HÖLDER CONTINUOUS POTENTIALS ON MANIFOLDS
WITH PARTIALLY POSITIVE CURVATURE

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Abstract   It is proved that solutions of the complex Monge–Ampère equation on compact Kähler manifolds with right hand side in L^p, $p > 1$, are uniformly Hölder continuous under the assumption on non-negative orthogonal bisectional curvature.

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1. Introduction

The regularity of weak solutions of the complex Monge–Ampère equation in the setting of compact Kähler manifolds is a very timely topic (see [**10**, **16**]). A strong motivation for this branch of research are the recent results in the theory of the Kähler–Ricci flow (see [**4**, **19**–**21**], to mention only a few). In particular, it is important to understand the limiting behaviour of the Kähler–Ricci flow near finite time singularities (if the canonical divisor K_X is big) or the limit at infinity if additionally K_X is numerically effective. In dimension 2 the picture is more or less clear and is very similar to the one in the minimal model program for surfaces (see [**4**, **21**]). In higher dimensions, as explained in [**19**] and [**20**], the successful termination of the program depends on understanding the following.

(1) The complex Monge–Ampère equation on singular varieties.

(2) The regularity of the solution of the corresponding Monge–Ampère equation in the limiting case.

The first task was studied recently in [**10**]. The difficulty with the second one is due to the fact that in the limiting cases the underlying metric is no more Kähler and thus the equation is much less tractable. Even in the *big* form case the continuity of solutions is still an unsolved problem, except in some special cases (see [**9**]).

However, even in the case of a Kähler metric the picture is far from being clear. Thus in this note we restrict our attention to the Kähler case.

The first natural question is obtaining Hölder regularity for the potentials. The basic result regarding this problem was obtained by Kołodziej in [**16**], where the following was proved.

Theorem 1.1. *Let* (X, ω) *be a compact* n-*dimensional Kähler manifold* ($n \geqslant 2$). *Consider the following Monge–Ampère equation:*

$$(\omega + \mathrm{i}\partial\bar{\partial}\phi)^n = f\omega^n, \quad \sup_X \phi = 0, \quad \phi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X), \qquad (1.1)$$

where $f \geqslant 0$ *is a given function satisfying the additional property that* $f \in L^p(\omega^n)$, $p > 1$ *(and the necessary normalization* $\int_X f\omega^n = \int_X \omega^n$*). Then the solution* ϕ *is Hölder continuous with Hölder exponent dependent on* X, n *and* p.

In [**10**] it was shown that that under the additional assumption that X is *rational homogeneous* compact Kähler manifold the Hölder exponent is actually independent of X and can be taken to be $2/(2 + nq + \epsilon)$, $\forall \epsilon > 0$ (q is the conjugate of p, i.e. $1/p + 1/q = 1$). It is unknown whether this exponent is sharp, however by modifying an argument of Pliś (see [**17**]) one can produce an explicit example showing that the exponent cannot be bigger than $2/nq$.

There are some results, however, which suggest that on a general manifold the exponent should also depend only on p and n.

First, the corresponding equation in a strictly pseudoconvex domain was studied in [**12**], and in this setting the exponent is indeed independent of the domain considered (strictly speaking there is a universal lower bound γ, such that any solution is at least γ-Hölder continuous).

Secondly, in [**1**] and [**2**] the same equation was studied, but instead of the condition $f \geqslant 0$, $f \in L^p(\omega^n)$, the author assumed that $f \geqslant 0$ and $f^{1/n}$ is Lipschitz. Then a Lipschitz regularity for ϕ was obtained (in [**1**] under the extra condition of non-negative bisectional curvature, which was later removed in [**2**]).

Finally, in [**9**] the following stability estimate was obtained (see also [**14**] for the original (slightly weaker) result).

Theorem 1.2. *Let* ϕ *and* ψ *solve the equations*

$$(\omega + \mathrm{i}\partial\bar{\partial}\phi)^n = f\omega^n, \quad (\omega + \mathrm{i}\partial\bar{\partial}\psi)^n = g\omega^n, \quad \phi, \psi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X),$$

with $f, g \geqslant 0$, $f, g \in L^p(\omega^n)$, $p > 1$. *If the solutions are normalized so that* $\sup_X(\phi - \psi) = \sup_X(\psi - \phi)$ *(recall that solutions are bounded under the assumption that the right-hand side is in* L^p, $p > 1$ *(see [**13**, **15**]))*, *then there exists a constant* $c = c(p, \epsilon, c_0)$, *where* c_0 *is an upper bound for* $\|f\|_p$ *and* $\|g\|_p$, *such that*

$$\|\phi - \psi\|_\infty \leqslant c\|f - g\|_1^{1/(n+\epsilon)}, \quad \forall \epsilon > 0.$$

Note that again stability exponent is independent of the manifold.

All these results lead to the following conjecture (posed in [**9**]).

Conjecture 1.3. *The Hölder exponent in Kołodziej's theorem in [16] depends only on n and p, not on X.*

The main result in this note confirms the conjecture above for a large class of manifolds (containing the homogeneous ones, so in particular it covers the case considered in [10], although we get a worse exponent). Let us state it.

