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A global stability estimate for the photo-acoustic inverse problem in layered media†

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This paper is concerned with the stability issue in determining absorption and diffusion coefficients in photoacoustic imaging. Assuming that the medium is layered and the acoustic wave speed is known, we derive global Hölder stability estimates of the photoacoustic inversion. These results show that the reconstruction is stable in the region close to the optical illumination source, and deteriorate exponentially far away. Several experimental pointed out that the resolution depth of the photoacoustic modality is about tens of millimeters. Our stability estimates confirm these observations and give a rigorous quantification of this depth resolution.

Key words: Inverse problems, wave equation, diffusion equation, Lipschitz stability.

1991 Mathematics Subject Classification Primary: 35R30

1 Introduction

Photoacoustic imaging (PAI) [6,7,12,32,36,44,49] is a recent hybrid imaging modality that couples diffusive optical waves with ultrasound waves to achieve high-resolution imaging of optical properties of heterogeneous media such as biological tissues.

In a typical PAI experiment, a short pulse of near infra-red photons is radiated into a medium of interest. A part of the photon energy is absorbed by the medium, which leads to the heating of the medium. The heating then results in a local temperature rise. The medium expanses due to this temperature rise. When the rest of the photons leave the medium, the temperature of the medium drops accordingly, which leads to the contraction of the medium. The expansion and contraction of the medium induce pressure changes, which then propagate in the form of ultrasound waves. Ultrasound transducers located on an observation surface, usually a part of the surface surrounding the object, measure the generated ultrasound waves over an interval of time $(0, T)$ with T large enough.

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The collected information is used to reconstruct the optical absorption and scattering properties of the medium.

Assuming that the ultrasound speed in the medium is known, the inversion procedure in PAI proceeds in two steps. In the first step, we reconstruct the initial pressure field, a quantity that is proportional to the local absorbed energy inside the medium, from measured pressure data. Mathematically speaking, this is a linear inverse source problem for the acoustic wave equation [2,3,5,19,21,23,25–27,30,31,33,40–42,45,46]. In the second step, we reconstruct the optical absorption and diffusion coefficients using the result of the first inversion as available *internal* data [4,16,17,38,39,43].

In theory, PAI provides both contrast and resolution. The contrast in PAI is mainly due to the sensitivity of the optical absorption and scattering properties of the media in the near infra-red regime. For instance, different biological tissues absorb Near-infrared (NIR) photons differently. The resolution in PAI comes in when the acoustic properties of the underlying medium are independent of its optical properties, and therefore the wavelength of the ultrasound generated provides good resolution (usually submillimeter).

In practice, it has been observed in various experiments that the imaging depth, i.e., the maximal depth of the medium at which structures can be resolved at expected resolution, of PAI is still fairly limited, usually on the order of millimeters. This is mainly due to the limitation on the penetration ability of diffusive NIR photons: optical signals are attenuated significantly by absorption and scattering. The same issue that is faced in optical tomography [11]. Therefore, the ultrasound signal generated decays very fast in the depth direction.

The objective of this work is to mathematically analyse the issue of imaging depth in PAI. To be more precise, assuming that the underlying medium is layered, we derive a stability estimate that shows that image reconstruction in PAI is stable in the region close to the optical illumination source, and deteriorate exponentially in the depth direction. This provides a rigorous explanation on the imaging depth issue of PAI.

In the first section, we introduce the PAI model and give the main global stability estimates in Theorem 2.1. Section 2 is devoted to the acoustic inversion, we derive observability inequalities corresponding to the internal data generated by well-chosen laser illuminations. We also provide an observability inequality from one side for general initial states in Theorem 3.2. In Section 3, we solve the optical inversion and show weighted stability estimates of the recovery of the optical coefficients from the knowledge of two internal data. Finally, the main global stability estimates are obtained by combining stability estimates from the acoustic and optical inversions.

2 The main results

In our model, we assume that the laser source and the ultrasound transducers are on the same side of the sample Γ_m; see Figure 1. This situation is quite realistic since in applications only a part of the boundary is accessible, and in the exiting prototypes a laser source acts trough a small hole in the transducers. We also assume that the optical parameters (D, μ_a), similar to the acoustic speed c, only depend on the variable y following the normal direction to Γ_m. We further consider the optical parameters $(D(y), \mu_a(y))$ within

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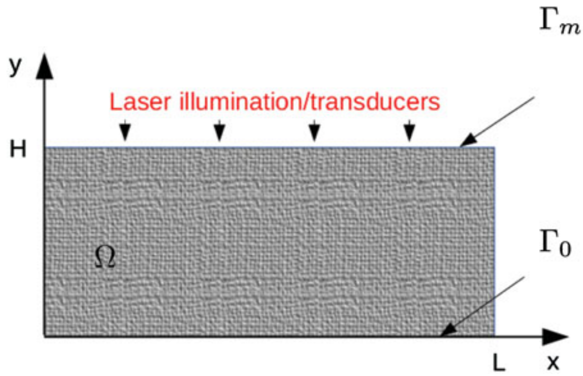

FIGURE 1. The geometry of the sample.

the set

$$\mathcal{O}_M = \{(D,\mu) \in C^3([0,H])^2;\ \ D > D_0,\ \mu > \mu_0;\ \ \|D\|_{C^3}, \|\mu\|_{C^3} \leqslant M\},$$

where $D_0 > 0, \mu_0 > 0$ and $M > \max(D_0, \mu_0)$ are fixed real constants.

The propagation of the optical wave in the sample is modelled by the following diffusion equation:

$$\begin{cases} -\nabla \cdot D(y)\nabla u(\mathbf{x}) + \mu_a(y)u(\mathbf{x}) = 0 & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Gamma_m, \\ u(\mathbf{x}) = 0 & \mathbf{x} \in \Gamma_0, \\ u(0,y) = u(L,y) & y \in (0,H), \end{cases} \tag{2.1}$$

where g is the laser illumination, D and μ_a are, respectively, the diffusion and absorption coefficients.

The part of the boundaries Γ_j are given by

$$\Gamma_m \ = \ (0,L) \times \{y = H\}, \Gamma_0 \ = \ (0,L) \times \{y = 0\}.$$

We further denote by Γ_p the complementary of $\overline{\Gamma_0} \cup \overline{\Gamma_m}$ in $\partial\Omega$.

We follow the approach taken in several papers [13, 14, 17], and consider two laser illuminations $g_j, j = 1, 2$. Denote $u_j, j = 1, 2$, the corresponding laser intensities.

Let

$$V \ := \ \{v \in H^1(\Omega); v(0,y) = v(L,y),\ \ y \in (0,H); u = 0\ \text{ on } \Gamma_0\}.$$

We further assume that $g \in V_{\Gamma_m}$, where

$$V_{\Gamma_m} \ := \ \{v|_{\Gamma_m} : v \in H^1(\Omega); v(0,y) = v(L,y),\ \ y \in (0,H); u = 0\ \text{ on } \Gamma_0\}.$$

Then, there exists a unique solution $u \in V$ satisfying the system (2.1). The proof uses techniques developed in [24]. The first step is to show that the set

$$V_0 \ := \ \{v \in H^1(\Omega); v(0,y) = v(L,y),\ \ y \in (0,H); u = 0\ \text{ on } \Gamma_0 \cup \Gamma_m\},$$

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is a closed sub space of $H^1(\Omega)$, using a specific trace theorem for regular curvilinear polygons ([24, Theorem 1.5.2.8, p. 50]). Then, applying the classical Lax–Milligram for elliptic operators in Lipschitz domain gives the existence and uniqueness of solution to the system (2.1).

Remark 2.1 *Using an explicit characterisation of the trace theorem obtained in [24], one can derive an optimal local regularity for $g \in H^{\frac{1}{2}}(\Gamma_m)$ that guarantees the existence and uniqueness of solutions to the system* (2.1) *(see also [8]).*

For simplicity, we will further consider $g_j = \varphi_{k_j}, j = 1, 2$, where $k_1 < k_2$, and $\varphi_k(x), k \in \mathbb{N}$, is the Fourier orthonormal basis of $L^2(0, L)$ satisfying $-\varphi_k''(y) = \lambda_k^2 \varphi_k(y)$, with $\lambda_k = \frac{2k\pi}{L}, k \in \mathbb{Z}$. Direct calculation gives $\varphi_k(x) = \frac{1}{\sqrt{L}} e^{i\lambda_k x}, k \in \mathbb{Z}$.

