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ARTICLE

The decumulation period of a personal pension with risk sharing: investment approach versus consumption approach

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Abstract
This paper models the decumulation period of a Personal Pension with Risk sharing (PPR). We derive several relationships between the contract parameters. Individuals can adopt two approaches to the decumulation period of a PPR: the investment approach and the consumption approach. In the investment approach, individuals specify how to invest wealth and how much wealth to withdraw. Retirement consumption follows endogenously. In the consumption approach, in contrast, individuals specify retirement consumption exogenously. Investment and withdrawal policies follow endogenously. We explore these two approaches in detail. Consistent with habit formation, we allow for excess smoothness and excess sensitivity in retirement consumption.

Key words: Consumption approach; investment approach; liability-driven investment; personal pension plan; smoothing of shocks

JEL Classification: D91; G11; G13; J32

Private pension provision is in transition, moving from employer-sponsored defined benefit (DB) pension plans towards individual defined contribution (DC) pension plans (Investment Company Institute (2017)). Many workers regard this trend as undesirable (Rhee and Boivie (2015)). Indeed, a DC plan focuses primarily on accumulating retirement wealth rather than providing a stable lifelong income stream. Recently, Bovenberg and Nijman (2015) propose a new pension contract called a Personal Pension with Risk sharing (PPR). A PPR unbundles the three main functions of variable annuity contracts: insurance, investment and withdrawal.[1] In particular, a PPR organizes the insurance function collectively and individualizes the investment and withdrawal functions.[2,3]

[1]A PPR differs from a variable annuity in three key aspects. First, a PPR defines property rights in terms of a personal investment account rather than an income stream. Second, a PPR allows for more flexibility in tailoring investment and withdrawal policies to individual needs. Third, a PPR integrates the accumulation period with the decumulation period.

[2]Individualization of the investment function is possible without any welfare loss. Indeed, pooling of systematic risks does not generate any welfare gain. In fact, individualization of the investment function typically leads to a welfare improvement, because pension providers can tailor the investment function to individual needs.

[3]Like a PPR, a pooled annuity fund allows individuals to pool idiosyncratic longevity risk and, at the same time, it allows individuals to take systematic risks; see, e.g., Piggott *et al.* (2005), Valdez *et al.* (2006), Stamos (2008), and Donnelly (2015).

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While the work by Bovenberg and Nijman (2015) is descriptive in nature, this paper formalizes the decumulation period of a PPR. We define the pension contract in terms of various parameters such as the investment policy and the median growth rate of the benefit payout. We show how the budget condition implies several relationships between the contract parameters. The budget condition does, however, not uniquely identify all parameters. As a consequence, individuals must specify some parameters exogenously. They can specify the parameters according to (at least) two alternative approaches: the investment approach and the consumption approach. We explore these two approaches in detail and show how they differ from each other.

In the investment approach, individuals specify, in each period, how to invest their accumulated retirement wealth and how much to withdraw. The value of the investment account at the start of the decumulation period (e.g., accumulated retirement wealth at the age of retirement) is also given exogenously. Many pension providers adopt the investment approach in practice. For example, when an individual purchases a variable annuity from a provider, the payout policy is (usually) a function of the adopted investment policy and the so-called Assumed Interest Rate (AIR). In particular, the investment policy and the AIR together determine the median growth rate of the benefit payout. Indeed, if a pension provider adopts a more conservative investment policy and leaves the AIR unaffected, then the benefit payout will, in expectation, grow at a slower rate. Changes in not only the investment policy but also the expected return on investments affect the median growth rate of the benefit payout.

In the consumption approach, individuals specify the entire retirement consumption stream exogenously.[4] More specifically, individuals define the desired median growth rate and the volatility of the benefit payout. They also specify the benefit payout at the start of the decumulation period exogenously. The initial value of the investment account, the investment policy and the withdrawal policy follow endogenously from these objectives. Individuals thus adopt the principle of liability-driven investment. In fact, the investment policy consists of two endogenous components: a speculative component and a hedging component. This decomposition of the investment policy is familiar from the literature on optimal consumption and portfolio choice under a stochastic investment opportunity set (see, e.g., Brennan and Xia (2002), Wachter (2002), Chacko and Viceira (2005), and Liu (2007)). The speculative portfolio allows individuals to take advantage of risk premia, while the hedging portfolio hedges changes in the investment opportunity set that affect the costs of future benefit payouts (Merton (1971)).[5] The hedging portfolio thus enables individuals to achieve a stable median payout stream in retirement.

Standard variable annuities fully reflect a speculative shock into the current benefit payout; that is, the current benefit payout responds one-to-one to an unexpected change in the value of the speculative portfolio (see, e.g., Chai *et al.* (2011) and Maurer *et al.* (2013)).[6] As a result, in a standard variable annuity, a speculative shock does not affect the AIR. A PPR, in contrast, allows a speculative shock to affect the AIR so as to reduce the year-on-year volatility of retirement consumption. This so-called smoothing of speculative shocks is optimal in the presence of internal habit formation and loss aversion (see, e.g., Fuhrer (2000), Pagel (2017), and Van Bilsen *et al.* (2017)).[7] As the individual ages and the duration of his pension liabilities declines, the AIR becomes less effective in absorbing speculative shocks. Accordingly, in order to prevent an extreme year-on-year volatility of retirement consumption at higher ages, the individual must reduce the riskiness of his investment portfolio over the course of his life.

[4]Brown *et al.* (2008, 2013) find that individuals value annuities more if framed in terms of consumption rather than investment.

[5]Changes in the investment opportunity set are due to shocks in, for instance, the real interest rate.

[6]Pension providers have also developed annuity products in which the current benefit payout responds sluggishly to a speculative shock (see, e.g., Guillén *et al.* (2006) and Maurer *et al.* (2016)).

[7]Smoothing of speculative shocks implies an excessively smooth and excessively sensitive payout stream. Aggregate consumption data also exhibit these properties (see, e.g., Flavin (1985), Deaton (1987), and Campbell and Deaton (1989)).

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1. Modelling the decumulation period of a PPR

1.1 Financial market

This section describes the financial market. Our specification of the financial market is closely related to Liu (2007). The only difference between his specification and ours is that he does not assume a complete financial market, whereas we do. Denote by X_t an N-dimensional vector of state variables. This vector characterizes the asset prices in the financial market. The vector of state variables could include the short-term real interest rate, the realized rate of inflation, or predictors of stock returns. We assume that X_t satisfies the following dynamic equation:

$$\mathrm{d}X_t = \mu^X(X_t)\mathrm{d}t + \Sigma^X(X_t)\mathrm{d}Z_t, \tag{1}$$

where the drift term $\mu^X(X_t)$ and the diffusion matrix $\Sigma^X(X_t)$ are an N-by-1 vector function and an N-by-N matrix function of X_t, respectively, and Z_t is an N-dimensional standard Brownian motion.

We consider a financial market consisting of N risky assets and one (locally) risk-free asset. The vector of risky asset prices $P_t = (P_{1t}, \ldots, P_{Nt})$ and the risk-free asset price P_{0t} satisfy, respectively, the following dynamic equations:[8]

$$\mathrm{d}P_t = \mu(X_t)P_t\mathrm{d}t + \Sigma(X_t)P_t\mathrm{d}Z_t, \tag{2}$$

$$\mathrm{d}P_{0t} = R^f(X_t)P_{0t}\mathrm{d}t. \tag{3}$$

The drift term $\mu(X_t)$, the diffusion matrix $\Sigma(X_t)$ and the risk-free interest rate $R^f(X_t)$ are functions of X_t. In what follows, we write $\mu^X(X_t) = \mu_t^X$, $\Sigma^X(X_t) = \Sigma_t^X$, $\mu(X_t) = \mu_t$, $\Sigma(X_t) = \Sigma_t$ and $R^f(X_t) = R_t^f$.

1.2 Survival probabilities

To protect individuals against the risk of outliving their accumulated retirement wealth, a PPR distributes the accumulated retirement wealth of someone who dies among the surviving individuals of the same age. Hence, a PPR pools individual longevity risk. We assume that the risk-sharing pool is sufficiently large so that the law of large numbers applies. Furthermore, we abstract away from macro longevity risk.

Denote by y the date of birth of an individual, by x_r the age at which individuals retire, and by $x_{\max}$ the maximum age individuals can reach. If the date of birth y falls between time $t - x_r$ and time $t - x_{\max}$ and the individual has survived up to time t, then this individual receives a pension payment at time t. We denote the probability that an individual aged $x = t - y$ will survive to age $x + h$ by

$${}_hp_x = \exp\left\{-\int_0^h \theta_{x+v}\mathrm{d}v\right\} \tag{4}$$

Here, θ_{x+v} represents the force of mortality (i.e., hazard rate) at age $x + v$.

1.3 Budget condition

The value of the individual's investment account should match the value of the individual's pension liabilities in every state of nature and at any date. Indeed, in a PPR, an individual finances its pension liabilities by its own investment account. Let $W_{t,y}$ and $V_{t,y}$ denote, respectively, the value of the

[8]For notational convenience, we often write a column vector in the form $z = (z_1, \ldots, z_N)$, where z_i represents the ith element of z.

investment account and the value of the pension liabilities at time t of an individual born at time y. Mathematically, budget balance implies that for each $t \in [y + x_r, y + x_{\max}]$

$$W_{t,y} = V_{t,y} \tag{5}$$

The budget condition (5) states that the (balance sheet) funding ratio $W_{t,y}/V_{t,y}$ is equal to unity in every state of nature and at any date. It follows from (5) that dlog $W_{t,y}$ = dlog $V_{t,y}$. We explore the dynamics of log $W_{t,y}$ and log $V_{t,y}$ in Sections 1.4 and 1.5, respectively. Section 1.6 derives several relationships between the contract parameters that follow from the budget condition (5).

1.4 Dynamics of the value of the investment account

The value of the investment account of a surviving individual satisfies the following dynamic equation:

$$\mathrm{d}W_{t,y} = \left(\theta_{t-y} + R_t^f + \omega_{t,y}^{\top}[\mu_t - R_t^f]\right) W_{t,y}\mathrm{d}t + \omega_{t,y}^{\top}\Sigma_t W_{t,y}\mathrm{d}Z_t - B_{t,y}\mathrm{d}t. \tag{6}$$

Here, $\omega_{t,y}$ and $B_{t,y}$ represent, respectively, the vector of portfolio weights and the (annualized) benefit payout at time t of an individual born at time y.[9] We can view θ_{t-y} as the biometric rate of return.[10] Indeed, because the accumulated retirement wealth of someone who dies goes to the surviving individuals (and not to its heirs), surviving individuals earn an additional return. The symbol '$\top$' denotes the transpose sign.

Application of Itô's lemma to log $W_{t,y}$ yields

$$d\log W_{t,y} = \left(\theta_{t-y} + \mu_{t,y}^W - c_{t,y}\right)\mathrm{d}t + \omega_{t,y}^{\top}\Sigma_t \mathrm{d}Z_t, \tag{7}$$

where $\mu_{t,y}^W$ and $c_{t,y}$ denote, respectively, the (geometric) expected financial return on accumulated retirement wealth and the withdrawal rate at time t of a person born at time y:[11]

$$\mu_{t,y}^W = R_t^f + \omega_{t,y}^{\top}\left(\mu_t - R_t^f\right) - \frac{1}{2}\omega_{t,y}^{\top}\Sigma_t\Sigma_t^{\top}\omega_{t,y}, \tag{8}$$

$$c_{t,y} = \frac{B_{t,y}}{W_{t,y}}. \tag{9}$$

The withdrawal rate (9) controls the speed at which accumulated retirement wealth $W_{t,y} = V_{t,y}$ is depleted. In fact, it models how the individual's accumulated retirement wealth is allocated between his current payout and his future payouts.

1.5 Dynamics of the value of the pension liabilities

1.5.1 A factorization

Let $C_{t,y}$ denote the conversion factor at time t of an individual born at time y. This factor is implicitly defined as follows:

$$C_{t,y}B_{t,y} = W_{t,y} = V_{t,y}. \tag{10}$$

[9]The portfolio weight $\omega_{it,y}$ denotes the share of accumulated retirement wealth invested in the ith risky asset at time t of an individual born at time y.

[10]In the absence of pooling of mortality risk, this term drops out.

[11]The last term on the right-hand side of (8) is due to Itô lemma. As a result, the geometric expected financial return $\mu_{t,y}^W$ (see (8)) differs from the arithmetic expected financial return $R_t^f + \omega_{t,y}^{\top}\left(\mu_t - R_t^f\right)$.

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It follows from (10) that

$$\mathrm{d}\log V_{t,y} = \mathrm{d}\log C_{t,y} + \mathrm{d}\log B_{t,y}. \tag{11}$$

Hence, to derive the dynamics of the log value of the individual's pension liabilities $V_{t,y}$, we first need to derive the dynamics of the log conversion factor $C_{t,y}$ and the dynamics of the log benefit payout $\log B_{t,y}$. Sections 1.5.2 and 1.5.3 explore the dynamics of $\log C_{t,y}$ and $\log B_{t,y}$, respectively.

1.5.2 Dynamics of the log conversion factor

Denote by $V_{t,y,h}$ the value of the future benefit payout $B_{t+h,y}$ We can write the conversion factor $C_{t,y}$ as follows:

$$C_{t,y} = \frac{V_{t,y}}{B_{t,y}} = \int_{0}^{x_{\max}-(t-y)} \frac{V_{t,y,h}}{B_{t,y}} \mathrm{d}h = \int_{0}^{x_{\max}-(t-y)} C_{t,y,h} \mathrm{d}h, \tag{12}$$

where $C_{t,y,h} = V_{t,y,h}/B_{t,y}$.

