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Borderline gradient estimates at the boundary in Carnot groups

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In this article, we prove the continuity of the horizontal gradient near a $C^{1,\mathrm{Dini}}$ non-characteristic portion of the boundary for solutions to $\Gamma^{0,\mathrm{Dini}}$ perturbations of horizontal Laplaceans as in (1.1) below, where the scalar term is in scaling critical Lorentz space $L(Q,1)$ with Q being the homogeneous dimension of the group. This result can be thought of both as a sharpening of the $\Gamma^{1,\alpha}$ boundary regularity result in [**4**] as well as a subelliptic analogue of the main result in [**1**] restricted to linear equations.

Keywords: Second-order subelliptic equations; borderline gradient continuity; Dini-domains; Carnot group

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1. Introduction

In this article, we consider the following boundary value problem:

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m} X_i^\star(a_{ij}X_j u) = \sum_{i=1}^{m} X_i^\star f_i + g \quad \text{in } \Omega \subset \mathbb{G}, \\ u = h \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\mathbb{A} = [a_{ij}]$ is an $m \times m$ real symmetric matrix satisfying the following ellipticity condition:

$$\lambda \mathbb{I}_m \leqslant \mathbb{A}(p) \leqslant \lambda^{-1}\mathbb{I}_m, \quad p \in \mathbb{G} \tag{1.2}$$

for some $\lambda > 0$. In (1.2), $\mathbb{I}_m$ stands for $m \times m$ identity matrix and $\mathbb{G}$ is a Carnot group of step k (see definition 2.1). The main importance of such Lie groups in the analysis of the hypoelliptic operators was established in the work of Rothschild and Stein on the so-called *lifting theorem*, see [**36, 38**]. The motive of this article is to obtain the pointwise gradient estimate for weak solutions to (1.1) upto the non-characteristic portion of the boundary under minimal regularity assumptions on $[a_{ij}]$, f_i, g, h and the boundary $\partial\Omega$.

The fundamental role of such borderline regularity results in the context of elliptic and parabolic equations is well known. By using the well-established theory of singular integral in the setting of Heisenberg group, interior Schauder estimates have been studied by many authors in [**11**, **12**, **22**, **31**, **37**, **41**, **42**] and the reference therein. They play an important role in the analysis of nonlinear PDE's.

In 1981, D. Jerison [**24**, **25**] addressed the question of Schauder estimate at the boundary for the horizontal Laplacian in the Heisenberg group $\mathbb{H}^n$. Jerison divided his analysis in two parts, according to whether or not the relevant portion of the boundary contains the so-called *characteristic points* (see definition 2.7). At such points, the vector fields that form the relevant differential operator become tangent to the boundary and thus one should expect a sudden loss of differentiability, somewhat akin to what happens in the classical setting with oblique derivative problems. In fact, Jerison proved that there exist no Schauder boundary estimates at characteristic points! He did so by constructing a domain in $\mathbb{H}^n$ with real-analytic boundary that support solutions of the horizontal Laplacian $\Delta_{\mathscr{H}} u = 0$ which vanish near a characteristic boundary point, and which near such point possess no better regularity than Hölder's. On the other hand, he established Schauder estimates at the non-characteristic portion of the boundary.

Very recently in [**3**], by suitably adapting the Levi's method of parametrix, A. Baldi, G. Citti and G. Cupini established $\Gamma^{2,\alpha}$ type Schauder estimate for non-divergence form operators upto the non-characteristic portion of a C^∞ boundary in more general Carnot groups. Subsequently in [**4**], by employing an alternate approach based on geometric compactness arguments, the authors showed the validity of $\Gamma^{1,\alpha}$ boundary Schauder estimate for divergence form operators as in (1.1) when boundary is $C^{1,\alpha}$ regular and $a_{ij}, f_i \in \Gamma^{0,\alpha}, h \in \Gamma^{1,\alpha}, g \in L^\infty$. The compactness argument first of all appears in the seminal work of Caffarelli [**9**] and is independent of the method of parametrix. In this article, we consider a similar framework as in [**4**] and prove the continuity of the horizontal gradient under weaker assumptions on the coefficients, domain and the scalar term g belongs to the scaling critical Lorentz space $L(Q,1)$, with Q being the homogeneous dimension of the Carnot group $\mathbb{G}$.

For historical note, E. Stein [**39**] showed the following limiting case of Sobolev embedding theorem.

THEOREM 1.1. *Let $L(n,1)$ denote the standard Lorentz space, then the following holds:*

$$\nabla v \in L(n,1) \implies v \textit{ is continuous.}$$

For the definition of Lorentz space, see definition 2.6. One of the interesting properties of the Lorentz space is the following:

$$L^{n+\epsilon} \subset L(n,1) \subset L^n$$

for any $\epsilon > 0$ with all the inclusions being strict. One can think theorem 1.1 as the limiting case of Sobolev-Morrey embedding theory which says that

$$\nabla v \in L^{n+\epsilon} \implies v \in C^{0,\epsilon/n+\epsilon}.$$

Now, by combining theorem 1.1 with the classical Calderon-Zygmund theory, we get the following interesting result.

THEOREM 1.2. $\Delta u \in L(n,1) \implies \nabla u$ *is continuous.*

In 2013, Kuusi and Mingione in [**27**], made a break through by generalizing theorem 1.2 for operators modelled after the p-Laplacian operator. Later on it has been generalised in the setting of more general nonlinear and possibly degenerate elliptic and parabolic equations by using complicated and powerful nonlinear potential theory (see for instance [**18**, **26**–**30**] and the references therein). Recently, theorem 1.2 has also been extended in the context of fully nonlinear elliptic equations. For instance, see [**16**, theorem 1.2] where authors have established the gradient potential estimate for fully nonlinear elliptic equations. We also refer to [**1**] for the boundary analogue of the regularity result in [**16**] and also to the more recent work [**5**] for similar borderline regularity results in the context of normalized p-Laplacian.

The main idea in order to establish such end point gradient continuity estimates is to employ the modified Riesz potential defined as follows:

$$\tilde{I}^g_q(p,R) = \int_0^R \left(\frac{1}{|\Omega \cap B(p,\tau)|} \int_{\Omega \cap B(p,\tau)} |g(x)|^q dx \right)^{1/q} d\tau, \tag{1.3}$$

where $B(p,\tau)$ is defined as in (2.12) below. In fact, one estimates the L^∞ norm of the gradient as well as a certain moduli of continuity estimate in terms of such modified Riesz potential. Then the continuity of the gradient follows from the fact that

$$\tilde{I}^g_q(p,R) \to 0 \text{ as } R \to 0 \tag{1.4}$$

provided $g \in L(Q,1)$ and $q < Q$, for the details, see [**16**, theorem 1.3]. We will follow a similar approach to prove our main result theorem 1.3. α-decreasing (see definition 2.11) property of the modulus of continuity will play an important role in our arguments.

Taking these considerations into account, we initiate the study of the regularity property of the solution of (1.1). In order to state the main theorem, we introduce a few relevant notations. Given an open set $\Omega \subset \mathbb{G}$, $p_0 \in \partial\Omega$ and $\tau > 0$, we set

$$\mathscr{W}_\tau = \Omega \cap B(p_0,\tau), \quad \mathscr{S}_\tau = \partial\Omega \cap B(p_0,\tau). \tag{1.5}$$

Here, $\Omega \subset \mathbb{G}$ is a $C^{1,\mathrm{Dini}}$ domain, see definition 2.13 and we also consider the data in the class $\Gamma^{k,\mathrm{Dini}}$ for $k \in \mathbb{N} \cup \{0\}$, see definition 2.12. We now state our main theorem.

THEOREM 1.3. *Let* $\Omega \subset \mathbb{G}$ *be of class* $C^{1,Dini}$ *and* $p_0 \in \partial\Omega$ *be such that for some* $\tau > 0$ *we have that the set* $\mathscr{S}_\tau$ *consists only of non-characteristic points . Let* $u \in$

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$\mathscr{L}^{1,2}_{loc}(\mathscr{W}_\tau) \cap C(\overline{\mathscr{W}_\tau})$ be a weak solution to (1.1), with a_{ij}, f_i, g and h satisfying the following hypothesis:

$$a_{ij} \in \Gamma^{0,Dini}(\overline{\mathscr{W}_\tau}), \quad f_i \in \Gamma^{0,Dini}(\overline{\mathscr{W}_\tau}), \quad g \in L(Q,1), \quad h \in \Gamma^{1,Dini}(\overline{\mathscr{W}_\tau}). \tag{1.6}$$

Furthermore, we also assume that the uniform ellipticity condition as in (1.2) holds. Then $\nabla_{\mathscr{H}} u$ is continuous in $\overline{\mathscr{W}_{\tau/2}}$. In particular, for any $p, q \in \overline{\mathscr{W}_{\tau/2}}$, there exists an universal constant C_0 such that the following estimate holds:

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(q)| \leqslant C_1 W(C_0 d(p,q)), \tag{1.7}$$

where $C_1 = C_1(\mathbb{G}, \lambda, [a_{ij}]_{\Gamma^{0,Dini}}, \Omega) > 0$, $d(\cdot,\cdot)$ is defined by (2.11), $\nabla_{\mathscr{H}} u$ stands for the horizontal gradient of u and W is a modulus of continuity given by (3.84) which depends on the Dini modulus of $(f_1, f_2, \cdots, f_m), h$, $\partial\Omega$ and $L(Q,1)$ character of g.

Our proof consists of five main steps. Though the idea of proof of our main theorem 1.3 is motivated by the work of Agnid et. al in [4], but due to the lack of the enough regularity on the data and boundary we obtain abstract modulus of continuity of the horizontal gradient instead of the Hölder modulus of continuity. The presence of the abstract modulus of continuity poses additional difficulty in the proof. For instance, one can see steps 3, 4 and 5 in the proof of theorem 1.3. In step 3, we prove the existence of Taylor polynomial at non-characteristic portion of the boundary, which follows from the mathematical induction in combination with compactness lemma 3.2. In order to apply the compactness lemma, we define a new rescaled function by (3.44) which contains the modulus of continuity ω. So in order to satisfy all the assumptions in the compactness lemma, we need many properties of the modulus of continuity, which are given in step 2 of the proof of theorem 1.3. Similarly, in the proof of continuity of the horizontal gradient on the non-characteristic portion of the boundary (see step 4 in the proof of theorem 1.3) and up to the boundary (see step 5 in the proof of theorem 1.3) we need a suitable scaling invariant version of the interior estimate, see corollary 3.5. This estimate is a suitable adaptation of corollary 3.2 in [4] in our setup. In step 5, we patch up the interior and boundary estimate to get the continuity of the horizontal gradient up to the boundary. In the process of patching, we crucially use α-decreasing property of the modulus of continuity.

The article is organized as follows. Section 2 consists of some basic definitions concerning Carnot groups. We also collect some known regularity results that will be used in the proof of theorem 1.3. Section 3 is devoted to the proof of our main result theorem 1.3.

2. Basic definitions and results

Before we proceed with the proof of our main theorem, we need to state some of the basic definitions concerning Carnot groups, modulus of continuity of functions, etc., and some of its properties that will be used throughout the article. In the last part of this section, some known regularity results also have been presented which will be needed in the proof of theorem 1.3. Most of the definitions related to Carnot groups, we refer [4] for the details. Let us start by defining Carnot groups.

DEFINITION 2.1. Given $k \in \mathbb{N}$, a Carnot group of step k is a simply-connected real Lie group $(\mathbb{G}, \circ)$ whose Lie algebra $\mathfrak{g}$ is stratified and k-nilpotent, that is, there exist vector spaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ such that the following holds:

1. $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$;
2. $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$, $j = 1, \ldots, k-1$, $[\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$.

First, we assume that a scalar product $\langle \cdot, \cdot \rangle$ is given on $\mathfrak{g}$ for which the $\mathfrak{g}_j'$s are mutually orthogonal. We let $m_j = \dim \mathfrak{g}_j$, $j = 1, \ldots, k$, and denote by $N = m_1 + \ldots + m_k$ the topological dimension of $\mathbb{G}$. For simplicity in the notation from here onwards we will write m for m_1. Since $\mathbb{G}$ is simply-connected, the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is a global analytic diffeomorphism onto, see for instance [**13**, **40**]. We will use this global chart to identify the point $p = \exp \xi \in \mathbb{G}$ with its logarithmic preimage $\xi \in \mathfrak{g}$.