We assume that point-like ultrasound transducers, located on an observation surface Γ_m, are used to detect the values of the pressure $p(\mathbf{x}, t)$, where $\mathbf{x} \in \Gamma_m$ is a detector location and $t \geqslant 0$ is the time of the observation. We also assume that the speed of sound in the sample occupying $\Omega = (0, L) \times (0, H)$ is a smooth function and depends only on the vertical variable y, that is, $c = c(y) > 0$. Then, the following model is known to describe correctly the propagating pressure wave $p(\mathbf{x}, t)$ generated by the photoacoustic effect

$$\begin{cases} \partial_{tt} p(\mathbf{x}, t) = c^2(y) \Delta p(\mathbf{x}, t) & \mathbf{x} \in \Omega, t \geqslant 0, \\ \partial_\nu p(\mathbf{x}, t) + \beta \partial_t p(\mathbf{x}, t) = 0 & \mathbf{x} \in \Gamma_m, t \geqslant 0, \\ p(\mathbf{x}, t) = 0 & \mathbf{x} \in \Gamma_0, t \geqslant 0, \\ p((0, y), t) = p((L, y), t) & y \in (0, H), t \geqslant 0, \\ p(\mathbf{x}, 0) = f_0(\mathbf{x}),\ \partial_t p(\mathbf{x}, 0) = f_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.2}$$

where ∂_ν is the derivative along ν, the unit normal vector pointing outward of Ω. We note that ν is everywhere defined except at the vertices of Ω.

Here, $\beta > 0$ is the damping coefficient, and $f_j(\mathbf{x}), j = 0, 1$, are the initial values of the acoustic pressure, which one needs to find in order to determine the optical parameters of the sample. The coupling between the optical and the acoustic waves is through the relation

$$f_0(\mathbf{x}) = \mu_a(y) u(\mathbf{x}), \qquad \mathbf{x} \in \Omega,$$

where $u(\mathbf{x})$ is the unique solution to the system (2.1).

Remark 2.2 *Notice that in most existing works in photoacoustic imaging, the initial state $f_0(\mathbf{x})$ is given by $\mu_a(\mathbf{x})u(\mathbf{x})$, and is assumed to be compactly supported inside Ω, while the initial speed $f_1(\mathbf{x})$ is zero everywhere [31,42,45]. Recovering initial values by boundary measurement in hyperbolic systems is a well-studied inverse problem [9,10,29,50]. In our model, the initial speed $\partial_t p(\mathbf{x}, 0) = f_1(\mathbf{x})$ can be considered as the correction of the photoacoustic effect generated by the heat at Γ_m.*

The PA inverse problem we consider in this paper is *to recover the optical coefficients $\mu_a(y)$ and $D(y)$ on $(0, H)$ from the knowledge of the acoustic pressure $p(\mathbf{x}, t)$, $\mathbf{x} \in \Gamma_m$, $t \in (0, T)$, for $T > 0$ large enough.*

The inversion can be divided into two main steps. The first step is to determine the coupling internal function $f_0(\mathbf{x}) = \mu_a(y)u(\mathbf{x})$, $\mathbf{x} \in \Omega$, from the boundary measurement

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$p(\mathbf{x},t)$, $\mathbf{x} \in \Gamma_m$, $t \in (0,T)$. This step will be referred to us as the acoustic inversion. The second step, called the optical inversion, is to determine the coefficients $(D(y),\mu_a(y))$, $y \in (0,H)$ from the knowledge of the internal data $\mu_a(y)u(\mathbf{x})$, $\mathbf{x} \in \Omega$ recovered by the first inversion.

The following global stability estimates are the main results of the paper, obtained by combining stability estimates from the acoustic and optical inversions.

Theorem 2.1 *Let (D,μ_a), $(\widetilde{D},\widetilde{\mu}_a)$ be in $\mathcal{O}_M$, and k_i, $i = 1,2$ be two distinct integers. Let $c(y) \in W^{1,\infty}(0,H)$ with $0 < c_m \leqslant c^{-2}(y)$, and set $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$. Denote u_{k_i}, $i = 1,2$, and $\widetilde{u}_{k_i}$, $i = 1,2$, the solutions to the system (2.1) for $g_i = \varphi_{k_i}, i = 1,2$, with coefficients (D,μ_a) and $(\widetilde{D},\widetilde{\mu}_a)$, respectively. Let p_i, $i = 1,2$ and $\widetilde{p}_i$, $i = 1,2$, be the acoustic waves, solutions to the system (2.2), generated, respectively, by the optical waves u_{k_i}, $i = 1,2$ and $\widetilde{u}_{k_i}$, $i = 1,2$. Assume that $D(H) = \widetilde{D}(H)$, $D'(H) = \widetilde{D}'(H), \mu_a(H) = \widetilde{\mu}_a(H), \mu_a'(H) = \widetilde{\mu}_a'(H)$, $k_1 < k_2$, and k_1 is large enough.*

Then, for $T > 2\theta H$, there exists a constant $C > 0$ that only depends on $\mu_0, D_0, k_1, k_2, M, L$, and H, such that the following stability estimates hold.

$$\|\underline{u}_m^5(\mu_a - \widetilde{\mu}_a)\|_{C^0} \leqslant C\left(\sum_{i=1}^{2}\int_0^T \left(\frac{C_M}{T-2\theta H} + \beta\right)\|\partial_t p_i - \partial_t \widetilde{p}_i\|^2_{L^2(\Gamma_m)} + \|\partial_x p_i - \partial_x \widetilde{p}_i\|^2_{L^2(\Gamma_m)} dt\right)^{\frac{1}{4}},$$

and

$$\|\underline{u}_m^5(D - \widetilde{D})\|_{C^0} \leqslant C\left(\sum_{i=1}^{2}\int_0^T \left(\frac{C_M}{T-2\theta H} + \beta\right)\|\partial_t p_i - \partial_t \widetilde{p}_i\|^2_{L^2(\Gamma_m)} + \|\partial_x p_i - \partial_x \widetilde{p}_i\|^2_{L^2(\Gamma_m)} dt\right)^{\frac{1}{4}},$$

where

$$C_M = He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H) + \beta^2),$$
$$\underline{u}_m(y) = \frac{D^{\frac{1}{2}}(H)}{D^{\frac{1}{2}}(y)}\frac{\sinh(\kappa_m^{\frac{1}{2}}y)}{\sinh(\kappa_m^{\frac{1}{2}}H)}, \ \kappa_m = \min_{0\leqslant y\leqslant H}\left(\frac{(D^{\frac{1}{2}})''}{D^{\frac{1}{2}}} + \frac{\mu_a}{D} + \lambda_k^2\right) > 0.$$

Since the function $\underline{u}_m(y)$ is exponentially decreasing between the value 1 on Γ_m to the value 0 on Γ_0, the stability estimates in Theorem 2.1 shows that the resolution deteriorate exponentially in the depth direction far from Γ_m.

3 The acoustic inversion

The data obtained by the point detectors located on the surface Γ_m are represented by the function

$$p(\mathbf{x},t) = d(\mathbf{x},t) \qquad \mathbf{x} \in \Gamma_m,\ t \geqslant 0.$$

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Thus, the first inversion in PAI is to find, using the data $d(\mathbf{x},t)$ measured by transducers, the initial value $f_0(\mathbf{x})$ at $t = 0$ of the solution $p(\mathbf{x},t)$ of (2.2). We will also recover the initial speed $f_1(\mathbf{x})$ inside Ω, but we will not use it in the second inversion.

We first focus on the direct problem and prove existence and uniqueness of the acoustic problem (2.2). Denote by $L^2_c(\Omega)$ the Sobolev space of square integrable functions with weight $\frac{1}{c^2(y)}$. Since the speed c^2 is lower and upper bounded, the norm corresponding to this weight is equivalent to the classical norm of $L^2(\Omega)$.

Let

$$V = \{p \in H^1(\Omega);\ p(0,y) = p(L,y),\ y \in (0,H); p = 0 \text{ on } \Gamma_0\},$$

and consider in $V \times L^2_c(\Omega)$ the unbounded linear operator A defined by

$$A(p,q) = (q, c^2\Delta p),\ D(A) = \{(p,q) \in V \times V;\ \Delta p \in L^2(\Omega);\ \partial_\nu p + \beta q = 0 \text{ on } \Gamma_m\}.$$

We have the following existence and uniqueness result.