Let $\delta_{t,y,h}$ denote individual y's discount rate (or AIR) at time t for a benefit payout occurring at time $t + h$. This discount rate is implicitly defined as follows:

$$C_{t,y,h} = {}_{h}p_{t-y} \exp\{-\delta_{t,y,h}h\}. \tag{13}$$

The discount rate models the speed at which an individual withdraws his accumulated retirement wealth. We allow the discount rate to depend on future expected financial rates of return and past speculative shocks.[12] Hence, we can divide $\delta_{t,y,h}$ into two parts:

$$\delta_{t,y,h} = \delta^{u}_{t,y,h} + \delta^{c}_{t,y,h}. \tag{14}$$

Here, $\delta^{u}_{t,y,h}$ models how the discount rate depends on future expected financial rates of returns. The individual's elasticity of intertemporal substitution determines the extent to which $\delta^{u}_{t,y,h}$ responds to a change in the investment opportunity set. In the special case where the individual has unit elasticity of intertemporal substitution, the discount rate $\delta^{u}_{t,y,h}$ is insensitive to changes in the investment opportunity set. The term $\delta^{c}_{t,y,h}$ models how the discount rate depends on past speculative shocks. By increasing (decreasing) $\delta^{c}_{t,y,h}$ following a negative (positive) speculative shock, the individual (partially) absorbs a shock into the future growth rates of the benefit payout. We call $\delta^{u}_{t,y,h}$ and $\delta^{c}_{t,y,h}$ the unconditional and the conditional part of $\delta_{t,y,h}$, respectively. Henceforth, we refer to $\delta^{u}_{t,y,h}$ as the unconditional discount rate. We note that at the start of the decumulation period, the unconditional discount rate coincides with the actual discount rate (i.e., $\delta^{u}_{y+x_r,y,h} = \delta_{y+x_r,y,h}$ for all h).

Using (13) and (14), we can write $C_{t,y,h}$ as follows:

$$C_{t,y,h} = A_{t,y,h}F_{t,y,h}, \tag{15}$$

where

$$A_{t,y,h} = {}_{h}p_{t-y} \exp\{-\delta^{u}_{t,y,h}h\}, \tag{16}$$

$$F_{t,y,h} = \exp\{-\delta^{c}_{t,y,h}h\}. \tag{17}$$

[12]To dampen the impact of a speculative shock on the current benefit payout, we allow an individual to adjust his conversion factor following a speculative shock.

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In what follows, we refer to $A_{t,y,h}$ and $F_{t,y,h}$ as the horizon-dependent annuity factor and the horizon-dependent funding ratio, respectively.

It follows from Itô's lemma, (12) and (15) that

$$
\begin{aligned}
\mathrm{d}\log C_{t,y} = & \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\mathrm{d}\log C_{t,y,h}\mathrm{d}h - c_{t,y}\mathrm{d}t \\
& + \frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\mathrm{d}\big[\log C_{t,y,h}, \log C_{t,y,h}\big]\mathrm{d}h \\
& - \frac{1}{2}\int_0^{x_{\max}-(t-y)}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,u}\alpha_{t,y,v}\mathrm{d}\big[\log C_{t,y,u}, \log C_{t,y,v}\big]\mathrm{d}u\mathrm{d}v,
\end{aligned}
\tag{18}
$$

and

$$
\mathrm{d}\log C_{t,y,h} = \mathrm{d}\log A_{t,y,h} + \mathrm{d}\log F_{t,y,h}. \tag{19}
$$

Here, $\alpha_{t,y,h} = C_{t,y,h}/C_{t,y}$ and [log $C_{t,y,u}$, log $C_{t,y,v}$] denotes the quadratic covariation between log $C_{t,y,u}$ and log $C_{t,y,v}$. We now derive the dynamics of $\log A_{t,y,h}$ and $\log F_{t,y,h}$, respectively.

Dynamics of the Log Horizon-Dependent Annuity Factor. To derive the dynamics of the log horizon-dependent annuity factor $\log A_{t,y,h}$, we assume that the unconditional discount rate $\delta^u_{t,y,h}$ depends on the vector of state variables X_t. By Itô's lemma, the log horizon-dependent annuity factor $\log A_{t,y,h} = \log {}_h p_{t-y} - \delta^u_{t,y,h}h$ (see also (16) for the definition of $A_{t,y,h}$) satisfies the following dynamic equation:

$$
\begin{aligned}
\mathrm{d}\log A_{t,y,h} = \theta_{t-y}\mathrm{d}t + & \frac{\partial\left(\delta^u_{t,y,h}h\right)}{\partial h}\mathrm{d}t - D^\top_{t,y,h}\mu^X_t\mathrm{d}t \\
& - \frac{1}{2}\mathrm{Tr}\left[\left(\Sigma^X_t\right)^\top H_{t,y,h}\Sigma^X_t\right]\mathrm{d}t - D^\top_{t,y,h}\Sigma^X_t\mathrm{d}Z_t.
\end{aligned}
\tag{20}
$$

Here, $D_{t,y,h} = \nabla_X\left(\delta^u_{t,y,h}h\right)$ and $H_{t,y,h} = H_X\left(\delta^u_{t,y,h}h\right)$ are the gradient and the Hessian matrix of $\delta^u_{t,y,h}h$ with respect to X_t, respectively, and Tr denotes the trace operator. The first term on the right-hand side of (20) denotes the change in the (log) survival probability. This term is positive. Indeed, the probability of surviving to age $t + h - y$ increases as time proceeds. The second term represents the unwinding of the discount rate. Finally, the last three terms model how the underlying state variables affect the log horizon-dependent annuity factor.

Dynamics of the Log Horizon-Dependent Funding Ratio. Equation (20) shows that the (aggregate) annuity factor $A_{t,y} = \int_0^{x_{\max}-(t-y)} A_{t,y,h}\mathrm{d}h$ is typically stochastic. The hedging portfolio aims at hedging stochastic variations in the annuity factor. If the hedging portfolio does not coincide with the actual portfolio, then there is a speculative risk. The individual can absorb a speculative shock in either the current benefit payout $B_{t,y}$ or the future growth rates of the benefit payout or a combination of both.

Denote by $\omega^S_{t,y}$ the N-dimensional vector of speculative portfolio weights at time t of an individual born at time y. The speculative shock at time t is thus given by $\left(\omega^S_{t,y}\right)^\top\Sigma_t\mathrm{d}Z_t$. The individual translates a fraction $q_{t,y,h}$ of a current speculative shock into the future benefit payout $B_{t+h,y}$. That is, the exposure of log $V_{t,y,h}$ to a current speculative shock equals $q_{t,y,h}$. We assume that $q_{t,y,h}$ (which we call the

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smoothing coefficient) weakly increases with the horizon h so that a current speculative shock does not yield a smaller impact on a benefit payout in the distant future than on a benefit payout in the near future. To absorb the entire speculative shock into the current payout and the future growth rates of the payout, we must have that

$$\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} q_{t,y,h} \mathrm{d}h = 1. \tag{21}$$

The exposure of the log horizon-dependent conversion factor $\log C_{t,y,h} = \log V_{t,y,h} - \log V_{t,y,0}$ to a current speculative shock equals $q_{t,y,h} - q_{t,y,0}$. Hence, the horizon-dependent funding ratio (which models how the conversion factor depends on past speculative shocks) is given by[13]

$$F_{t,y,h} = \exp\left\{ \int_{y+x_r}^{t} \left(q_{s,y,t+h-s} - q_{s,y,t-s}\right)\left(\omega_{s,y}^S\right)^\top \Sigma_s \mathrm{d}Z_s \right\}. \tag{22}$$

By comparing (17) with (22), we arrive at

$$\delta_{t,y,h}^c h = -\int_{y+x_r}^{t} \left(q_{s,y,t+h-s} - q_{s,y,t-s}\right)\left(\omega_{s,y}^S\right)^\top \Sigma_s \mathrm{d}Z_s = -\log F_{t,y,h}. \tag{23}$$

Equation (23) shows how past speculative shocks affect the discount rate. In the case of no horizon differentiation in risk exposures (i.e., $q_{t,y,h}$ is equal to unity for every h), past speculative shocks do not affect the conversion factor $C_{t,y}$. Indeed, in the absence of horizon differentiation in risk exposures, the individual fully translates a speculative shock into the current benefit payout. The conditional part of the discount rate, i.e., $\delta_{t,y,h}^c$, is thus the consequence of the gradual adjustment of the current benefit payout to a speculative shock. The log horizon-dependent funding ratio obeys the following equation (this follows from (22)):

$$\mathrm{d}\log F_{t,y,h} = \left(q_{t,y,h} - q_{t,y,0}\right)\left(\omega_{t,y}^S\right)^\top \Sigma_t \mathrm{d}Z_t - \int_{y+x_r}^{t} \mathrm{d}q_{s,y,t-s}\left(\omega_{s,y}^S\right)^\top \Sigma_s \mathrm{d}Z_s. \tag{24}$$

The first term on the right-hand side of (24) represents the impact of a current speculative shock on the horizon-dependent funding ratio. The second term denotes past speculative shocks that are absorbed into the current benefit payout so that they are no longer included in the horizon-dependent funding ratio.

Also, Guillén *et al.* (2006) and Maurer *et al.* (2016) consider a pension product in which a speculative shock has less impact on the current benefit payout than on the future benefit payouts.[14] Their pension product works as follows. In the case of a positive investment return, only a fraction of the positive return will be added to the benefit payout. The insurer retains the remainder of the return. In the case of a negative investment return, only a fraction of the negative return will be subtracted from the benefit payout. The insurer confers the additional benefit payout. A potential drawback of

[13]We note that the so-called balance sheet funding ratio $W_{t,y}/V_{t,y}$ is equal to unity in every state of nature and at any date (see (5)). In contrast, the so-called cash-flow funding ratio $F_{t,y} = \int_0^{x_{\max}-(t-y)} F_{t,y,h} A_{t,y,h} \mathrm{d}h / A_{t,y} = W_{t,y}/(B_{t,y} A_{t,y})$ can deviate from unity.

[14]See also Jørgensen and Linnemann (2011), Guillén *et al.* (2013), and Linnemann *et al.* (2014).

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this shock absorbing mechanism is that the pension provider runs the risk of ending up with a negative reserve. Our specification, in contrast, implies individual buffers so that investment shocks do not cause transfers between individuals and the provider.

1.5.3 Dynamics of the log benefit payout

We specify the benefit payout at time $t+h$ of an individual born at time y as follows:

$$B_{t+h,y} = B_{y+x_r,y} \exp\left\{ \int_{y+x_r}^{t+h} \gamma^{\mu}_{s,y} \mathrm{d}s + \int_{y+x_r}^{t+h} \left(\Sigma^{B}_{s,y,t+h-s}\right)^{\top} \mathrm{d}Z_s \right\}. \tag{25}$$

Here, $\gamma^{\mu}_{t,y}$ denotes the unconditional median growth rate of the benefit payout at time t of an individual born at time y and $\Sigma^{B}_{t,y,h}$ models the exposure of the future benefit payout $\log B_{t+h,y}$ to a current Brownian shock $\mathrm{d}Z_t$. We require that $\Sigma^{B}_{t,y,h}$ weakly increases with the horizon h so that a current Brownian shock does not yield a smaller impact on a benefit payout in the distant future than on a benefit payout in the near future.

It follows from (25) that

$$B_{t+h,y} = B_{t,y} F_{t,y,h} \exp\left\{ \int_{t}^{t+h} \gamma^{\mu}_{s,y} \mathrm{d}s + \int_{t}^{t+h} \left(\Sigma^{B}_{s,y,t+h-s}\right)^{\top} \mathrm{d}Z_s \right\}, \tag{26}$$

where

$$F_{t,y,h} = \exp\left\{ \int_{y+x_r}^{t} \left(\Sigma^{B}_{s,y,t+h-s} - \Sigma^{B}_{s,y,t-s}\right)^{\top} \mathrm{d}Z_s \right\}. \tag{27}$$

Comparison of (27) with (22) yields

$$\Sigma^{B}_{t,y,h} = q_{t,y,h} \Sigma^{\top}_{t} \omega^{S}_{t,y}. \tag{28}$$

Equation (28) expresses how the vector of speculative portfolio weights $\omega^{S}_{t,y}$ and the smoothing coefficient $q_{t,y,h}$ together determine the vector of payout exposures $\Sigma^{B}_{t,y,h}$.

The log benefit payout $\log B_{t,y}$ evolves according to (this follows from (25)):

$$\mathrm{d}\log B_{t,y} = \gamma^{\mu}_{t,y}\mathrm{d}t + \int_{y+x_r}^{t} \mathrm{d}\left(\Sigma^{B}_{s,y,t-s}\right)^{\top} \mathrm{d}Z_s + \left(\Sigma^{B}_{t,y,0}\right)^{\top} \mathrm{d}Z_t = \gamma_{t,y}\mathrm{d}t + \left(\Sigma^{B}_{t,y,0}\right)^{\top} \mathrm{d}Z_t. \tag{29}$$

Here, $\gamma_{t,y}$ denotes the actual median growth rate of the current benefit payout. This rate depends on current expected financial rates of return (i.e., if expected returns change, the individual may want to reallocate consumption over time) and past Brownian shocks. Hence, we can divide $\gamma_{t,y}$ into two parts:

$$\gamma_{t,y} = \gamma^{\mu}_{t,y} + \gamma^{c}_{t,y}, \tag{30}$$

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where

$$\gamma_{t,y}^{c} = \int\limits_{y+x_r}^{t} \mathrm{d}\left(\Sigma_{s,y,t-s}^{B}\right)^{\top} \mathrm{d}Z_s. \tag{31}$$

We refer to $\gamma_{t,y}^{c}$ as the conditional part of $\gamma_{t,y}$. Equation (31) represents past Brownian shocks that affect the median growth rate of the current benefit payout.

