Compositio Math. 144 (2008) 1497–1503
doi:10.1112/S0010437X08003722

On the independence of the generators of tautological rings

D. Arcara and Y.-P. Lee

Abstract

We prove that all monomials of κ-classes and ψ-classes are independent in $R^k(\overline{\mathcal{M}}_{g,n})/R^k(\partial\overline{\mathcal{M}}_{g,n})$ for all $k \leqslant [g/3]$. We also give a simple argument for $\kappa_l \neq 0$ in $R^l(\mathcal{M}_g)$ for $l \leqslant g-2$.

1. Introduction

1.1 Tautological rings

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stacks of stable curves. Then $\overline{\mathcal{M}}_{g,n}$ are proper, irreducible, smooth Deligne–Mumford stacks. The Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ over $\mathbb{Q}$ are isomorphic to the Chow rings of their coarse moduli spaces. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are subrings of $A^*(\overline{\mathcal{M}}_{g,n})$, or subrings of $H^{2*}(\overline{\mathcal{M}}_{g,n})$ via cycle maps, generated by some 'geometric classes', which will be described below.

The first type of geometric class is the *boundary strata*; $\overline{\mathcal{M}}_{g,n}$ have natural stratification by topological types. The second type of geometric class is the Chern classes of tautological vector bundles; these include cotangent classes ψ_i, Hodge classes λ_k and κ-classes κ_l.

To give a precise definition of tautological rings, some natural morphisms between moduli stacks of curves will be used. The *forgetful morphisms* $\mathrm{ft}_i : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forget one of the $n+1$ marked points. The *gluing morphisms* $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ and $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ glue two marked points to form a curve with a new node. Note that the boundary strata are the images (of the repeated applications) of the gluing morphisms, up to factors in $\mathbb{Q}$ due to automorphisms.

The *system of tautological rings* $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ is the smallest system of $\mathbb{Q}$-unital subalgebra (containing classes of type one and two, and is) closed under the forgetful and gluing morphisms.

The study of tautological rings is one of the central problems in moduli of curves. The readers are referred to [Vak06] and references therein for many examples and motivation.

1.2 Main result

Let $\mathcal{M}_{g,n}$ be the moduli stack of smooth n-pointed curves. Let $\partial\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. The main result of this paper is the following theorem.

Theorem 1. *The monomials generated by the κ and ψ have no relations in $R^k(\overline{\mathcal{M}}_{g,n})/R^k(\partial\overline{\mathcal{M}}_{g,n})$ for $k \leqslant [g/3]$ and all n.*

We define $R^*(\partial\overline{\mathcal{M}}_{g,n})$ and the quotient $R^k(\overline{\mathcal{M}})/R^k(\partial\overline{\mathcal{M}})$ in (1) and the paragraph following it.

Received 28 May 2006, accepted in final form 16 May 2008.
2000 Mathematics Subject Classification 14H10.
Keywords: moduli of curves, tautological rings.

1.3 Motivation: Faber's conjecture

The formulation of Theorem 1 is motivated by Faber's conjecture on the structure of tautological rings. One of the guiding problems in the study of tautological rings of moduli of curves is the set of conjectures proposed by Faber, Pandharipande, Looijenga, ... (For a survey, the readers are referred to [Vak06].) In [Fab99], Faber conjectures that (Conjecture 1b) $\kappa_1, \kappa_2, \ldots, \kappa_{[g/3]}$ generate the ring $R^*(\mathcal{M}_g)$, with no relations in degree $k \leqslant [g/3]$. This statement will be referred to as *Faber's conjecture* in this paper. The generation statement has been proved by Morita [Mor03] and Ionel [Ion05]. Therefore, the remaining part of Faber's conjecture would be the independence statement.

An expert in tautological rings can immediately notice the relation between Faber's conjecture and Mumford's conjecture on the *stable* cohomology ring of $\mathcal{M}_{g,n}$. In particular, the following theorem of Madsen and Weiss [MW07] (Mumford's conjecture) and its generalization by Looijenga [Loo96, Proposition 2.1] are similar to Faber's conjecture.

Theorem 2 [MW07, Loo96]. *One has $H^*(\mathcal{M}_{g,n}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \psi_1, \ldots, \psi_n]$ in the stable range.*

Therefore, it is natural to generalize Faber's conjecture accordingly.