Proposition 3.1 *For* $(f_0, f_1) \in D(A)$*, the problem* (2.2) *has a unique solution* $p(x,t)$ *satisfying*

$$(p, \partial_t p) \in C\left([0,+\infty), D(A)\right) \cap C^1\left([0,+\infty), V \times L^2_c(\Omega)\right).$$

Proof There are various methods for proving well posedness of evolution problems: variational methods, the Laplace transform method and the semi-group method. Here, we will consider the semi-group method [47], and prove that the operator A is m-dissipative on the Hilbert space $V \times L^2_c(\Omega)$.

Denote by $\langle \cdot,\cdot \rangle$ the scalar product in $V \times L^2_c(\Omega)$, that is, for $(p_i, q_i) \in V \times L^2_c(\Omega)$ with $i = 1, 2$,

$$\langle (p_1,q_1),(p_2,q_2)\rangle = \int_\Omega \nabla p_1 \nabla \overline{p}_2 d\mathbf{x} + \int_\Omega q_1 \overline{q}_2 \frac{d\mathbf{x}}{c^2}.$$

Now let $(p,q) \in D(A)$. We have

$$\langle A(p,q),(p,q)\rangle = \int_\Omega \nabla q \nabla \overline{p} d\mathbf{x} + \int_\Omega \Delta p \overline{q} d\mathbf{x}.$$

Since $\Delta p \in L^2(\Omega)$ and $\partial_\nu p + \beta q = 0$ on Γ_m, applying Green formula leads to

$$\langle A(p,q),(p,q)\rangle = \int_\Omega \nabla q \nabla \overline{p} d\mathbf{x} - \int_\Omega \nabla \overline{q} \nabla p d\mathbf{x} - \beta \int_{\Gamma_m} |q|^2 d\sigma(\mathbf{x}).$$

Consequently,

$$\Re(\langle A(p,q),(p,q)\rangle) = -\beta \int_{\Gamma_m} |q|^2 d\sigma(\mathbf{x}).$$

Therefore, the operator A is dissipative. The fact that 0 is in the resolvent of A is straightforward. Then, A is m-dissipative and hence, it is the generator of a strongly

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continuous semigroup of contractions [47]. Consequently, for $(f_0, f_1) \in D(A)$ there exists a unique strong solution to the problem (2.2). □

Now, back to the inverse problem of reconstructing the initial data (f_0, f_1). We further assume that the initial data is generated by a finite number of Fourier modes, that is,

$$f_j(x, y) = \sum_{|k| \leqslant N} f_{jk}(y)\varphi_k(x) \qquad (x, y) \in \Omega \qquad j = 0, 1, \tag{3.1}$$

with N being a fixed positive integer.

As it was already remarked in many works, this linear initial-to-boundary inverse problem is strongly related to boundary observability of the source from the set Γ_m (see, for instance [35, 45, 47, 51]). We will emphasise on the links between our findings and known results in this context later. Here, we will use a different approach taking advantage of the fact that the wave speed $c(y)$ only depends on the vertical variable y.

Since $p(\mathbf{x})$ is L-periodic in the y variable, it has the following discrete Fourier decomposition:

$$p(x, y) = \sum_{|k| \leqslant N} p_k(y, t)\varphi_k(x) \qquad (x, y) \in \Omega.$$

One can check that $p_k(y, t)\varphi_k(x)$ is exactly the solution to the problem (2.2) with initial data $(f_{0k}(y)\varphi_k(x), f_{1k}(y)\varphi_k(x))$. Precisely, if $\lambda_k = \frac{2k\pi}{L}$, the functions $p_k(y, t)$ satisfy the following one-dimensional (1-D) wave equation:

$$\begin{cases} \frac{1}{c^2(y)}\partial_{tt} p(y, t) = \partial_{yy} p(y, t) - \lambda_k^2 p(y, t), & y \in (0, H), t \geqslant 0, \\ \partial_y p(H, t) + \beta \partial_t p(H, t) = 0 & t \geqslant 0, \\ p(0, t) = 0 & t \geqslant 0, \\ p(y, 0) = f_{0k}(y),\ \partial_t p(y, 0) = f_{1k}(y), & y \in (0, H). \end{cases} \tag{3.2}$$

Next, we will focus on the boundary observability problem of the initial data f_k at the extremity $y = H$. Taking advantage of the fact that the equation is 1-D, we will derive a boundary observability inequality with a sharp constant. Define $E(t)$ the total energy of the system (3.2) by

$$E(t) = \int_0^H \left(c^{-2}(y)|\partial_t p(y, t)|^2 + |\partial_y p(y, t)|^2 + \lambda_k^2 |p(y, t)|^2 \right) dy. \tag{3.3}$$

Multiplying the first equation in the system (3.2) by $\partial_t p(y, t)$ and integrating over $(0, H)$ leads to

$$E'(t) = -\beta |\partial_t p(H, t)|^2 \qquad \text{for} \quad t \geqslant 0. \tag{3.4}$$

Consequently, $E(t)$ is a non-increasing function, and the decay is clearly related to the magnitude of the dissipation on the boundary Γ_m.

It is well known that the system (3.2) has a unique solution. Here, we establish an estimate of the continuity constant.

Proposition 3.2 *Assume that $c(y) \in W^{1,\infty}(0,H)$ with $0 < c_m \leqslant c^{-2}(y)$. Then, for any $T > 0$, we have*

$$\beta^2 \int_0^T |\partial_t p_k(H,t)|^2 dt \leqslant ((C_m^1 + C_m^2 \lambda_k)T + C_m^3)E_k(0),$$

for $k \in \mathbb{N}$, where

$$\begin{aligned} E_k(0) &= \int_0^H \left(c^{-2}(y)|f_{1k}(y)|^2 + |f'_{0k}(y)|^2 + \lambda_k^2 |f_{0k}(y)|^2\right) dy, \\ C_m^1 &= (1 + Hc^{-2}(H))^{-1} \left(1 + (1 + \frac{H}{c_m})\|c^{-2}\|_{W^{1,\infty}}\right), \\ C_m^3 &= (1 + Hc^{-2}(H))^{-1} \left(1 + 2H\|c^{-2}\|_{L^\infty}^{1/2}\right), \\ C_m^2 &= H(1 + Hc^{-2}(H))^{-1}. \end{aligned}$$

Proposition 3.3 *Assume that $c(y) \in W^{1,\infty}(0,H)$ with $0 < c_m \leqslant c^{-2}(y)$. Let $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$ and $T > 2\theta H$. Then, the following inequalities hold*

$$\lambda_k^2 \int_0^H |f_{0k}(y)|^2 dy \leqslant \left(\frac{C_M}{T - 2\theta H} + \beta\right) \int_0^T |\partial_t p_k(H,t)|^2 dt + \lambda_k^2 \int_0^T |p_k(H,t)|^2 dt,$$

for $k \in \mathbb{N} \setminus \{0\}$,

$$\int_0^H c^{-2}(y)|f_{1k}(y)|^2 + |f'_{0k}(y)|^2 dy \leqslant \left(\frac{C_M}{T - 2\theta H} + \beta\right) \int_0^T |\partial_t p_k(H,t)|^2 dt + \lambda_k^2 \int_0^T |p_k(H,t)|^2 dt,$$

for $k \in \mathbb{N}$, with

$$C_M = He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H) + \beta^2).$$

The proofs of these results are given, respectively, in Sections 6 and 7.

The main result of this section is the following.

Theorem 3.1 *Assume that $c(y) \in W^{1,\infty}(0,1)$ with $0 < c_m \leqslant c^{-2}(y)$, and f_0, f_1 have a finite Fourier expansion* (3.1). *Let $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$ and $T > 2\theta H$. Then,*

$$\int_\Omega |\nabla f_0(\mathbf{x})|^2 d\mathbf{x} \leqslant \left(\frac{C_M}{T - 2\theta H} + \beta\right) \int_0^T \|\partial_t p(\mathbf{x},t)\|_{L^2(\Gamma_m)}^2 dt + \int_0^T \|\partial_x p(\mathbf{x},t)\|_{L^2(\Gamma_m)}^2 dt,$$

and

$$\int_\Omega c^{-2}(y)|f_1(\mathbf{x})|^2 d\mathbf{x} \leqslant \left(\frac{C_M}{T-2\theta H}+\beta\right)\int_0^T \|\partial_t p_k(\mathbf{x},t)\|^2_{L^2(\Gamma_m)}dt + \int_0^T \|\partial_x p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)}dt,$$

with

$$C_M = He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H)+\beta^2).$$

Proof The estimates are direct consequences of Propositions 3.2 and 3.3. The fact that the Fourier series of $p(\mathbf{x},t)$ has a finite number of terms justifies the regularity of the solution $p(\mathbf{x},t)$, and allow interchanging the order between the Fourier series and the integral over $(0,T)$. □

Using microlocal analysis techniques, it is known that the boundary observability in a rectangle holds if the set of boundary observation necessarily contains at least two adjacent sides [18,20]. Then, we expect that the the Lipschitz stability estimate in Theorem 3.1 will deteriorate when the number of modes N becomes larger. In fact the series on the right side does not converge because $\partial_x p(\mathbf{x},t)$ does not belong in general to $L^2(\Gamma_m\times(0,T))$. We here provide a hölder stability estimate that corresponds to the boundary observability on only one side of the rectangle.