1.6 Relationships between the contract parameters

The previous sections have modelled the decumulation period of a PPR in terms of seven parameters. The first column of Table 1 lists these parameters. This section derives several relationships between the contract parameters that follow from the budget condition (5). We can write this condition as follows (use dlog $V_{t,y}$ = dlog $C_{t,y}$ + dlog $B_{t,y}$):

$$\mathrm{d}\log W_{t,y} = \mathrm{d}\log C_{t,y} + \mathrm{d}\log B_{t,y}. \tag{32}$$

Appendix A.1 shows that (32) is equivalent to:

$$\mu_{t,y}^{W}\mathrm{d}t + \omega_{t,y}^{\top}\Sigma_t \mathrm{d}Z_t = \left(\mu_{t,y}^{C} + \gamma_{t,y}\right)\mathrm{d}t + [\left(\omega_{t,y}^{S}\right)^{\top}\Sigma_t - \hat{D}_{t,y}^{\top}\Sigma_t^{X}]\mathrm{d}Z_t. \tag{33}$$

Here, $\hat{D}_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{t,y,h}\mathrm{d}h$ models the sensitivity of the conversion factor with respect to the underlying state variables and $\mu_{t,y}^{C}$ is the expected financial rate of return on the conversion factor; see Appendix A.1 for the expression of $\mu_{t,y}^{C}$.

Using (5) and (33), we find the following system of equations:[15]

$$W_{y+x_r,y} = C_{y+x_r,y}\, B_{y+x_r,y}, \tag{34}$$

$$\gamma_{t,y} = \mu_{t,y}^{W} - \mu_{t,y}^{C}, \tag{35}$$

$$\omega_{t,y} = \omega_{t,y}^{S} - \left(\Sigma_t^{X}\Sigma_t^{-1}\right)^{\top}\hat{D}_{t,y}. \tag{36}$$

Individuals can use this system of equations to determine the contract parameters. Condition (or relationship) (35) shows that the median growth rate of the current benefit payout $\gamma_{t,y}$ equals the difference between the expected financial rate of return on accumulated retirement wealth $\mu_{t,y}^{W}$ and the expected financial rate of return on the conversion factor $\mu_{t,y}^{C}$. Condition (36) shows that the vector of portfolio weights $\omega_{t,y}$ is equal to the sum of the vector of speculative portfolio weights $\omega_{t,y}^{S}$ and the vector of hedge portfolio weights $\omega_{t,y}^{H} = -\left(\Sigma_t^{X}\Sigma_t^{-1}\right)^{\top}\hat{D}_{t,y}$. We also have a condition for every horizon h (see also (28)):

$$\Sigma_{t,y,h}^{B} = q_{t,y,h}\Sigma_t^{\top}\omega_{t,y}^{S}. \tag{37}$$

[15]In the absence of pooling of mortality risk, the right-hand side of (35) includes the biometric rate of return, i.e., $\gamma_{t,y} = \mu_{t,y}^{W} - \mu_{t,y}^{C} - \theta_{t-y}$

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Table 1. Investment approach versus consumption approach. The second (third) column of this table summarizes the exogenous and endogenous parameters of the investment (consumption) approach

Parameter	Investment approach	Consumption approach
Value of investment account at start of period	Exogenous	Endogenous
Vector of portfolio weights	Exogenous	Endogenous
Unconditional discount rate	Exogenous	Endogenous
Smoothing coefficient	Exogenous	Endogenous
Benefit payout at start of period	Endogenous	Exogenous
Unconditional median growth rate	Endogenous	Exogenous
Vector of payout exposures	Endogenous	Exogenous

Conditions (34)–(37) do not uniquely identify all contract parameters. Individuals must thus specify some parameters exogenously. They can specify the parameters according to (at least) two alternative approaches: the investment approach and the consumption approach. In the investment approach individuals specify the initial value of the investment account, the unconditional discount rate (as a function of t and h), the vector of portfolio weights (as a function of t) and the smoothing coefficient (as a function of t and h) exogenously; see also the second column of Table 1. In the consumption approach individuals specify the benefit payout at the start of the decumulation period, the unconditional median growth rate of the current benefit payout (as a function of t) and the vector of payout exposures (as a function of t and h) exogenously; see also the third column of Table 1. The next sections explore these two approaches in more detail and show how they differ from each other.

2. Investment approach

This section explores the investment approach which is commonly adopted by pension providers. In this approach the benefit payout at the start of the decumulation period, the unconditional median growth rate of the current benefit payout and the vector of payout exposures are endogenously determined; see also the second column of Table 1. Section 2.1 specifies the vector of state variables and its dynamics. Section 2.2 considers the investment approach, with the restriction that the smoothing coefficient $q_{t,y,h}$ is equal to unity for all h. Hence, past speculative shocks do not affect the discount rate. We relax this restriction in Section 2.3.

2.1 State variables

For the sake of simplicity, we characterize asset prices by three state variables: the inflation rate π_t, the real interest rate r_t, and the stock price S_t. Hence, $X_t = (\pi_t, r_t, S_t)$. Following Brennan and Xia (2002), the inflation rate and the real interest rate follow Ornstein–Uhlenbeck processes and the stock price follows a geometric Brownian motion. The drift term μ_t^X and the diffusion coefficient Σ_t^X are thus specified as follows:[16]

$$\mu_t^X = \begin{pmatrix} \eta(\bar{\pi} - \pi_t) \\ \kappa(\bar{r} - r_t) \\ S_t R_t^f + S_t \lambda_S \sigma_S \end{pmatrix}, \Sigma_t^X = \begin{pmatrix} \sigma_\pi & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & S_t \sigma_S \end{pmatrix}. \tag{38}$$

Here, $\eta > 0$ and $\kappa > 0$ are mean reversion coefficients, $\bar{\pi}$ and $\bar{r}$ denote long-term means, λ_S is the equity risk premium per unit of risk, and $\sigma_\pi > 0$, $\sigma_r > 0$ and $\sigma_S > 0$ correspond to diffusion coefficients.[17]

The individual invests his retirement wealth in three risky assets: two nominal zero-coupon bonds with times to maturity h_1 and h_2, and a stock. We find the following expressions for the expected excess return $\mu_t - R_t^f$ and the diffusion matrix Σ_t (see Appendix A.2):

[16]We assume that the state variables are *un*correlated. It is straightforward to extend our analysis to the case where the state variables are correlated.

[17]In the standard model with neither interest rate risk nor inflation risk, we have $\sigma_\pi = \sigma_r = 0$.

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Table 2. Parameter values. This table reports the parameter values that we use in our numerical illustrations

Parameter	Value	Parameter	Value
$\bar{\pi}$	0.02	$\bar{r}$	0.01
η	0.2	κ	0.1
σ_π	0.01	σ_r	0.015
λ_π	−0.05	λ_r	−0.15

$$\mu_t - R_t^f = \begin{pmatrix} -\lambda_\pi\sigma_\pi K_{h_1} - \lambda_r\sigma_r L_{h_1} \\ -\lambda_\pi\sigma_\pi K_{h_2} - \lambda_r\sigma_r L_{h_2} \\ \lambda_S\sigma_S \end{pmatrix}, \Sigma_t = \begin{pmatrix} -\sigma_\pi K_{h_1} & -\sigma_r L_{h_1} & 0 \\ -\sigma_\pi K_{h_2} & -\sigma_r L_{h_2} & 0 \\ 0 & 0 & \sigma_S \end{pmatrix}. \tag{39}$$

Here, $\lambda = (\lambda_\pi, \lambda_r, \lambda_S)$ is the vector of market prices of risk, $K_{h_1} = (1 - e^{-\eta h_1})/\eta$ and $L_{h_1} = (1 - e^{-\kappa h_1})/\kappa$. Table 2 reports the parameter values that we use in our numerical illustrations.

2.2 No Smoothing of speculative shocks

Figure 1 illustrates the investment approach, with the following two restrictions. First, the smoothing coefficient $q_{t,y,h}$ is equal to unity for all h. Second, the (unconditional) discount rate is constant (i.e., $\delta^u_{t,y,h} = \delta^u = \delta$ for all t and h). The latter restriction implies that the vector of hedge portfolio weights is equal to zero:

$$\omega^H_{t,y} = 0. \tag{40}$$

The benefit payout at the start of the decumulation period, the (unconditional) median growth rate of the current benefit payout and the vector of payout exposures follow from (34), (35) and (37), respectively. We find[18]

$$B_{y+x_r,y} = \frac{W_{y+x_r,y}}{C_{y+x_r,y}}, \tag{41}$$

$$\gamma^u_{t,y} = \mu^W_{t,y} - \delta^u, \tag{42}$$

$$\Sigma^B_{t,y,h} = \Sigma^B_{t,y} = \Sigma_t^\top \omega^S_{t,y}. \tag{43}$$

Equation (42) shows that the expected rate of return on accumulated retirement wealth $\mu^W_{t,y} = R_t^f + \omega^\top_{t,y}\left(\mu_t - R_t^f\right) - 1/2\omega^\top_{t,y}\Sigma_t\Sigma_t^\top\omega_{t,y}$ and the (unconditional) discount rate δ^u together determine the (unconditional) median growth rate of the current benefit payout $\gamma^u_{t,y}$. As a result, the median growth rate of the current benefit payout is not constant but rather depends on the inflation rate and the real interest rate.[19] To obtain a constant median growth rate, the individual must determine the discount rate endogenously. Section 3 derives the discount rate under the assumption that the individual specifies the median growth rate of the current benefit payout exogenously. We will

[18]We note that the conditional part of the median growth rate of the current benefit payout is equal to zero. Furthermore, in the absence of pooling of mortality risk, the right-hand side of (42) includes the biometric rate of return, i.e., $\gamma^u_{t,y} = \mu^W_{t,y} - \delta^u - \theta_{t-y}$.

[19]This assumes the presence of interest rate risk and/or inflation risk. In the absence of both risks (i.e., $\sigma_\pi = \sigma_r = 0$), the median growth rate of the current benefit payout will be constant.

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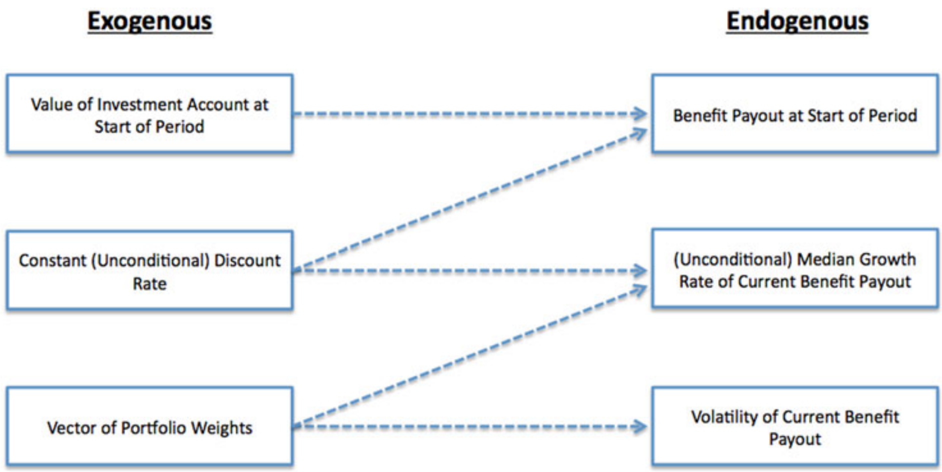

Figure 1. Illustration of the Investment Approach: A Special Case. The figure illustrates the investment approach, with the restriction that the smoothing coefficient $q_{t,y,h}$ is equal to unity for all h and the (unconditional) discount rate is constant. The left-hand side of the figure shows the exogenous parameters of the contract. These exogenous parameters determine the parameters of the contract on the right-hand side of the figure.

see that the discount rate then depends on the current inflation rate, the current real interest rate and the horizon. In fact, the discount rate has an endogenous term structure.

2.3 Smoothing of speculative shocks

This section assumes that the individual now adjusts the conversion factor following a speculative shock; in other words, the smoothing coefficient $q_{t,y,h}$ increases with the investment horizon h. As a direct consequence, the conditional part of the discount rate, i.e., $\delta^c_{t,y,h}$, differs from zero (see (23)). We still assume that the unconditional discount rate is constant (i.e., $\delta^u_{t,y,h} = \delta^u$ for all t and h). The unconditional median growth rate of the current benefit payout, i.e., $\gamma^u_{t,y}$ is given by (42). However, equation (43) no longer applies. The exposure of the future log benefit payout $\log B_{t+h,y}$ to a current Brownian shock is now given by

$$\Sigma^B_{t,y,h} = q_{t,y,h}\Sigma_t^\top \omega^S_{t,y}. \tag{44}$$

It follows from (21) that $q_{t,y,0}$ increases and converges to one when the individual becomes older because his remaining life expectancy declines and the value weights of the shorter horizons $\alpha_{t,y,h}$ increase. Hence, under the condition that the vector of speculative portfolio weights $\omega^S_{t,y}$ does not change over time, the vector of payout exposures $\Sigma^B_{t,y,0}$ becomes larger as the individual ages (see (44)). Intuitively, a relatively old individual with a short remaining life expectancy can no longer smooth the investment results over a long remaining lifetime. Hence, the year-on-year volatility of retirement consumption increases as the individual ages. To arrive at a constant year-on-year volatility of retirement consumption, the individual must determine the investment policy endogenously. Section 3.2.2 below derives an endogenous investment policy such that the year-on-year volatility of retirement consumption is constant.

3. Consumption approach

This section explores the consumption approach. We consider the same setting as in Section 2.1. In the consumption approach, individuals specify the entire retirement consumption stream exogenously.