Theorem 1.4. *Let (X, ω) be a compact n-dimensional Kähler manifold $(n \geqslant 2)$. Assume additionally that X has non-negative orthogonal bisectional curvature.* * Consider the following Monge–Ampère equation:*

$$(\omega + \mathrm{i}\partial\bar\partial\phi)^n = f\omega^n, \quad \sup_X \phi = 0, \quad \phi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X),$$

where $f \geqslant 0$ is a given function satisfying $\int_X f\omega^n = \int_X \omega^n$, $f \in L^p(\omega^n)$, $p > 1$. Then the solution ϕ is Hölder continuous with Hölder exponent at least $1/(nq + 1 + \epsilon)$ for any $\epsilon > 0$.

Note that in [1] also an additional assumption on the bisectional curvature played a role.

We would like to point out that the non-negativity of the orthogonal bisectional curvature is needed in our approach, but we do not know whether this geometric condition has any impact on the considered problem. In fact we still believe that the conjecture is true in general and, perhaps as in [2], a new argument would allow to remove this technical assumption.

Recently, in [11] (see also [5]) a characterization of all compact Kähler manifolds with non-negative orthogonal bisectional curvature was obtained (in terms of the universal covering spaces). The universal covering space of such a manifold should be isometrically biholomorphic to an element in one of the following two families:

$$(\mathbb{C}^k, h_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (\mathbb{P}^{n_1}, \omega_1) \times \cdots \times (\mathbb{P}^{n_r}, \omega_r), \qquad (1.2)$$

where h_0 is the Euclidean metric, h_i are canonical metrics on irreducible compact Hermitian symmetric spaces M_i of rank at least 2, and ω_j are Kähler metrics on $\mathbb{P}^{n_j}$ carrying non-negative orthogonal bisectional curvature;

$$(Y, g_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (\mathbb{P}^{n_1}, \omega_1) \times \cdots \times (\mathbb{P}^{n_r}, \omega_r), \qquad (1.3)$$

with M_i, h_i, ω_j as above and Y being either a simply connected Riemann surface with Gauss curvature negative somewhere, or a non-compact simply connected Kähler manifold $(\dim Y \geqslant 2)$ with non-negative orthogonal bisectional curvature and the minimum of the holomorphic sectional curvature is less than zero somewhere. In the latter case also the holomorphic sectional curvatures of M_i and $\mathbb{P}^{n_j}$ are less than

$$-\min\{\text{holomorphic sectional curvature of } Y\} > 0.$$

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* For simplicity we assume in the note that the metric carrying non-negative orthogonal bisectional curvature coincides with ω, however the same proof, with minor modifications, carries over if this assumption is with respect to another Kähler metric χ.

Notation

Throughout the note C will denote different constants depending only on the relevant quantities. If there is a possibility of confusion, these different Cs will be indexed. We will use the standard notation for (local) partial derivatives $h_j := \partial h/\partial z_j$, $h_{\bar{j}} := \partial h/\partial \bar{z}_j$. In order to make the note more readable for analysts we have decided to avoid the usage of Einstein summation.

2. Preliminaries

Throughout the note we shall work on a fixed compact n-dimensional Kähler manifold X equipped with a fundamental Kähler form ω (that is d-closed strictly positive globally defined form) given in local coordinates by

$$\omega = \tfrac{1}{2}\mathrm{i} \sum_{k,j=1}^{n} g_{k\bar{j}}\, \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^j.$$

We assume that the metric is normalized so that

$$\int_X \omega^n = 1.$$

With ω one naturally associates the Kähler metric

$$g = \sum_{k,j=1}^{n} g_{k\bar{j}}\, \mathrm{d}z^k \otimes \mathrm{d}\bar{z}^j$$

acting on the tangent bundle TX, i.e. it defines a hermitian product on each tangent space $T_z X$, $z \in X$.

Let D denote the Levi-Civita connection associated to the Kähler metric g. Let also $\Theta = \Theta(D) = D^2$ be the associated curvature tensor. It is known (see [**8**]) that $\mathrm{i}\Theta$ is a section of the bundle $\mathcal{C}_{1,1}^{\infty}(X, \mathrm{Herm}(TX, TX))$ which analytically means that $\mathrm{i}\Theta$ is (locally) a matrix with $(1,1)$-forms as entries. Explicitly, in a local basis $\partial/\partial z_i$, $i = 1, \ldots, n$, of TX the curvature tensor Θ reads

$$\mathrm{i}\Theta = \mathrm{i} \sum_{1 \leqslant k,l \leqslant n,\, 1 \leqslant i,j \leqslant n} \mathcal{C}_{klij}\, \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_l \otimes \left(\frac{\partial}{\partial z_i}\right)^* \otimes \frac{\partial}{\partial z_j},$$

for some coefficients $\mathcal{C}_{klij} \in \mathbb{C}$.