Conjecture 1. *The classes $\kappa_1, \ldots, \kappa_{[g/3]}, \psi_1, \ldots, \psi_n$ generate the ring $R^*(\mathcal{M}_{g,n})$, with no relations in degree $k \leqslant [g/3]$.*

The generation statement follows immediately from Morita and Ionel's results cited above. What is in question is the independence statement. Note that there is a natural sequence of tautological rings

$$R^k(\partial\overline{\mathcal{M}}_{g,n}) \longrightarrow R^k(\overline{\mathcal{M}}_{g,n}) \longrightarrow R^k(\mathcal{M}_{g,n}) \longrightarrow 0, \tag{1}$$

where $R^k(\mathcal{M}_{g,n})$ is defined to be the restriction of $R^k(\overline{\mathcal{M}}_{g,n})$, and $R^k(\partial\overline{\mathcal{M}}_{g,n})$ is defined to be the pushforward of the normalized boundary divisors via gluing morphisms. The exactness in the middle of this sequence was conjectured by Faber and Pandharipande in [FP05], and proved in some cases. This statement will be referred to as the *boundary tautological class conjecture*. Theorem 1 implies Conjecture 1 if the boundary tautological class conjecture holds.

1.4 Invariance constraints

The main tool employed is the invariance constraints [Lee08a, Lee08b]. More recently, Faber, Shadrin and Zvonkine, and independently Pandharipande and the second author, have given a very simple geometric proof of this statement; see [FSZ06, § 3].

Theorem 3. *There exist a series of linear operators*

$$\mathfrak{r}_l : R^k(\overline{\mathcal{M}}_{g,n}) \to R^{k-l+1}(\overline{\mathcal{M}}^\bullet_{g-1,n+2}), \tag{2}$$

where $^\bullet$ denotes moduli of possibly disconnected curves with at most two connected components.

All tautological classes in $R^k(\overline{\mathcal{M}}_{g,n})$ can be represented by a $\mathbb{Q}$-linear combination of *decorated graphs*, and each $\mathfrak{r}_l$ is defined by graph operations. To each stable curve C with marked points, one can associate a dual graph Γ. Vertices of Γ correspond to irreducible components of C. They are labeled by their geometric genus. Assign an edge joining two vertices each time the two components intersect. To each marked point, one draws a half-edge incident to the vertex, with the same label as the point. Now, the stratum corresponding to Γ is the closure of the subset of all stable curves in $\overline{\mathcal{M}}_{g,n}$ that have the same topological type as C. Each dual graph Γ can be decorated by assigning a monomial of ψ classes to each half-edge and κ classes to each vertex.

Let us describe $\mathfrak{r}_1$. Define three graph operations on the spaces of decorated graphs $\{\Gamma\}$.

(i) *Cutting edges.* Cut one edge and create two new half-edges. Label two new half-edges with $i, j \notin \{1, 2, \ldots, n\}$ in two different ways. Produce a formal sum of four graphs by decorating extra ψ^l to i-labeled new half-edges with coefficient $1/2$ and by decorating extra ψ to j-labeled new half-edges with coefficient $-1/2$. (By 'extra' decoration we mean that ψ is multiplied by whatever decorations are already there.) Do this to all edges.

(ii) *Genus reduction.* For each vertex, reduce the genus of this given vertex by one and add two new half-edges. Label two new half-edges with i, j.

(iii) *Splitting vertices.* Split one vertex into two. Add one new half-edge to each of the two new vertices and label them with i, j. Produce new graphs by splitting the genus g between the two new vertices (g_1, g_2 such that $g_1 + g_2 = g$), and distributing to the two new vertices the (old) half-edges that belong to the original chosen vertex, in all possible ways. Split the κ-classes on the given vertex in all possible ways between the two new vertices.

Take the outputs of the above operations, and remove unstable graphs. Multiply the outputs of (ii) and (iii) by $1/2$. Then $\mathfrak{r}_1(\Gamma)$ is defined to be the formal sum of the outputs of the above three operations. These operations actually descend to $R^k(\overline{\mathcal{M}}_{g,n})$. Indeed, *the operation* $\mathfrak{r}_1$ *is nothing but the pull back to the double cover of the normalized boundary divisors*; see [FSZ06, § 3].

2. The proof

Theorem 1 will be proved by induction on (g, n), in the lexicographic order.

The case $g \leqslant 2$ is obvious. Assume now that the statement holds for all genera up to $g - 1$ and for all n. The following proposition will be used in the proof.

PROPOSITION 1. *One has* $\kappa_l \neq 0$ *in* $R^*(\mathcal{M}_g)$ *for all* $l \leqslant g - 2$.