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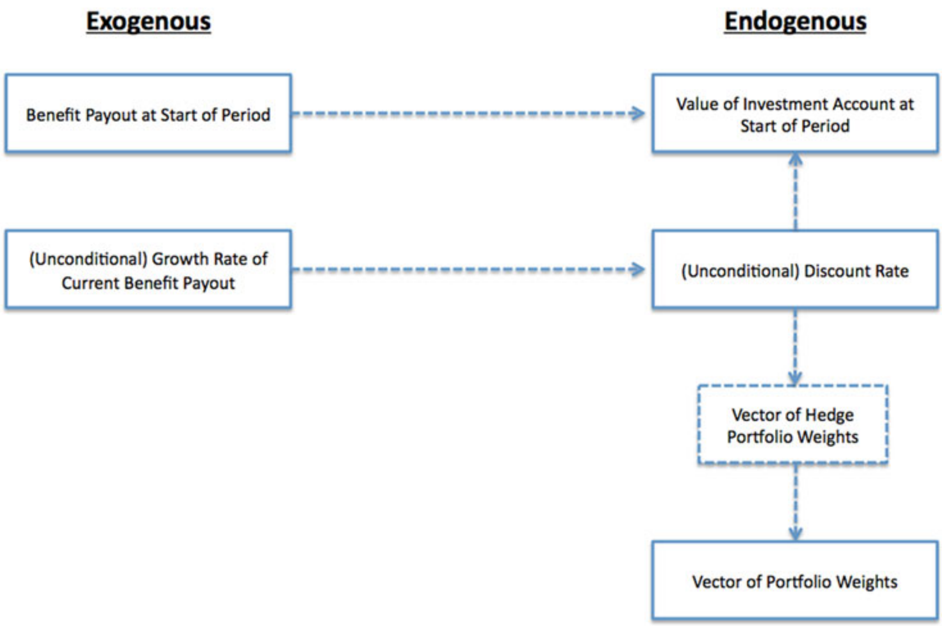

Figure 2. Illustration of the DB Approach. The figure illustrates the DB approach. The left-hand side of the figure shows the exogenous parameters of the contract. These exogenous parameters determine the parameters of the contract on the right-hand side of the figure.

Section 3.1 examines the DB approach in which retirement consumption is constant in either nominal or real terms. Section 3.2 extends the DB approach to stochastic benefit payouts.

3.1 DB approach

In the DB approach individuals specify the benefit payout at the start of the decumulation period and the rate at which the benefit payout grows over time. The benefit payout either grows with the inflation rate (inflation-linked annuity) or does not grow at all (nominal annuity). This section derives, in line with the principle of liability-driven investment, the (unconditional) forward discount rate and the vector of portfolio weights endogenously from the liabilities of the contract. Figure 2 illustrates the DB approach.

The DB approach specifies the (unconditional) growth rate of the current benefit payout as follows:

$$\gamma^u_{t,y} = g_0 + g_1 \cdot \pi_t, \tag{45}$$

where $g_1 \in \{0, 1\}$. If g_1 equals zero (unity), retirement consumption is constant in nominal (real) terms.

Let $\breve{\delta}^u_{t,y,v}$ denote individual y's (unconditional) forward discount rate at time t for horizon v. The (unconditional) forward discount rate is implicitly defined as follows:

$$\delta^u_{t,y,h} = \frac{1}{h}\int_0^h \breve{\delta}^u_{t,y,v}\mathrm{d}v. \tag{46}$$

We find that the forward discount rate $\breve{\delta}^u_{t,y,v}$ is specified as follows (see Appendix A.3):

$$\breve{\delta}^u_{t,y,v} = \mathbb{E}_t[(1-g_1)\pi_{t+v} + r_{t+v}] - g_0 - D_{1v}\sigma_\pi\left(\lambda_\pi + \frac{1}{2}D_{1v}\sigma_\pi\right) - D_{2v}\sigma_r\left(\lambda_r + \frac{1}{2}D_{2v}\sigma_r\right), \tag{47}$$

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where $D_{1v} = (1-g_1)1/\eta(1-e^{-\eta v})$, $D_{2v} = (1/\kappa)(1-e^{-\kappa v})$ and $\mathbb{E}_t[(1-g_1)\pi_{t+v} + r_{t+v}] = (1-g_1)[\pi_t + (1-e^{-\eta v})(\bar{\pi} - \pi_t)] + r_t + (1-e^{-\kappa v})(\bar{r} - r_t)$.

We observe that the coefficients g_0 and g_1, which model the median growth rate of the current benefit payout, determine the forward discount rate (47). Intuitively, the higher the median growth rate of the current benefit payout is, the higher the costs of future benefit payouts are and the lower the forward discount rate will be.

Mean reversion in the state variables also affects the forward discount rate and is captured by the term $\mathbb{E}_t[(1-g_1)\pi_{t+v} + r_{t+v}]$. Indeed, if the current inflation rate and the current real interest rate exceed their long-term means (i.e.,$\pi_t > \bar{\pi}$ and $r_t > \bar{r}$), then returns are expected to decline over time (i.e., $\mathbb{E}_t[(1-g_1)\pi_{t+v} + r_{t+v}]$ decreases with v), and hence it may be relatively more expensive to finance long-term benefit payouts than to finance short-term benefit payouts depending on the size of the second term, i.e., $-D_{1v}\sigma_\pi(\lambda_\pi + (1/2)D_{1v}\sigma_\pi) - D_{2v}\sigma_r(\lambda_r + (1/2)D_{2v}\sigma_r)$.

The latter term reflects the expected excess return on the underlying hedging portfolio and typically increases with the horizon. Indeed, larger horizons are more exposed to inflation risk and real interest rate risk and thus benefit from higher risk premia (i.e., D_{1v} and D_{2v} increase with v and λ_π, $\lambda_r < 0$). Hence, the later the payout date is, the larger the second term will be. The discount rate thus exhibits an endogenous term structure, which depends both on current state variables (i.e., current inflation rate and the current real interest rate) and the horizon. In particular, the term structure is upward sloping unless the current inflation rate and the current real interest rate are substantially above their long-term means.

3.1.1 Guaranteed nominal benefit payouts

The individual receives guaranteed flat nominal benefit payouts if $g_0 = g_1 = 0$. We find the following expression for the (unconditional) forward discount rate (substitute $g_0 = g_1 = 0$ into (47)):

$$\check{\delta}^u_{t,y,v} = R_{t,v}, \tag{48}$$

where $R_{t,v}$ denotes the nominal forward interest rate:

$$R_{t,v} = \mathbb{E}_t[\pi_{t+v} + r_{t+v}] - D_{1v}\sigma_\pi\left(\lambda_\pi + \frac{1}{2}D_{1v}\sigma_\pi\right) - D_{2v}\sigma_r\left(\lambda_r + \frac{1}{2}D_{2v}\sigma_r\right). \tag{49}$$

Figure 3 illustrates the forward discount rate $\check{\delta}^u_{t,y,v}$ for various values of the current inflation rate π_t and the current real interest rate r_t as a function of the horizon v. The solid line shows the case in which the current inflation rate and the current real interest rate are equal to their long-term means (i.e., $\mathbb{E}_t[\pi_{t+v}] = \bar{\pi}$ and $\mathbb{E}_t[r_{t+v}] = \bar{r}$ for all v). As shown by Figure 3, the solid line is not horizontal, but rather rises with the investment horizon v. Indeed, the longer the investment horizon, the more a nominal benefit payout is exposed to inflation risk and real interest rate risk, and hence the larger the nominal interest rate risk premium $\check{\delta}^u_{t,y,v} - \mathbb{E}_t[\pi_{t+v} + r_{t+v}] = -D_{1v}\sigma_\pi(\lambda_\pi + (1/2)D_{1v}\sigma_\pi) - D_{2v}\sigma_r(\lambda_r + (1/2)D_{2v}\sigma_r)$.[20]

The dash-dotted and the dashed line show the case in which the current inflation rate and the current real interest rate deviate from their long-term means. If the current inflation rate and the current real interest rate exceed their long-term means, then the term $\mathbb{E}_t[\pi_{t+v} + r_{t+v}]$ in (49) decreases with the investment horizon v. Indeed, in a situation where returns are expected to decline over time, it may be relatively more expensive to finance long-term benefit payouts than to finance short-term benefit payouts if the mean-reversion term $\mathbb{E}_t[\pi_{t+v} + r_{t+v}]$ dominates the risk premium term $-D_{1v}\sigma_\pi(\lambda_\pi + (1/2)D_{1v}\sigma_\pi) - D_{2v}\sigma_r(\lambda_r + (1/2)D_{2v}\sigma_r)$

[20] In the absence of interest rate risk and inflation risk, the nominal interest rate would be fixed so that the solid line in Figure 3 would be flat.

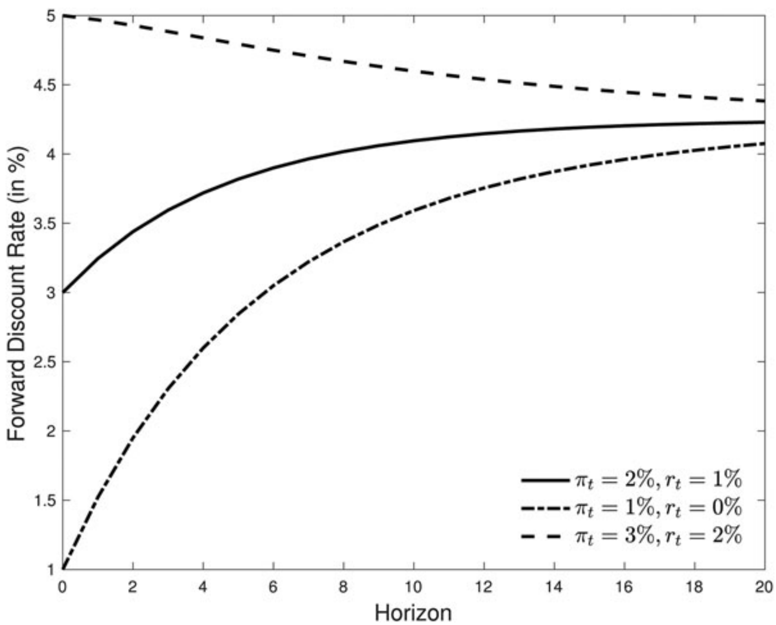

Figure 3. Illustration of the Nominal Forward Interest Rate. The figure illustrates the nominal forward interest rate for various values of the current inflation rate π_t and the current real interest rate r_t as a function of the horizon v. The benchmark parameter values are given in Table 2.

The inflation sensitivity and the real interest rate sensitivity of the value of the pension liabilities are, respectively, given by

$$\hat{D}_{1t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{1h} \mathrm{d}h = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}(1 - e^{-\eta h})\mathrm{d}h/\eta, \tag{50}$$

$$\hat{D}_{2t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{2h} \mathrm{d}h = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}(1 - e^{-\kappa h})\mathrm{d}h/\kappa. \tag{51}$$

Pension providers can replicate the pension contract by investing in a portfolio of nominal bonds with inflation sensitivity $\hat{D}_{1t,y}$ and real interest rate sensitivity $\hat{D}_{2t,y}$; that is, the portfolio weights $\omega_{1t,y}$ and $\omega_{2t,y}$ solve the following system of equations:

$$\hat{D}_{1t,y} = \omega_{1t,y} K_{h_1} + \omega_{2t,y} K_{h_2}, \tag{52}$$

$$\hat{D}_{2t,y} = \omega_{1t,y} L_{h_1} + \omega_{2t,y} L_{h_2}. \tag{53}$$

Equations (52) and (53) show that the portfolio weights $\omega_{1t,y}$ and $\omega_{2t,y}$ are not constant (as is usually the case under the investment approach), but rather depend on the inflation sensitivity and the real interest rate sensitivity of the value of the liabilities. In particular, the larger the inflation sensitivity

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and the real interest rate sensitivity of the value of the liabilities, the greater the extent to which the value of the bond portfolio responds to a change in the inflation rate and the real interest rate.

3.1.2 Guaranteed inflation-linked benefit payouts

The individual receives guaranteed flat inflation-linked benefit payouts if $g_0 = 0$ and $g_1 = 1$. We find the following expression for the (unconditional) forward discount rate (substitute $g_0 = 0$ and $g_1 = 1$ into (47)):

$$\breve{\delta}^{u}_{t,y,v} = r_{t,v}, \tag{54}$$

where $r_{t,v}$ denotes the real forward interest rate:

$$r_{t,v} = \mathbb{E}_t[r_{t+v}] - D_{2v}\sigma_r\left(\lambda_r + \frac{1}{2}D_{2v}\sigma_r\right). \tag{55}$$

Figure 4 illustrates the forward discount rate $\breve{\delta}^{u}_{t,y,v}$ for various values of the current real interest rate r_t as a function of the horizon v. The solid line shows the case in which the current real interest rate is equal to its long-term mean. This line increases with the investment horizon v. Indeed, the longer the investment horizon, the more an inflation-linked benefit payout is exposed to real interest rate risk, and hence the larger the real interest rate risk premium $\breve{\delta}^{u}_{t,y,v} - \mathbb{E}_t[r_{t+v}] = -D_{2v}\sigma_r(\lambda_r + (1/2)D_{2v}\sigma_r)$. Also here, the slope of the term structure depends on the risk premium term, which is typically upward sloping, and the mean-reversion term, which may be both upward and downward sloping depending on how the current short-term real interest rate compares with its long-term mean.