These coefficients can be computed according to the formula*

$$\mathrm{i}\Theta = \mathrm{i}\bar{\partial}[(\bar{g})^{-1}\partial\bar{g}],$$

where g is the local matrix of the coefficients of the Kähler metric, $(\bar{g})^{-1}$ is the inverse transposed matrix of g and the computation is done componentwise (we use matrix

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* This formula varies in the literature (compare with [**18**], for example). The difference comes simply from reversal of the row/column notation.

notation in the above equation). In terms of the local coordinates the (h, l)th entry is the $(1, 1)$-form

$$\mathrm{i} \sum_{j,k=1}^{n} \left(- \sum_{p=1}^{n} g^{h\bar{p}} g_{l\bar{p}j\bar{k}} + \sum_{r,s,p=1}^{n} g^{h\bar{s}} g^{r\bar{p}} g_{r\bar{s}\bar{k}} g_{l\bar{p}j} \right) \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k, \tag{2.1}$$

where, as usual, $g^{a\bar{b}}$ is the entry of the inverse transposed matrix of $(g_{a\bar{b}})$ (the minus sign comes from the fact that we have interchanged the order of $\mathrm{d}z_j$ and $\mathrm{d}\bar{z}_k$).

With the curvature tensor one associates the Chern curvature form which is a bilinear form on $TX \otimes TX$ defined by

$$\mathrm{i}\Theta(\zeta \otimes \eta, \zeta \otimes \eta) := \sum_{j,k,m,l=1}^{n} c_{j\bar{k}m\bar{l}} \zeta_j \eta_m \bar{\zeta}_k \bar{\eta}_l,$$

where

$$\begin{aligned} c_{j\bar{k}m\bar{l}} &:= - \sum_{h=1}^{n} \sum_{p=1}^{n} g_{h\bar{m}} g^{h\bar{p}} g_{l\bar{p}j\bar{k}} + \sum_{h=1}^{n} \sum_{r,s,p=1}^{n} g_{h\bar{m}} g^{h\bar{s}} g^{r\bar{p}} g_{r\bar{s}\bar{k}} g_{l\bar{p}j} \\ &= -g_{l\bar{m}j\bar{k}} + \sum_{r,p=1}^{n} g^{r\bar{p}} g_{r\bar{m}\bar{k}} g_{l\bar{p}j}. \end{aligned} \tag{2.2}$$

Geometrically, this form arises as a contraction of the curvature tensor with the metric $g_{i\bar{j}}$. The hermitian property of $\mathrm{i}\Theta$ gives the formulae

$$\bar{c}_{k\bar{l}i\bar{j}} = c_{l\bar{k}j\bar{\imath}}. \tag{2.3}$$

The Kählerness of g gives us the commutation identities

$$g_{i\bar{j}k} = g_{k\bar{j}i}, \qquad g_{i\bar{j}\bar{k}} = g_{i\bar{k}\bar{j}},$$

from which one easily deduces the well-known fact that in the Kähler case

$$c_{j\bar{k}h\bar{l}} = R_{j\bar{k}h\bar{l}}, \tag{2.4}$$

where R denotes the bisectional curvature tensor. Therefore, non-negative bisectional curvature (in the Kähler case) is equivalent to Griffiths semi-positivity of the tangent bundle, while non-negative orthogonal bisectional curvature is equivalent to partial positivity of the curvature of the tangent bundle in the sense of Demailly (see [7]).[*]

Recall that

$$\mathrm{PSH}(X, \omega) := \{\phi \in L^1(X, \omega) : \mathrm{i}\partial\bar{\partial}\phi \geqslant -\omega, \ \phi \in \mathcal{C}^{\uparrow}(X)\},$$

where $\mathcal{C}^{\uparrow}(X)$ denotes the space of upper semicontinuous functions. We call the functions that belong to $\mathrm{PSH}(X, \omega)$ ω-plurisubharmonic (ω-psh for short).

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[*] Recall that in [7] a more general situation was considered. In the non-Kähler case this equivalence fails.

The main topic of this note are the weak solutions of the following nonlinear problem:

$$(\omega + i\partial\bar{\partial}\phi)^n = f\omega^n, \quad f \geqslant 0,\ f \in L^p(\omega^n),\ p > 1, \qquad \sup_X \phi = 0, \quad \phi \in \mathrm{PSH}(X, \omega), \ (2.5)$$

where the wedge product on the left-hand side is defined using pluripotential theory (see [**14**, **15**]). We have the following stability theorem for the solutions of this problem.

Theorem 2.1 (Eyssidieux *et al.* [10]). *Let* $u, v \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ *solve the equations*

$$(\omega + i\partial\bar{\partial}u)^n = f\omega^n, \qquad (\omega + i\partial\bar{\partial}v)^n = g\omega^n,$$

where f *and* g *are non-negative functions with the right total integral and moreover* $f, g \in L^p(\omega^n)$, $p > 1$. *Then there exists a constant* C *dependent on* X, p, s, ϵ, $\|f\|_{L^p}$, $\|g\|_{L^p}$ *such that*

$$\|u - v\|_\infty \leqslant C\|u - v\|_{L^s(\omega^n)}^{s/(nq+s+\epsilon)}, \quad \forall s > 0,\ \epsilon > 0.$$

By analysing the proof of this theorem in [**10**] one obtains the following corollary.