Theorem 3.2 *Assume that $c(y)\in W^{1,\infty}(0,1)$ with $0<c_m\leqslant c^{-2}(y)$, and $(f_0,f_1)\in \big(H^2(\Omega)\times H^1(\Omega)\big)\cap D(A)$ satisfying $\|f_0\|_{H^2},\|f_1\|_{H^1}\leqslant \widetilde{M}$. Let $\theta=\sqrt{\|c^{-2}\|_{L^\infty}}$ and $T>2\theta H$. Then,*

$$\int_\Omega |\nabla f_0(\mathbf{x})|^2 d\mathbf{x} \leqslant \left(\frac{C_M}{T-2\theta H}+\beta\right)\int_0^T \|\partial_t p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)}dt + C_{\widetilde{M}}\left(\int_0^T \|p(\mathbf{x},t)\|^2_{H^{\frac{1}{2}}(\Gamma_m)}dt\right)^{\frac{2}{3}},$$

and

$$\int_\Omega c^{-2}(y)|f_1(\mathbf{x})|^2 d\mathbf{x} \leqslant \left(\frac{C_M}{T-2\theta H}+\beta\right)\int_0^T \|\partial_t p_k(\mathbf{x},t)\|^2_{L^2(\Gamma_m)}dt + C_{\widetilde{M}}\theta^{\frac{2}{3}}\left(\int_0^T \|p(\mathbf{x},t)\|^2_{H^{\frac{1}{2}}(\Gamma_m)}dt\right)^{\frac{2}{3}},$$

with

$$C_M = He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H)+\beta^2),\ C_{\widetilde{M}} = 2\widetilde{M}^{\frac{2}{3}}.$$

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Proof The proof is again based on the results of Proposition 3.3. We first deduce from Proposition 3.1 that $\partial_t p(\mathbf{x}, t) \in L^2(\Gamma_m)$. Now, define

$$f_j^N(\mathbf{x}) = \sum_{|k|\leqslant N} f_{jk}(y)\varphi_k(x) \qquad \mathbf{x} \in \Omega \qquad j = 0, 1,$$

with $f_{jk}(y)$ are the Fourier coefficients of $f_j(\mathbf{x})$, and N being a large positive integer.

Consequently,

$$\int_\Omega |\partial_x f_0(\mathbf{x})|^2 d\mathbf{x} = \int_\Omega |\partial_x f_0^N(\mathbf{x})|^2 d\mathbf{x} + \int_0^H \sum_{|k|\geqslant N+1} |(\partial_x f_0)_k(y)|^2 dy.$$

Using the fact that $y \to f_0(x, y)$ is $L-$periodic, and integrating by parts in the integral defining $(\partial_x f_0)_k(y)$, we find

$$|(\partial_x f_0)_k(y)|^2 \leqslant \frac{1}{\lambda_k^2} |(\partial_{yx} f_0)_k(y)|^2,$$

which implies

$$\begin{aligned}\int_\Omega |\partial_x f_0(\mathbf{x})|^2 d\mathbf{x} &\leqslant \int_\Omega |\partial_x f_0^N(\mathbf{x})|^2 d\mathbf{x} + \frac{1}{\lambda_{N+1}^2} \int_\Omega |\partial_{yx} f_0(\mathbf{x})|^2 d\mathbf{x} \\ &\leqslant \int_\Omega |\partial_x f_0^N(\mathbf{x})|^2 d\mathbf{x} + \frac{\widetilde{M}^2}{\lambda_{N+1}^2}.\end{aligned}$$

Repeating the same argument with $\partial_y f_0$, we finally obtain

$$\int_\Omega |\nabla f_0(\mathbf{x})|^2 d\mathbf{x} \leqslant \int_\Omega |\nabla f_0^N(\mathbf{x})|^2 d\mathbf{x} + \frac{\widetilde{M}^2}{\lambda_{N+1}^2}, \tag{3.5}$$

Similarly, we have

$$\int_\Omega c^{-2}(y)|f_1(\mathbf{x})|^2 d\mathbf{x} \leqslant \int_\Omega c^{-2}(y)|f_1^N(\mathbf{x})|^2 d\mathbf{x} + \frac{\theta^2 \widetilde{M}^2}{\lambda_{N+1}^2}, \tag{3.6}$$

for N large. Applying now Proposition 3.3 to (f_0^N, f_1^N), gives

$$\begin{aligned}\int_\Omega |\nabla f_0(\mathbf{x})|^2 d\mathbf{x} &\leqslant \left(\frac{C_M}{T - 2\theta H} + \beta\right) \int_0^T \|\partial_t p(\mathbf{x}, t)\|_{L^2(\Gamma_m)}^2 dt \\ &\quad + \lambda_N \int_0^T \|p(\mathbf{x}, t)\|_{H^{\frac{1}{2}}(\Gamma_m)}^2 dt + \frac{\widetilde{M}^2}{\lambda_{N+1}^2}, \\ \int_\Omega c^{-2}(y)|f_1(\mathbf{x})|^2 d\mathbf{x} &\leqslant \left(\frac{C_M}{T - 2\theta H} + \beta\right) \int_0^T \|\partial_t p(\mathbf{x}, t)\|_{L^2(\Gamma_m)}^2 dt \\ &\quad + \lambda_N \int_0^T \|p(\mathbf{x}, t)\|_{H^{\frac{1}{2}}(\Gamma_m)}^2 dt + \frac{\theta^2 \widetilde{M}^2}{\lambda_{N+1}^2}.\end{aligned}$$

By minimising the right-hand terms with respect to the value N, we obtain the desired results. □

4 The optical inversion

Once the initial pressure $f_0(\mathbf{x})$, generated by the optical wave has been reconstructed, a second step consists of determining the optical properties in the sample. Although this second step has not been well studied in biomedical literature due to its complexity, it is of importance in applications. In fact the optical parameters are very sensitive to the tissue condition and their values for healthy and unhealthy tissues are extremely different.

The second inversion is to determine the coefficients $(D(y), \mu_a(y))$ from the initial pressures recovered in the first inversion, that is, $h_j(\mathbf{x}) = \mu_a(y)u_j(\mathbf{x}),\ \mathbf{x} \in \Omega,\ j = 1, 2$.

For simplicity, we will consider $g_j(\mathbf{x}) = \varphi_{k_j}(x),\ j = 1, 2$ with k_1 and k_2 are two distinct Fourier eigenvalues that are large enough. We specify how large they should be later in the analysis.

The main result of this section is the following.

Theorem 4.1 *Let (D, μ_a), $(\widetilde{D}, \widetilde{\mu}_a)$ in $\mathcal{O}_M$, and k_i, $i = 1, 2$ be two distinct integers. Denote u_{k_i}, $i = 1, 2$ and $\widetilde{u}_{k_i}$, $i = 1, 2$ the solutions to the system* (4.1) *for $g_i = \varphi_{k_i}, i = 1, 2$, with coefficients (D, μ_a) and $(\widetilde{D}, \widetilde{\mu}_a)$, respectively. Assume that $D(H) = \widetilde{D}(H)$, $D'(H) = \widetilde{D}'(H), \mu_a(H) = \widetilde{\mu}_a(H), \mu_a'(H) = \widetilde{\mu}_a'(H)$, $k_1 < k_2$, and k_1 is large enough. Then, there exists a constant $C > 0$ that only depends on $(\mu_0, D_0, k_1, k_2, M, L, H)$, such that the following stability estimates hold.*

$$\|\underline{u}_m^5(D - \widetilde{D})\|_{C^0} \leqslant C \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right),$$
$$\|\underline{u}_m^5(\mu_a - \widetilde{\mu}_a)\|_{C^0} \leqslant C \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right).$$

Classical elliptic operator theory implies the following result for the direct problem [37].