Now, we will define the *translations* and *dilations* available in Carnot groups. First, we define the left and right translations in $\mathbb{G}$ by an element $p' \in \mathbb{G}$ with the help of the group law $\circ$ as follows:

$$L_{p'}(p) = p' \circ p, \quad R_{p'}(p) = p \circ p'. \tag{2.1}$$

Given a function $f : \mathbb{G} \to \mathbb{R}$, the action of $L_{p'}$ and $R_{p'}$ on f is defined by:

$$L_{p'} f(p) = f(L_{p'}(p)), \quad R_{p'} f(p) = f(R_{p'}(p)), \quad p \in \mathbb{G}.$$

In order to define the dilations in a Carnot group $\mathbb{G}$, we first assign the formal degree j to the j-th layer $\mathfrak{g}_j$ of the Lie algebra. Then a family of non-isotropic dilations $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is given by

$$\Delta_\lambda \xi = \lambda \xi_1 + \cdots + \lambda^k \xi_k, \tag{2.2}$$

where $\xi = \xi_1 + \cdots + \xi_k \in \mathfrak{g}$, with $\xi_j \in \mathfrak{g}_j$, $j = 1, \ldots, k$. We then use the exponential mapping to lift (2.2) to a one-parameter family of dilations $\{\delta_\lambda\}_{\lambda > 0}$ in the group $\mathbb{G}$. The dilations $\{\delta_\lambda\}_{\lambda > 0}$ in the group $\mathbb{G}$ is given by

$$\delta_\lambda(p) = \exp \circ \Delta_\lambda \circ \exp^{-1}(p), \quad p \in \mathbb{G}. \tag{2.3}$$

The *homogeneous dimension* of $\mathbb{G}$ with respect to the dilations (2.3) is given by

$$Q = \sum_{j=1}^{k} j \dim \mathfrak{g}_j.$$

Let us introduce analytic maps $\xi_j : \mathbb{G} \to \mathfrak{g}_j$, $j = 1, \ldots, k$, by $p = \exp(\xi_1(p) + \ldots + \xi_k(p))$. For $p \in \mathbb{G}$, the projection of the *logarithmic coordinates* of p onto the layer $\mathfrak{g}_j$, $j = 1, \ldots, k$, is defined by

$$x_{j,s}(p) = \langle \xi_j(p), e_{j,s} \rangle, \quad s = 1, \ldots, m_j, \tag{2.4}$$

where $(x_1(p), \ldots, x_m(p)) = (x_{1,1}(p), \ldots, x_{1,m}(p))$ are the horizontal coordinates of p and the sets $\{e_{j,1}, \ldots, e_{j,m_j}\}$, $j = 1, \ldots, k$, are a fixed orthonormal basis of the

j-th layer $\mathfrak{g}_j$ of the Lie algebra $\mathfrak{g}$. Sometimes, we will omit the dependence in p, and identify p with its logarithmic coordinates

$$p \cong (x_1, \ldots, x_m, x_{2,1}, \ldots, x_{2,m_2}, \ldots, x_{k,1}, \ldots, x_{k,m_k}). \tag{2.5}$$

In order to simplify the notation, let us set

$$\xi_1 = (x_1, \ldots, x_m),\ \xi_2 = (x_{2,1}, \ldots, x_{2,m_2}), \ldots, \xi_k = (x_{k,1}, \ldots, x_{k,m_k}). \tag{2.6}$$

Furthermore, we write $x = x(p) \cong \xi_1 = (x_1, \ldots, x_m)$, and $y = y(p)$ the $(N-m)-$dimensional vector

$$y \cong (\xi_2, \ldots, \xi_k) = (x_{2,1}, \ldots, x_{2,m_2}, \ldots, x_{k,1}, \ldots, x_{k,m_k}).$$

In this case, we will write $z = (x, y)$, see [20]. For every $j = 1, \ldots, k$ we also use the following multi-index notation $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,m_j}) \in (\mathbb{N} \cup \{0\})^{m_j}$.

In this article, we assume that $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $\mathfrak{g}_1$. The family of left-invariant vector fields $\{X_1, \ldots, X_m\}$ on $\mathbb{G}$ is given by

$$X_j(p) = dL_p(e_j), \quad j = 1, \ldots, m, \quad p \in \mathbb{G},$$

where dL_p denotes the differential of L_p. Note that, the vector fields $\{X_1, \ldots, X_m\}$ form a basis for the so-called horizontal sub-bundle $\mathscr{H}$ of the tangent bundle $T\mathbb{G}$. Given a point $p \in \mathbb{G}$, the fibre of $\mathscr{H}$ at p is given by

$$\mathscr{H}_p = d\mathcal{L}_p(\mathfrak{g}_1). \tag{2.7}$$

DEFINITION 2.2 Horizontal Laplacean. The *horizontal Laplacean* associated with an orthonormal basis $\{e_1, \ldots, e_m\}$ of the horizontal layer $\mathfrak{g}_1$ is the left-invariant second-order partial differential operator in $\mathbb{G}$ defined by

$$\Delta_{\mathscr{H}} = -\sum_{j=1}^{m} X_j^{\star} X_j = \sum_{j=1}^{m} X_j^2, \tag{2.8}$$

where $\{X_1, \ldots, X_m\}$ are left-invariant vector fields on $\mathbb{G}$ and the formal adjoint of X_j in $L^2(\mathbb{G})$ is given by $X_j^{\star} = -X_j$.

2.1. Gauge pseudo-distance

In a Carnot group there exists a left-invariant distance $d_C(p, p_0)$ associated with the horizontal subbundle $\mathscr{H}$, see for instance [6, 35] and chapter 4 in [21]. A piecewise $\mathcal{C}^1$ curve $\alpha : [0, T] \to \mathbb{G}$ is called *horizontal* if there exist piecewise continuous functions $b_i : [0, T] \to \mathbb{G}$ with $\sum_{i=1}^{m} |b_i| \leqslant 1$ such that

$$\alpha'(t) = \sum_{i=1}^{m} b_i(t) X_i(\alpha(t)).$$

We define the *horizontal length* of α as $\ell_{\mathscr{H}}(\alpha) = T$ and the metric

$$d_C(p, p_0) = \inf_{\alpha \in \Gamma(p, p_0)} \ell_{\mathscr{H}}(\alpha),\ p, p_0 \in \mathbb{G}$$

where $\Gamma(p, p_0)$ is the collection of all horizontal curves $\alpha : [0, T] \to \mathbb{G}$ such that $\alpha(0) = p$ and $\alpha(T) = p_0$. The metric $d_C(p, p_0)$ is called the *Carnot-Carathéodory*

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distance. By Chow's theorem [7], any two points can be connected by a horizontal curve, which makes d_C a metric on $\mathbb{G}$.

The Carnot-Carathéodory metric $d_C(p,p')$ is equivalent to a more explicitly defined pseudo-distance function, called the *gauge pseudo-distance*, defined as follows. Let $||\cdot||$ denote the Euclidean distance to the origin in $\mathfrak{g}$. For $\xi = \xi_1 + \cdots + \xi_k \in \mathfrak{g}$, $\xi_j \in \mathfrak{g}_j$, $j = 1, \ldots, k$, we define

$$|\xi|_{\mathfrak{g}} = \left(\sum_{j=1}^{k} ||\xi_j||^{\frac{2k!}{j}}\right)^{2k!}, \quad |p|_{\mathbb{G}} = |\exp^{-1} p|_{\mathfrak{g}} \quad p \in \mathbb{G}. \tag{2.9}$$

The function $p \to |p|_{\mathbb{G}}$ is called the non-isotropic group gauge and satisfies for any $\lambda > 0$

$$|\delta_\lambda(p)| = \lambda |p|, \tag{2.10}$$

where dilations $\{\delta_\lambda\}_{\lambda>0}$ are group automorphisms (see [4]) and $|p| = |p|_{\mathbb{G}}$. The *gauge pseudo-distance* in $\mathbb{G}$ is defined by

$$d(p, p_0) = |p^{-1} \circ p_0|. \tag{2.11}$$

Now, we define the metric and the gauge pseudo ball centred at p with radius R

$$B_C(p,R) = \{p_0 \in \mathbb{G} \mid d_C(p_0,p) < R\}, \quad B(p,R) = \{p_0 \in \mathbb{G} \mid d(p_0,p) < R\}, \tag{2.12}$$

respectively. When the centre is the group identity e, we will write $B_C(R)$ and $B(R)$ instead of $B_C(e,R)$ and $B(e,R)$. Note that if $\mathcal{L}$ is Lebesgue measure on $\mathfrak{g}$, then $\mathcal{L} \circ \exp^{-1}$ is a bi-invariant Haar measure on $\mathbb{G}$. Now, we denote $|E| = \int_E dp$ the Haar measure of a set $E \subset \mathbb{G}$. Observe that $\omega_C = \omega_C(\mathbb{G}) = |B_C(1)| > 0$ and $\omega = \omega(\mathbb{G}) = |B(1)| > 0$, and hence for every $p \in \mathbb{G}$ and $R > 0$,

$$|B_C(p,R)| = \omega_C R^Q, \quad |B(p,R)| = \omega R^Q. \tag{2.13}$$

LEMMA 2.3 [35]. *For every connected $\Omega \subset\subset \mathbb{G}$ there exist $C,\ \epsilon > 0$ such that*

$$C d_{\mathscr{R}}(p,p_0) \leqslant d_C(p,p_0) \leqslant C^{-1} d_{\mathscr{R}}(p,p_0)^{\epsilon}, \tag{2.14}$$

where $d_{\mathscr{R}}(x,y)$ is the left-invariant Riemannian distance in $\mathbb{G}$ and $p, p_0 \in \Omega$.

2.2. The Folland-Stein Hölder classes

Now, we recall the intrinsic Hölder classes $\Gamma^{\kappa,\alpha}$ introduced by Folland and Stein in [19] and especially [20], see also chapter 20 in [8].

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Definition 2.4. Let $0 < \alpha \leqslant 1$. Given an open set $\Omega \subset \mathbb{G}$ we say that $u : \Omega \to \mathbb{R}$ belongs to $\Gamma^{0,\alpha}(\Omega)$ if there exists a positive constant M such that for every $p, p_0 \in \Omega$,

$$|u(p) - u(p_0)| \leqslant M\ d(p, p_0)^{\alpha}.$$

We define the semi-norm

$$[u]_{\Gamma^{0,\alpha}(\Omega)} = \sup_{\substack{p,p_0 \in \Omega \\ p \neq p_0}} \frac{|u(p) - u(p_0)|}{d(p, p_0)^{\alpha}}. \tag{2.15}$$

Given $\kappa \in \mathbb{N}$, the spaces $\Gamma^{\kappa,\alpha}(\Omega)$ are defined inductively: we say that $u \in \Gamma^{\kappa,\alpha}(\Omega)$ if $X_i u \in \Gamma^{\kappa-1,\alpha}(\Omega)$ for every $i = 1, .., m$.

Note that for any $\lambda > 0$, $[\delta_\lambda u]_{\Gamma^{0,\alpha}(\delta_{\lambda^{-1}}(\Omega))} = \lambda^{\alpha}[u]_{\Gamma^{0,\alpha}(\Omega)}$, where dilations $\{\delta_\lambda\}_{\lambda>0}$ are group automorphisms, see [4] for more details.

Definition 2.5 Sobolev space. For an open set $\Omega \subset \mathbb{G}$ we denote by $\mathscr{L}^{1,p}(\Omega)$, where $1 \leqslant p \leqslant \infty$, the Sobolev space $\{f \in L^p(\Omega) \mid X_j f \in L^p(\Omega), j = 1, \ldots, m\}$ endowed with its natural norm

$$\|f\|_{\mathscr{L}^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{j=1}^{m} \|X_j f\|_{L^p(\Omega)}.$$

The local space $\mathscr{L}^{1,p}_{loc}(\Omega)$ has the usual meaning. We also denote by $\mathscr{L}^{1,p}_0(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{\mathscr{L}^{1,p}(\Omega)}}$. Let λ denote the distribution function of f defined on $\mathbb{G}$, then the non-increasing rearrangement $f^\star$ is defined for $t > 0$ by letting

$$f^\star(t) = \inf\{s > 0 : \lambda(s) \leqslant t\}.$$

Definition 2.6 Lorentz spaces [17]. Let Q be strictly positive number such that $Q > 1$. The Lorentz space $L(Q, 1)(\mathbb{G})$ is defined as the set of real valued measurable functions f, defined on $\mathbb{G}$ such that:

$$\|f\|_{L(Q,1)(\mathbb{G})} = \int_0^\infty (f^\star(t)\, t^{1/Q}) \frac{\mathrm{d}t}{t} < \infty.$$

Note that, Carnot group $\mathbb{G}$ endowed with the Carnot gauge $\|x\|_C = d_C(x, 0)$ or with a smooth gauge $x \to |x|_{\mathfrak{g}}$ together with the Lebesgue measure $\mathcal{L}$ forms a real variable rearrangement structure. For more details one refer to [34, theorem 3.1], see also [2].