Corollary 2.2. *In the setting as above if we additionally know that* $u > v$, *then the same conclusion holds if merely* g *is in* L^p, *i.e. we put an assumption only on the Monge–Ampère measure of the smaller function.*

3. Proof

Let ϕ be the given function with $(\omega + i\partial\bar{\partial}\phi)^n = f\omega^n$, $f \in L^p(\omega^n)$, $p > 1$. The main idea of the proof is to use an approximating technique, due to Demailly (see [**7**]), to produce an approximating sequence ϕ_ϵ which looks similarly to the convenient regularization by convolution.

If $z \in X$ then the exponential mapping

$$\exp_z : T_z X \ni \zeta \to \exp_z(\zeta) \in X,$$

is defined by $\exp_z(\zeta) := \gamma(1)$, where $\gamma : [0, 1] \to X$ is the geodesic starting from $z = \gamma(0)$ with the initial velocity $(\mathrm{d}\gamma/\mathrm{d}t)(0) = \zeta$.

In $\mathbb{C}^n$ endowed with the Euclidean metric exp reads $\exp_z(\zeta) = z + \zeta$. However, on a general complex manifold the exponential mapping is not holomorphic. To circumvent this difficulty Demailly (in [**7**]) introduced the new mapping exph as follows.

Definition 3.1. $\mathrm{exph} : TX \ni (z, \zeta) \to \mathrm{exph}_z(\zeta) \in X$, $\zeta \in T_z X$ is defined by

(1) exph is a $\mathcal{C}^\infty$ smooth mapping;

(2) $\forall z \in X$, $\mathrm{exph}_z(0) = x$ and $d_\zeta \mathrm{exph}(0) = \mathrm{Id}_{T_z X}$;

(3) $\forall z \in X$ the map $\zeta \to \mathrm{exph}_z \zeta$ has a holomorphic Taylor expansion at $\zeta = 0$.

An intuitive way to see this mapping is to take the ordinary $\zeta \to \exp_z(\zeta)$ mapping, Taylor expand it at 0 and erase all non-holomorphic terms in ζ. By Borel's theorem there exists a smooth mapping with Taylor expansion matching this erased data. Note that the mapping is not uniquely defined, however any two mappings differ by a $\mathcal{C}^\infty$ functions flat along the zero section of TX which is naturally identified with X.

Of course, the exph mapping is still not holomorphic in ζ (the Taylor expansion at 0 may well be divergent), but it does share some properties of holomorphic functions that will be satisfactory for our needs.

Let $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a cut-off function with $\chi(t) = 0$ for $t \geqslant 1$, $\chi(t) > 0$, for $t < 1$, such that

$$\int_{\mathbb{C}^n} \chi(\|z\|^2)\, \mathrm{d}\lambda(z) = 1, \tag{3.1}$$

with λ the Lebesgue measure on $\mathbb{C}^n$. We define ϕ_ϵ to be

$$\phi_\epsilon(z) = \frac{1}{\epsilon^{2n}} \int_{\zeta \in T_z X} \phi(\mathrm{exph}_z(\zeta)) \chi\left(\frac{|\zeta|^2}{\epsilon^2}\right) \mathrm{d}\lambda_\omega(\zeta), \quad \epsilon > 0. \tag{3.2}$$

Here $|\zeta|^2$ stands for

$$\sum_{i,j=1}^n g_{i\bar{j}}(z)\zeta_i\bar{\zeta}_j,$$

and $\mathrm{d}\lambda_\omega(\zeta)$ is the induced measure $(1/2^n n!)(i\partial\bar{\partial}|\zeta|^2)^n$. Intuitively this corresponds to the familiar convolution with smoothing kernel (actually in the case of $\mathbb{C}^n$ endowed with the Euclidean metric this is exactly the smoothing convolution).

Demailly introduced yet another function, namely for $w \in \mathbb{C}$, $|w| = \epsilon$ he defines $\Phi(z, w) := \phi_\epsilon(z)$. One has the following equality:

$$\Phi(z, w) = \int_{\zeta \in T_z X} \phi(\mathrm{exph}_z(w\zeta)) \chi(|\zeta|^2)\, \mathrm{d}\lambda_\omega(\zeta). \tag{3.3}$$

The reason for introducing the new variable w (as far as the author understands Demailly's intentions) is twofold.

- First, Demailly proves that (a modification of) $\Phi(z, w)$ is subharmonic in the w variable and, since Φ is a radial function in w, one obtains that (modifications of) ϕ_ϵ are increasing with ϵ (like it is the case when convoluting ordinary plurisubharmonic functions)—a fact that is probably hard to achieve by merely working with ϕ_ϵ.

- Secondly, Demailly's motivation was to obtain an approximation of $(1,1)$-currents and he was mainly interested in highly singular ones, i.e. those with non-vanishing

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* In his proof Demailly made the explicit choice

$$\chi(t) = \frac{C}{(1-t)^2} \exp\left(\frac{1}{t-1}\right)$$

for $t < 1$ with a suitable constant C.

Lelong numbers. By introducing the new variable Demailly made possible applications of Kiselman minimum principle so that one can kill the Lelong numbers up to certain level (see [**7**] for the details). Since our motivation is different (and since $i\partial\bar{\partial}\phi$ for a bounded ϕ has all Lelong numbers zero) we shall at places proceed in a slightly different manner, for example putting attention to quantities that were immaterial in Demailly's approach.