Proof. Recall that a nodal genus-g curve is said to have rational tails if one of the irreducible components is smooth of genus-g, and $\mathcal{M}^{rt}_{g,n}$ is the moduli stack of genus-g nodal curves with rational tails. If $l = g - 2$, then the result is true because $\kappa_{g-2} \neq 0$ (see [Fab99, Theorem 2]). If $l < g - 2$, then

$$\kappa_l \kappa_{g-l-2} + \kappa_{g-2} = \pi_*(\psi_1^{l+1} \psi_2^{g-l-1}) = c\kappa_{g-2}, \tag{3}$$

where π is the forgetful morphism $\mathcal{M}^{rt}_{g,2} \to \mathcal{M}_g$ and

$$c = \frac{(2g-1)!!}{(2l+1)!!(2g-2l-3)!!}.$$

The first equality is well known (see, for example, [Lee08b, Lemma 2]). For the second one, see [GJV06, Theorem 2.5]. Therefore,

$$\kappa_l \kappa_{g-l-2} = (c-1)\kappa_{g-2} \neq 0,$$

which implies that $\kappa_l \neq 0$. □

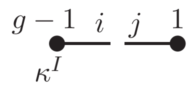
Remark. The second equality in (3) is part of Faber's *intersection number conjecture*, first established by Getzler and Pandharipande in [GP98] conditional to Virasoro conjecture of $\mathbb{P}^2$, which was later established by Givental [Giv01].

Case I: $n = 0$. Assume that

$$E = \sum_I c_I \kappa^I + \sum_m c_m \Gamma_m = 0$$

is a tautological equation in $R^k(\overline{\mathcal{M}}_g)$ for $k \leqslant [g/3]$. Here κ^I are (distinct) monomials of κ-classes, and Γ_m are in the image of $R^k(\partial\overline{\mathcal{M}}_{g,n})$ via (1). The goal is to show that $c_I = 0$ for all I.

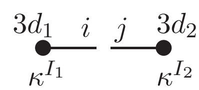
By Theorem 3, we have that $\mathfrak{r}_1(E) = 0$ in $R^k(\overline{\mathcal{M}}^{\bullet}_{g-1,2})$. Let κ^I be a monomial in the κ of degree k. Note that the degree used here is the *Chow degree*. Now we will analyze the output $\mathfrak{r}_1(\kappa^I)$. It is easy to see that the following term appears in $\mathfrak{r}_1(\kappa^I)$ (splitting the vertex) and does not appear in $\mathfrak{r}_1(\kappa^J)$ for $J \neq I$ or $\mathfrak{r}_1(\Gamma_m)$:



where $g-1$, 1 are the genera and i, j are the new half-edges.

Suppose that $k < [g/3]$. The combination of the following three facts implies that $c_I = 0$: (1) it only appears in $\mathfrak{r}_1(\kappa^I)$, (2) all monomials in the κ of degree k are independent in $R^k(\overline{\mathcal{M}}_{g-1,1})$ by induction hypothesis, and (3) $\mathfrak{r}_1(E) = 0$.

Let us now assume that $k = [g/3]$. If κ^I is a monomial in the κ of degree $[g/3]$ such that $\kappa^I \neq \kappa_{[g/3]}$, one can write $\kappa^I = \kappa^{I_1}\kappa^{I_2}$ with κ^{I_a} a monomial in the κ of degree $d_a > 0$ $(a = 1, 2)$ such that $d_1 + d_2 = [g/3]$. Since all monomials in the κ of degree d_a are independent in $R^{d_a}(\overline{\mathcal{M}}_{3d_a,n})$ $(a = 1, 2)$ for all n by induction, the term



in $\mathfrak{r}_1(\kappa^I)$ is independent of any other term appearing in $\mathfrak{r}_1(E)$. Again, since this term only appears in $\mathfrak{r}_1(\kappa^I)$, and not in $\mathfrak{r}_1$ of any other element of $R^{[g/3]}(\overline{\mathcal{M}}_g)$, the fact that its coefficient in $\mathfrak{r}_1(E)$ must be zero implies that the coefficient of κ^I in E is also zero.