2.3. The characteristic set

We start with an open set $\Omega \subset \mathbb{G}$ which belongs to a class C^1, that is, for every $p_0 \in \partial\Omega$ there exist a neighbourhood U_{p_0} of p_0, and a function $\varphi_{p_0} \in C^1(U_{p_0})$ with

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$|\nabla\varphi_{p_0}| \geqslant \alpha > 0$ in U_{p_0}, such that

$$\Omega \cap U_{p_0} = \{p \in U_{p_0} \mid \varphi_{p_0}(p) < 0\}, \quad \partial\Omega \cap U_{p_0} = \{p \in U_{p_0} \mid \varphi_{p_0}(p) = 0\}. \tag{2.16}$$

At every point $p \in \partial\Omega \cap U_{p_0}$ the outer unit normal is given by

$$\nu(p) = \frac{\nabla\varphi_{p_0}(p)}{|\nabla\varphi_{p_0}(p)|},$$

where ∇ denotes the Riemannian gradient.

Definition 2.7. Let $\Omega \subset \mathbb{G}$ be an open set of class C^1. A point $p_0 \in \partial\Omega$ is called *characteristic* if

$$\nu(p_0) \perp \mathscr{H}_{p_0}, \tag{2.17}$$

where $\mathscr{H}_{p_0}$ is as in (2.7). The *characteristic set* $\Sigma = \Sigma_\Omega$ is the collection of all characteristic points of Ω. A boundary point $p_0 \in \partial\Omega \setminus \Sigma$ is called *non-characteristic* boundary point. For more details, we refer to [**10**].

2.4. Modulus of continuity and its properties

Definition 2.8. A function $\Phi(s)$ for $0 \leqslant s \leqslant R_0$ is called a modulus of continuity if the following properties are satisfied:

1. $\Phi(s) \to 0 \quad \text{as} \quad s \to 0$.
2. $\Phi(s)$ is positive and increasing as a function of s.
3. Φ is sub-additive, i.e. $\Phi(s_1 + s_2) \leqslant \Phi(s_1) + \Phi(s_2)$
4. Φ is continuous.

Let us define the notion of Dini-continuity.

Definition 2.9. Suppose that $\Omega \subset \mathbb{G}$ and $f : \Omega \longrightarrow \mathbb{R}$ is a given function. Then we define the modulus of continuity of f as follows:

$$\omega_f(s) = \sup_{d(p,\overline{p}) \leqslant s} |f(p) - f(\overline{p})|. \tag{2.18}$$

We say that the function f is Dini-continuous if

$$\int_0^1 \frac{\omega_f(s)}{s} ds < \infty. \tag{2.19}$$

Notice that for a continuous function f, ω_f satisfies all properties (1)–(4) mentioned in definition 2.8. Similarly, for a vector valued function $(f_1, f_2, \cdots, f_m) : \Omega \longrightarrow \mathbb{R}^m$ we define the modulus of continuity as follows:

$$\omega_f(s) = \sup_{d(p,\overline{p}) \leqslant s} |f(p) - f(\overline{p})|. \tag{2.20}$$

So, as above the function $(f_1, f_2, \cdots, f_m)$ is called Dini-continuous if (2.19) holds. From [**33**, Page 44], we see that any continuous, increasing function $\Phi(s)$ on the interval $[0, R_0]$ which satisfies $\Phi(0) = 0$ is modulus of continuity if it is concave. From this, we have the following important result proved in [**33**, theorem 8]:

THEOREM 2.10. *For each modulus of continuity $\Psi(s)$ on $[0, R_0]$, there is a concave modulus of continuity $\tilde{\Psi}(s)$ with the property*

$$\Psi(s) \leqslant \tilde{\Psi}(s) \leqslant 2\Psi(s) \quad \textit{for all} \quad s \in [0, R_0]. \tag{2.21}$$

DEFINITION 2.11. Given $\alpha > 0$, we say that the modulus of continuity Ψ is α-decreasing if for any $t_1, t_2 \in (0, R_0]$ satisfying $t_1 \leqslant t_2$, we have

$$\frac{\Psi(t_1)}{t_1^\alpha} \geqslant \frac{\Psi(t_2)}{t_2^\alpha}.$$

DEFINITION 2.12. Given an open set $\Omega \subset \mathbb{G}$ we say that $u : \Omega \to \mathbb{R}$ belongs to $\Gamma^{0,\mathrm{Dini}}(\Omega)$ if there exists a positive constant M such that for every $p, p_0 \in \Omega$,

$$|u(p) - u(p_0)| \leqslant M\, \omega_u(d(p, p_0)),$$

where ω_u satisfies the Dini-integrability condition in (2.19). Correspondingly, we define the semi-norm in the following way:

$$[u]_{\Gamma^{0,\mathrm{Dini}}(\Omega)} = \sup_{\substack{p,p_0\in\Omega \\ p\neq p_0}} \frac{|u(p) - u(p_0)|}{\omega_u(d(p, p_0))}.$$

Furthermore, such a space is equipped with the following norm:

$$\|u\|_{\Gamma^{0,\mathrm{Dini}}(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{\Gamma^{0,\mathrm{Dini}}(\Omega)}.$$

Given $\kappa \in \mathbb{N}$, the spaces $\Gamma^{\kappa,\mathrm{Dini}}(\Omega)$ are defined inductively: we say that $u \in \Gamma^{\kappa,\mathrm{Dini}}(\Omega)$ if $X_i u \in \Gamma^{\kappa-1,\mathrm{Dini}}(\Omega)$ for every $i = 1, \ldots, m$.

DEFINITION 2.13. We start with an open set $\Omega \subset \mathbb{G}$ which belongs to a class $C^{1,\mathrm{Dini}}$, that is, after translation, rotation and scaling, we may assume that $p_0 = e \in \partial\Omega$ and in the logarithmic coordinates $\mathscr{W}_\tau = \Omega \cap B(e, \tau)$, $\tau > 0$ is given by

$$\mathscr{W}_\tau := \{(x, y) \in \mathbb{R}^N \mid x_m > \psi(x', y)\},$$

where $\psi(0, 0) = 0$, $\nabla_{x'}\psi(0, 0) = 0$, $x' = (x_1, \ldots, x_{m-1})$ and $\psi \in C^{1,\mathrm{Dini}}$. In particular, $\nabla\psi$ belongs to $C^{0,\mathrm{Dini}}$, where the Banach space $C^{0,\mathrm{Dini}}(D)$ is the set of all bounded and continuous vector valued function f on $D \subset \mathbb{R}^{N-1}$ for which

$$[f]_{\omega_f;D} := \sup_{z,\bar{z}\in D, z\neq\bar{z}} \frac{|f(z) - f(\bar{z})|}{\omega_f(|z - \bar{z}|)} < \infty,$$

where $f = (f_1, \ldots, f_{\tilde{m}})$ and ω_f satisfies the Dini-integrability condition in (2.19). It is equipped with the norm

$$\|f\|_{C^{0,\mathrm{Dini}}(D)} = \|f\|_{C^0(D)} + [f]_{\omega_f;D}.$$

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2.5. Some known results

In this subsection, we recall the following smoothness result at the non-characteristic portion of the boundary, see theorem 3.5 [4].

Theorem 2.14. *Let $\mathbb{A} = [a_{ij}]$ be a symmetric constant-coefficient matrix. Assume that Ω is a C^∞ domain, and let $u \in \mathscr{L}^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ be a weak solution of (1.1) with $f_i, g \equiv 0$. Let $p_0 \in \partial\Omega$ be a non-characteristic point and assume that for some neighbourhood $W = B_{\mathscr{R}}(p_0, r_0)$ of p_0, we have that $u \equiv 0$ in $\partial\Omega \cap W$. Then there exists an open neighbourhood V of p_0 depending on W and Ω and a positive constant $C^\star = C^\star(M, p_0) > 0$, depending on p_0 and $M = \sup_\Omega |u|$, such that*

$$\|u\|_{C^2(\overline{\Omega}\cap V)} \leqslant C^\star. \tag{2.22}$$

Next, we state a Hölder continuity result near a $C^{1,\mathrm{Dini}}$ non-characteristic portion of the boundary that is direct consequence of the results in [14].

Proposition 2.15. *Let $\Omega \subset \mathbb{G}$ be a $C^{1,Dini}$ domain such that $p_0 \in \partial\Omega$ is a non-characteristic point. Suppose $u \in \mathscr{L}^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ is a weak solution of*

$$\begin{cases} \sum_{i,j=1}^m X_i^\star(a_{ij}X_j u) = \sum_{i=1}^m X_i^\star f_i + g, \\ u = h \text{ on } \partial\Omega, \end{cases} \tag{2.23}$$

where $\mathbb{A} = [a_{ij}]$ is a symmetric matrix satisfying (1.2), for all $p \in \Omega$. Furthermore, assume that $f^i \in L^\infty(\Omega)$, $g \in L^q(\Omega)$, $Q < 2q < 2Q$ and $h \in \Gamma^{0,\gamma}(\partial\Omega)$ for some $\gamma > 0$. Then, there exist $r_0, C > 0$ and $\beta \in (0,1)$, depending on Ω, λ, γ and $M \overset{def}{=} \sup_\Omega |u| < \infty$, such that

$$\sup_{\substack{p,p' \in \overline{\Omega\cap B(p_0,r)} \\ p\neq p'}} \frac{|u(p) - u(p')|}{d(p,p')^\beta} \leqslant C. \tag{2.24}$$

3. Proof of main result

In this section, we will prove our main result, theorem 1.3. Given a bounded open set $\Omega \subset \mathbb{G}$, with $p_0 \in \partial\Omega$ we will use the notations $\mathscr{W}_\tau$ and $\mathscr{S}_\tau$ as in (1.5). The proof of the theorem 1.3 follows in several steps. The first step is to establish the compactness lemma. In the proof of the compactness lemma we need the following Caccioppoli type inequality. This type of inequality has different applications in the PDE's. So, we are presenting it as an independent result.

Lemma 3.1. *Suppose that (1.2) holds. Let $u \in \mathscr{L}^{1,2}_{loc}(\mathscr{W}_1) \cap C(\overline{\mathscr{W}_1})$ be a weak solution to (1.1) in $\mathscr{W}_1$ with $\|u\|_{L^\infty(\mathscr{W}_1)} \leqslant 1$. Furthermore, assume that $f^i \in L^\infty(\Omega)$, $g \in L^q(\Omega)$, $Q < 2q$ and there is an $R > 0$ such that $B(p, 2R) \subset \mathscr{W}_1$, then the following*

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estimate holds:

$$\int_{B(p,R)} |\nabla_{\mathscr{H}} u|^2 \leqslant C \left[\sum_{i=1}^{m} \|f^i\|_{L^\infty(B(p,2R))} + \|g\|_{L^q(B(p,2R))} \right], \tag{3.1}$$

for some universal $C(Q, \lambda)$.

Proof. Let ϕ be a smooth cut-off function such that $\phi \equiv 1$ in $B(p,R)$ and vanishes outside $B(p,2R)$. Now, by taking $\eta = \phi^2 u$ as a test function in the weak formulation, we obtain the following equality:

$$\begin{aligned} \int_{B(p,2R)} \phi^2 \langle \mathbb{A} \nabla_{\mathscr{H}} u, \nabla_{\mathscr{H}} u \rangle = &\int_{B(p,2R)} \phi^2 \langle f, \nabla_{\mathscr{H}} u \rangle + 2 \int_{B(p,2R)} \phi u \langle f, \nabla_{\mathscr{H}} \phi \rangle \\ &- \int_{B(p,2R)} g \phi^2 u - 2 \int_{B(p,2R)} \phi u \langle \mathbb{A} \nabla_{\mathscr{H}} u, \nabla_{\mathscr{H}} \phi \rangle, \end{aligned}$$

where $f = (f_1, \ldots, f_m)$. Now, by applying Cauchy Schwartz inequality and the fact that $\|u\|_{L^\infty(\mathscr{W}_1)} \leqslant 1$, we obtain

$$\begin{aligned} \lambda \int_{B(p,2R)} \phi^2 |\nabla_{\mathscr{H}} u|^2 \leqslant C \Bigg[&\sum_i \|f_i\|_{L^\infty(B(p,2R))} \|\phi\|^2_{L^2(B(p,2R))} \\ &+ \frac{\lambda}{2} \int_{B(p,2R)} \phi^2 |\nabla_{\mathscr{H}} u|^2 + \|\nabla_{\mathscr{H}} \phi\|_{L^2(B(p,2R))} \\ &+ \|g\|_{L^q(B(p,2R))} \|\phi\|^2_{L^{2q/(q-1)}(B(p,2R))} \Bigg]. \end{aligned} \tag{3.2}$$

By subtracting off the second integral in the right hand side of (3.2) from the left hand side in (3.2), we obtain that the desired conclusion follows by using bounds on ϕ and the fact that $\phi \equiv 1$ in $B(p,R)$. □

3.1. Compactness lemma

Now, we are ready to prove the compactness lemma 3.2. This lemma states that if the coefficient matrix $[a_{ij}]$ in (1.1) is very close to the constant matrix in certain norm and the other data are sufficiently small then the solutions of (1.1) can be approximated by a sufficiently smooth functions, in fact by the solutions of uniformly elliptic equation with constant coefficient.