Below we state the fundamental technical estimate for the (complex) Hessian of $\Phi(z, w)$ [**7**, Proposition 3.8].

Proposition 3.2. *We have the following estimate:*

$$i\partial\bar{\partial}\Phi(z, w)[\theta, \eta]^2$$
$$\geqslant \frac{1}{\pi}|w|^2 \int_{\mathbb{C}^n} -\chi_1(|\zeta|^2) \sum_{j,k,l,m=1}^{n} \frac{\partial^2\phi}{\partial\bar{z}_l\partial z_m}(\exp\mathrm{h}_z(w\zeta))\left(c_{j\bar{k}l\bar{m}} + \frac{1}{|w|^2}\delta_{jm}\delta_{kl}\right)\tau_j\bar{\tau}_k\,\mathrm{d}\lambda(\zeta)$$
$$- K'(|\theta|\,|\eta| + |\eta|^2).$$

Here $i\partial\bar{\partial}$ *acts on* $X \times \mathbb{C}$ *(i.e. also on the* w *variable) and the symbol* $[\theta, \eta]^2$ *means that we compute the Levi form on the vector* $(\theta, \eta) \in T_z X \times \mathbb{C}$ *(we follow Demailly's notation for the ease of the reader). The constants* $c_{j\bar{k}l\bar{m}}$ *are the coefficients in the Chern curvature form (alternatively the coefficients of the bisectional curvature tensor). The vector* τ *is of the form* $\theta + \eta\zeta + O(|w|)^*$ *and* $\chi_1(t) := -\int_t^\infty \chi(u)\,\mathrm{d}u$. K' *denotes a constant dependent merely on* X *and finally* δ_{jm} *is the Kronecker delta.*

We shall not reproduce the technical details of the (computationally involved) proof—we refer to [**7**] (there is a similar computation in [**6**] made on the 'ordinary' exponential mapping). Instead we shall explain a detail that *a priori* seems to make such a result impossible.

Indeed, since exph is not holomorphic in general, when differentiating under the integral (one first parametrizes the tangent space $T_z X$ so that we pull back the computation of the 'vector part' onto $\mathbb{C}^n$) one inevitably gets terms where elements like $(\partial^2\phi/\partial z_i\partial z_j)(\exp\mathrm{h}_z(w\zeta))$ appear! Since for a plurisubhamonic function we have information *only* on the mixed complex derivatives (i.e. of type $\partial^2\phi/\partial z_i\partial\bar{z}_j$) one might feel that there is no hope of getting any estimates.

Note however that $\exp\mathrm{h}_z$ is a local diffeomorphism (when viewed as a real mapping) and its real Jacobian is invertible. So, if in a coordinate chart exph reads $\exp\mathrm{h}_z = (F^1, \ldots, F^n)$, then the matrix

$$\begin{pmatrix} \dfrac{\partial F^{(i)}}{\partial z_j} & \dfrac{\partial F^{(i)}}{\partial \bar{z}_j} \\[3ex] \dfrac{\partial \bar{F}^{(i)}}{\partial z_j} & \dfrac{\partial \bar{F}^{(i)}}{\partial \bar{z}_j} \end{pmatrix}_{i,j}$$

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* For its definition and geometric meaning we refer to [**7**], these are immaterial in our approach.

is also invertible. In the matrix notation we have

$$(\phi(F)'_1, \ldots, \phi(F)'_n, \phi(F)'_{\bar 1}, \ldots, \phi(F)'_{\bar n}) = (\phi'_1, \ldots, \phi'_n, \phi'_{\bar 1}, \ldots, \phi'_{\bar n})_F \times \begin{pmatrix} \dfrac{\partial F^{(i)}}{\partial z_j} & \dfrac{\partial F^{(i)}}{\partial \bar z_j} \\[2mm] \dfrac{\partial \bar F^{(i)}}{\partial z_j} & \dfrac{\partial \bar F^{(i)}}{\partial \bar z_j} \end{pmatrix}.$$

$$(3.4)$$

So, if

$$\begin{pmatrix} a_{ij} & b_{i\bar j} \\ c_{\bar i j} & d_{\bar i \bar j} \end{pmatrix} := \begin{pmatrix} \dfrac{\partial F^{(i)}}{\partial z_j} & \dfrac{\partial F^{(i)}}{\partial \bar z_j} \\[2mm] \dfrac{\partial \bar F^{(i)}}{\partial z_j} & \dfrac{\partial \bar F^{(i)}}{\partial \bar z_j} \end{pmatrix}^{-1},$$

then each of the coefficients among a, b, c and d is a smooth function and one can write down

$$\phi'_p(F) = \sum_{i=1}^{n} \phi(F)_i a_{ip} + \sum_{i=1}^{n} \phi(F)_{\bar i} c_{\bar i p},$$

$$\phi'_{\bar p}(F) = \sum_{i=1}^{n} \phi(F)_i b_{i\bar p} + \sum_{i=1}^{n} \phi(F)_{\bar i} d_{\bar i \bar p}.$$

These formulae actually allow one to integrate by parts.