The real interest rate sensitivity of the value of the pension liabilities is given by

$$\hat{D}_{2t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{2h} \mathrm{d}h = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}(1 - e^{-\kappa h})\mathrm{d}h/\kappa. \tag{56}$$

Budget balance requires the pension provider to invest in an investment portfolio that is insensitive to changes in the inflation rate and, furthermore, has the same real interest rate sensitivity as the value of the liabilities, i.e.,

$$0 = \omega_{1t,y} K_{h_1} + \omega_{2t,y} K_{h_2}, \tag{57}$$

$$\hat{D}_{2t,y} = \omega_{1t,y} L_{h_1} + \omega_{2t,y} L_{h_2}. \tag{58}$$

The investment portfolio is thus continuously rebalanced over time.

3.2 Defined Ambition (DA) Approach

The DA approach generalizes the DB approach to stochastic benefit payouts. In the DA approach individuals specify the benefit payout at the start of the decumulation period, the unconditional median growth rate of the current benefit payout and the vector of payout exposures. As in the DB approach, the unconditional forward discount rate and the vector of portfolio weights follow endogenously from the liabilities of the contract. The DA approach generalizes the DB approach in two directions.

First, the unconditional median growth rate of the current benefit payout (45) may depend on the real interest rate r_t, i.e.,

$$\gamma^{u}_{t,y} = g_0 + g_1 \cdot \pi_t + g_2 \cdot r_t, \tag{59}$$

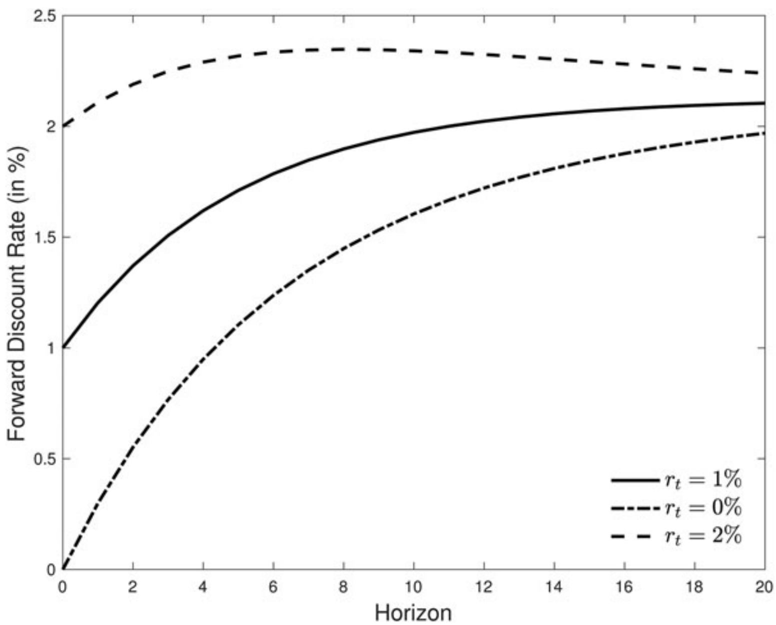

Figure 4. Illustration of the Real Forward Interest Rate. The figure illustrates the real forward interest rate for various values of the current real interest rate r_t as a function of the horizon v. The benchmark parameter values are given in Table 2.

where the coefficients g_0, g_1 and g_2 are given exogenously.[21] In fact, the coefficient g_2 models the preference for intertemporal substitution (i.e., the extent to which the drawdown policy changes in response to a real interest rate shock). We note that if $g_1 \in \{0, 1\}$ and $g_2 = 0$, (59) reduces to (45).

Second, the individual is allowed to take the speculative risk (i.e., $\Sigma^B_{t,y,h} \neq 0$). Section 3.2.1 assumes that the exposure of a future benefit payout to a current Brownian shock does not depend on the investment horizon h; that is, $\Sigma^B_{t,y,h} = \Sigma^B_{t,y}$. Figure 5 illustrates this case. Section 3.2.2 considers the DA approach with gradual adjustment of the current benefit payout to a current Brownian shock; that is, $\Sigma^B_{t,y,h}$ increases with the horizon h. This case is illustrated by Figure 6.

3.2.1 Direct adjustment of the current benefit payout

We find that the forward discount rate $\breve{\delta}^u_{t,y,v}$ is given by (see Appendix A.3):

$$\begin{aligned}\breve{\delta}^u_{t,y,v} = {} & \mathbb{E}_t[(1-g_1)\pi_{t+v} + (1-g_2)r_{t+v}] - g_0 - D_{1v}\sigma_\pi\left(\lambda_\pi + \frac{1}{2}D_{1v}\sigma_\pi\right) \\ & - D_{2v}\sigma_r\left(\lambda_r + \frac{1}{2}D_{2v}\sigma_r\right) + \left(\omega^S_{t,y}\right)^\top\left[\left(\mu_t - R^f_t\right) - \frac{1}{2}\Sigma_t(\Sigma_t)^\top\omega^S_{t,y}\right] \\ & + \left(\omega^S_{t,y}\right)^\top\Sigma_t\left(D_v^\top\Sigma^X_t\right)^\top,\end{aligned} \tag{60}$$

where the vector of (efficient) speculative portfolio weights $\omega^S_{t,y}$ follows from the vector of payout exposures $\Sigma^B_{t,y}$:

$$\omega^S_{t,y} = \left(\Sigma_t^{-1}\right)^\top\Sigma^B_{t,y}. \tag{61}$$

[21]Appendix B derives the optimal coefficients g_0, g_1 and g_2 in case the individual aims to maximize constant relative risk aversion utility. This appendix also derives the optimal median growth rate of the current benefit payout in case mortality risk is not pooled.

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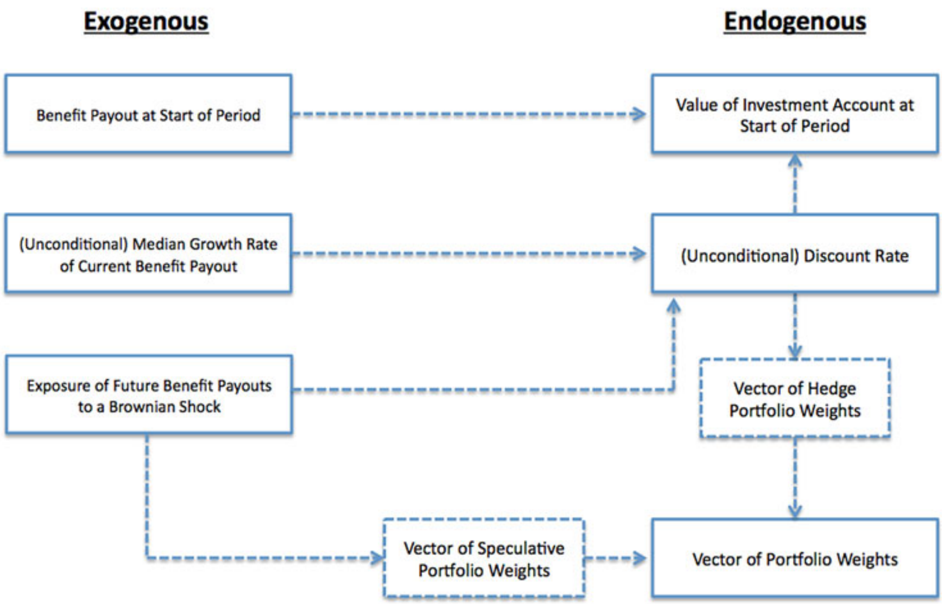

Figure 5. Illustration of the DA Approach: Direct Adjustment of the Current Benefit Payout. The figure illustrates the DA approach with direct adjustment of the current benefit payout. The left-hand side of the figure shows the exogenous parameters of the contract. These exogenous parameters determine the parameters of the contract on the right-hand side of the figure.

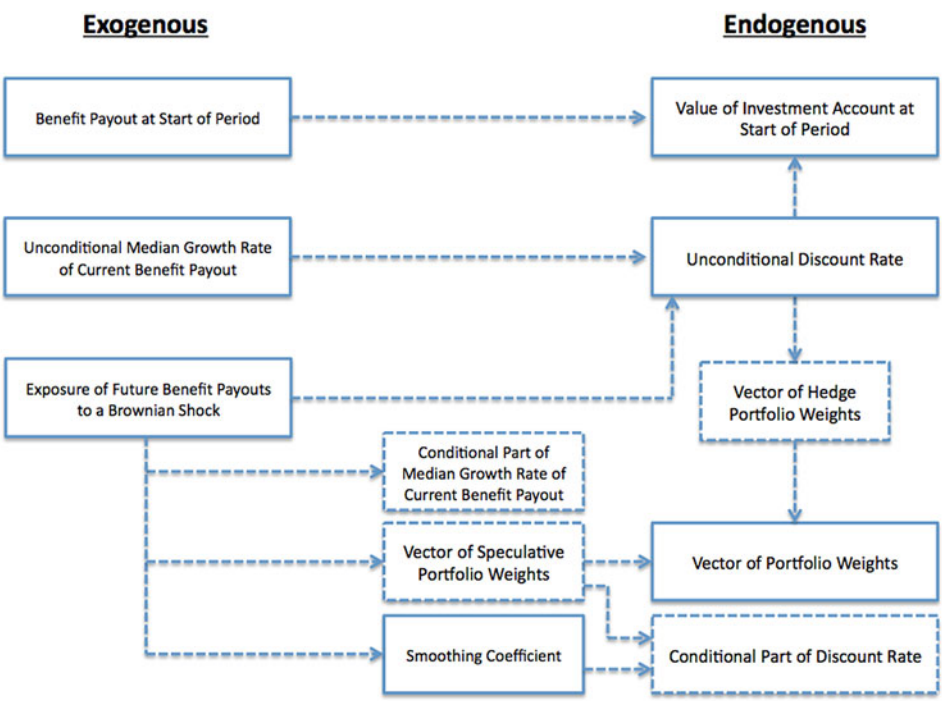

Figure 6. Illustration of the DA Approach: Gradual Adjustment of the Current Benefit Payout. The figure illustrates the DA approach with gradual adjustment of the current benefit payout. The left-hand side of the figure shows the exogenous parameters of the contract. These exogenous parameters determine the parameters of the contract on the right-hand side of the figure.

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Comparing (60) with (47), we observe that the forward discount rate $\breve{\delta}^{u}_{t,y,v}$ now includes two additional terms. The first additional term

$$\left(\omega^S_{t,y}\right)^\top \left[\left(\mu_t - R^f_t\right) - \frac{1}{2}\Sigma_t(\Sigma_t)^\top \omega^S_{t,y}\right] = \left(\Sigma^B_{t,y}\right)^\top \left[\Sigma_t^{-1}\left(\mu_t - R^f_t\right) - \frac{1}{2}\Sigma^B_{t,y}\right] \tag{62}$$

is due to taking the speculative risk. By taking the speculative risk, pension providers are able to offer an adequate expected payout stream at an affordable price.

The last additional term in (60)

$$\left(\omega^S_{t,y}\right)^\top \Sigma_t \left(D_v^\top \Sigma^X_t\right)^\top = -\sigma^2_\pi\left(\omega^S_{1t,y}K_{h_1} + \omega^S_{2t,y}K_{h_2}\right)D_{1v} - \sigma^2_r\left(\omega^S_{1t,y}L_{h_1} + \omega^S_{2t,y}L_{h_2}\right)D_{2v} \tag{63}$$

is due to the interaction between the speculative portfolio and the hedging portfolio. Appendix A.4 shows that $\omega^S_{1t,y}$ and $\omega^S_{2t,y}$ are positive if the individual invests efficiently. Hence, the longer the investment horizon is, the larger D_{1v} and D_{2v} are and thus the more negative the interaction term $\left(\omega^S_{t,y}\right)^\top \Sigma_t \left(D_v^\top \Sigma^X_t\right)^\top$ becomes. Intuitively, long investment horizons benefit less from speculative risk premia because lower interest rates raising the value of the speculative portfolio go together with a higher value of the liabilities. This applies especially to longer horizons for which the value of the liabilities is relatively more sensitive to interest rates. Figure 7 illustrates the forward discount rate (60) (panel (a)) and the interaction term (panel (b)) as a function of the horizon v. This figure assumes that the current inflation rate and the current real interest rate equal their long-term means.

The inflation sensitivity and the real interest rate sensitivity of the value of the pension liabilities are, respectively, given by

$$\hat{D}_{1t,y} = (1-g_1)\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}(1-e^{-\eta h})\mathrm{d}h/\eta, \tag{64}$$

$$\hat{D}_{2t,y} = (1-g_2)\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}(1-e^{-\kappa h})\mathrm{d}h/\kappa. \tag{65}$$

To replicate the benefit payouts of the contract, the pension provider should choose a hedging portfolio with the same sensitivities.