LEMMA 3.2. *Suppose that* (1.2) *holds. Assume that for a given* $p_0 = e \in \partial\Omega$ *the set* $\mathscr{S}_1$ *is non-characteristic, and that in the logarithmic coordinates* $\mathscr{W}_1$ *is given by* $\{(x,y) \mid x_m > \psi(x',y)\}$, *where* $\psi \in C^{1,Dini}$, *and* $x' = (x_1, \ldots, x_{m-1})$. *Let* $u \in \mathscr{L}^{1,2}_{loc}(\mathscr{W}_1) \cap C(\overline{\mathscr{W}_1})$ *be a weak solution to* (1.1) *in* $\mathscr{W}_1$ *with* $\|u\|_{L^\infty(\mathscr{W}_1)} \leqslant 1$. *Then,* *for a given* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that if*

$$\|\psi\|_{C^{1,Dini}} \leqslant \delta,\ ||a_{ij} - a^0_{ij}||_{L^\infty(\mathscr{W}_1)} \leqslant \delta,\ ||h||_{\Gamma^{0,\alpha}(\mathscr{S}_1)} \leqslant \delta,$$
$$||f_i||_{L^\infty(\mathscr{W}_1)} \leqslant \delta,\ ||g||_{L^q(\mathscr{W}_1)} \leqslant \delta, \tag{3.3}$$

where $a^0_{ij} = a_{ij}(e)$ *and* q *as in lemma 3.1, we can find* $w \in C^2(\overline{\mathscr{W}_{1/2}})$ *such that*

$$\|u - w\|_{L^\infty(\mathscr{W}_{1/2})} \leqslant \epsilon,$$

with

$$\|w\|_{C^2(\overline{\mathscr{W}_{1/2}})} \leqslant CC^\star.$$

Here, the constant $C > 0$ *is a universal constant, whereas* $C^\star$ *can be taken as that in the estimate* (2.22) *in lemma 2.14, corresponding to* $p_0 = e$ *and* $M = 1$.

Proof. The proof of the lemma follows by the standard contradiction argument as in the work [**9**]. Suppose that there exists an $\epsilon_0 > 0$ such that for every $\nu \in \mathbb{N}$ we can find:

1. a matrix-valued function $\mathbb{A}^\nu = [a^\nu_{ij}]$ with continuous entries in $\mathbb{G}$ and satisfying (1.2),
2. a domain Ω_ν with $\mathscr{W}^\nu_1 = \Omega_\nu \cap B(1)$ and $\mathscr{S}^\nu_1 = \partial\Omega_\nu \cap B(1)$,
3. a solution u_ν to the problem

$$\sum_{i,j=1}^m X_i^\star(a^\nu_{ij} X_j u_\nu) = \sum_{i=1}^m X_i^\star f^\nu_i + g_\nu \quad \text{in } \mathscr{W}^\nu_1, \quad u_\nu = h_\nu \quad \text{on } \mathscr{S}^\nu_1, \tag{3.4}$$

along with

$$\|u_\nu\|_{L^\infty(\mathscr{W}^\nu_1)} \leqslant 1,$$
$$\|\psi_\nu\|_{C^{1,\mathrm{Dini}}} \leqslant \frac{1}{\nu},\ ||a^\nu_{ij} - a^0_{ij}||_{L^\infty(\mathscr{W}^\nu_1)} \leqslant \frac{1}{\nu},\ ||h_\nu||_{\Gamma^{0,\alpha}(\mathscr{S}^\nu_1)} \leqslant \frac{1}{\nu}, \tag{3.5}$$
$$||f^\nu_i||_{L^\infty(\mathscr{W}^\nu_1)} \leqslant \frac{1}{\nu},\ ||g_\nu||_{L^q(\mathscr{W}^\nu_1)} \leqslant \frac{1}{\nu},$$

but for every $w \in C^2(\overline{\mathscr{W}^\nu_{1/2}})$ and $\|w\|_{C^2(\overline{\mathscr{W}^\nu_{1/2}})} \leqslant CC^\star$ we have

$$\|u_\nu - w\|_{L^\infty(\mathscr{W}^\nu_{1/2})} \geqslant \epsilon_0. \tag{3.6}$$

Note that the sets $\mathscr{W}^\nu_1$ above are described in the logarithmic coordinates by the functions $\psi_\nu \in C^{1,\mathrm{Dini}}$, that is, $\{(x, y) \mid x_m > \psi_\nu(x', y)\}$. Now, we will show that the validity of (3.6) leads to a contradiction. We proceed by observing that the uniform bounds in (3.5), combined with proposition 2.15, produces constants $C, \beta > 0$, depending on λ, α, but not on ν, such that

$$\|u_\nu\|_{\Gamma^{0,\beta}(\mathscr{W}^\nu_{4/5})} \leqslant C.$$

Since u_ν's are defined on varying domains $\mathscr{W}^\nu_1$, we need to work with functions defined on the same domain. To do this, we now use an idea similar to that in the proof of [4, lemma 4.1]. Let $\tilde{p} = \Phi_\nu(p)$ be the $C^{1,\mathrm{Dini}}$ local diffeomorphism that straightens the portion $\mathscr{S}_1^\nu$ of $\partial\Omega_\nu$. More precisely, $\mathscr{S}_1^\nu$ can be locally expressed in the logarithmic coordinates as

$$\Phi_\nu(x, y) = (x', x_m - \psi_\nu(x', y), y).$$

Let $v_\nu(\tilde{p}) = u_\nu \circ \Phi_\nu^{-1}(\tilde{p})$ and we denote by $\tilde{p} = (\tilde{x}', \tilde{x}_m, \tilde{y})$ the logarithmic coordinates of $\tilde{p}$. The function v_ν is now defined for $\tilde{x}_m \geqslant 0$. Then by the classical method of extension in terms of reflection, we define the extension of v_ν to the region $\{\tilde{x}_m < 0\}$ as follows:

$$V_\nu(\tilde{x}', \tilde{x}_m, \tilde{y}) = \begin{cases} v_\nu(\tilde{x}', \tilde{x}_m, \tilde{y}) & \tilde{x}_m \geqslant 0, \\ \displaystyle\sum_{i=1}^{3} c_i v_\nu\left(\tilde{x}', -\frac{\tilde{x}_m}{i}, \tilde{y}\right) & \tilde{x}_m < 0, \end{cases} \tag{3.7}$$

where the constants c_1, c_2 and c_3 are determined by the system of equations,

$$\sum_{i=1}^{3} c_i(-1/i)^m = 1, \quad m = 0, 1, 2, \tag{3.8}$$

see e.g. p. 14 in [32]. We now define the extension U_ν of u_ν by setting $U_\nu = V_\nu \circ \Phi_\nu$. It is clear that the following bound holds,

$$\|U_\nu\|_{\Gamma^{0,\beta}(B(\frac{4}{5}))} \leqslant C' \|u_\nu\|_{\Gamma^{0,\beta}(\mathscr{W}_{4/5}^\nu)} \leqslant C_1,$$

for some $C_1 > 0$. As a consequence, we have the following convergence results.

1. By applying Arzela-Ascoli theorem, we obtain a subsequence, that we will still denote by $\{U_\nu\}_{\nu\in\mathbb{N}}$, that converges uniformly to a function $U_0 \in \Gamma^{0,\beta}(B(4/5))$. Clearly, U_0 satisfies

$$U_0(x', x_m, y) = \begin{cases} U_0(x', x_m, y) & x_m \geqslant 0, \\ \displaystyle\sum_{i=1}^{3} c_i U_0(x', -x_m/i, y) & x_m < 0, \end{cases} \tag{3.9}$$

where the constants c_1, c_2 and c_3 are given by the system (3.8).

2. From (3.5), we see that $f_\nu \to 0$ as $\nu \to \infty$.

3. Since by (3.5) we have $||\psi_\nu||_{\Gamma^{0,1}(\mathscr{W}_1^\nu)} \leqslant \frac{1}{\nu}$ for every ν so we get

$$U_0(x', 0, y) = 0. \tag{3.10}$$

Now, we will show that $U_0 \in \mathscr{L}_{loc}^{1,2}(B(4/5) \cap \{x_m > 0\}) \cap C(\overline{B(4/5) \cap \{x_m > 0\}})$. Moreover U_0 is a weak solution to the problem

$$\sum_{i,j=1}^{m} a_{ij}^0 X_i X_j U_0 = 0 \ \text{ in } B(4/5) \cap \{x_m > 0\}, \quad U_0 = 0 \ \text{ on } B(4/5) \cap \{x_m = 0\}. \tag{3.11}$$

To see this, let us observe that $\|\psi_\nu\|_{C^{1,\mathrm{Dini}}} \leqslant 1/\nu \to 0$, so for a given $p \in B(4/5) \cap \{x_m > 0\}$, there exist $\eta > 0$ and $\nu_0(p) \in \mathbb{N}$ such that for all $\nu \geqslant \nu_0(p)$ we have

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$B(p, 2\eta) \subset \mathscr{W}_1^\nu$. By the Caccioppoli inequality (see lemma 3.1 with $R = \eta$) for the problem (3.4) combined with the uniform bounds in (3.5), we find that for all $\nu \geqslant \nu_0(p)$ following inequality holds:

$$\int_{B(p,\eta)} |\nabla_{\mathscr{H}} u_\nu|^2 \leqslant C, \tag{3.12}$$

for some $C(\lambda, \eta) > 0$ independent of ν. Therefore, $\{u_\nu\}_{\nu\in\mathbb{N}}$ has a subsequence, which we still denote by $\{u_\nu\}_{\nu\in\mathbb{N}}$, such that

$$u_\nu \to w \text{ weakly in } \mathscr{L}^{1,2}(B(p,\eta)), \quad \text{and} \quad u_\nu \to w \text{ strongly in } L^2(B(p,\eta)).$$

Since $\{U_\nu\}_{\nu\in\mathbb{N}}$ converges to U_0 uniformly, by uniqueness of limits we can assert that $w = U_0$ in $B(p,\eta)$. Moreover, using the uniform energy estimate for the u_ν's in (3.12) and (3.5) it follows by standard weak type arguments that U_0 is a weak solution to

$$\sum_{i,j=1}^{m} a_{ij}^0 X_i X_j U_0 = 0$$

in $B(p,\eta)$, and hence a classical solution by Hörmander's hypoellipticity theorem in [**23**]. By the arbitrariness of $p \in B(4/5) \cap \{x_m > 0\}$ and (3.10), we conclude that (3.11) holds. We can now make use of the estimate from theorem 2.14 to obtain

$$\|U_0\|_{C^2(\overline{B(1/2)\cap\{x_m>0\}})} \leqslant C^\star$$

for some universal $C^\star > 0$. This follows since $[a_{ij}^0]$ is a constant coefficient matrix, and the portion $B(4/5) \cap \{x_m = 0\}$ of the boundary of $B(4/5) \cap \{x_m > 0\}$ is non-characteristic and C^∞. Now, from the expression of U_0 in (3.9) we see that the second derivatives in x_m are continuous across $x_m = 0$, and thus in fact $U_0 \in C^2(B(1/2))$, and

$$\|U_0\|_{C^2(\overline{\mathscr{W}_{1/2}^\nu})} \leqslant \|U_0\|_{C^2(B(1/2))} \leqslant CC^\star,$$

where $C > 0$ is a universal constant. This shows that $w = U_0$ is an admissible candidate for the estimate (3.6). In particular, we have for $\nu \in \mathbb{N}$

$$0 < \epsilon_0 \leqslant \|u_\nu - U_0\|_{L^\infty(\mathscr{W}_{1/2}^\nu)},$$

which is a contradiction for large enough ν's, since $u_\nu \to U_0$ uniformly. This completes the proof of the lemma. □

Having proved the compactness lemma, we are now ready to prove main theorem 1.3. Since proof of the theorem is long so we have divided it in many steps.