Thus, for example, doing this operation twice an element of type

$$\int \phi''_{ps}(F) F_{\bar l}^{p'} F_k^{s'} G$$

can be exchanged with a sum of elements of type $\int \phi(F) H$ for some smooth function H. Thus the analysis is reduced to estimation of these error terms, which indeed can be done as Demailly has shown.

Heuristically the argument above is nothing but the multilinear analogue of the well known integration-by-parts formula

$$\int u'(f(t))g(t)\,\mathrm{d}t = \int u'(f(t))f'(t)\left(\frac{g(t)}{f'(t)}\right)\mathrm{d}t = -\int u(f(t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{f'(t)}\right)\mathrm{d}t,$$

which is justified in the case when $f' \neq 0$.

In order to estimate further Demailly has shown a simple lemma [**7**, Lemma 4.4]). Here we prove a slightly different version, since we shall use the lemma in a bit different context (strictly speaking we are interested in the quantity $\delta(|w|)$ in the original proof (see [**7**, Theorem 4.1]), which was more or less immaterial for Demailly, whose main concern was the quantity $\lambda(z, |w|)$ there).

Lemma 3.3. *Suppose X has non-negative orthogonal bisectional curvature, i.e.*

$$\sum_{j,k,l,m=1}^{n} \frac{1}{2\pi} c_{j\bar k l\bar m} \tau_j \bar\tau_k \xi_l \bar\xi_m \geqslant 0$$

for any tangent vectors τ and ξ, satisfying $\sum_{i,j=1}^{n} g_{i\bar{j}}\tau_i\bar{\xi}_j = 0$. At a point z we choose geodesic coordinates, so that $g_{i\bar{j}} = \delta_{ij}$ (δ is the Kronecker delta). Then there exists a constant C dependent only on the geometry of X such that at z

$$\sum_{j,k,l,m=1}^{n} \frac{1}{2\pi}\left(c_{j\bar{k}l\bar{m}} + \frac{1}{|w|^2}\delta_{jm}\delta_{kl}\right)\tau_j\bar{\tau}_k\xi_l\bar{\xi}_m + C|w|\,\|\tau\|^2\,\|\xi\|^2 \geqslant 0$$

for all tangent vectors τ, ξ and $|w| \in \mathbb{R}_{>0}$.

Remark 3.4. Actually instead of merely the Chern curvature form Demailly considered the form

$$(\mathrm{i}\Theta + u \otimes \mathrm{Id}_{TX})(\tau \otimes \xi, \tau \otimes \xi)$$

for some semi-positive $(1,1)$-form u on X. In local coordinates it reads

$$\sum_{j,k,l,m=1}^{n} \frac{1}{2\pi}c_{j\bar{k}l\bar{m}}\tau_j\bar{\tau}_k\xi_l\bar{\xi}_m + \sum_{j,k,l=1}^{n} u_{j\bar{k}}\tau_j\bar{\tau}_k\xi_l\bar{\xi}_l.$$

This notion is close to the orthogonal bisectional curvature with conformal factor (see [**18**, Chapter 5]). For our application, however, it is crucial that u vanishes identically.

Proof. Let

$$\mu := \sup_{\|\tau\|=1=\|\xi\|} \left|\mathrm{i}\Theta(\tau \otimes \xi, \tau \otimes \xi)\right|$$

(this quantity depends only on the geometry of X). We will show that actually one can take $C = 5\mu\sqrt{\mu}$ although we shall not use this explicit bound.

If $\mu = 0$ there is nothing to prove, so in the sequel we assume $\mu > 0$. Note that, by assumption, $\mathrm{i}\Theta(\tau \otimes \xi, \tau \otimes \xi) \geqslant 0$ whenever $\tau \perp \xi$. Consider two cases.

First case. Suppose $|\sum_{i=1}^{n} \tau_i\bar{\xi}_i| > |w|\sqrt{\mu}$. Then

$$\mathrm{i}\Theta(\tau \otimes \xi, \tau \otimes \xi) + \frac{1}{|w|^2}\left|\sum_i \tau_i\bar{\xi}_i\right|^2 + 5\mu\sqrt{\mu}|w|\,\|\tau\|^2\,\|\xi\|^2$$
$$\geqslant -\mu + \mu + 5\mu\sqrt{\mu}|w|\,\|\tau\|^2\,\|\xi\|^2 \geqslant 0.$$

Second case. Let now $|\sum_{i=1}^{n} \tau_i\bar{\xi}_i| \leqslant |w|\sqrt{\mu}$. By homogeneity it is enough to prove the inequality for all unit vectors τ and ξ. Let $\tau = a\xi + b\vartheta$ for some $a, b \in \mathbb{C}$ and some unit vector ϑ perpendicular to ξ^* (so $|a|^2 + |b|^2 = 1$ and $|a| = |\sum_{i=1}^{n} \tau_i\bar{\xi}_i| \leqslant |w|\sqrt{\mu}$). Then we calculate

$$\mathrm{i}\Theta(\tau \otimes \xi, \tau \otimes \xi) + \frac{1}{|w|^2}\left|\sum_i \tau_i\bar{\xi}_i\right|^2 + 5\mu\sqrt{\mu}|w|\,\|\tau\|^2\,\|\xi\|^2$$
$$\geqslant |a|^2\mathrm{i}\Theta(\xi \otimes \xi, \xi \otimes \xi) + |b|^2\mathrm{i}\Theta(\vartheta \otimes \xi, \vartheta \otimes \xi) + 2\,\mathrm{Re}(a\bar{b}\mathrm{i}\Theta(\xi \otimes \xi, \vartheta \otimes \xi)) + 5\mu\sqrt{\mu}|w|.$$

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* In the sense that $\sum_{j=1}^{n} \xi_j\bar{\vartheta}_j = 0$.