Proposition 4.1 *Assume (D, μ_a) be in $\mathcal{O}_M$ and $g \in V_{\Gamma_m}$. Then, there exists a unique solution $u \in V$ to the system* (2.1). *It verifies*

$$\|u\|_{H^1(\Omega)} \leqslant C_0 \|g\|_{H^{\frac{1}{2}}(\Gamma_m)},$$

where $C_0 = C_0(\mu_0, D_0, M, L, H) > 0$.

For $g(\mathbf{x}) = \varphi_k(x)$, the unique solution u has the following decomposition:

$$u(\mathbf{x}) = u_k(y)\varphi_k(x) \qquad \mathbf{x} \in \Omega,$$

where $u_k(y)$ satisfies the following 1-D elliptic equation:

$$\begin{cases} -\left(D(y)u'(y)\right)' + (\mu_a(y) + \lambda_k^2 D(y))u(y) = 0 & y \in (0, H), \\ u(H) = 1, \quad u(0) = 0, \end{cases} \tag{4.1}$$

Next, we will derive some useful properties of the solution to the system (4.1).

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Lemma 4.1 *Let $u(y)$ be the unique solution to the system (4.1). Then, $u(y) \in C^2([0, H])$ and there exists a constant $b = b(\mu_0, D_0, M, L, H) > 0$ such that $\|u\|_{C^2} \leqslant b$ for all $(D, \mu_a) \in \mathcal{O}_M$. In addition, the following inequalities hold for k large enough:*

$$\underline{u}_m(y) \leqslant u(y) \leqslant \overline{u}_M(y),$$

for $0 \leqslant y \leqslant H$, where

$$\underline{u}_m(y) = \frac{D^{\frac{1}{2}}(H)}{D^{\frac{1}{2}}(y)} \frac{\sinh(\kappa_m^{\frac{1}{2}} y)}{\sinh(\kappa_m^{\frac{1}{2}} H)}, \quad \overline{u}_M(y) = \frac{D^{\frac{1}{2}}(H)}{D^{\frac{1}{2}}(y)} \frac{\sinh(\kappa_M^{\frac{1}{2}} y)}{\sinh(\kappa_M^{\frac{1}{2}} H)},$$
$$\kappa_m = \min_{0 \leqslant y \leqslant H} \left(\frac{(D^{\frac{1}{2}})''}{D^{\frac{1}{2}}} + \frac{\mu_a}{D} + \lambda_k^2 \right), \quad \kappa_M = \max_{0 \leqslant y \leqslant H} \left(\frac{(D^{\frac{1}{2}})''}{D^{\frac{1}{2}}} + \frac{\mu_a}{D} + \lambda_k^2 \right).$$

Proof We first make the Liouville change of variables and introduce the function

$$v(y) = \frac{D^{\frac{1}{2}}(y)}{D^{\frac{1}{2}}(H)} u(y).$$

Forward calculations show that $v(y)$ is the unique solution to the following system:

$$\begin{cases} -v''(y) + \kappa(y) v(y) = 0 & y \in (0, H), \\ v(H) = 1, \quad v(0) = 0, \end{cases} \tag{4.2}$$

where

$$\kappa(y) = \frac{(D^{\frac{1}{2}})''}{D^{\frac{1}{2}}} + \frac{\mu_a}{D} + \lambda_k^2.$$

Assume now that k is large enough such that $\kappa_m > 0$, and let $\underline{v}_m(y)$ and $\overline{v}_M(y)$ be the solutions to the system (4.2) when we replace $\kappa(y)$ by, respectively, the constants κ_m and κ_M. They are explicitly given by

$$\underline{v}_m(y) = \frac{\sinh(\kappa_m^{\frac{1}{2}} y)}{\sinh(\kappa_m^{\frac{1}{2}} H)},$$
$$\overline{v}_M(y) = \frac{\sinh(\kappa_M^{\frac{1}{2}} y)}{\sinh(\kappa_M^{\frac{1}{2}} H)}.$$

The maximum principle [37] implies that $0 < v(y), \underline{v}_m(y), \overline{v}_M(y) < 1$ for $0 < y < H$.

By applying again, the maximum principle on the differences $v - \underline{v}_m$ and $v - \overline{v}_M$, we deduce that $\underline{v}_m(y) < v(y) < \overline{v}_M(y)$ for $0 \leqslant y \leqslant H$, which leads to the desired lower and upper bounds.

We deduce from the regularity of the coefficients D and μ_a and the classical elliptic regularity [37] that $u \in H^3(0, H)$. Moreover, there exist a constant $b >$ that only depends on (μ_0, D_0, M, L, H) such that

$$\|u\|_{H^3} \leqslant b. \tag{4.3}$$

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Consequently, the uniform C^2 bound of u can be obtained using the continuous Sobolev embedding of $H^3(0,H)$ into $C^2([0,H])$ [1]. □

Lemma 4.2 *Let $(D,\mu_a) \in \mathcal{O}_M$, and $u(y)$ be the unique solution to the system (4.1). Then, for k large enough there exists a constant $\varrho = \varrho(D_0,\mu_0,M,k) > 0$ such that*

$$u'(y) \geqslant \varrho,$$

for $0 \leqslant y \leqslant H$.

Proof Since 0 is the global minimum of $u(y)$, we have $u'(0) > 0$. Moreover, for k large enough, Lemma 4.1 implies that

$$u(y) \geqslant \frac{D^{\frac{1}{2}}(H)}{D^{\frac{1}{2}}(y)} \frac{\sinh(\kappa_m^{\frac{1}{2}} y)}{\sinh(\kappa_m^{\frac{1}{2}} H)},$$

for all $y \in [0,H]$. Therefore,

$$u'(0) \geqslant \frac{D^{\frac{1}{2}}(H)}{\|D\|_{L^\infty}^{\frac{1}{2}}} \frac{\kappa_m^{\frac{1}{2}}}{\sinh(\kappa_m^{\frac{1}{2}} H)}.$$

Now, integrating equation (4.1) over $(0,y)$, we obtain

$$D(y)u'(y) = D(0)u'(0) + \int_0^y (\mu_a(s) + \lambda_k^2 D(s))u(s)ds$$

$$\begin{aligned} D(y)u'(y) &\geqslant D(0)u'(0) + \int_0^y (\mu_a(s) + \lambda_k^2 D(s)) \frac{D^{\frac{1}{2}}(H)}{D^{\frac{1}{2}}(s)} \frac{\sinh(\kappa_m^{\frac{1}{2}} s)}{\sinh(\kappa_m^{\frac{1}{2}} H)} ds \\ &\geqslant \frac{D^{\frac{1}{2}}(H)}{\|D\|_{L^\infty}^{\frac{1}{2}}} \frac{\kappa_m^{\frac{1}{2}} D_0}{\sinh(\kappa_m^{\frac{1}{2}} H)} + (\mu_0 + \lambda_k^2 D_0) \frac{D^{\frac{1}{2}}(H)}{\|D\|_{L^\infty}^{\frac{1}{2}}} \frac{\cosh(\kappa_m^{\frac{1}{2}} y) - 1}{\sinh(\kappa_m^{\frac{1}{2}} H)}. \end{aligned}$$

Taking into account the explicit expression of κ_m finishes the proof. □

Since the illumination are chosen to coincide with the Fourier basis functions φ_{k_j}, $j = 1,2$, the data $\mathbf{h}_j(\mathbf{x})$, $j = 1,2$, can be rewritten as $\mathbf{h}_j(\mathbf{x}) = h_j(y)\varphi_{k_j}(x)$, $j = 1,2$, where $h_j(y) = \mu_a(y)u_{k_j}(y)$.

Therefore, the optical inversion is reduced to the problem of identifying the optical pair (D,μ_a) from the knowledge of the pair $(h_1(y),h_2(y))$ over $(0,H)$.

Let (D,μ_a), $(\widetilde{D},\widetilde{\mu}_a)$ be two different pairs in $\mathcal{O}_M$, and denote u_k and $\widetilde{u}_k$ the solutions to the system (4.1), with coefficients (D,μ_a) and $(\widetilde{D},\widetilde{\mu}_a)$, respectively.