Proof of Theorem 1.3. We divide the proof into five steps:

1. Preliminary reductions.

2. Setting modulus of continuity.

3. Existence of the first-order Taylor polynomial at every $\overline{p} \in \mathscr{S}_{1/2}$.

4. Continuity of the horizontal gradient on $\mathscr{S}_{1/2}$.

5. Patching the interior and boundary estimate (modulus of continuity of the horizontal gradient upto the boundary).

(1) Preliminary reductions Let us make some observations. **(a)** First we consider $\hat{u} = u - h$ which solves:

$$\sum_{i,j=1}^{m} X_i^\star(a_{ij} X_j \hat{u}) = \sum_{i=1}^{m} X_i^\star \hat{f}_i + g \quad \text{in } \mathscr{W}_\tau, \quad \hat{u} = 0 \text{ on } \mathscr{S}_\tau, \tag{3.13}$$

where $\hat{f}_i = f_i - \sum_{j=1}^m a_{ij} X_j h$, which is again Dini-continuous with the modulus of continuity depending on the modulus of continuity of $A = [a_{ij}]$, h and f_i. More precisely, for any $p, q \in \Omega$ we have:

$$\begin{aligned} |\hat{f}_i(p) - \hat{f}_i(q)| \leqslant \omega_{f_i}(d(p,q)) + \|A\|_{L^\infty(\Omega)} \omega_{\nabla_{\mathscr{H}} h}(d(p,q)) \\ + \|\nabla_{\mathscr{H}} h\|_{L^\infty(\Omega)} \omega_A(d(p,q)). \end{aligned}$$

Therefore, $\hat{f}_i$'s are Dini-continuous functions and hence, without loss of generality, we can assume that $h \equiv 0$. **(b)** In view of the left translation we may assume that $p_0 = e$. Furthermore, by scaling with respect to the family of dilations $\{\delta_\lambda\}_{\lambda>0}$ and suitable rotation of the horizontal layer $\mathfrak{g}_1$, without loss of generality we may assume that

1. $\tau = 1$.

2. $p_0 = e$.

3. In the logarithmic coordinates, $\mathscr{W}_1 = \Omega \cap B(1)$ can be expressed as

$$\big\{(x', x_m, y) \mid x_m > \psi(x', y)\big\} \tag{3.14}$$

with $\psi(0,0) = 0$, $\nabla_{x'}\psi(0,0) = 0$ and $\|\psi\|_{C^{1,\mathrm{Dini}}} \leqslant 1$.

(c) In view of the scaling we may assume that the data are sufficiently small (satisfying (3.21)), so that we can employ lemma 3.2. Indeed, for every $0 < \tau \leqslant 1$ consider the domain $\Omega_\tau = \delta_{\tau^{-1}}(\Omega)$. In the logarithmic coordinates Ω_τ can be expressed as follows:

$$\Omega_\tau = \{(x', x_m, y_2, y_3, \cdots y_k) \mid (\tau x', \tau x_m, \tau^2 y_2, \cdots \tau^k y_k) \in \Omega\}. \tag{3.15}$$

Observe that $\partial\Omega_\tau$ is given by:

$$x_m = \psi_\tau(x', y) = \psi_\tau(x', y_2, \cdots y_k) := \frac{1}{\tau}\psi(\tau x', \tau^2 y_2, \cdots \tau^k y_k). \tag{3.16}$$

We set

$$\mathscr{W}_\tau = \Omega_\tau \cap B(\tau^{-1}), \quad \mathscr{T}_\tau = \partial\Omega_\tau \cap B(\tau^{-1}).$$

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Let us observe that:

$$\begin{cases} \nabla_{x'}\psi_\tau(x',y) = \nabla_{x'}\psi(\tau x', \tau^2 y_2, \cdots \tau^k y_k) \\ \nabla_{y_j}\psi_\tau(x',y) = \tau^{j-1}\nabla_{y_j}\psi(\tau x', \tau^2 y_2, \cdots \tau^k y_k), \quad \text{for } j = 2\cdots k. \end{cases}$$

Thus, $\nabla\psi_\tau(x',y) \to \langle \nabla_{x'}\psi(0,0), 0\rangle$ as $\tau \to 0$. Therefore, by Taylor's theorem we get

$$\psi_\tau(x',y) \to \langle \nabla_{x'}\psi(0,0), x'\rangle = 0 \quad \text{as} \quad \tau \to 0, \tag{3.17}$$

consequently,

$$\partial\Omega_\tau \cap B(1) \longrightarrow \{x_m = 0\} \cap B(1). \tag{3.18}$$

It is also easy to see that for any $(x',y), (\bar{x'},\bar{y}) \in \Omega_\tau \cap B(1)$, we have:

$$\begin{aligned} &|\nabla\psi_\tau(x',y) - \nabla\psi_\tau(\bar{x'},\bar{y})| \\ &\leqslant (1 + \tau + \tau^2 + \cdots + \tau^{k-1})\omega_{\nabla\psi}(\tau|x' - \bar{x'}| + \cdots + \tau^k|y_k - \bar{y_k}|) \to 0, \end{aligned} \tag{3.19}$$

as $\tau \to 0$. In addition, we also observe that $u_\tau(p) = u(\delta_\tau p)$ solves the following problem:

$$\sum_{i,j=1}^{m} X_i^\star(a_{ij,\rho} X_j u_\tau) = \sum_{i=1}^{m} X_i^\star f_{i,\tau} + g_\tau \quad \text{in } \mathscr{W}_\tau, \quad u_\tau = h_\tau \quad \text{on } \mathscr{S}_\tau, \tag{3.20}$$

where

$$a_{ij,\tau}(p) = a_{ij}(\delta_\tau p), \quad f_{i,\tau}(p) = \tau f_i(\delta_\tau p),\ g_\tau(p) = \tau^2 g(\delta_\tau p)\ \ h_\tau(p) = h(\delta_\tau p).$$

Consequently, we have the following relations:

1. $|a_{ij,\tau}(p) - a_{ij,\tau}(q)| = |a_{ij}(\delta_\tau p) - a_{ij}(\delta_\tau q)| \leqslant \omega_A(\tau d(p,q)) \to 0$ as $\tau \to 0$.
2. $\|f_{i,\tau}\|_{L^\infty(((\mathscr{W}_\tau)))} \leqslant \tau\|f_i\|_{L^\infty(((\mathscr{W}_1)))}$.
3. $|f_{i,\tau}(p) - f_{i,\tau}(q)| = \tau|f_i(\delta_\tau p) - f_{i,}(\delta_\tau q)| \leqslant \tau\omega_f(\tau d(p,q)) \to 0$ as $\tau \to 0$.
4. $\|g_\tau\|_{L^q(\mathscr{W}_\tau)} = \tau^{2-\frac{Q}{q}}\|g\|_{L^q((\mathscr{W}_1))}$.
5. $\|\nabla_{\mathscr{H}} h_\tau\|_{L^\infty(\mathscr{W}_\tau)} \leqslant \tau\|\nabla_{\mathscr{H}} h_\tau\|_{L^\infty(\mathscr{W}_1)}$.
6. $|\nabla_{\mathscr{H}} h_\tau(p) - \nabla_{\mathscr{H}} h_\tau(q)| \leqslant \tau\omega_{\nabla_{\mathscr{H}} h}(\tau d(p,q))$.

Remark 3.3. In view of (3.19) and the above relations, it is clear that by choosing τ sufficiently small, say τ_0, we can make all the data sufficiently small so that the compactness lemma is applicable provided we consider u_τ, $a_{ij,\tau}$, $f_{i,\tau}$ g_τ, h_τ and Ω_τ instead of corresponding terms u, $a_{i,j}$, f_i g, h and Ω. Therefore, without loss of generality, from here onwards in the proof of this theorem we assume that

$$\begin{aligned} &\|a_{ij} - a_{ij}(e)\|_{L^\infty(\Omega)\cap B(1)} \leqslant \tilde{\delta}, \quad \|\psi\|_{C^{1,\text{Dini}}} \leqslant \tilde{\delta}, \quad \|h\|_{\Gamma^{0,\alpha}(\mathscr{S}_1)} \leqslant \tilde{\delta}, \quad \|f_i\|_{L^\infty(\mathscr{W}_1)} \leqslant \tilde{\delta} \\ &\quad \text{and} \quad \|g\|_{L^q(\mathscr{W}_1)} \leqslant \tilde{\delta}. \end{aligned} \tag{3.21}$$

where $\tilde{\delta}$ is given by (3.55).

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(2) Setting modulus of continuity. Let us first fix a constant α (see also corollary 3.5) such that $0 < \alpha < 1$ and consider the function

$$\tilde{\omega}_1(\sigma) = \max\{\omega_{\nabla\psi}(\sigma), \sigma^{\alpha}\}. \tag{3.22}$$

After normalization and using theorem 2.10, we can assume that $\tilde{\omega}_1$ is concave and $\tilde{\omega}_1(1) = 1$. With the help of the above function we can define a new function $\omega_1(\sigma) = \tilde{\omega}_1(\sigma^{\alpha})$. Then this function becomes α-decreasing (see definition 2.11) and ω_1 is still Dini-continuous, see [**1**] for more details. Now, let us define

$$\tilde{\omega}_2(\sigma) = \max\{\sigma^{\alpha}, \omega_f(\sigma)\}. \tag{3.23}$$

Again following the similar argument as above for ω_1, without loss of generality we can assume that $\tilde{\omega}_2$ concave and α-decreasing. Having defined $\tilde{\omega}_2$, let us define a new function

$$\omega_2(\sigma) := \max\left\{C_{II}\sigma\left(\frac{1}{|\Omega\cap B(\sigma)|}\int_{\Omega\cap B(\sigma)}|g|^q\right)^{1/q}, \quad \tilde{\omega}_2(\sigma)\right\}. \tag{3.24}$$

Having defined ω_1 and ω_2, let us define another function as follows:

$$\omega_3(\sigma^l) := \frac{1}{\tilde{\delta}}\sum_{j=0}^{l}\omega_1(\sigma^{l-j})\omega_2(\sigma^j), \tag{3.25}$$

where $\tilde{\delta}$ is given by (3.21). Finally, let us set

$$\omega(\sigma^l) := \max\{\omega_3(\sigma^l), \sigma^{l\alpha}\}. \tag{3.26}$$

We will be using some of the properties of the modulus of continuous functions defined above. So for the sake of completeness we list the required properties and sketch their proofs here.

Lemma 3.4.

1. *We have the following estimate:*

$$\sum_{j=0}^{\infty}\omega(\sigma^j) \leqslant C_b. \tag{3.27}$$

2. *For any fixed positive integer* $\nu \in \mathbb{N}$, *the following estimate holds:*

$$\sigma^{\alpha}\omega(\sigma^{\nu}) \leqslant \omega(\sigma^{\nu+1}). \tag{3.28}$$

3. ω_1 *is monotone.*

4. $1 \leqslant \omega(1)$.

5. *It is also clear that*

$$\frac{1}{\tilde{\delta}}\omega_{f_i}(\sigma) \leqslant \omega(\sigma) \quad \textit{and} \quad \frac{1}{\tilde{\delta}}\omega_2(\sigma^{\nu}) \leqslant \omega(\sigma^{\nu}). \tag{3.29}$$

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6.

$$\sigma^{\alpha} \leqslant \omega(\sigma) \tag{3.30}$$

7. *ω is α-decreasing.*

We prove (1), (7) and rest follows from the definition of the respective modulus of continuity. For details, we refer to lemmas 4.5 and 4.7 in [**1**].

Proof of (1). In order to estimate the sum in the left-hand side of (3.27), we first need to estimate the following sum:

$$\begin{aligned}\sum_{j=0}^{\infty} \omega_3(\sigma^j) &= \frac{1}{\tilde{\delta}} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \omega_1(\sigma^{j-i}) \omega_2(\sigma^i) \\ &= \frac{1}{\tilde{\delta}} \left(\sum_{j=0}^{\infty} \omega_1(\sigma^j) \right) \left(\sum_{j=0}^{\infty} \omega_2(\sigma^j) \right).\end{aligned} \tag{3.31}$$

Thus, from (3.31), it is clear that, in order to estimate the above sum we need to estimate $\sum_{j=1}^{\infty} \omega_1(\sigma^j)$ and $\sum_{j=1}^{\infty} \omega_2(\sigma^j)$. The sum involving the term ω_1 is finite because of the Dini-continuity of $\bigtriangledown\psi$. More precisely, we have the following estimate:

$$\sum_{j=1}^{\infty} \omega_1(\sigma^j) \leqslant \frac{1}{-\log \sigma} \sum_{j=1}^{\infty} \int_{\sigma^j}^{\sigma^{j-1}} \frac{\omega_1(t)}{t} dt \leqslant \int_0^1 \frac{\omega_1(t)}{t} dt < \infty. \tag{3.32}$$

