frac{1}{l-1} + \frac{1}{2}\left(1 + \frac{2}{l-1} + t\right)\right) \\ &\quad + (1+t)^{-(1/l-1)} \left(\frac{M}{2} \left(\frac{1 + \frac{2}{l-1} + t}{1+t}\right)^{-(1/l-1)} - C_1\right) \\ &\geqslant M \left(1 + \frac{2}{l-1} + t\right)^{-(1/l-1)-1} \left(-\frac{1}{l-1} + \frac{1}{2}\frac{2}{l-1}\right) \\ &\quad + (1+t)^{-(1/l-1)} \left(\frac{M}{2} \left(1 + \frac{2}{l-1}\right)^{-(1/l-1)} - C_1\right) \\ &\geqslant 0 \end{aligned}$$

for all $t \in (0, +\infty)$. By the comparison principle of ODE, we get $y(t) \leqslant \bar{y}(t)$ for all $t \in (0, +\infty)$ which directly implies the estimate of the L^1 norm of v. □

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Proof of theorem 1.2. Let $\psi_0 = C\gamma'^2(v)|\nabla v|^2$, $\psi_1 = \gamma'(v)\nabla v$ and $\psi_2 = \lambda_+ u + \mu u^l$ in conditions (A_1), (A_2), (A_3) of [**14**]. With an application of [**14**] to the solution of the first equation of (1.4) and the boundedness of v, we get that there exist $\theta_1 \in (0,1)$ and $C_1 > 0$ such that

$$\|u\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C_1. \tag{4.1}$$

Due to the well-known Hölder regularity in scalar parabolic equation, we get that there exist $\theta_2 \in (0,1)$ and $C_2 > 0$ such that

$$\|v\|_{C^{\theta_2,\frac{\theta_2}{2}}(\overline{\Omega}\times[t,t+1])} \leqslant C_2$$

for all $t > 1$.

Supposing that (1.8) is false, we can find $C_3 > 0$ and $\{t_j\}_{j\in\mathbb{N}}$ such that

$$t_j \to +\infty \quad \text{as} \quad j \to +\infty$$

and

$$\|u(\cdot,t_j)\|_{L^\infty} \geqslant C_3 \quad \text{for all} \quad j \in \mathbb{N}. \tag{4.2}$$

Since $\{u(\cdot,t_j)\}_{j\in\mathbb{N}}$ is relatively compact in $C^0(\overline{\Omega})$ according to (4.1) and the Arzelà–Ascoli theorem, we may assume that

$$u(\cdot,t_j) \to \overline{u} \quad \text{in} \quad L^\infty(\Omega)$$

as $j \to +\infty$ with some nonnegative $\overline{u} \in C^0(\overline{\Omega})$. But from lemma 4.1 we already know that

$$u(\cdot,t) \to 0 \quad \text{in} \quad L^1(\Omega)$$

as $t \to +\infty$. Hence, it holds that $\overline{u} = 0$ which is incompatible with (4.2) and therefore we prove

$$u(\cdot,t) \to 0 \quad \text{in} \quad L^\infty(\Omega).$$

In quite a similar manner, the convergence results on v can be obtained. □

5. Large time behaviour: the case $\rho > 0$

In this section, the large time behaviour of the solution in the case $\rho > 0$ can be obtained by constructing a Lyapunov function of system (1.4). Our idea comes from the references [**19**, **34**]. Moreover, we show that the solution of system (1.4) will converge to the spatially homogeneous equilibrium

$$\left(\left(\frac{\rho_+}{\mu}\right)^{1/(l-1)}, \left(\frac{\rho_+}{\mu}\right)^{1/(l-1)}\right).$$

Given a positive number u_*, we let $\varphi_{u_*} : (0,\infty) \to \mathbb{R}$ be defined by

$$\varphi_{u_*}(x) := x - u_* - u_* \ln \frac{x}{u_*}, \quad x > 0.$$

Then φ_{u_*} is convex with $\varphi_{u_*}(u_*) = \varphi'_{u_*}(u_*) = 0$, so that $\varphi_{u_*}(x) \geqslant 0$ for all $x > 0$. Define an energy functional for the system (1.4) as follows, for $B > 0$ and any

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nonnegative continuous $u : \overline{\Omega} \to (0, \infty)$ and $v : \overline{\Omega} \to (0, \infty)$,

$$\mathcal{F}_{u_*,B}(u, v) := \int_\Omega \varphi_{u_*}(u) + \frac{B}{2} \int_\Omega (v - u_*)^2 . \tag{5.1}$$

It satisfies $\mathcal{F}_{u_*,B}(u_*, u_*) = 0$.