3.2.2 Gradual adjustment of the current benefit payout

We find that the forward discount rate $\breve{\delta}^{u}_{t,y,v}$ is given by (see Appendix A.3):

$$\begin{aligned}\breve{\delta}^{u}_{t,y,v} &= \mathbb{E}_t[(1-g_1)\pi_{t+v} + (1-g_2)r_{t+v}] - g_0 - D_{1v}\sigma_\pi\left(\lambda_\pi + \frac{1}{2}D_{1v}\sigma_\pi\right) \\ &\quad - D_{2v}\sigma_r\left(\lambda r + \frac{1}{2}D_{2v}\sigma_r\right) + q_{t,y,v}\left(\omega^S_{t,y}\right)^\top\left[\left(\mu_t - R^f_t\right) - \frac{1}{2}q_{t,y,v}\Sigma_t\Sigma_t^\top \omega^S_{t,y}\right] \\ &\quad + q_{t,y,v}\left(\omega^S_{t,y}\right)^\top \Sigma_t\left(D_v^\top \Sigma^X_t\right)^\top,\end{aligned} \tag{66}$$

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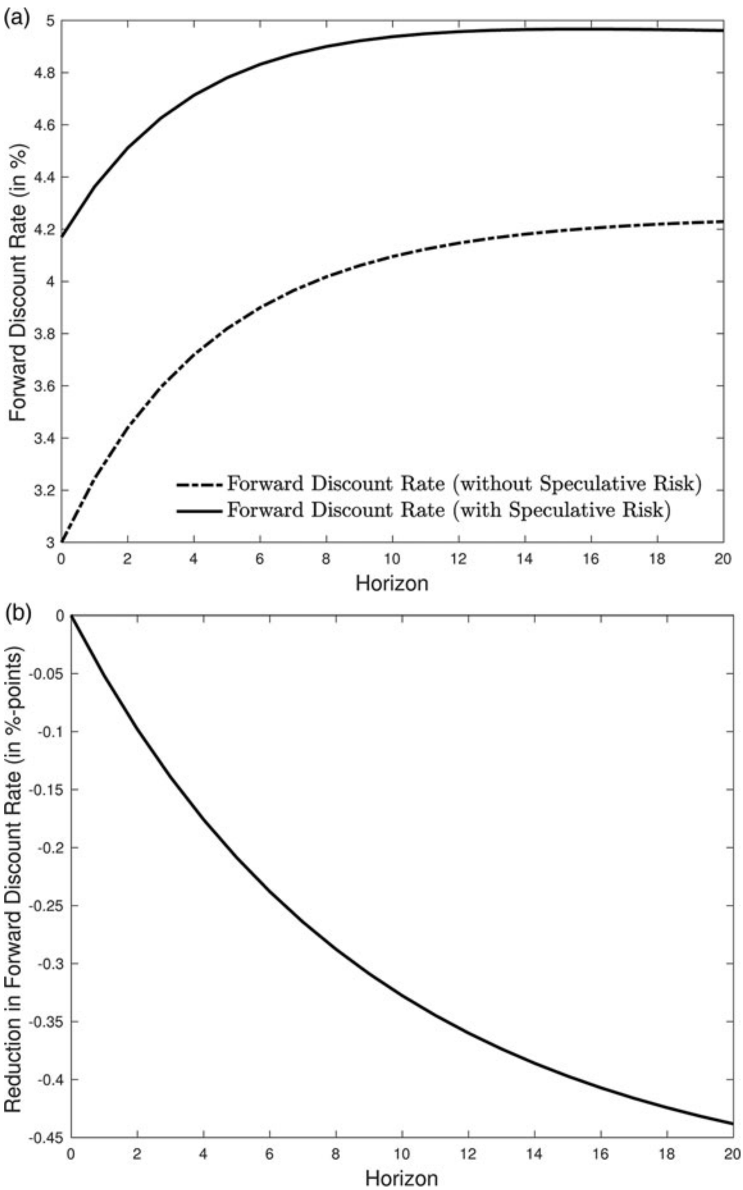

Figure 7. Illustration of the Forward Discount Rate. Panel (a) illustrates the forward discount rate in the presence of speculative risk. We assume that $\pi_t = \bar{\pi} = 2\%,\ r_t = \bar{r} = 1\%$ $g_0 = g_1 = g_2 = 0, \Sigma^B_{1t,y} = \lambda_\pi/5,\ \Sigma^B_{2t,y} = \lambda_r/5$ and $\Sigma^B_{3t,y} = \lambda_S/5$ Panel (b) illustrates the reduction in the forward discount rate (in %-points) as a result of the interaction between the hedging portfolio and the speculative portfolio. The benchmark parameter values are given in Table 2.

where the vector of speculative weights is given by

$$\hat{\Sigma}^B_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} \Sigma^B_{t,y,h} \mathrm{d}h. \tag{67}$$

with

$$\hat{\Sigma}^B_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} \Sigma^B_{t,y,h} \mathrm{d}h. \tag{68}$$

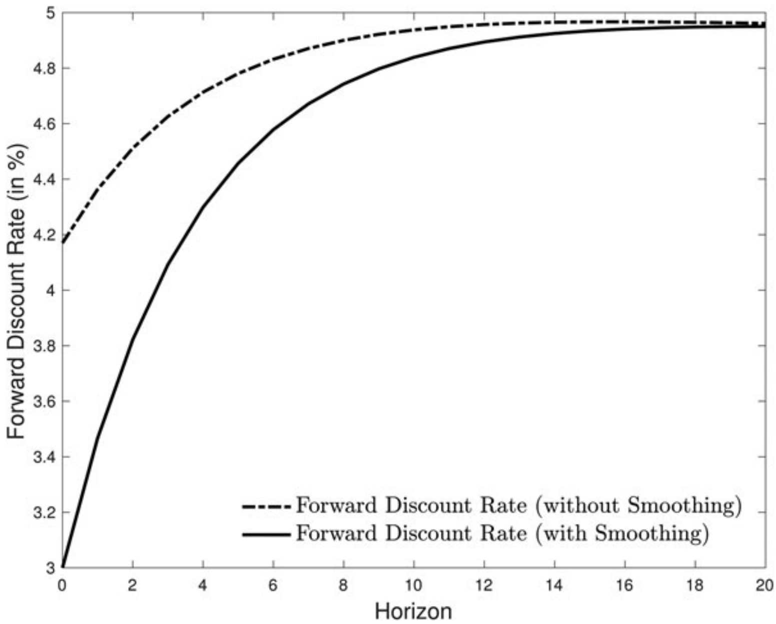

Figure 8. Illustration of the Forward Discount Rate. The figure illustrates the forward discount rate in the case of smoothing of speculative shocks. We assume that $\pi_t = \bar{\pi} = 2\%$, $r_t = \bar{r} = 1\%$, $g_0 = g_1 = g_2 = 0$, $\Sigma^B_{1t,y,v} = (1 - e^{-0.2v})\lambda_\pi/5$, $\Sigma^B_{2t,y,v} = (1 - e^{-0.2v})\lambda_r/5$ and $\Sigma^B_{3t,y,v} = (1 - e^{-0.2v})\lambda_S/5$. The benchmark parameter values are given in Table 2. We also illustrate the case in which the individual does not smooth speculative shocks, i.e., $\Sigma^B_{1t,y} = \lambda_\pi/5$, $\Sigma^B_{2t,y} = \lambda_r/5$ and $\Sigma^B_{3t,y} = \lambda_S/5$. Hence, the exposure of a future benefit payout to a current Brownian shock is always strictly smaller in the case of smoothing than in the case of no smoothing.

Equations (67) and (68) show that the individual implements a life-cycle investment strategy if the vector of payout exposures $\Sigma^B_{t,y,h}$ – which is exogenously given – increases with the horizon h. Indeed, when the remaining life expectancy declines, the value weights $\alpha_{t,y,h}$ of the shorter horizons become larger. As a result, $\hat{\Sigma}^B_{t,y}$ decreases as the individual ages. Hence, the vector of speculative portfolio weights $\omega^S_{t,y}$ is also a decreasing function of age.

Equation (66) shows that gradual adjustment of the current benefit payout to speculative shocks (i.e., $q_{t,y,v}$ increases with v) is another reason why the forward discount rate $\breve{\delta}^u_{t,y,v}$ depends on the horizon v. Indeed, the further into the future a benefit payout occurs, the larger the exposure of the benefit payout to a current speculative shock is, and hence the higher the discount rate will be.

We note that gradual adjustment of the current benefit payout also impacts the interaction term $q_{t,y,v}\left(\omega^S_{t,y}\right)^\top \Sigma_t \left(D_v^\top \Sigma_t^X\right)^\top$. This term causes the term structure to become less upward sloping. Hence, although gradual adjustment causes the forward discount rate to increase more rapidly with the horizon, the interaction term somewhat mitigates the increase in the slope of the term structure of the forward discount rates.

We illustrate the forward discount rate (66) in Figure 8. This figure assumes that the current inflation rate and the current real interest rate equal their long-term means. Moreover, the vector of payout exposures is given by $\Sigma^B_{t,y,v} = (1 - e^{-0.2v})(\lambda_\pi/5, \lambda_r/5, \lambda_S/5)$. Hence, it takes 3 1/2 years before 50% of the Brownian shocks $\lambda_\pi/5 dZ_{1t}$, $\lambda_r/5 dZ_{2t}$, and $\lambda_S/5 dZ_{3t}$ are reflected in the current benefit payout.

4. Concluding remarks

This paper has explored how to model the decumulation period of a PPR. We have derived a system of restrictions on the contract parameters. These restrictions do not uniquely identify all parameters. As a

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result, individuals must specify some parameters exogenously. They can specify the parameters according to (at least) two alternative approaches: the investment approach and the consumption approach. In the investment approach, individuals specify the speed of decumulation and the investment policy exogenously. We have shown how these exogenous parameters determine the median growth rate and the volatility of retirement consumption. The consumption approach, in contrast, specifies the entire retirement consumption stream exogenously. We have shown how to derive the discount rate and the investment policy given a particular exogenous consumption profile.

If the median retirement consumption is assumed flat in real terms and benefit payouts respond gradually to a current speculative shock, then the discount rate depends on the investment horizon because of four reasons.[22] First, mean reversion in the state variables affects the discount rate. Indeed, if the real interest rate exceeds its long-term mean, then returns are expected to decline, and hence long-term benefit payouts benefit less from the current high real interest rate than short-term benefit payouts. Second, the degree of real interest rate exposure varies across horizons. Typically, longer horizons are more exposed to real interest rate risk than shorter horizons and thus benefit from a higher real interest rate risk premium. Hence, the discount rate for a short horizon is lower than the discount rate for a long horizon unless the current real interest rate is substantially above its long-term mean. Third, if the individual invests efficiently, then the hedging portfolio and the speculative portfolio are positively correlated with each other. In particular, longer horizons benefit less from speculative risk premia because lower interest rates raising the value of the speculative portfolio go together with a larger value of the liabilities. Fourth, since a speculative shock is smoothed, such a shock has a larger impact on benefit payouts in the far future than on benefit payouts in the near future. With a larger risk premium for longer horizons, the discount rate increases with the horizon.

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Conflicts of interest. None.

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[22]Without interest rate risk and inflation risk, only the fourth reason is relevant.

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Appendix

A.1 Derivation of (A1.33)

The log horizon-dependent conversion factor $\log C_{t,y,h}$ satisfies the following dynamic equation (the second equality follows from (A1.20) and (A1.24)):

$$\begin{aligned} \mathrm{d}\log C_{t,y,h} &= \mathrm{d}\log A_{t,y,h} + \mathrm{d}\log F_{t,y,h} \\ &= \theta_{t-y}\mathrm{d}t + \frac{\partial\left(\delta^u_{t,y,h}h\right)}{\partial h}\mathrm{d}t - \int_{y+x_r}^{t} \mathrm{d}q_{s,y,t-s}\left(\omega^S_{s,y}\right)^\top \Sigma_s \mathrm{d}Z_s \\ &\quad \left(D^\top_{t,y,h}\mu^X_t + \frac{1}{2}\mathrm{Tr}\left[\left(\Sigma^X_t\right)^\top H_{t,y,h}\Sigma^X_t\right]\right)\mathrm{d}t \\ &\quad + \left(\left[q_{t,y,h} - q_{t,y,0}\right]\left(\omega^S_{t,y}\right)^\top \Sigma_t - D^\top_{t,y,h}\Sigma^X_t\right)\mathrm{d}Z_t. \end{aligned} \tag{A1.69}$$

The dynamic equation of the log conversion factor log $C_{t,y}$ is given by (this follows from (A1.12), (A1.21) and (A1.69))

$$\mathrm{d}\log C_{t,y} = \left(\theta_{t-y} + \mu^C_{t,y}\right)\mathrm{d}t + \left(\left[1 - q_{t,y,0}\right]\left(\omega^S_{t,y}\right)^\top \Sigma_t - \hat{D}^\top_{t,y}\Sigma^X_t\right)\mathrm{d}Z_t - c_{t,y}\mathrm{d}t, \tag{A170}$$

where $\mu^C_{t,y}\mathrm{d}t$ is defined as follows:

$$\begin{aligned} \mu^C_{t,y}\mathrm{d}t &= -\,\theta_{t-y}\mathrm{d}t + \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\mathbb{E}_t[\mathrm{d}\log C_{t,y,h}]\mathrm{d}h \\ &\quad + \frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\mathrm{d}[\log C_{t,y,h}, \log C_{t,y,h}]\mathrm{d}h \\ &\quad - \frac{1}{2}\int_0^{x_{\max}-(t-y)}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,u}\alpha_{t,y,v}\mathrm{d}[\log C_{t,y,u}, \log C_{t,y,v}]\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{A1.71}$$

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Here, $\mathrm{d}[\log C_{t,y,u}, \log C_{t,y,v}]$ is given by

$$\begin{aligned}\mathrm{d}[\log C_{t,y,u}, \log C_{t,y,v}] = &\left(\left[q_{t,y,u} - q_{t,y,0}\right]\left(\omega_{t,y}^{S}\right)^{\top}\Sigma_t - D_{y,u}^{\top}\Sigma_t^{X}\right)\\ &\times\left(\left[q_{t,y,v} - q_{t,y,0}\right]\left(\omega_{t,y}^{S}\right)^{\top}\Sigma_t - D_{y,v}^{\top}\Sigma_t^{X}\right)^{\top}\mathrm{d}t.\end{aligned} \tag{A1.72}$$

Substitution of (A1.7), (A1.70) and (A1.29) into (A1.32) yields (A1.33).