The last case to consider is the coefficient of $\kappa_{[g/3]}$. Suppose that it is nonzero, i.e. suppose that

$$E = a\kappa_{[g/3]} + \sum_m c_m \Gamma_m = 0$$

is a tautological equation in $R^{[g/3]}(\overline{\mathcal{M}}_g)$ with $a \neq 0$. Then, taking the image of this equation in $R^{[g/3]}(\mathcal{M}_g)$, we would obtain that $\kappa_{[g/3]} = 0$, which is not true by Proposition 1 as $g \geqslant 3$. Therefore the coefficient of $\kappa_{[g/3]}$ must also be zero. This concludes the proof of the $(g, 0)$ case.

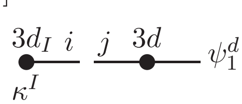
Case II: $n = 1$. Let

$$E = \sum_{I,d} c_{I,d}\kappa^I \psi_1^d + \sum_m c_m \Gamma_m = 0$$

be a tautological equation in $R^k(\overline{\mathcal{M}}_{g,1})$. Theorem 3 implies that $\mathfrak{r}_1(E) = 0$ in $R^k(\overline{\mathcal{M}}^{\bullet}_{g-1,3})$. The goal is to show that all $c_{I,d} = 0$.

As in Case I, we can deduce that $c_{I,d} = 0$ if either (a) $k < [g/3]$ or (b) $d = 0$ and $\kappa^I \neq \kappa_{[g/3]}$. Indeed, it does not matter where we attach the extra half-edge (marking), since the induction hypothesis says the statement holds for genus at most $g-1$ and all n.

Let us now consider terms of the form $\kappa^I\psi_1^d \in R^{[g/3]}(\overline{\mathcal{M}}_{g,1})$, with κ^I a monomial of degree $d_I = [g/3] - d$. Suppose that $0 < d < [g/3]$. The term



in $\mathfrak{r}_1(\kappa^I\psi_1^d)$ is independent of any other term appearing in $\mathfrak{r}_1(E)$. As before, this term only appears in $\mathfrak{r}_1(\kappa^I\psi_1^d)$ (note that it is important here that $\psi_1^d \neq \psi_2^d$ in $R^d(\overline{\mathcal{M}}_{3d,2})$, which follows by induction because $0 < d < [g/3]$), and therefore the coefficient of $\kappa^I\psi_1^d$ in E is zero.

Therefore, we have proved that, if $E = 0$ is a tautological equation in $R^k(\overline{\mathcal{M}}_{g,1})$, then all coefficients of monomials in the κ and ψ_1 are zero, except possibly for the coefficients of $\kappa_{[g/3]}$ and $\psi_1^{[g/3]}$ in $R^{[g/3]}(\overline{\mathcal{M}}_{g,1})$.

Suppose that

$$E = a\kappa_{[g/3]} + b\psi_1^{[g/3]} + \sum_m c_m \Gamma_m = 0$$

is a tautological equation in $R^{[g/3]}(\overline{\mathcal{M}}_{g,1})$, since all other coefficients must vanish. We will perform two operations on E to obtain two linear equations on the coefficients of E.

(i) Multiply E by ψ_1 and push forward $\psi_1 E$ from $R^{[g/3]+1}(\overline{\mathcal{M}}_{g,1})$ to $R^{[g/3]}(\overline{\mathcal{M}}_g)$. The push-forward is easy to perform via the substitution

$$\kappa_{[g/3]} = \pi^* \kappa_{[g/3]} + \psi_1^{[g/3]}. \tag{4}$$

As $E = 0$, the push-forward of $\psi_1 E$ must vanish. Thus

$$(2g-2)a\kappa_{[g/3]} + a\kappa_{[g/3]} + b\kappa_{[g/3]} = 0$$

in $R^{[g/3]}(\overline{\mathcal{M}}_g)/R^{[g/3]}(\partial\overline{\mathcal{M}}_g)$. Since $\kappa_{[g/3]} \neq 0$, we obtain that $(2g-2)a + a + b = 0$.

(ii) Push forward E from $R^{[g/3]}(\overline{\mathcal{M}}_{g,1})$ to $R^{[g/3]-1}(\overline{\mathcal{M}}_g)$. Equation (4) implies that

$$a\kappa_{[g/3]-1} + b\kappa_{[g/3]-1} = 0$$

in $R^{[g/3]-1}(\overline{\mathcal{M}}_g)/R^{[g/3]-1}(\partial\overline{\mathcal{M}}_g)$. Since $\kappa_{[g/3]-1} \neq 0$, we have that $a + b = 0$.

Operations (i) and (ii) together imply that $a = b = 0$. This completes the $n = 1$ case.