Now, let us estimate the sum involving ω_2. It is easy to see that there exists a constant $\overline{C}$ such that

$$\begin{aligned}\sum_{j=1}^{\infty} \omega_2(\sigma^j) &\leqslant \overline{C} \int_0^2 \left(\frac{1}{|\Omega \cap B(\tau)|} \int_{\Omega \cap B(\tau)} |g(x)|^q dx \right)^{1/q} d\tau + \sum_{j=1}^{\infty} \tilde{\omega}_2(\sigma^j) \\ &=: \overline{C} \tilde{\mathbf{I}_q^g}(e, 2) + \sum_{j=1}^{\infty} \sigma^{\alpha j} + \sum_{j=1}^{\infty} \omega_f(\sigma^j) \\ &= I + II + III,\end{aligned} \tag{3.33}$$

where $\tilde{\mathbf{I}_q^g}$ is defined in (1.3). Note that I is finite because $g \in L(Q, 1)$ (see definition 2.6) so making use of the result in [**16**, equation (3.13)], we get

$$\sup_p \tilde{\mathbf{I}_q^g}(p, r) \leqslant \frac{1}{|B(1)|^{1/Q}} \int_0^{|B(r)|} \left[g^{**}(\tau) \tau^{\frac{q}{Q}} \right]^{1/q} \frac{\mathrm{d}\tau}{\tau}. \tag{3.34}$$

II is finite because it is geometric sum. While III is finite because f is Dini-continuous as in (3.32) the sum containing ω_1 is finite. Thus, by using (3.34) (with $f = g$ there), (3.33) and (3.32) in (3.31), we find that the sum in (3.31) is finite. □

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Proof of (7). From (3.26), if $\omega(\sigma^\nu) = \sigma^{\nu\alpha}$, then we get

$$\sigma^\alpha \omega(\sigma^\nu) = \sigma^{\alpha(1+\nu)} \leqslant \omega(\sigma^{1+\nu}). \tag{3.35}$$

Now, suppose that $\omega(\sigma^\nu) = \omega_3(\sigma^\nu)$. In this case, let us proceed as follows:

$$\begin{aligned}\sigma^\alpha \omega_3(\sigma^\nu) = \frac{1}{\tilde{\delta}} \sum_{j=0}^{\nu} \sigma^\alpha \omega_1(\sigma^{\nu-j})\omega_2(\sigma^j) &\leqslant \frac{1}{\tilde{\delta}} \sum_{j=0}^{\nu} \omega_1(\sigma^{1+\nu-j})\omega_2(\sigma^j) \\ &\quad \times \text{(since } \omega_1(\cdot) \text{ is } \alpha \text{ decreasing)} \\ &\leqslant \omega_3(\sigma^{1+\nu}) \quad \text{(by definition of } \omega_3) \leqslant \omega(\sigma^{1+\nu}) \quad \text{by (3.26)}.\end{aligned} \tag{3.36}$$

□

(3) Existence of the first-order Taylor polynomial at every $\overline{p} \in \mathscr{S}_{1/2}$. The aim of this section is to establish that u is $\Gamma^1(\overline{p})$ for every $\overline{p} \in \mathscr{S}_{1/2}$. More precisely, we want to establish the estimate (3.76), which will be accomplished in two sub-steps. In the first sub-step, we show that for any $\overline{p} \in \mathscr{S}_{1/2}$ there exists a sequence of first-order polynomial approximating u near $\overline{p}$. Later on in the next sub-step, we show that the limiting polynomial will give the affine approximation to the solution at $\overline{p}$. **(a)** Let $\overline{p} \in \mathscr{S}_{1/2}$ be a non-characteristic point. In view of translation and rotation without loss of generality, we can assume that $\overline{p} = e \in \mathscr{S}_{1/2}$. Also by normalizing the solution if necessary, we can assume that $\|u\|_{L^\infty(\mathscr{W}_1)} \leqslant 1$. Denote the constant CC^* in the compactness lemma 3.2 by θ and fix $\sigma > 0$ such that

$$0 < \sigma < (4\theta)^{-(1/1-\alpha)}. \tag{3.37}$$

We also let

$$\epsilon = \frac{\sigma^{1+\alpha}}{2}. \tag{3.38}$$

Suppose that $\delta(\epsilon)$ be the number in the compactness lemma 3.2 corresponding to ϵ defined above. Let us take another number $\tilde{\delta} \in (0, \delta)$ which will be fixed later. In view of the remark 3.3, it is clear that by choosing the scaling parameter τ sufficiently small we may assume that the smallness condition in (3.21) with such an $\tilde{\delta}$ can be ensured. For any $\kappa \in \mathbb{N} \cup \{0\}$, first we denote by $\mathfrak{P}_\kappa$ the set of homogeneous polynomials in $\mathbb{G}$ of homogeneous degree less or equal to κ. Now, we use induction to show that there exists a sequence of polynomials $\{L_\nu\}_{\nu \in \mathbb{N}\cup\{-1,0\}}$ in $\mathfrak{P}_1$ such that for every $\nu \in \mathbb{N} \cup \{-1, 0\}$ the following holds:

$$\|u - L_\nu\|_{L^\infty(\Omega \cap B(\sigma^\nu))} \leqslant \sigma^\nu \omega(\sigma^\nu), \tag{3.39}$$

$$\|L_{\nu+1} - L_\nu\|_{L^\infty(B(\sigma^\nu))} \leqslant C\sigma^\nu \omega(\sigma^\nu), \tag{3.40}$$

$$|L_\nu| \leqslant C_b\theta \quad (\text{where } \ C_b \ \text{ is from } \ (3.27)), \tag{3.41}$$

$$\|L_\nu \circ \delta_{\sigma^\nu}\|_{\Gamma^{0,1}(\partial\Omega_{\sigma^\nu} \cap B(1))} \leqslant \delta\sigma^\nu \omega(\sigma^\nu), \tag{3.42}$$

where $\Omega_\nu = \delta_{\nu^{-1}}(\Omega)$ is defined in (3.15). We prove the above assertion by mathematical induction. Let us set $a_{-1} = a_0 = 0$ and by definning the corresponding

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polynomials $L_0 = L_{-1} = 0$, we get:

$$\|u\|_{L^\infty(\Omega\cap B(1))} \leqslant 1 \leqslant w(1) \text{ by lemma 3.4(4)}. \tag{3.43}$$

As we want to establish the continuity of the horizontal gradient at the boundary so we consider the polynomial L_ν of the form $L_\nu(p) = l_\nu x_m$, where (x', x_m, y) denotes the logarithmic coordinates of p. Thus, the result follows for $\nu = -1, 0$. Now, assume that for some fixed $\nu \in \mathbb{N}$, the polynomials $L_1, L_{2,} \cdots L_\nu$ have been constructed satisfying (3.39)–(3.42). In order to complete the mathematical induction, we need to construct $L_{\nu+1}$ such that (3.39)–(3.42) hold for $\nu + 1$. This will be accomplished by using the compactness lemma 3.2. Let us consider the following rescaled function:

$$\tilde{u}(p) := \frac{(u - L_\nu)(\delta_{\sigma^\nu}(p))}{\sigma^\nu \omega(\sigma^\nu)}, \quad \text{for } p \in \tilde{\Omega} \cap B(1), \tag{3.44}$$

where $\tilde{\Omega} = \Omega_{\sigma^\nu}$. It is easy to observe that $\tilde{u}$ satisfies the following problem:

$$\begin{cases} \sum_{i,j=1}^{m} X_i^\star(a_{ij} X_j \tilde{u}) = \sum_{i=1}^{m} X_i^\star \tilde{\tilde{f}}_i + \tilde{\tilde{g}} \text{ in } \tilde{\Omega} \cap B(1), \\ \tilde{u} = \tilde{\tilde{\Phi}} \quad \text{on } \partial\tilde{\Omega} \cap B(1), \end{cases}$$

where

$$\tilde{\tilde{f}}_i = \frac{\tilde{f}_i - \sum_j^m \tilde{a}_{ij} X_j \tilde{L}_\nu}{\omega(\sigma^\nu)}, \quad \tilde{\tilde{g}} = \frac{\sigma^\nu \tilde{g}}{\omega(\sigma^\nu)}, \quad \tilde{\tilde{\Phi}} = -\frac{\tilde{L}_\nu}{\sigma^\nu \omega(\sigma^\nu)}, \tag{3.45}$$

and

$$\tilde{a}_{ij}(p) = a_{ij}(\delta_{\sigma^\nu} p), \;\; \tilde{f}_i(p) = f_i(\delta_{\sigma^\nu} p), \;\; \tilde{g}(p) = g(\delta_{\sigma^\nu} p), \;\; \tilde{L}_\nu(p) = L_\nu(\delta_{\sigma^\nu} p). \tag{3.46}$$

Since the result follows for ν, so in view of (3.39), we have

$$\|\tilde{u}\|_{L^\infty(\tilde{\Omega}\cap B(1))} \leqslant 1. \tag{3.47}$$

It is also easy to observe the following points: since L_ν is a polynomial of degree 1, so we have

$$\sum_{i,j=1}^{m} X_i^\star X_j(\tilde{a}_{ij}^0 \tilde{L}_\nu) = 0, \tag{3.48}$$

where $a_{ij}^0 = a_{ij}(e)$ and $\tilde{a}_{ij}^0 = \tilde{a}_{ij}(e) = a_{ij}(\delta_{\sigma^\nu}(e)) = a_{ij}(e)$. Consequently,

$$X_i^\star(\tilde{a}_{ij} X_j \tilde{L}_\nu) = \sum_{i,j=1}^{m} X_i^\star\big((\tilde{a}_{ij} - \tilde{a}_{ij}^0) X_j \tilde{L}_\nu\big) \tag{3.49}$$

and also

$$X_i^\star(\tilde{f}_i - \tilde{f}_i(e)) = X_i^\star \tilde{f}_i, \tag{3.50}$$

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since $X_i^\star \tilde{f}_i(e) = 0$. Therefore, we find that $\tilde{u}$ satisfies the following equation:

$$\begin{cases} \sum_{i,j=1}^m X_i^\star(\tilde{a}_{ij}X_j\tilde{u}) = \sum_{i=1}^m X_i^\star F_i + \tilde{\tilde{g}} \;\text{ in }\; \tilde{\Omega}\cap B(1), \\ \tilde{u} = \tilde{\tilde{\Phi}} \;\text{ on }\; \partial\tilde{\Omega}\cap B(1), \end{cases}$$

where

$$F_i = \frac{\tilde{f}_i - \tilde{f}_i(e) - \sum_{j=1}^m(\tilde{a}_{ij} - \tilde{a_{ij}^0})X_j\tilde{L}_\nu}{\omega(\sigma^\nu)}. \tag{3.51}$$

Now, we show that all the hypotheses in the compactness lemma are satisfied. Indeed, let us observe that:

$$\tilde{a}_{ij}^0 = \tilde{a}_{ij}(e) = a_{ij}(\delta_{\sigma^\nu}e) = a_{ij}(e). \tag{3.52}$$

Thus, we have

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}^0\|_{L^\infty(\tilde{\Omega}\cap B(1))} = \|a_{ij} - a_{ij}^0\|_{L^\infty(\Omega\cap B(\sigma^\nu))} \leqslant \omega_{a_{ij}}(\sigma^\nu) \leqslant \omega_A(\sigma^\nu). \tag{3.53}$$

Therefore, in view of remark 3.3 and the discussion in the beginning of this section, we have

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}^0\|_{L^\infty(\tilde{\Omega}\cap B(1))} \leqslant \tilde{\delta}.$$

It follows from (3.42) that

$$\|\tilde{\tilde{\Phi}}\|_{\Gamma^{0,\alpha}(\partial\tilde{\Omega}\cap B(1))} \leqslant \delta. \tag{3.54}$$

For any $q \in \mathscr{W}_1$, we have:

$$\begin{aligned} |F_i(q)| &= \frac{|\tilde{f}_i(q) - \tilde{f}_e - \sum_{j=1}^m(\tilde{a}_{ij}(q) - \tilde{a^0}_{ij})X_j\tilde{L}_\nu(q)|}{\omega(\sigma^\nu)} \\ &\leqslant \frac{|f_i(\delta_{\sigma^\nu}(q)) - f_i(e)| + \sum_{j=1}^m \left|(\tilde{a}_{ij}(q) - \tilde{a^0}_{ij})X_j\tilde{L}_\nu(q)\right|}{\omega(\sigma^\nu)} \\ &\leqslant \frac{\omega_{f_i}(\sigma^\nu) + \sigma^\nu\left|(a_{im}(\sigma^\nu q) - a_{im}(e))l_\nu\right|}{\omega(\sigma^\nu)}, \quad \text{where } \; L_\nu(x) = l_\nu x_m. \\ &\leqslant \frac{\omega_{f_i}(\sigma^\nu) + \sigma^\nu\omega_A(\sigma^\nu)|l_\nu|}{\omega(\sigma^\nu)} \leqslant (1 + C_b\theta)\tilde{\delta}, \end{aligned}$$

where C_b and θ are from (3.27) and (3.37). In concluding the last line we have used (3.29), that is, $\frac{1}{\tilde{\delta}}\omega_{f_i}(\nu) \leqslant \omega(\sigma^\nu)$, $\omega_{a_{im}}(\sigma^\nu) \leqslant \omega_A(1) \leqslant \tilde{\delta}$, $\sigma^{\nu\alpha} \leqslant \omega(\sigma^\nu)$ and $\alpha < 1$.