A.2 Derivation of (A2.39)

We start by deriving the analytical solution to the stochastic differential equation (SDE) for the Ornstein–Uhlenbeck process. After applying Itô's lemma to the function $f(t, \pi_t) = e^{\eta t}(\pi_t - \bar{\pi})$, we find

$$\begin{aligned}\mathrm{d}f(t, \pi_t) &= \eta e^{\eta t}(\pi_t - \bar{\pi})\mathrm{d}t + e^{\eta t}\mathrm{d}\pi_t\\ &= \eta e^{\eta t}(\pi_t - \bar{\pi})\mathrm{d}t - e^{\eta t}\eta(\pi_t - \bar{\pi})\mathrm{d}t + e^{\eta t}\sigma_\pi \mathrm{d}Z_{1t} = \sigma_\pi e_{\mathrm{d}}^{\eta t} Z_{1t}\end{aligned} \tag{A2.73}$$

The solution of (A2.73) is given by

$$f(t, \pi_{t+v}) = f(t, \pi_t) + \sigma_\pi \int_t^{t+v} e^{\eta u}\mathrm{d}Z_{1u}. \tag{A2.74}$$

The inflation rate at time $t + v > t$ is given by (the first and third equality follow from the definition of $f(t, \pi_t)$, and the second equality follows from (A2.74))

$$\begin{aligned}\pi_{t+v} &= \bar{\pi} + e^{-\eta(t+v)}f(t, \pi_{t+v}) = \bar{\pi} + e^{-\eta(t+v)}f(t, \pi_t) + \sigma_\pi \int_t^{t+v} e^{-\eta(t+v-u)}\mathrm{d}Z_{1u}\\ &= \bar{\pi} + e^{-\eta v}(\pi_t - \bar{\pi}) + \sigma_\pi \int_0^{v} e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)}\\ &= \pi_t + (1 - e^{-\eta v})(\bar{\pi} - \pi_t) + \sigma_\pi \int_0^{v} e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)}.\end{aligned} \tag{A2.75}$$

In a similar fashion, we find

$$r_{t+v} = r_t + (1 - e^{-\kappa v})(\bar{r} - r_t) + \sigma_r \int_0^{v} e^{-\kappa(v-u)}\mathrm{d}Z_{2(t+u)}. \tag{A2.76}$$

The (conditional) expectation of the inflation rate $\mathbb{E}_t[\pi_{t+v}]$ and the (conditional) expectation of the real interest rate $\mathbb{E}_t[r_{t+v}]$ are given by

$$\mathbb{E}_t[\pi_{t+v}] = \pi_t + \eta(\bar{\pi} - \pi_t)K_v, \tag{A2.77}$$

$$\mathbb{E}_t[r_{t+v}] = r_t + \kappa(\bar{r} - r_t)L_v. \tag{A2.78}$$

The aggregate inflation rate $\bar{\pi}_{t,h} = \int_0^h \pi_{t+v}\mathrm{d}v$ and the aggregate real interest rate $\bar{r}_{t,h} = \int_0^h r_{t+v}\mathrm{d}v$ play a key role in determining the yield to maturity. We find (the first equality follows from substituting (A2.75) to eliminate π_{t+v})

$$\begin{aligned}
\bar{\pi}_{t,h} &= \int_0^h \pi_{t+v}\mathrm{d}v \\
&= \int_0^h (\pi_t + (\bar{\pi} - \pi_t)(1 - e^{-\eta v}))\mathrm{d}v + \sigma_\pi \int_0^h \int_0^v e^{-\eta(v-u)}\mathrm{d}Z_{1(t+u)}\mathrm{d}v \\
&= \int_0^h (\pi_t + (\bar{\pi} - \pi_t)(1 - e^{-\eta v}))\mathrm{d}v + \sigma_\pi \int_0^h \int_v^h e^{-\eta(h-u)}\mathrm{d}u\mathrm{d}Z_{1(t+v)} \\
&= \int_0^h (\pi_t + (\bar{\pi} - \pi_t)\eta K_v)\mathrm{d}v + \frac{\sigma_\pi}{\eta} \int_0^h (1 - e^{-\eta(h-v)})\mathrm{d}Z_{1(t+v)} \\
&= \int_0^h \mathbb{E}_t[\pi_{t+v}]\mathrm{d}v + \sigma_\pi \int_0^h K_{h-v}\mathrm{d}Z_{1(t+v)}.
\end{aligned} \tag{A2.79}$$

In a similar fashion, we find that the aggregate real interest rate $\bar{r}_{t,h}$ is given by

$$\bar{r}_{t,h} = \int_0^h r_{t+v}\mathrm{d}v = \int_0^h \mathbb{E}_t[r_{t+v}]\mathrm{d}v + \sigma_r \int_0^h L_{h-v}\mathrm{d}Z_{2(t+v)}. \tag{A2.80}$$

The pricing kernel is given by (see, e.g., Brennan and Xia (2002)):

$$\xi_t = \exp\left\{ -\int_0^t \left(\pi_s + r_s + \frac{1}{2}\lambda^\top \lambda \mathrm{d}s - \lambda^\top Z_t \right) \right\}. \tag{A2.81}$$

Here, $\lambda = (\lambda_\pi, \lambda_r, \lambda_S)$ represents the vector of market prices of risk. Denote by P_{1t} the price of a bond with time to maturity h_1. We can determine P_{1t} as follows:

$$P_{1t} = \mathbb{E}_t\left[\frac{\xi_{t+h_1}}{\xi_t} \right], = \mathbb{E}_t\left[\exp\left\{ -\int_0^{h_1} \left(\pi_{t+v} + r_{t+v} + \frac{1}{2}\lambda^\top \lambda \mathrm{d}v - \lambda\left(Z_{t+h_1} - Z_t\right) \right) \right\} \right]. \tag{A2.82}$$

Substituting (A2.79) and (A2.80) into the pricing formula (A2.82) to eliminate $\int_0^h \pi_{t+v}\mathrm{d}v$ and $\int_0^h r_{t+v}\mathrm{d}v$, we arrive at

$$\begin{aligned}
P_{1t} = {} & \exp\left\{ -\int_0^{h_1} \left(\mathbb{E}_t[\pi_{t+v} + r_{t+v}] + \frac{1}{2}\lambda^\top \lambda \mathrm{d}v \right) \right\} \mathbb{E}_t\left[\exp\left\{ -\int_0^{h_1} \lambda_S \mathrm{d}Z_{3(t+v)} \right\} \right. \\
& \left. \exp\left\{ \int_0^{h_1} (-\lambda_\pi - \sigma_\pi K_{h_1-v})\mathrm{d}Z_{1(t+v)} + \int_0^{h_1} (-\lambda_r - \sigma_r L_{h_1-v})\mathrm{d}Z_{2(t+v)} \right\} \right] \\
& \exp\left\{ -\int_0^{h_1} \left(\mathbb{E}_t[\pi_{t+v} + r_{t+v}] - \lambda_\pi \sigma_\pi K_v - \lambda_r \sigma_r L_v - \frac{1}{2}(\sigma_\pi K_v)^2 - \frac{1}{2}(\sigma_r L_v)^2 \right) \mathrm{d}v \right\} \\
= {} & \exp\left\{ -\int_0^{h_1} R_{t,v}\mathrm{d}v \right\}.
\end{aligned} \tag{A2.83}$$

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The instantaneous nominal forward interest rate $R_{t,v}$ is defined as follows:

$$R_{t,v} = \mathbb{E}_t[\pi_{t+v} + r_{t+v}] - \lambda_\pi\sigma_\pi K_v - \lambda_r\sigma_r L_v - \frac{1}{2}(\sigma_\pi K_v)^2 - \frac{1}{2}(\sigma_r L_v)^2. \tag{A2.84}$$

The log bond price is given by (this follows from (A2.77), (A2.78), (A2.83) and (A2.84))

$$\begin{aligned}\log P_{1t} = -\int_0^{h_1} & (-\pi_t + \eta(\bar{\pi} - \pi_t)K_v + r_t + \kappa(\bar{r} - r_t)L_v - \lambda_\pi\sigma_\pi K_v \\ & - \lambda_r\sigma_r L_v - \frac{1}{2}(\sigma_\pi K_v)^2 - \frac{1}{2}(\sigma_r L_v)^2)\mathrm{d}v.\end{aligned} \tag{A2.85}$$

Solving the integral (A2.85), we arrive at[23]

$$\begin{aligned}\log P_{1t} &= -\pi_t h_1 - (\bar{\pi} - \pi_t)\big(h_1 - K_{h_1}\big) - r_t h_1 - (\bar{r} - r_t)\big(h_1 - L_{h_1}\big) \\ &+ \frac{\lambda_\pi\sigma_\pi}{\eta}\big(h_1 - K_{h_1}\big) + \frac{\lambda_r\sigma_r}{\kappa}\big(h_1 - L_{h_1}\big) \\ &+ \frac{1}{2}\left(\frac{\sigma_\pi}{\eta}\right)^2\left(h_1 - 2K_{h_1} + \frac{1}{2}K_{2h_1}\right) + \frac{1}{2}\left(\frac{\sigma_r}{\kappa}\right)^2\left(h_1 - 2L_{h_1} + \frac{1}{2}L_{2h_1}\right) \\ &= -\pi_t K_{h_1} - r_t L_{h_1} - M_{h_1}.\end{aligned} \tag{A2.86}$$

Here, the horizon-dependent constant M_{h_1} is defined as follows:

$$\begin{aligned}M_{h_1} = &\left(\bar{\pi} - \frac{\lambda_\pi\sigma_\pi}{\eta} - \frac{1}{2}\left[\frac{\sigma_\pi}{\eta}\right]^2\right)\big(h_1 - K_{h_1}\big) + \frac{1}{4\eta}\big(\sigma_\pi K_{h_1}\big)^2 \\ &+ \left(\bar{r} - \frac{\lambda_r\sigma_r}{\kappa} - \frac{1}{2}\left[\frac{\sigma_r}{\kappa}\right]^2\right)\big(h_1 - L_{h_1}\big) + \frac{1}{4\kappa}\big(\sigma_r L_{h_1}\big)^2.\end{aligned} \tag{A2.87}$$

To calculate how the value of the bond with a fixed maturity date $t + h_1$ develops as time proceeds (i.e., $t + h_1$ is fixed but t changes), we apply Itô's lemma to

$$P_{1t} = \exp\{-\pi_t K_{h_1} - r_t L_{h_1} - M_{h_1}\}. \tag{A2.88}$$

We find

$$\begin{aligned}\frac{\mathrm{d}P_{1t}}{P_{1t}} &= (R_{t,h_1} - \eta(\bar{\pi} - \pi_t)K_{h_1} - \kappa(\bar{r} - r_t)L_{h_1} + \frac{1}{2}\big(\sigma_\pi K_{h_1}\big)^2 + \frac{1}{2}\big(\sigma_r L_{h_1}\big)^2)\mathrm{d}t \\ &\quad \sigma_\pi K_{h_1}\mathrm{d}Z_{1t} - \sigma_r L_{h_1}\mathrm{d}Z_{2t} \\ &= \big(r_t + \pi_t - \lambda_\pi\sigma_\pi K_{h_1} - \lambda_r\sigma_r L_{h_1}\big)\mathrm{d}t - \sigma_\pi K_{h_1}\mathrm{d}Z_{1t} - \sigma_r L_{h_1}\mathrm{d}Z_{2t}.\end{aligned} \tag{A2.89}$$

A.3 Derivation of (A3.47), (A3.60) and (A.66)

We show that the following specification of the (unconditional) forward discount rate yields budget balance:

$$\breve{\delta}^u_{t,y,v} = d_{0v} + d_{1v}\cdot\pi_t + d_{2v}\cdot r_t. \tag{A3.90}$$

[23] The first equality follows from $K_v^2 = (1 - 2e^{-\eta v} + e^{-2\eta v})/\eta^2$ and the second equality follows from $K_{h_1}^2 = (2K_{h_1} - K_{2h_1})/\eta$.

Budget balance implies that (see also (A3.35))

$$\gamma_{t,y}\mathrm{d}t = \mu_{t,y}^W \mathrm{d}t - \mu_{t,y}^C \mathrm{d}t.$$

Substituting (A3.59) and (A3.31) into (A3.91), we arrive at

$$(g_0 + g_1 \cdot \pi_t + g_2 \cdot r_t)\mathrm{d}t + \int_{y+x_r}^{t} \mathrm{d}\left(\Sigma_{s,y,t-s}^B\right)^\top \mathrm{d}Z_s = \mu_{t,y}^W \mathrm{d}t - \mu_{t,y}^C \mathrm{d}t. \tag{A3.92}$$

The vector of portfolio weights depends on $\hat{D}_{t,y} = \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} D_{t,y,h} \mathrm{d}h$ and is given by

$$\omega_{t,y} = \omega_{t,y}^H + \omega_{t,y}^S = -\left(\Sigma_t^X \Sigma_t^{-1}\right)^\top \hat{D}_{t,y} + \left(\Sigma_t^{-1}\right)^\top \hat{\Sigma}_{t,y}^B. \tag{A3.93}$$

Here, $D_{t,y,h} = \nabla_X\left(\delta_{t,y,h} h\right) = \left(\int_0^h d_{1v}\mathrm{d}v, \int_0^h d_{2v}\mathrm{d}v, 0\right) = (D_{1h}, D_{2h}, D_{3h}) = D_h$ models the sensitivity of the discount rate with respect to the underlying state variables.