The second term is non-negative. We estimate the third one using a polarization. So, we get

$$|a|^2\mathrm{i}\Theta(\xi\otimes\xi,\xi\otimes\xi) + |b|^2\mathrm{i}\Theta(\vartheta\otimes\xi,\vartheta\otimes\xi) + 2\operatorname{Re}(a\bar{b}\mathrm{i}\Theta(\xi\otimes\xi,\vartheta\otimes\xi)) + 5\mu\sqrt{\mu}|w|$$

$$\geqslant -|a|^2\mu - |a|\,|b|[2|\operatorname{Re}(\mathrm{i}\Theta(\xi\otimes\xi,\vartheta\otimes\xi))| + 2|\operatorname{Im}(\mathrm{i}\Theta(\xi\otimes\xi,\vartheta\otimes\xi))|] + 5\mu\sqrt{\mu}|w|$$

$$\geqslant -|a|^2\mu - |a|\,|b|\left|\mathrm{i}\Theta\left(\frac{(\xi+\vartheta)}{\sqrt{2}}\otimes\xi,\frac{(\xi+\vartheta)}{\sqrt{2}}\otimes\xi\right) - \mathrm{i}\Theta\left(\frac{(\xi-\vartheta)}{\sqrt{2}}\otimes\xi,\frac{(\xi-\vartheta)}{\sqrt{2}}\otimes\xi\right)\right|$$

$$- |a|\,|b|\left|\mathrm{i}\Theta\left(\frac{(\xi+\mathrm{i}\vartheta)}{\sqrt{2}}\otimes\xi,\frac{(\xi+\mathrm{i}\vartheta)}{\sqrt{2}}\otimes\xi\right) - \mathrm{i}\Theta\left(\frac{(\xi-\mathrm{i}\vartheta)}{\sqrt{2}}\otimes\xi,\frac{(\xi-\mathrm{i}\vartheta)}{\sqrt{2}}\otimes\xi\right)\right|$$

$$+ 5\mu\sqrt{\mu}|w|$$

$$\geqslant -|a|^2\mu - 4|a|\,|b|\mu + 5\mu\sqrt{\mu}|w|$$

$$\geqslant -5|a|\mu + 5\mu\sqrt{\mu}|w|$$

$$\geqslant 0$$

(recall that ϑ is perpendicular to ξ, so all the vectors $(\xi\pm\vartheta)/\sqrt{2}$, $(\xi\pm\mathrm{i}\vartheta)/\sqrt{2}$ are unitary). $\qquad\square$

Now, following Demailly [7] if we apply the above inequality to each vector ξ in a basis of eigenvectors of $\mathrm{i}\partial\bar{\partial}\phi$, multiply by the corresponding non-negative eigenvalue and take the sum, we get

$$\frac{1}{2\pi}\sum_{j,k,l,m=1}^{n}\frac{\partial^2\phi}{\partial\bar{z}_l\partial z_m}(\operatorname{exph}_z(w\zeta))\left(c_{j\bar{k}l\bar{m}} + \frac{1}{|w|^2}\delta_{jm}\delta_{kl}\right)\tau_j\bar{\tau}_k$$

$$+ \sum_{l=1}^{n}\frac{\partial^2\phi}{\partial z_l\partial\bar{z}_l}(\operatorname{exph}_z(w\zeta))C|w|\,\|\tau\|^2 \geqslant 0.$$

(Strictly speaking one has to apply an approximation procedure to get this estimate, since *a priori* $\mathrm{i}\partial\bar{\partial}\phi$ is merely a current, and we cannot apply pointwise linear algebra.)

Plugging this into the Demailly's Proposition 3.2 we infer

$$\frac{1}{\pi}\mathrm{i}\partial\bar{\partial}\Phi(z,w)[\theta,\eta]^2$$

$$\geqslant -2|w|^2\int_{\mathbb{C}^n}-\chi_1(|\zeta|^2)\sum_{l=1}^{n}\frac{\partial^2\phi}{\partial z_l\partial\bar{z}_l}(\operatorname{exph}_z(w\zeta))C|w|\,\|\tau\|^2\,\mathrm{d}\lambda(\zeta) - K'(|\theta|\,|\eta| + |\eta|^2)$$

$$\geqslant -\left[2|w|^2\int_{\mathbb{C}^n}-\chi_1(|\zeta|^2)\sum_{l=1}^{n}\frac{\partial^2\phi}{\partial z_l\partial\bar{z}_l}(\operatorname{exph}_z(w\zeta))\,\mathrm{d}\lambda(\zeta)\right]C|w|\,|\theta|^2 - K''(|\theta|\,|\eta| + |\eta|^2),$$

where we have used $\tau = \theta + \eta\zeta + O(|w|)$. Since the element in the square brackets is uniformly bounded (roughly speaking it is comparable with the integral of the Laplacian over a ball with radius $|w|$) we obtain

$$\frac{1}{\pi}\mathrm{i}\partial\bar{\partial}\Phi(z,w)[\theta,\eta]^2 \geqslant -C|w|\,|\theta|^2 - K(|\theta|\,|\eta| + |\eta|^2), \tag{3.5}$$

for some constant K dependent only on X.