Lemma 5.1. *Assume* $\rho > 0$, $\mu > 0$ *and*

$$l > \max\left\{\frac{n+2}{2}, 2\right\}$$

are constants. If

$$K := \sup_{0 \leqslant v \leqslant +\infty} \frac{|\gamma'(v)|^2}{\gamma(v)} < 16\mu \left(\frac{\rho}{\mu}\right)^{(l-3)/(l-1)}, \tag{5.2}$$

then there exist $B > 0$ *and* $C > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{u_*,B}(u, v) + C\left\{\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} + \int_\Omega |\nabla v|^2 + \int_\Omega (v - u_*)^2 + \int_\Omega (u - u_*)^2\right\} \leqslant 0$$

for all $t > 0$, *where* $u_* = (\rho/\mu)^{1/(l-1)}$ *and* $\mathcal{F}_{u_*,B}$ *defined as in* (5.1). *Moreover, we have*

$$\mathcal{F}_{u_*,B}(u(t), v(t)) \leqslant \mathcal{F}_{u_*,B}(u(t_0), v(t_0))$$

whenever $0 \leqslant t_0 \leqslant t$ *and there exists* $C > 0$ *such that*

$$\int_0^{+\infty} \int_\Omega (v - u_*)^2 + \int_0^{+\infty} \int_\Omega (u - u_*)^2 \leqslant C.$$

Proof. According to the condition (5.2), we have

$$\frac{u_* K}{4} < 4\mu u_*^{l-2}.$$

Hence there exists a suitably small $\eta \in (0, 1)$ such that

$$\frac{u_* K}{4(1-\eta)} < 4\mu u_*^{l-2},$$

this implies the existence of $B > 0$ fulfilling

$$\frac{u_* K}{4(1-\eta)} < B < 4\mu u_*^{l-2}.$$

So

$$B > \frac{u_* K}{4(1-\eta)} \tag{5.3}$$

and

$$\frac{B}{4} < \mu u_*^{l-2},$$

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which ensure that there exists some suitably small $\theta \in (0,1)$ such that

$$\frac{B}{4(1-\theta)} < \mu u_*^{l-2}. \tag{5.4}$$

With the above chosen B, we use the system (1.4) to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{u_*,B}(u,v)\\
&= \int_\Omega u_t - u_* \int_\Omega \frac{u_t}{u} + B\int_\Omega (v-u_*)v_t\\
&= \rho\int_\Omega u - \mu\int_\Omega u^l - u_*\int_\Omega \frac{1}{u}\left(\Delta(\gamma(v)u) + \rho u - \mu u^l\right)\\
&\quad + B\int_\Omega (v-u_*)\left(\Delta v - v + u\right)\\
&= \rho\int_\Omega u - \mu\int_\Omega u^l - \rho u_*|\Omega| + \mu u_*\int_\Omega u^{l-1} - u_*\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2}\\
&\quad - u_*\int_\Omega \gamma'(v)\frac{\nabla u\cdot\nabla v}{u}\\
&\quad - B\int_\Omega |\nabla v|^2 - B\int_\Omega (v-u_*)^2 + B\int_\Omega (v-u_*)(u-u_*)
\end{aligned} \tag{5.5}$$