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So, if we choose

$$\tilde{\delta} < \frac{\delta}{1 + C_b\theta} \tag{3.55}$$

we get

$$\|F_i\|_{L^\infty(\tilde{\Omega}\cap B(1))} \leqslant \delta. \tag{3.56}$$

Since $\partial\tilde{\Omega} \cap B(1)$ can be expressed as follows:

$$x_m = \psi_{\sigma^\nu}(x', y_2, \ldots, y_k) = \frac{\psi(\sigma^\nu x', \sigma^{2\nu} y_2, \ldots, \sigma^{k\nu} y_k)}{\sigma^\nu}. \tag{3.57}$$

Let us denote ψ_{σ^ν} by $\tilde{\psi}$. Therefore, for any $p, \overline{p} \in \tilde{\Omega} \cap B(1)$ with $p = (x', x_m, y_2, \ldots, y_k)$ and $\overline{p} = (\overline{x'}, \overline{x}_m, \overline{y}_2, \ldots, \overline{y}_k)$, we have

$$\begin{aligned} |\nabla\tilde{\psi}(p) - \nabla\tilde{\psi}(\overline{p})| &\leqslant (1 + \tau + \tau^2 + \ldots \tau^{k-1}) \\ &\quad \times \omega_{\nabla\psi}(\tau|x' - \overline{x'}| + \tau^2|y_2 - \overline{y_2}| + \cdots \tau^k|y_k - \overline{y_k}|), \end{aligned} \tag{3.58}$$

where $\tau = \sigma^\nu$. Since $\tau < 1$ and $\psi(0, 0) = 0$ so by remark 3.3, we have

$$\|\tilde{\psi}\|_{C^{1,\mathrm{Dini}}} \leqslant \delta. \tag{3.59}$$

Now, let us consider

$$\begin{aligned} |\tilde{\tilde{g}}\|_{L^q(\tilde{\Omega}\cap B(1))} &= \left(\int_{\tilde{\Omega}\cap B(1)} |\tilde{\tilde{g}}(p)|^q dq\right)^{1/q} = \frac{\sigma^\nu}{\omega(\sigma^\nu)}\left(\int_{\tilde{\Omega}\cap B(1)} |\tilde{g}(p)|^q dp\right)^{1/q} \\ &\leqslant \frac{\sigma^\nu}{\omega(\sigma^\nu)}\left(\frac{1}{|\Omega \cap B(\sigma^\nu)|}\int_{\Omega\cap B(\sigma^\nu)} |g(p)|^q dp\right)^{1/q} \\ &\leqslant \frac{\omega_2(\sigma^\nu)}{C_{II}\omega(\sigma^\nu)} \leqslant \frac{\tilde{\delta}}{C_{II}} \quad \text{in view of } (3.29). \end{aligned}$$

Therefore, by the compactness lemma 3.2, there exists a $v \in C^2(B(\frac{1}{2}))$ such that $\|v\|_{C^2(B(\frac{1}{2}))} \leqslant \theta$ and

$$\|\tilde{u} - v\|_{L^\infty(\tilde{\Omega}\cap B(\frac{1}{2}))} \leqslant \epsilon. \tag{3.60}$$

Moreover, since $v = 0$ on $B(4/5) \cap \{x_m = 0\}$ so by Taylor's formula and the fact that $\|v\|_{C^2(B(1/2))} \leqslant \theta$ there exists $l \in \mathbb{R}$ with $|l| \leqslant \theta$ such that

$$\|v - lx_m\|_{L^\infty(B(\sigma))} \leqslant \theta\sigma^2 < \frac{\sigma^{1+\alpha}}{4}, \tag{3.61}$$

where the last inequality follows from the choice of σ in (3.37). From (3.60), (3.61) and the choice of ϵ (see (3.38)) along with the triangle inequality we get the following

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inequality:

$$\|\tilde{u} - lx_m\|_{L^\infty(B(\sigma))} \leqslant \sigma^{1+\alpha}. \tag{3.62}$$

Let us denote by $L(p) = lx_m \in \mathfrak{P}_1$ so (3.62) implies that

$$\begin{aligned}\|\tilde{u} - L\|_{L^\infty(\tilde{\Omega}\cap B(\sigma))} &= \sup_{p\in\tilde{\Omega}\cap B(\sigma)} \left|\frac{(u - L_\nu)(\delta_{\sigma^\nu}(p))}{\sigma^\nu\omega(\sigma^\nu)} - L(p)\right| \\ &= \frac{1}{\sigma^\nu\omega(\sigma^\nu)}\|u - L_{\nu+1}\|_{L^\infty(\Omega\cap B(\sigma^{\nu+1}))},\end{aligned} \tag{3.63}$$

where

$$L_{\nu+1}(p) := L_\nu(p) + \sigma^\nu\omega(\sigma^\nu)L(\delta_{\sigma^{-\nu}}(p)), \quad \text{for} \;\; p \in \Omega\cap B(\sigma^{\nu+1}). \tag{3.64}$$

It follows from (3.62) and (3.63) that

$$\|u - L_{\nu+1}\|_{L^\infty(\Omega\cap B(\sigma^{\nu+1}))} \leqslant \sigma^{\nu+1}\sigma^\alpha\omega(\sigma^\nu) \leqslant \; \sigma^{\nu+1}\omega(\sigma^{\nu+1}) \quad (\text{by} \;\; (3.28)). \tag{3.65}$$

Also, from (3.64),

$$\|L_{\nu+1} - L_\nu\|_{L^\infty(B(\sigma^\nu))} \leqslant C\sigma^\nu\omega(\sigma^\nu), \tag{3.66}$$

where $C = \|L\|_{L^\infty(B(1))}$. Moreover, from the expression of $L_{\nu+1}$ in terms of L_ν as in (3.64) we can infer by induction that in the logarithmic coordinates the polynomials L_ν are of the form

$$L_\nu(p) = l_\nu x_m, \tag{3.67}$$

where

$$|l_\nu| \leqslant \sum_{j=0}^{\nu}\theta\omega(\sigma^j) \leqslant \theta\sum_{j=0}^{\infty}\omega(\sigma^j) \leqslant C_b\theta. \tag{3.68}$$

Therefore, (3.41) follows. In order to prove (3.42), let us consider points $p, \overline{p} \in \partial\tilde{\Omega}\cap B(1)$, where $\tilde{\Omega} = \Omega_{\sigma^{-(\nu+1)}} = \delta_{\sigma^{-(\nu+1)}}\Omega$. Let (x, y) and $(\overline{x}, \overline{y})$ denote the logarithmic coordinates of p and $\overline{p}$, respectively. With $\tau = \sigma^{\nu+1}$, we have

$$x_m = \frac{\psi(\tau x', \tau^2 y_2, \cdots, \tau^k y_k)}{\tau} \quad \text{and} \quad \overline{x}_m = \frac{\psi(\tau\overline{x}', \tau^2\overline{y}_2, \cdots \tau^k\overline{y}_k)}{\tau}. \tag{3.69}$$

This gives

$$\begin{aligned}|L_{\nu+1}(\delta_\tau p) - L_{\nu+1}(\delta_\tau\overline{p})| &= |l_{\nu+1}||\tau x_m - \tau\overline{x}_m| \\ &= |l_{\nu+1}|\big|\psi(\tau x', \tau^2 y_2\cdots, \tau^k y_k) - \psi(\tau\overline{x}', \tau^2\overline{y}_2\cdots, \tau^k\overline{y}_k)\big| \\ &\leqslant \; C_b\theta\big|\psi(\tau\overline{x'}, \tau^2\overline{y_2}\cdots, \tau^k\overline{y_k}) - \psi(\tau\underline{x'}, \tau^2\underline{y_2}\cdots, \tau^k\underline{y_k})\big| \\ &\leqslant C_b\theta\big|\psi(\tau x', \tau^2 y_2\cdots, \tau^k y_k) - \psi(\tau\overline{x'}, \tau^2 y_2\cdots, \tau^k y_k)\big| \\ &\quad + C_b\theta\big|\psi(\tau\overline{x'}, \tau^2 y_2\cdots, \tau^k y_k) - \psi(\tau\overline{x'}, \tau^2\overline{y}_2\cdots, \tau^k\overline{y}_k)\big|.\end{aligned} \tag{3.70}$$

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In order to estimate the right-hand side of inequation (3.70), let us observe that the following holds:

$$\|\nabla_{x'}\psi\|_{L^\infty(B(s))} \leqslant \tilde{\delta}\omega_{\nabla\psi}(s), \tag{3.71}$$

because

$$\|\psi\|_{C^{1,\mathrm{Dini}}} \leqslant \tilde{\delta}, \quad \psi(0,0) = 0 \quad \text{and} \quad \nabla_{x'}\psi(0,0) = 0. \tag{3.72}$$

So, in view of (3.71) and Taylor's formula, the first term of the rightmost extreme inequality in (3.70) can be estimated as follows:

$$\begin{aligned} \left|\psi(\tau x', \tau^2 y_2 \cdots, \tau^k y_k) - \psi(\tau\overline{x}', \tau^2 y_2 \cdots, \tau^k y_k)\right| &\leqslant \tilde{\delta}\tau|x' - \overline{x}'|\omega_{\nabla\psi}(\tau) \\ &\leqslant C_2\tilde{\delta}\tau\omega_{\nabla\psi}(\tau)d(p,\overline{p}) \\ &\leqslant C_2\tilde{\delta}\tau\omega(\tau)d(p,\overline{p}) \end{aligned} \tag{3.73}$$

where, we use $|x' - \overline{x'}| \leqslant C_2 d(p,\overline{p})$ and $\omega_{\nabla\psi}(\tau) \leqslant \omega(\tau)$. Now, by using the mean value theorem, we can also estimate the second term of the rightmost extreme inequality in (3.70) as follows:

$$\begin{aligned} &\left|\psi(\tau\overline{x'}, \tau^2 y_2 \cdots, \tau^k y_k) - \psi(\tau\overline{x'}, \tau^2 \overline{y}_2 \cdots, \tau^k \overline{y}_k)\right| \\ &\quad \leqslant C_3\tilde{\delta}\tau^{1+\alpha}d(p,\overline{p}) \leqslant C_3\tau\omega(\tau)\tilde{\delta}d(p,\overline{p}). \end{aligned} \tag{3.74}$$

The first inequality follows since $\tau^i \leqslant \tau^{1+\alpha}$ for any $2 \leqslant i$. In (3.74), we have used $\tau^\alpha \leqslant \omega(\tau)$ and $\tau < 1$. Now, let us take $\tilde{C} = \max\{C_2,\ C_3\}$ and choose

$$\tilde{\delta} = \min\left\{\frac{\delta}{2C_b\tilde{C}\theta},\ \ \frac{\delta}{C_b 2m^2\theta}\right\}. \tag{3.75}$$

Therefore, by the above choice of $\tilde{\delta}$ and using the inequalities from (3.73), (3.74) in (3.70), we get (3.42). **(3-(b))** ***Affine approximation of the solution*** u ***on the non-characteristic portion of the boundary.*** Now, we show that $\{L_\nu\}$ the sequence of polynomial converges to linear function $\mathbb{L}$ as $\nu \to \infty$. Moreover, $\mathbb{L}$ is an affine approximation of solution to (1.1) on $e \in \partial\Omega$. By translation, in a similar way one can show that at each point of the non-characteristic portion of the boundary, there is an affine approximation of solution to (1.1). More precisely, given any non-characteristic point $p_0 \in \partial\Omega$ there exists an affine function L_{p_0} such that

$$|u(p) - L_{p_0}(p)| \leqslant C_{aff}\, d(p,p_0)W(d(p,p_0)). \tag{3.76}$$

Moreover, W can be chosen to be α-decreasing in the sense of definition 2.11. Now, let us try to prove (3.76) for $p_0 = e \in \partial\Omega$ by assuming that all the previous step holds at e. Let us take an arbitrary $p \in \partial\Omega \cap B(1)$ and choose an integer $\nu \in \mathbb{N}$ such that $\sigma^{\nu+1} \leqslant |p| \leqslant \sigma^\nu$. Let us define $\mathbb{L} = \lim_{\nu\to\infty} L_\nu$, where L_ν is from above

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step and consider

$$\begin{aligned} |u(p)-\mathbb{L}(p)| &\leqslant |u(p)-L_\nu(p)|+|L_\nu(p)-\mathbb{L}(p)| \\ &\leqslant \sigma^\nu\omega(\sigma^\nu)+\sum_{j=0}^{\infty}|L_{\nu+j}-L_{\nu+j+1}|_{\Omega\cap B(\sigma^\nu)} \\ &\leqslant \sigma^\nu\omega(\sigma^\nu)+C_b\sigma^\nu\sum_{j=\nu}^{\infty}\omega(\sigma^j). \end{aligned} \tag{3.77}$$