We deduce from Lemma 4.1 that $\frac{1}{u_k}$ and $\frac{1}{\widetilde{u}_k}$ lie in $L^p(0,H)$ for $0 < p < 1$. Unfortunately, for $0 < p < 1$, the usual $\|\cdot\|_{L^p}$ is not anymore a norm on the vector space $L^p(0,H)$ because it does not satisfy the triangle inequality (see, for instance [1]. In contrast with

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triangle inequality Hölder inequality holds for $0 < p < 1$, and we have

$$\|\frac{v}{u_k}\|_{L^r} \leqslant \|\frac{1}{u_k}\|_{L^p}\|v\|_{L^q}, \tag{4.4}$$

for all $v \in L^q(0,H)$ with $p,q > 0$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Consequently, $h = \frac{h_2}{h_1} = \frac{u_{k_2}}{u_{k_1}}$ can be considered as a distribution that coincides with a C^2 function over $(0,H)$. A forward calculation shows that h satisfies the equation

$$-\left(Du_{k_1}^2 h'\right)' + Du_{k_1}^2 h(\lambda_2^2 - \lambda_1^2) = 0, \tag{4.5}$$

over $(0,H)$.

Since u_{k_j}, $j = 1,2$ are in $C^2([0,H])$, an asymptotic analysis of $D(y)u_{k_1}^2(y)h'(y)$ at 0 and the results of Lemma 4.1, gives

$$\lim_{y\to 0} Du_{k_1}^2 h' = 0.$$

Similarly, we have

$$\lim_{y\to 1} h = 1.$$

Integrating the equation (4.5) over $(0,y)$, we get

$$D(y)u_{k_1}^2(y)h'(y) = (\lambda_2^2 - \lambda_1^2)\int_0^y D(s)u_{k_1}(s)u_{k_2}(s)ds.$$

Dividing both sides by $D(y)u_{k_1}^2(y)$, and using again Lemma 4.1, imply

$$(\lambda_2^2 - \lambda_1^2)MD_0\overline{u}_M^{-2}(y)\int_0^y \underline{u}_m^2(s)ds \leqslant h'(y) \leqslant (\lambda_2^2 - \lambda_1^2)MD_0\overline{u}_m^{-2}(y)\int_0^y \underline{u}_M^2(s)ds,$$

over $0,H$, which leads to

$$c_h^- y \leqslant h'(y) \leqslant c_h^+ y, \qquad y \in (0,H), \tag{4.6}$$

where the constants $c_h^{\pm}$ are strictly positive and only depend on D_0, μ_0, M, H, L, k_1, and k_2.

Now, back to the optical inversion. Dividing the equation (4.5) by $Du_{k_1}^2 h$, and integrating over (y,H), we obtain

$$h'(y)D(y)u_{k_1}^2(y) = D(H)h'(H)e^{-(\lambda_2^2-\lambda_1^2)\int_y^H \frac{h}{h'}ds}, \tag{4.7}$$

for $y \in (0,H)$.

The identity (4.7) allows us to derive the following result.

Lemma 4.3 *Under the assumptions of Theorem 4.1, there exists a constant* $C = C(\mu_0, D_0, k_1, k_2, M, L, H) > 0$ *such that the following inequality holds.*

$$\|y^2(Du_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2)\|_{C^0} \leqslant C\left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1}\right).$$

Proof Recall that the relation (4.7) is also valid for the pair $(\widetilde{D}, \widetilde{\mu}_a)$, that is,

$$\widetilde{h}'(y)\widetilde{D}(y)\widetilde{u}_{k_1}^2(y) = \widetilde{D}(H)\widetilde{h}'(H)e^{-(\lambda_2^2-\lambda_1^2)\int_y^H \frac{\widetilde{h}}{\widetilde{h}'}ds}, \tag{4.8}$$

for $y \in (0, H)$.

Taking the difference between the equations (4.7) and (4.8), we find

$$h'w = \widetilde{D}\widetilde{u}_{k_1}^2(\widetilde{h}'(y) - h'(y)) + D(H)(h'(H) - \widetilde{h}'(H))e^{-(\lambda_2^2-\lambda_1^2)\int_y^H \frac{\widetilde{h}}{\widetilde{h}'}ds} + D(H)\widetilde{h}'(H)\mathcal{I}, \tag{4.9}$$

where $w = (D(y)u_{k_1}^2(y)h'(y) - \widetilde{D}(y)\widetilde{u}_{k_1}^2(y))$, and

$$\mathcal{I} = e^{-(\lambda_2^2-\lambda_1^2)\int_y^H \frac{h}{h'}ds} - e^{-(\lambda_2^2-\lambda_1^2)\int_y^H \frac{\widetilde{h}}{\widetilde{h}'}ds}.$$

A forward calculations leads to

$$\begin{aligned} y|\mathcal{I}| &\leqslant (\lambda_2^2 - \lambda_1^2)\left|y\int_y^H (\frac{h}{h'} - \frac{\widetilde{h}}{\widetilde{h}'})ds\right|, \\ &\leqslant y\int_y^H \frac{1}{h'}|h - \widetilde{h}| + \frac{\widetilde{h}}{\widetilde{h}'h'}|\widetilde{h}' - h'|ds, \\ &\leqslant \left(y\int_y^H \frac{1}{h'} + \frac{\widetilde{h}}{\widetilde{h}'h'}ds\right)\|h - \widetilde{h}\|_{C^1}, \end{aligned}$$

which combined with (4.6) yield

$$y|\mathcal{I}| \leqslant c_1\|h - \widetilde{h}\|_{C^1}, \tag{4.10}$$

where $c_1 = c_1(\mu_0, D_0, k_1, k_2, M, L, H) > 0$.

Multiplying now the identity (4.9) by y and considering the behaviour of h' described in (4.6), we obtain the desired result. □

Now, we are ready to prove the main stability result of this section. We remark as in [17], that $\frac{1}{u_{k_1}}$ is a solution to the following equation:

$$-\left(Du_{k_1}^2 \frac{1}{u_{k_1}}'\right)' + \lambda_{k_1}^2 Du_{k_1}^2 \frac{1}{u_{k_1}} = h_1,\ y \in (0, H).$$

Since $\frac{1}{\widetilde{u}_{k_1}}$ solves the same type of equation, we obtain that $w = \frac{1}{u_{k_1}} - \frac{1}{\widetilde{u}_{k_1}}$, is the solution to the following system:

$$\begin{cases} -\left(Du_{k_1}^2 w'\right)' + \lambda_{k_1}^2 Du_{k_1}^2 w = e, & y \in (0, H), \\ w(H) = 0,\ w'(H) = \frac{1}{\mu_a(H)}(\widetilde{h}_1'(H) - h_1'(H)), & u_{k_1}^2 w' \in L^2(0, H), \end{cases} \tag{4.11}$$

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where

$$e = -\left((Du_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2)\frac{1}{\widetilde{u}_{k_1}}'\right)' + \lambda_{k_1}^2(Du_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2)\frac{1}{\widetilde{u}_{k_1}} + h_1 - \widetilde{h}_1.$$

We remark that to solve this system, we have to deal with two main difficulties, the first is that the operator is elliptic degenerate, and the second is that the solution $w(y)$ may be unbounded at $y = 0$.

Integrating over (s, H), for $s \in (0, H)$, the first equation of the system, leads to

$$D(s)u_{k_1}^2(s)w'(s) = D(H)\frac{1}{\mu_a(H)}(\widetilde{h}_1'(H) - h_1'(H)) + \int_s^H \lambda_{k_1}^2 Du_{k_1}^2 w dr - \int_s^H e(r)dr.$$

Integrating again over (y, H), we find

$$\int_y^H Du_{k_1}^2 w' ds = (H - y)D(H)\frac{1}{\mu_a(H)}(\widetilde{h}_1'(H) - h_1'(H)) + \int_y^H \int_s^H \lambda_{k_1}^2 Du_{k_1}^2 w drds - \int_y^H \int_s^H e(r)drds.$$

Integrating by parts the integral on the left-hand side, yields

$$D(y)u_{k_1}^2(y)w(y) = \int_y^H (Du_{k_1}^2)' w ds - (H - y)D(H)\frac{1}{\mu_a(H)}(\widetilde{h}_1'(H) - h_1'(H)) - \int_y^H \int_s^H \lambda_{k_1}^2 Du_{k_1}^2 w drds + \int_y^H \int_s^H e(r)drds.$$

Hence,

$$D_0 u_{k_1}^2(y)|w(y)| \leqslant \int_y^H |(Du_{k_1}^2)'||w|ds + (H - y)D(H)\frac{1}{\mu_a(H)}|\widetilde{h}_1'(H) - h_1'(H)| + \int_y^H \int_s^H \lambda_{k_1}^2 Du_{k_1}^2 |w| drds + \left|\int_y^H \int_s^H e(r)drds\right|. \tag{4.12}$$

We deduce from Lemma 4.1 that there exists a constant $c_2 = c_2(\mu_0, D_0, k_1, k_2, M, L, H) > 0$, such that

$$0 < \frac{u_k}{\widetilde{u}_k}, \frac{\widetilde{u}_k}{u_k} \leqslant c_2. \tag{4.13}$$

Substituting w by $w_0 u_{k_1}^{-1}\widetilde{u}_{k_1}^{-1}$, where $w_0(y) = \widetilde{u}_{k_1}(y) - u_{k_1}(y)$, and multiplying both sides by $u_{k_1}^3(y)$ in the inequality (4.12) give

$$u_{k_1}^4(y)|w_0(y)| \leqslant c_3|\widetilde{h}_1'(H) - h_1'(H)| + c_4 u_{k_1}^3(y)\widetilde{u}_{k_1}(y)\left|\int_y^H \int_s^H e(r)drds\right| + c_5 u_{k_1}^3(y)\int_y^H u_{k_1}|w_0|ds,$$

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where $c_i, i = 3, 4, 5$, are strictly positive constants that only depend on $\mu_0, D_0, k_1, k_2, M, L$, and H.