Let us stress that this estimate holds for plurisubharmonic functions as well as for ω-psh ones. This is due to the fact that the estimate, as well as all the quantities appearing in it are local. Thus instead of working with ϕ one can apply the construction to $\rho + \phi$ with ρ a local potential for the Kähler metric ω. Everything will be well defined for small enough ϵ. Of course we would have to take into account the difference between $\rho(z)$ and $\rho_\epsilon(z)$, but, since the potential is smooth, this difference is easily estimated to be of order ϵ^2, i.e. it can be absorbed harmlessly into the estimate.

Note that the estimate shows that $\Phi(z, w) + K|w|^2$ is subharmonic in w. Since it depends on $|w|$ it must be an increasing function. Therefore, $\phi_\epsilon + K\epsilon^2$ is increasing with ϵ. Also, for a given bounded ω-psh function ϕ on a complex Kähler manifold with non-negative orthogonal bisectional curvature the Demailly regularization gives us a quasi-psh function ϕ_ϵ satisfying

$$i\partial\bar{\partial}\phi_\epsilon \geqslant -(1 + C\epsilon)\omega,$$

for some constant C depending on the geometry of X (just take $\eta = 0$ and $|w| = \epsilon$).

Without loss of generality we may assume $\phi \leqslant -1$. Then $\phi_\epsilon \leqslant 0$ for small ϵ (by construction) and

$$\varphi_\epsilon := \frac{\phi_\epsilon + C_1\epsilon^2}{1 + C\epsilon}$$

is a family of smooth ω-psh functions decreasing towards ϕ.[*] By the corollary after Theorem 2.1 we have the estimate

$$\sup_X |\varphi_\epsilon - \phi| \leqslant C\|\varphi_\epsilon - \phi\|_{L^1(\omega^n)}^{1/(nq+1+\epsilon)}. \tag{3.6}$$

Note, however, that

$$\|\varphi_\epsilon - \phi\|_\infty = \left\|\frac{\phi_\epsilon - \phi + C_1\epsilon^2 - C\epsilon\phi}{1 + C\epsilon}\right\|_\infty \geqslant \frac{\|\phi_\epsilon - \phi\|_\infty - C_2\epsilon}{1 + C\epsilon},$$

for some C_2 dependent on X and $\|\phi\|_\infty$. Similarly,

$$\|\varphi_\epsilon - \phi\|_{L^1(\omega^n)} \leqslant \frac{\|\phi_\epsilon - \phi\|_{L^1(\omega^n)} + C_3\epsilon}{1 + C\epsilon},$$

for some constant C_3 also dependent on X and $\|\phi\|_\infty$.

Note however that $\|\phi_\epsilon - \phi\|_{L^1(\omega^n)} \leqslant C\epsilon^2$. Indeed, to see this, consider a double covering $\Omega_i \Subset \Omega_i^1$, $i = 1, \ldots, M$. In each Ω_i^1 we fix a Kähler potential ρ_i. For ϵ small enough we have the inclusion

$$\{\exp\mathrm{h}_z(\zeta) \mid z \in \Omega_i, \|\zeta\| \leqslant \epsilon\} \subset \Omega_i^1.$$

As we have already mentioned, instead of ϕ one can make approximation with $\psi_i := \rho_i + \phi_i$ whenever the base point varies in Ω_i. Also it is enough to estimate $\int_{\Omega_i} (\psi_i)_\epsilon(z) - \psi_i(z)\, \mathrm{d}z$ (recall that the error term coming from the regularization of the Kähler potential is of order ϵ^2). But then an application of Jensen type formula as in Lemma 4.3 in [**12**] gives us the desired statement.

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[*] A similar idea was used in [**3**].

Coupling all these estimates one obtains

$$\|\phi_\epsilon - \phi\|_\infty \leqslant C\epsilon^{1/(nq+1+\epsilon)} \tag{3.7}$$

for some C dependent on $\|\phi\|_\infty$, X, $\|f\|_p$, n and p.* Now, in a local chart, after addition of Kähler potential (so that we obtain true local plurisubharmonic functions) we may follow the proof of Lemma 4.2 in [**12**] to conclude that ϕ is Hölder continuous with exponent $1/(nq+1+\epsilon)$ (roughly speaking this follows from the fact that for plurisubharmonic functions the maximum over a small ball and the average over it are comparable). This completes the proof of Hölder continuity.

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* Actually, by Kołodziej's theorem $\|\phi\|_\infty$ is controlled by the other quantities.

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