Substitution of (A3.93) into (A3.8) yields

$$\begin{aligned}\mu_{t,y}^W \mathrm{d}t = \Bigg(&\pi_t + r_t + \left[\left(\hat{\Sigma}_{t,y}^B\right)^\top - \hat{D}_{t,y}^\top \Sigma_t^X\right]\Sigma_t^{-1}\left(\mu_t - R_t^f\right) \\ &-\frac{1}{2}\left(\hat{\Sigma}_{t,y}^B\right)^\top \hat{\Sigma}_{t,y}^B - \frac{1}{2}\hat{D}_{t,y}^\top \Sigma_t^X \left(\Sigma_t^X\right)^\top \hat{D}_{t,y} + \hat{D}_{t,y}^\top \Sigma_t^X \hat{\Sigma}_{t,y}^B\Bigg)\mathrm{d}t.\end{aligned} \tag{A3.94}$$

Substituting (A3.90) and the expression for μ_t^X (see (A3.38)) into (A3.71), we arrive at

$$\begin{aligned}\mu_{t,y}^C \mathrm{d}t = &\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{0h} \mathrm{d}h \mathrm{d}t + \pi_t \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{1h} \mathrm{d}h \mathrm{d}t \\ &+ r_t \int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{2h} \mathrm{d}h \mathrm{d}t - \hat{D}_{1t,y}^\top \eta(\bar{\pi} - \pi_t)\mathrm{d}t - \hat{D}_{2t,y}^\top \kappa(\bar{r} - r_t)\mathrm{d}t \\ &- \int_{y+x_r}^{t} \mathrm{d}q_{s,y,t-s}\left(\omega_{s,y}^S\right)^\top \Sigma_s \mathrm{d}Z_s + \frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} \mathrm{d}\left[\log C_{t,y,h}, \log C_{t,y,h}\right]\mathrm{d}h \\ &- \frac{1}{2}\int_0^{x_{\max}-(t-y)} \int_0^{x_{\max}-(t-y)} \alpha_{t,y,u}\alpha_{t,y,v} \mathrm{d}\left[\log C_{t,y,u}, \log C_{t,y,v}\right]\mathrm{d}u\mathrm{d}v.\end{aligned} \tag{A3.95}$$

It follows from substituting (A3.94) and (A3.95) into (A3.92) that d_{1h} and d_{2h} must satisfy the following two conditions:

$$d_{2h} = 1 - g_2 + \kappa D_{2h}. \tag{A3.96}$$

$$d_{2h} = 1 - g_2 + \kappa D_{2h}. \tag{A3.97}$$

Solving these two equations, we find $d_{1v} = (1 - g_1)e^{-\eta v}$ and $d_{2v} = (1 - g_2)e^{-\kappa v}$.

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The coefficient d_{0h} must satisfy

$$\begin{aligned}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h} d_{0h} \mathrm{d}h\mathrm{d}t = & -g_0 \mathrm{d}t + \left[\left(\hat{\Sigma}_{t,y}^B\right)^\top - \hat{D}_{t,y}^\top \Sigma_t^X\right]\Sigma_t^{-1}\left(\mu_t - R_t^f\right)\mathrm{d}t \\ & -\frac{1}{2}\left(\hat{\Sigma}_{t,y}^B\right)^\top \hat{\Sigma}_{t,y}^B \mathrm{d}t - \frac{1}{2}\hat{D}_{t,y}^\top \Sigma_t^X\left(\Sigma_t^X\right)^\top \hat{D}_{t,y}\mathrm{d}t + \hat{D}_{t,y}^\top \Sigma_t^X \hat{\Sigma}_{t,y}^B \mathrm{d}t \\ & + \hat{D}_{1t,y}^\top \eta\bar{\pi}\mathrm{d}t + \hat{D}_{2t,y}^\top \kappa\bar{r}\mathrm{d}t \\ & -\frac{1}{2}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,h}\mathrm{d}\left[\log C_{t,y,h}, \log C_{t,y,h}\right]\mathrm{d}h \\ & +\frac{1}{2}\int_0^{x_{\max}-(t-y)}\int_0^{x_{\max}-(t-y)} \alpha_{t,y,u}\alpha_{t,y,v}\mathrm{d}\left[\log C_{t,y,u}, \log C_{t,y,v}\right]\mathrm{d}u\mathrm{d}v.\end{aligned} \tag{A3.98}$$

We note that

$$\int_{y+x_r}^{t} \mathrm{d}\left(\Sigma_{s,y,t-s}^B\right)^\top \mathrm{d}Z_s = \int_{y+x_r}^{t} \mathrm{d}q_{s,y,t-s}\left(\omega_{s,y}^S\right)^\top \Sigma_s \mathrm{d}Z_s. \tag{A3.99}$$

Straightforward computations yield (A3.66). Equations (A3.47) and (A3.60) emerge as a special case of (A3.66).

A.4 Derivation of the Efficient Speculative Investment Portfolio

This appendix derives the efficient speculative investment portfolio. In particular, we show that $\Sigma_1^B = -\sigma_\pi\left(\omega_{1t,y}^S K_{h_1} + \omega_{2t,y}^S K_{h_2}\right)$ and $\Sigma_2^B = -\sigma_r\left(\omega_{1t,y}^S L_{h_1} + \omega_{2t,y}^S L_{h_2}\right)$ are (typically) negative. The individual aims to minimize the variance of the change in the log benefit payout, i.e., $\mathbb{V}_t[\mathrm{dlog}\, B_{t,y}]$, subject to a given expected excess return on the speculative portfolio $c \geq 0$. Hence, the individual's optimization problem is given by

$$\begin{aligned} & \min_{\Sigma_1^B, \Sigma_2^B, \Sigma_3^B} \left(\Sigma_1^B\right)^2 + \left(\Sigma_2^B\right)^2 + \left(\Sigma_3^B\right)^3 \\ & \text{s.t.} \quad \Sigma_1^B \lambda_\pi + \Sigma_2^B \lambda_r + \Sigma_3^B \lambda_S = c.\end{aligned} \tag{A4.100}$$

The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \left(\Sigma_1^B\right)^2 + \left(\Sigma_2^B\right)^2 + \left(\Sigma_3^B\right)^3 + y\left(c - \Sigma_1^B\lambda_\pi + \ \Sigma_2^B\lambda_r + \Sigma_3^B\lambda_S\right). \tag{A4.101}$$

Here, $y \geq 0$ denotes the Lagrange multiplier.

The first-order optimality conditions are given by

$$2\Sigma_1^{B^*} = y\lambda_\pi, \tag{A4.102}$$

$$2\Sigma_2^{B^*} = y\lambda_r, \tag{A4.103}$$

$$2\Sigma_3^{B^*} = y\lambda_S. \tag{A4.104}$$

It follows from (A4.102)–(A4.104) that $\Sigma_1^{B*} \propto \lambda_\pi$, $\Sigma_2^{B*} \propto \lambda_r$ and $\Sigma_S^{B*} \propto \lambda_S$. Since λ_r and λ_π are typically negative, it follows that Σ_1^{B*} and Σ_2^{B*} are also typically negative.

Optimal Benefit Policy

This appendix derives the optimal benefit policy with and without pooling of mortality risk. We assume the same economy as in Section 2.1. Denote by x_D the age at which the individual dies. In case mortality risk is pooled, we can determine the optimal benefit policy by assuming x_D is known in advance. Hence, we have the following maximization problem:

$$\begin{aligned} &\max_{B_{t,y}:x_r+y\le t\le x_D+y} \int_{x_r+y}^{x_D+y} e^{-\rho(t-x_r-y)}\mathbb{E}_{x_r+y}\left[\frac{1}{1-\gamma}\left(\frac{B_{t,y}}{\Pi_t}\right)^{1-\gamma}\right]\mathrm{d}t \\ &\text{s.t. } \mathbb{E}_{x_r+y}\left[\int_{x_r+y}^{x_D+y} \frac{\xi_t}{\xi_{x_r+y}}B_{t,y}\mathrm{d}t\right] \le W_{x_r+y}. \end{aligned} \tag{A4.105}$$

Here, ξ_t represents the nominal pricing kernel (or stochastic discount factor) at time t (see (A4.81) for an explicit analytical expression of ξ_t), $\Pi_t = \exp\left\{\int_0^t \pi_s \mathrm{d}s\right\}$ corresponds to the consumer price index at time t, γ stands for the coefficient of relative risk aversion, and ρ denotes the rate of time preference.

Maximizing (A4.105), we arrive at

$$B^*_{t,y} = \Pi_t\left(e^{\rho(t-x_r-y)}y\Pi_t\frac{\xi_t}{\xi_{x_r+y}}\right)^{-\frac{1}{\gamma}}, \tag{A4.106}$$

where $y > 0$ denotes the Lagrange multiplier.

Substituting (A4.81) into (A4.106), we find

$$\begin{aligned} B^*_{t,y} = B^*_{x_r+y,y}\exp\Bigg\{ &\int_{x_r+y}^{t} \pi_s \mathrm{d}s - \frac{\rho-(1/2)\lambda^\top\lambda}{\gamma}(t-x_r-y) \\ &+\frac{1}{\gamma}\int_{x_r+y}^{t} r_s\mathrm{d}s + \frac{\lambda_\pi}{\gamma}\int_{x_r+y}^{t}\mathrm{d}Z_{1s} + \frac{\lambda_r}{\gamma}\int_{x_r+y}^{t}\mathrm{d}Z_{2s} + \frac{\lambda_S}{\gamma}\int_{x_r+y}^{t}\mathrm{d}Z_{3s}\Bigg\}. \end{aligned} \tag{A4.107}$$

Hence, the optimal g_0, g_1 and g_2 are given by

$$g_0 = \frac{(1/2)\lambda^\top\lambda - \rho}{\gamma}, \tag{A4.108}$$

$$g_1 = 1, \tag{A4.109}$$

$$g_2 = \frac{1}{\gamma}. \tag{A4.110}$$

We now consider the case where mortality risk is *not* pooled. We have the following maximization problem in case the individual does not have bequest motives:

$$\begin{aligned} &\max_{B_{t,y}:x_r+y\le t\le x_{\max}+y} \int_{x_r+y}^{x_{\max}+y} e^{-\rho(t-x_r-y)}\mathbb{E}_{x_r+y}\left[\mathbb{1}_{1t<x_D+y}\right]\mathbb{E}_{x_r+y}\left[\frac{1}{1-\gamma}\left(\frac{B_{t,y}}{\Pi_t}\right)^{1-\gamma}\right]\mathrm{d}t \\ &\text{s.t. } \mathbb{E}_{x_r+y}\left[\int_{x_r+y}^{x_{\max}+y}\frac{\xi_t}{\xi_{x_r+y}}B_{t,y}\mathrm{d}t\right] \le W_{x_r+y}. \end{aligned} \tag{A4.111}$$

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Assuming independence between the age of death and financial returns, we can write (A4.111) as follows:

$$\text{s.t. } \mathbb{E}_{x_r+y}\left[\int_{x_r+y}^{x_{\max}+y} \frac{\xi_t}{\xi_{x_r+y}} B_{t,y}\mathrm{d}t\right] \leq W_{x_r+y}. \tag{A4.112}$$

Note that $\mathbb{E}_{x_r+y}[\mathbb{1}t < x_D + y]$ represents the probability that an individual aged x_r at time x_{r+y} will survive at least $t-y-x_r$ years, i.e.,

$$\mathbb{E}_{x_r+y}[\mathbb{1}t < x_D + y] = {}_{t-y-x_r}\, p_{x_r}. \tag{A4.113}$$

Problem (A4.112) is thus equivalent to:

$$\begin{aligned}
&\max_{B_{t,y}: x_r+y \leq t \leq x_{\max}+y} \int_{x_r+y}^{x_{\max}+y} e^{-\rho(t-x_r-y)}\, {}_{t-y-x_r}p_{x_r}\mathbb{E}_{x_r+y}\left[\frac{1}{1-\gamma}\left(\frac{B_{t,y}}{\Pi_t}\right)^{1-\gamma}\right]\mathrm{d}t \\
&\text{s.t. } \mathbb{E}_{x_r+y}\left[\int_{x_r+y}^{x_{\max}+y} \frac{\xi_t}{\xi_{x_r+y}} B_{t,y}\mathrm{d}t\right] \leq W_{x_r+y}.
\end{aligned} \tag{A4.114}$$

Maximizing (A4.114), we arrive at

$$B^*_{t,y} = \Pi_t\left(\frac{e^{\rho(t-x_r-y)}}{t-y-x_r p_{x_r}} y \Pi_t \frac{\xi_t}{\xi_{x_r+y}}\right)^{-\frac{1}{\gamma}}, \tag{A4.115}$$

where $y > 0$ denotes the Lagrange multiplier.

Note that the following holds (see also (A4.4)):

$${}_{t-y-x_r}p_{x_r} = \exp\left\{-\int_0^{t-y-x_r} \theta_{x_r+v}\mathrm{d}v\right\} \tag{A4.116}$$

Substituting (A4.81) into (A4.115) and using (A4.116), we find

$$\begin{aligned}
B^*_{t,y} = B^*_{x_r+y,y}\exp\Bigg\{&\int_{x_r+y}^{t} \pi_s\mathrm{d}s - \frac{\rho-(1/2)\lambda^\top\lambda}{\gamma}(t-x_r-y) \\
&+\frac{1}{\gamma}\int_{x_r+y}^{t}(r_s-\theta_{s-y})\mathrm{d}s + \frac{\lambda_\pi}{\gamma}\int_{x_r+y}^{t}\mathrm{d}Z_{1s} + \frac{\lambda_r}{\gamma}\int_{x_r+y}^{t}\mathrm{d}Z_{2s} + \frac{\lambda_S}{\gamma}\int_{x_r+t}^{t}\mathrm{d}Z_{3s}\Bigg\}.
\end{aligned} \tag{A4.117}$$

In case of no pooling of mortality risk, the optimal median growth rate of the benefit payout is thus given by:

$$\gamma^\mu_{t,y} = \pi_t + \frac{1}{\gamma}\cdot\left(r_t + \frac{1}{2}\lambda^\top\lambda - \rho - \theta_{t-y}\right).$$

Compared to the case where mortality risk is pooled, the biometric rate of return θ_{t-y} reduces the effective median growth rate of the current benefit payout.

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