Since $u_k, \widetilde{u}_k$ are increasing functions, we obtain using inequalities (4.13), the following estimates:

$$u_{k_1}^4(y)|w_0(y)| \leqslant c_3|\widetilde{h}_1'(H) - h_1'(H)| + c_4' \left| \int_y^H \int_s^H \widetilde{u}_{k_1}^4(r)e(r)drds \right| + c_5 \int_y^H u_{k_1}^4 |w_0| ds, \tag{4.14}$$

where c_4' is a strictly positive constant that only depends on $\mu_0, D_0, k_1, k_2, M, L$, and H.

Now, we focus on the second term on the right-hand side of the last inequality.

$$\begin{aligned} &\left| \int_y^H \int_s^H \widetilde{u}_{k_1}^4(r)e(r)drds \right| \\ &\leqslant \int_y^H \widetilde{u}_{k_1}^2 |Du_{k_1}^2 - \widetilde{D}||\widetilde{u}_{k_1}'| ds + 4 \int_y^H \int_s^H \widetilde{u}_{k_1}^2 |Du_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2| \frac{|\widetilde{u}_{k_1}'|^2}{\widetilde{u}_{k_1}} ds \\ &+ \lambda_{k_1}^2 H \int_0^H |Du_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2| \widetilde{u}_{k_1}^4 dy + c_6 \|h_1 - \widetilde{h}_1\|_{C^0}, \end{aligned} \tag{4.15}$$

where c_6 is a strictly positive constant that only depends on $\mu_0, D_0, k_1, k_2, M, L$, and H.

Using the estimates in Lemma 4.3, we find

$$\int_y^H \left| \int_s^H \widetilde{u}_{k_1}^4(r)e(r)dr \right| ds \leqslant c_7 \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right), \tag{4.16}$$

$c_7 > 0$ only depends on $\mu_0, D_0, k_1, k_2, M, L$, and H. Combining inequalities (4.14) and (4.16), leads to

$$\begin{aligned} &u_{k_1}^4(t)|w_0|(t) \\ &\leqslant C_1 \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right) + C_2 \int_t^H u_{k_1}^4(y)|w_0|(y)dy, \end{aligned} \tag{4.17}$$

for $0 \leqslant t \leqslant H$, where C_i, $i = 1, 2$, are strictly positive constants that only depend on $\mu_0, D_0, k_1, k_2, M, L$, and H.

Using Gronwall's inequality, we finally get

$$u_{k_1}^4(t)|w_0|(t) \leqslant C_1 e^{C_2 \int_0^H u_{k_1}^4(y)dy} \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right), \tag{4.18}$$

for $0 \leqslant t \leqslant H$.

The following Lemma is a direct consequence of the previous inequality.

Lemma 4.4 *Under the assumptions of Theorem 4.1, there exists a constant* $C = C(\mu_0, D_0, k_1, k_2, M, L, H) > 0$ *such that the following inequality holds.*

$$\|u_{k_1}^4(u_{k_1} - \widetilde{u}_{k_1})\|_{C^0} \leqslant C \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right).$$

Proof (Theorem 4.1) Recall that $h_1 = \mu_a u_{k_1}$ and $\widetilde{h}_1 = \widetilde{\mu}_a \widetilde{u}_{k_1}$ over $(0, H)$. Therefore,

$$u_{k_1}^5 |\mu_a - \widetilde{\mu}_a| \leqslant u_{k_1}^4 |h_1 - \widetilde{h}_1| + \widetilde{\mu}_a u_{k_1}^4 |u_{k_1} - \widetilde{u}_{k_1}|.$$

Lemma 4.4 implies

$$\|u_{k_1}^5 (\mu_a - \widetilde{\mu}_a)\|_{C^0} \leqslant C \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right). \tag{4.19}$$

A simple calculation yields

$$u_{k_1}^5 |D - \widetilde{D}| \leqslant \widetilde{D}\widetilde{u}_{k_1}^2 |u_{k_1}^3 - \widetilde{u}_{k_1}^3| + u_{k_1}^3 |D u_{k_1}^2 - \widetilde{D}\widetilde{u}_{k_1}^2|,$$

over $(0, H)$.

Inequalities (4.13), Lemmas 4.3 and 4.4 lead to

$$\|u_{k_1}^5 (D - \widetilde{D})\|_{C^0} \leqslant C \left(\|h_1 - \widetilde{h}_1\|_{C^1} + \|h_2 - \widetilde{h}_2\|_{C^1} \right). \tag{4.20}$$

Applying the bounds in Lemma 4.1, we obtain the wanted results. □

5 Proof of theorem 2.1

The main idea here is to combine the stability results of the acoustic and optic inversions in a result that shows how the reconstruction of the optical coefficients is sensitive to the noise in the measurements of the acoustic waves.

The principal difficulty is that the vector spaces used in both stability estimates are not the same due to the difference in the techniques used to derive them. We will use interpolation inequality between Sobolev spaces to overcome this difficulty.

We deduce from the uniform bound on the solutions u_i, $i = 1, 2$ (see, for instance (4.3) in the proof of Lemma 4.1) that

$$\|h_i\|_{H^3}, \|\widetilde{h}_i\|_{H^3} \leqslant Mb, \quad i = 1, 2, \tag{5.1}$$

for all pairs $(D, \mu_a)$and $(\widetilde{D}, \mu_a)$ in $\mathcal{O}_M$.

The Sobolev interpolation inequalities and embedding theorems [1] imply

$$\|h_i - \widetilde{h}_i\|_{C^1} \leqslant C \|h_i - \widetilde{h}_i\|_{H^2} \leqslant \widetilde{C} \|h_i - \widetilde{h}_i\|_{H^1}^{\frac{1}{2}} \|h_i - \widetilde{h}_i\|_{H^3}^{\frac{1}{2}}, \quad i = 1, 2,$$

which combined with (5.1) gives

$$\|h_i - \widetilde{h}_i\|_{C^1} \leqslant \widetilde{\widetilde{C}} \|h_i - \widetilde{h}_i\|_{H^1}^{\frac{1}{2}}, \quad i = 1, 2. \tag{5.2}$$

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Since the acoustic inversion is linear, we obtain from Theorem 3.1 (or Proposition 3.3) that,

$$\lambda_{k_i}^2 \int_0^H |h_i - \widetilde{h}_i|^2 dy \leqslant \left(\frac{C_M}{T-2\theta H} + \beta\right) \int_0^T |\partial_t p_i(H,t) - \partial_t \widetilde{p}_i(H,t)|^2 dt + \lambda_{k_i}^2 \int_0^T |p_i(H,t) - \widetilde{p}_i(H,t)|^2 dt,$$

for $i = 1, 2$, and

$$\int_0^H c^{-2}(y) |h_i' - \widetilde{h}_i'|^2 dy \leqslant \left(\frac{C_M}{T-2\theta H} + \beta\right) \int_0^T |\partial_t p_i(H,t) - \partial_t \widetilde{p}_i(H,t)|^2 dt + \lambda_{k_i}^2 \int_0^T |p_i(H,t) - \widetilde{p}_i(H,t)|^2 dt,$$

for $i = 1, 2$.