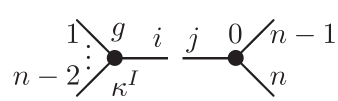
Case III: $n \geqslant 2$. Suppose now that the statement is proved for $(g, \leqslant n-1)$ and for all genera up to $g-1$. We will show that it holds for (g, n), with $n \geqslant 2$. Let

$$E = \sum_{I,J} c_{IJ} \kappa^I \psi^J + \sum_m c_m \Gamma_m = 0$$

be a tautological equation in $R^k(\overline{\mathcal{M}}_{g,n})$. Theorem 3 implies that $\mathfrak{r}_1(E) = 0$ in $R^k(\overline{\mathcal{M}}^{\bullet}_{g-1,n+2})$. The goal is to show that $c_{IJ} = 0$ for all I, J.

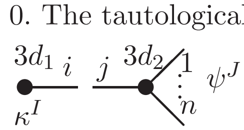
Again, the case $k < [g/3]$ is very similar to the previous cases and is left to the readers.

Suppose that $k = [g/3]$. If $J = 0$, the term



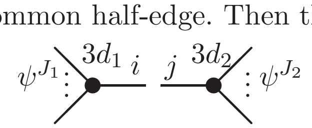
only appears in the output $\mathfrak{r}_1(\kappa^I)$. It is independent of other classes in the output of $\mathfrak{r}_1(E)$ by induction hypothesis. Therefore, the induction hypothesis implies that $c_{I0} = 0$ for all I.

If both $I \neq 0$ and $J \neq 0$, the monomial is of the form $\kappa^I \psi^J$, with κ^I degree d_1 and ψ^J degree d_2, such that $d_1 + d_2 = [g/3]$ and $d_1, d_2 \neq 0$. The tautological class



only appears in the output of $\mathfrak{r}_1(\kappa^I \psi^J)$, and is independent of other output classes by induction hypothesis. Therefore the coefficient c_{IJ} of $\kappa^I \psi^J$ in E is zero.

The next case is $I = 0$. If ψ^J is a monomial in the ψ of degree $[g/3]$, and $\psi^J \neq \psi_l^{[g/3]}$ for any l, one can write $\psi^J = \psi^{J_1}\psi^{J_2}$, with ψ^{J_a} a monomial in the ψ of degree d_a $(a = 1, 2)$, $d_1 + d_2 = [g/3]$, such that J_1 and J_2 do not share a common half-edge. Then the term



in $\mathfrak{r}_1(\psi^J)$ implies that the coefficient of ψ^J in E is zero.

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Therefore, we have proved that, if $E = 0$ is a tautological equation in $R^k(\overline{\mathcal{M}}_{g,n})$, then all coefficients of monomials in the κ and ψ are zero, except possibly for the coefficients of $\psi_1^{[g/3]}$, $\psi_2^{[g/3]}, \ldots, \psi_n^{[g/3]}$ in $R^{[g/3]}(\overline{\mathcal{M}}_{g,n})$.

Suppose that

$$E = \sum_{l=1}^{n} a_l \psi_l^{[g/3]} + \sum_m b_m \Gamma_m = 0$$

is a tautological relation in $R^{[g/3]}(\overline{\mathcal{M}}_{g,n})$, where the Γ_m are elements of $R^{[g/3]}(\partial\overline{\mathcal{M}}_{g,n})$.

If we multiply by ψ_n and then push-forward to $R^{[g/3]}(\overline{\mathcal{M}}_{g,n-1})$, we obtain that

$$\sum_{l=1}^{n-1}(2g-2+n-1)a_l\psi_l^{[g/3]} + a_n\kappa_{[g/3]} = 0$$

in $R^{[g/3]}(\overline{\mathcal{M}}_{g,n-1})/R^{[g/3]}(\partial\overline{\mathcal{M}}_{g,n-1})$. Since all monomials in the κ and ψ of degree $[g/3]$ are independent in $R^{[g/3]}(\overline{\mathcal{M}}_{g,n-1})/R^{[g/3]}(\partial\overline{\mathcal{M}}_{g,n-1})$ by induction hypothesis, one has $a_1 = a_2 = \cdots = a_n = 0$. This completes the proof of Theorem 1.

Acknowledgements

We wish to thank R. Cavalieri, C. Faber, E. Looijenga, I. Madsen, R. Pandharipande, and R. Vakil for helpful discussions. The second author is partially supported by NSF and an AMS Centennial Fellowship.

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