for all $t > 0$. Applying Young's inequality, we get

$$\begin{aligned}
&- u_*\int_\Omega \gamma'(v)\frac{\nabla u\cdot\nabla v}{u}\\
&\quad \leqslant (1-\eta)u_*\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} + \frac{u_*}{4(1-\eta)}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}|\nabla v|^2\\
&\quad \leqslant (1-\eta)u_*\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} + \frac{u_* K}{4(1-\eta)}\int_\Omega |\nabla v|^2
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&B\int_\Omega (v-u_*)(u-u_*)\\
&\quad \leqslant (1-\theta)B\int_\Omega (v-u_*)^2 + \frac{B}{4(1-\theta)}\int_\Omega (u-u_*)^2
\end{aligned} \tag{5.7}$$

for all $t > 0$. Simple calculus implies

$$\begin{aligned}
&\rho\int_\Omega u - \mu\int_\Omega u^l - \rho u_*|\Omega| + \mu u_*\int_\Omega u^{l-1}\\
&\quad = -\mu\int_\Omega (u-u_*)(u^{l-1} - u_*^{l-1})\\
&\quad \leqslant -\mu\int_\Omega (u-u_*)^2 u_*^{l-2}
\end{aligned} \tag{5.8}$$

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for all $t > 0$ where we have used the elementary inequality: If $m \geqslant 1$, then for all $x \geqslant 0$, $y \geqslant 0$ with $x \neq y$ we have

$$\frac{x^m - y^m}{x - y} \geqslant y^{m-1}.$$

Collecting (5.5), (5.6), (5.7) and (5.8), we thus infer that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{u_*,B}(u,v) \\ &\leqslant -\eta u_* \int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2} - \left(B - \frac{u_* K}{4(1-\eta)}\right)\int_\Omega |\nabla v|^2 \\ &\quad - \theta B \int_\Omega (v-u_*)^2 - \left(\mu u_*^{l-2} - \frac{B}{4(1-\theta)}\right)\int_\Omega (u-u_*)^2 \end{aligned}$$

for all $t > 0$. As (5.3) and (5.4) ensure that $B - u_* K/4(1-\eta)$ and $\mu u_*^{l-2} - B/4(1-\theta)$ are positive, this concludes the result easily. □

Proof of theorem 1.3. We use the contradiction method to prove the theorem as before. Supposing that (1.9) is false, we can find $C_1 > 0$, some sequences $\{t_j\}_{j\in\mathbb{N}} \subset (1, +\infty)$ and $\{x_j\}_{j\in\mathbb{N}} \subset \overline{\Omega}$ such that $t_j \to +\infty$ and

$$|u(x_j, t_j) - u_*| \geqslant C_1$$

for all $j \in \mathbb{N}$. (4.1) implies u is uniformly continuous in $\Omega \times (1, +\infty)$. Therefore, there exists $r > 0$ such that for any $j \in \mathbb{N}$,

$$|u(x,t) - u_*| \geqslant \frac{C_1}{2}$$

for all $x \in B_r(x_j) \cap \Omega$ and $t \in (t_j, t_j + 1)$. Owing to the smoothness of $\partial\Omega$, we can find $C_2 > 0$ which satisfies

$$|B_r(x_j) \cap \Omega| \geqslant C_2$$

for all $x_j \in \Omega$. Then we infer that for all $j \in \mathbb{N}$,

$$\begin{aligned} \int_{t_j}^{t_j+1}\int_\Omega |u(x,t) - u_*|^2 &\geqslant \int_{t_j}^{t_j+1}\int_{B_r(x_j)\cap\Omega} |u(x,t) - u_*|^2 \\ &\geqslant \int_{t_j}^{t_j+1} |B_r(x_j)\cap\Omega| \left(\frac{C_1}{2}\right)^2 \\ &\geqslant \left(\frac{C_1}{2}\right)^2 C_2. \end{aligned} \tag{5.9}$$

From lemma 5.1, we derive

$$\int_{t_j}^{t_j+T}\int_\Omega |u(x,t) - u_*|^2 \leqslant \int_{t_j}^{+\infty}\int_\Omega |u(x,t) - u_*|^2 \to 0$$

as $j \to +\infty$ which contradicts (5.9). Hence, (1.9) is verified.

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The properties on v can be derived similarly. □

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