Consequently,

$$\|h_i - \widetilde{h}_i\|_{C^1} \leqslant \widetilde{C} \left(\int_0^T \left(\frac{C_M}{T-2\theta H} + \beta\right) |\partial_t p_i(H,t) - \partial_t \widetilde{p}_i(H,t)|^2 + \lambda_{k_i}^2 |p_i(H,t) - \widetilde{p}_i(H,t)|^2 dt \right)^{\frac{1}{4}},$$

for $i = 1, 2$.

Using the optical stability estimates in Theorem 4.1, we obtain

$$\|\underline{u}_m^5 (\mu_a - \widetilde{\mu}_a)\|_{C^0} \leqslant \widetilde{\widetilde{C}} \left(\sum_{i=1}^2 \int_0^T \left(\frac{C_M}{T-2\theta H} + \beta\right) |\partial_t p_i(H,t) - \partial_t \widetilde{p}_i(H,t)|^2 + \lambda_{k_i}^2 |p_i(H,t) - \widetilde{p}_i(H,t)|^2 dt \right)^{\frac{1}{4}},$$

and

$$\|\underline{u}_m^5 (D - \widetilde{D})\|_{C^0} \leqslant \widetilde{\widetilde{C}} \left(\sum_{i=1}^2 \int_0^T \left(\frac{C_M}{T-2\theta H} + \beta\right) |\partial_t p_i(H,t) - \partial_t \widetilde{p}_i(H,t)|^2 + \lambda_{k_i}^2 |p_i(H,t) - \widetilde{p}_i(H,t)|^2 dt \right)^{\frac{1}{4}},$$

which ends the proof. □

6 Proof of proposition 3.2

Multiplying the first equation of the system (3.2) by $y\partial_y p(y,t)$ and integrating by part one time over $(0,T)$, we obtain

$$\int_0^T |\partial_y p(H,t)|^2 dt = \int_0^T \int_0^H |\partial_y p(y,t)|^2 dy dt - 2\int_0^T \int_0^H c^{-2}\partial_{tt} p(y,t) y \partial_y p(y,t) dy dt$$
$$- 2\lambda_k^2 \int_0^T \int_0^H p(y,t) y \partial_y p(y,t) dy dt = A_1 + A_2 + A_3.$$

In the rest of the proof, we shall derive bounds of each of the constants $A_i, i = 1,2,3$, in terms of the energy $E(0)$. Due to the energy decay (3.4), we have

$$|A_1| \leqslant TE(0).$$

Integrating by part again over $(0,T)$ in the integral A_2, we get

$$A_2 = -\int_0^T \int_0^H y c^{-2} \partial_y |\partial_t p(y,t)|^2 dy dt$$
$$+ 2\int_0^H y c^{-2} \partial_t p(y,T) \partial_y p(y,T) dy - 2\int_0^H y c^{-2} \partial_t p(y,0) \partial_y p(y,0) dy.$$

Integrating by part now over $(0,H)$, we find

$$A_2 + Hc^{-2}(H)\int_0^T |\partial_t p(H,t)|^2 dt = \int_0^T \int_0^H \partial_y(yc^{-2})|\partial_t p(y,t)|^2 dy dt$$
$$+ 2\int_0^H y c^{-2} \partial_t p(y,T) \partial_y p(y,T) dy - 2\int_0^H y c^{-2} \partial_t p(y,0) \partial_y p(y,0) dy,$$

which leads to the following inequality:

$$\left| A_2 + Hc^{-2}(H)\int_0^T |\partial_t p(H,t)|^2 dt \right|$$
$$\leqslant \left\| c^2 \partial_y (y c^{-2}(y)) \right\|_{L^\infty} TE(0) + H\|c^{-1}(y)\|(E(T) + E(0)).$$

Using the energy decay (3.4), we finally obtain

$$\left| A_2 + Hc^{-2}(H)\int_0^T |\partial_t p(H,t)|^2 dt \right|$$
$$\leqslant \left((1 + (1 + \frac{H}{c_m})\|c^{-2}\|_{W^{1,\infty}})T + 2H\|c^{-2}\|_{L^\infty}^{1/2} \right) E(0).$$

Similar arguments dealing with the integral A_3 show that

$$|A_3| \leqslant H\lambda_k TE(0).$$

Combining all the previous estimates on the constants $A_i, i = 1,2,3$, achieve the proof.

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7 Proof of proposition 3.3

Let $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$ and $T > 2\theta H$, and introduce the following function:

$$\Phi(y) = \int_{\theta y}^{T-\theta y} \left(c^{-2}(H-y)|\partial_t p(H-y,t)|^2 + |\partial_y p(H-y,t)|^2 + \lambda_k^2 |p(H-y,t)|^2\right) dt,$$
$$= \int_{\theta y}^{T-\theta y} \varphi(y,t)dt,$$

for $0 \leqslant y \leqslant H$. We remark that

$$\Phi(0) = (c^{-2}(H) + \beta^2) \int_0^T |\partial_t p(H,t)|^2 dt + \lambda_k^2 \int_0^T |p(H,t)|^2 dt. \tag{7.1}$$

On the other hand, a forward calculation of the derivative of $\Phi(y)$ gives

$$\Phi'(y) = \int_{\theta y}^{T-\theta y} \partial_y \varphi(y,t)dt - \theta\varphi(y, T-\theta y) - \theta\varphi(y,\theta y).$$

Integrating by parts in the integral, we deduce that

$$\Phi'(y) = B_\theta(y) + \partial_y(c^{-2}(H-y)) \int_{\theta y}^{T-\theta y} |\partial_t p(H-y,t)|^2 dt,$$

where

$$B_\theta(y) = \left(-2c^{-2}(H-y)\partial_t p(H-y,t)\partial_y p(H-y,t)\right)\Big|_{t=\theta y}^{t=T-\theta y}$$
$$- \theta\left(c^{-2}(H-y)|\partial_t p(H-y,t)|^2 + |\partial_y p(H-y,t)|^2 + \lambda_k^2|p(H-y,t)|^2\right)\Big|_{t=\theta y}^{t=T-\theta y}.$$

The choice of θ implies $B_\theta(y) < 0$ for $0 \leqslant y \leqslant H$. Hence, we obtain

$$\Phi'(y) \leqslant c^2(H-y)|\partial_y(c^{-2}(H-y))| \int_{\theta y}^{T-\theta y} c^{-2}(H-y)|\partial_t p(H-y,t)|^2 dt$$
$$\leqslant c^2(H-y)|\partial_y(c^{-2}(H-y))|\Phi(y).$$

Using Gronwall's inequality, we get

$$\Phi(y) \leqslant e^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}\Phi(0), \tag{7.2}$$

for $0 \leqslant y \leqslant H$.

We deduce from the energy decay (3.4) that

$$(T-2\theta H)E(T) \leqslant (T-2\theta H)E(T-\theta H) \leqslant \int_{\theta H}^{T-\theta H} E(t)dt. \tag{7.3}$$

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Rewriting now the right-hand side in terms of the function φ, we found

$$\int_{\theta H}^{T-\theta H} E(t)dt = \int_0^H \int_{\theta H}^{T-\theta H} \varphi(y,t)dtdy.$$

Since $(\theta H, T-\theta H) \subset (\theta y, T-\theta y)$ for all $0 \leqslant y \leqslant H$, we have

$$\int_{\theta H}^{T-\theta H} E(t)dt \leqslant \int_0^H \Phi(y)dy. \tag{7.4}$$

Combining inequalities (7.2)–(7.4), we find

$$(T-2\theta H)E(T) \leqslant He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}\Phi(0). \tag{7.5}$$

Back again to the energy derivative (3.4), and integrating the equality over $(0,T)$, we obtain

$$E(0) = E(T) + \beta \int_0^T |\partial_t p(H,t)|^2 dt.$$

The last equality and energy estimate (7.5) give

$$E(0) \leqslant (T-2\theta H)^{-1} He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}\Phi(0) + \beta \int_0^T |\partial_t p(H,t)|^2 dt.$$

Substituting $\Phi(0)$ by its expression in (7.1), we finally find

$$E(0) \leqslant \left((T-2\theta H)^{-1} He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H)+\beta^2)+\beta\right) \int_0^T |\partial_t p(H,t)|^2 dt$$
$$+ \lambda_k^2 \int_0^T |p(H,t)|^2 dt,$$

which combined with the fact that

$$E(0) = \int_0^H \left(c^{-2}(y)|f_1(y)|^2 + |f_0'(y)|^2 + \lambda_k^2 |f_0(y)|^2\right) dy,$$

finishes the proof.

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