The last step follows from (3.39) and (3.40). In order to estimate the sum in the last line of (3.77), let us observe that for any fixed $j\in\mathbb{N}$, it follows from (3.26) that

$$\omega(\sigma^j)\leqslant\frac{1}{\tilde{\delta}}\sum_{l=0}^{j/2}\omega_1(\sigma^{j-l})\omega_2(\sigma^l)+\frac{1}{\tilde{\delta}}\sum_{l=j/2}^{j}\omega_1(\sigma^{j-l})\omega_2(\sigma^l)+\sigma^{j\alpha}. \tag{3.78}$$

Therefore, we have

$$\begin{aligned} \sum_{j=\nu}^{\infty}\omega(\sigma^j) &\leqslant \frac{1}{\tilde{\delta}}\sum_{j=\nu}^{\infty}\sum_{l=0}^{j/2}\omega_1(\sigma^{j-l})\omega_2(\sigma^l)+\frac{1}{\tilde{\delta}}\sum_{j=\nu}^{\infty}\sum_{l=j/2}^{j}\omega_1(\sigma^{j-l})\omega_2(\sigma^l)+\sum_{j=\nu}^{\infty}\sigma^{j\alpha} \\ &\leqslant \frac{C}{\tilde{\delta}}\sum_{j=\nu}^{\infty}\omega_1(\sigma^{j/2})+\frac{1}{\tilde{\delta}}\sum_{j=\nu}^{\infty}\sum_{j=l/2}^{j}\omega_1(\sigma^{j-l})\omega_2(\sigma^l)+\sum_{j=\nu}^{\infty}\sigma^{j\alpha} \\ &= D+E+F. \end{aligned} \tag{3.79}$$

In the second line we have used $\sum_{j=0}^{\infty}\omega_2(\sigma^j)\leqslant C$, see (3.33). In order to estimate D, E and F in (3.79), let us define

$$\begin{aligned} &W_1(\epsilon):=\sup_{a\geqslant 0}\int_a^{a+\epsilon^{1/2}}\frac{\omega_1(s)}{s}ds,\quad W_2(\epsilon):=\epsilon^{\alpha/2}, \\ &W_3(\epsilon):=\sup_{a\geqslant 0}\int_a^{a+\epsilon^{1/2}}[g^{**}(s)s^{q/Q}]^{1/q}\frac{\mathrm{d}s}{s}\ \text{ and }\ W_4(\epsilon)=\sup_{a\geqslant 0}\int_a^{a+\epsilon^{1/2}}\frac{\tilde{\omega}_2(s)}{s}ds. \end{aligned} \tag{3.80}$$

Estimate for D: We estimate D as follows:

$$D\leqslant C\int_0^{\sigma^{\nu/2}}\frac{\omega_1(s)}{s}ds\leqslant CW_1(\sigma^\nu)\ \text{ in view of definition of }\ W_1. \tag{3.81}$$

Estimate for F: We use the standard formula for geometric series to get:

$$F\leqslant C\sigma^{\nu\alpha}=CW_2(\sigma^{2\nu})\leqslant CW_2(\sigma^\nu)\quad\text{in view of definition of } W_2, \tag{3.82}$$

where the last inequality follows because $\sigma < 1$ and so $\sigma^{2\nu} < \sigma^{\nu}$.

Estimate for E: Let us observe

$$
\begin{aligned}
E &\leqslant C\left(\sum_{j=\frac{\nu}{2}}^{\infty}\omega_2(\sigma^j)\right)\left(\sum_{j=1}^{\infty}\omega_1(\sigma^j)\right) \leqslant C\left(\sum_{j=\nu/2}^{\infty}\omega_2(\sigma^j)\right) \quad (\text{by } (3.32)) \\
&= \left[C_{II}\sum_{j=\nu/2}^{\infty}\sigma^j\left(\frac{1}{|\Omega\cap B(\sigma^j)|}\int_{\Omega\cap B(\sigma^j)}|g|^q\right)^{1/q} + \underbrace{\sum_{j=\nu/2}^{\infty}\omega_f(\sigma^j)}_{I} + \sum_{j=\nu/2}^{\infty}\sigma^{j\alpha}\right] \\
&\leqslant \left[\underbrace{\tilde{C}\int_0^{\sigma^{\nu Q/2}}\left[g^{**}(s)s^{q/Q}\right]^{1/q}\frac{\mathrm{d}s}{s}}_{III} + \underbrace{\sum_{j=\nu/2}^{\infty}\tilde{\omega}_2(\sigma^j)}_{II} + \sum_{j=\nu/2}^{\infty}\sigma^{j\alpha}\right] \\
&\leqslant C_1\underbrace{W_3(\sigma^\nu)}_{IV} + C_2W_4(\sigma^\nu) + C_3W_2(\sigma^\nu),
\end{aligned} \tag{3.83}
$$

where we have used the fact that $\sigma^{(Q\nu/2)} \leqslant \sigma^{(\nu/2)}$ in deducing II from I (which follows since $\sigma < 1$ and $Q \geqslant 2$) and $\omega_f(s) \leqslant \tilde{\omega}_2(s)$ in deducing IV from III. From (3.81)–(3.83) and the choice of $|p| \approx \sigma^\nu$, we find that D, E and $F \to 0$ as $|p| \to 0$. It is also clear from the definition of W_2 that it is non-decreasing. Moreover, we can also assume that each W_i is non-decreasing. Without loss of generality we can assume that $W_j(\cdot)$, for $j = 1 \cdots 4$, are α-decreasing in the sense of definition 2.11. Indeed, let us first consider the case W_1. From the fact that $\omega_1(\cdot)$ is a modulus of continuity and concave, we have that $W_1(\cdot)$ satisfies all the properties of the definition 2.8 and hence is also a modulus of continuity. Using theorem 2.10, without loss of generality, we can assume W_1 is also concave. Now, we can replace $W_1(s)$ by $W_1(s^\alpha)$, if necessary, we can assume $W_1(\cdot)$ is α-decreasing. Since $W_4(\cdot)$ is same as W_1 so the assertion for W_4 also follows. Now, let us consider the case of W_3. From definition 2.8, it is clear that W_3 is a modulus of continuity. Using theorem 2.10, without loss of generality, we can assume $W_3(\cdot)$ is also concave. Now, replacing $W_3(s)$ by $W_3(s^\alpha)$, if necessary, we can assume that $W_3(\cdot)$ is α-decreasing. Without loss of generality, we will denote the changed W_i with the same notion and assume that these are α-decreasing. With the above $W_i(\cdot)$ in the hand we define a new α-decreasing function $W(\cdot)$ as follows:

$$W(s) := W_1(s) + W_2(s) + W_3(s) + W_4(s), \tag{3.84}$$

which is again α-decreasing. So, in view of $|p| \approx \sigma^\nu$, (3.79), (3.81)–(3.83) along with (3.77), we have

$$|u(p) - \mathbb{L}(p)| \leqslant C\sigma^\nu W(\sigma^\nu) = C|p|W(|p|), \tag{3.85}$$

and this completes the proof of this step.

3.2. Interior estimate

In the next two steps, we prove the continuity of the horizontal gradient on the non-characteristic portion of the boundary and up to the boundary, respectively. In the proof of these results we need a scale-invariant interior estimate, see corollary 3.5. This estimate is a suitable adaptation of [4, corollary 3.2] to our set up. Since the proof follows on the same line as of the boundary case, therefore, we just sketch the proof instead of giving the complete details.

COROLLARY 3.5. *Given* $0 < \tau \leqslant 1$*, let* $u \in \mathscr{L}^{1,2}_{loc}(B(\tau)) \cap C(\overline{B(\tau)})$ *be a weak solution to*

$$\sum_{i,j=1}^{m} X_i^\star(a_{ij}X_j u) = \sum_{i=1}^{m} X_i^\star f_i + g \quad in \quad B(\tau), \tag{3.86}$$

where $f = (f_1, \ldots, f_m) \in \Gamma^{0,Dini}(B(\tau))$*,* $a_{ij} \in \Gamma^{0,Dini}(B(\tau))$*,* a_{ij} *satisfies* (1.2) *and* $g \in L^q(B(\tau))$ *with* $2q > Q$*. Then,* $u \in \Gamma^1(B(\tau/2))$*. Moreover, we have the following estimates:*

$$|\nabla_{\mathscr{H}} u(e)| \leqslant \frac{C\|u\|_{L^\infty(B(\tau))}}{\tau}\big(1 + W(\tau)\big), \tag{3.87}$$

and

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(e)| \leqslant C\|u\|_{L^\infty(B(\tau))}\big(W(|p|) + \frac{|p|^\alpha}{\tau^{1+\alpha}}\big), \tag{3.88}$$

$p \in B(\tau/2)$*, where* $C = C(\mathbb{G}, \lambda, [a_{ij}]_{\Gamma^{0,Dini}}, \Omega) > 0$ *and* $W(\cdot)$ *is a given by* (3.84)*.*

Proof. Given a function u let us define a new function $v(p) = u(\delta_\tau(p))$ for $p \in B(1)$. It is clear that v satisfies the following equation:

$$\sum_{i,j=1}^{m} X_i^\star(a_{ij,\tau}X_j v) = \sum_{i=1}^{m} X_i^\star f_{i,\tau} + g_\tau \quad \text{in} \quad B(1), \tag{3.89}$$

where $f_{i,\tau}(p) = \tau f_i(\delta_\tau(p))$ and $g_\tau(p) = \tau^2 g(\delta_\tau(p))$. Without loss of generality, we can assume that $\|v\|_{L^\infty(B(1))} \leqslant 1$, since otherwise we consider the function $v(p) = u(\delta_\tau(p))/\|u\|_{L^\infty(B(\tau))}$. In order to prove (3.87), it is sufficient to prove that there exists a sequence of polynomials $\{L_\nu\}$ of the form $L_\nu(p) = a_\nu + \langle b_\nu, x\rangle$, where $(x, y_2, \cdots, y_k)$ denote the logarithmic coordinate of p, such that

$$\begin{aligned} \|v - L_\nu\|_{L^\infty(B(\sigma^\nu))} \leqslant \sigma^\nu\omega(\sigma^\nu) \quad \text{and} \quad |b_\nu| \leqslant C, \\ |a_{\nu+1} - a_\nu| \leqslant C\sigma^\nu\omega(\sigma^\nu), \quad |b_{\nu+1} - b_\nu| \leqslant C\omega(\sigma^\nu). \end{aligned} \tag{3.90}$$

As in the proof of step (3), the above inequalities (3.90) follow by the induction argument. Here, we skip the details. Hence, using the estimates from before (adapted to the interior case), one sees that

$$|\nabla_{\mathscr{H}} v(e)| \leqslant C(1 + W(\tau)). \tag{3.91}$$

Therefore, scaling back to u we get

$$|\nabla_{\mathscr{H}} u(e)| \leqslant \frac{C\|u\|_{L^\infty(B(\tau))}}{\tau}\big(1 + W(\tau)\big). \tag{3.92}$$

Analogously, we also get

$$|\nabla_{\mathscr{H}} v(p) - \nabla_{\mathscr{H}} v(e)| \leqslant C\big(\tau W(\tau|p|) + |p|^\alpha\big), \tag{3.93}$$

for all $p \in B(1/2)$. Re-scaling the inequality (3.93) back to u, we get the following inequality:

$$|\nabla_{\mathscr{H}} u(\delta_\tau(p)) - \nabla_{\mathscr{H}} u(e)| \leqslant C\frac{\|u\|_{L^\infty(B(\tau))}}{\tau}\big(\tau W(\tau|p|) + |p|^\alpha\big),$$

that is,

$$|\nabla_{\mathscr{H}} u(\delta_\tau(p)) - \nabla_{\mathscr{H}} u(e)| \leqslant C\|u\|_{L^\infty(B(\tau))}\left(W(\tau|p|) + \frac{|p|^\alpha}{\tau}\right).$$

Now, putting back $q = \delta_\tau p$ we get

$$|\nabla_{\mathscr{H}} u(q) - \nabla_{\mathscr{H}} u(e)| \leqslant C\|u\|_{L^\infty(B(\tau))}\left(W(|q|) + \frac{|q|^\alpha}{\tau^{1+\alpha}}\right),$$

which completes the proof of the corollary. □

Having finished the interior estimate, let us now move to the next step.

Step-(4) Continuity of the horizontal gradient on $\mathscr{S}_{1/2}$. In step (3), we have shown that for any $p \in \mathscr{S}_{1/2}$, there is a Taylor polynomial L_p of u at p. In this step, our objective is to show that for any (non-characteristic) points $p_1, p_2 \in \mathscr{S}_{1/2}$, the following estimate holds:

$$|\nabla_{\mathscr{H}} L_{p_1} - \nabla_{\mathscr{H}} L_{p_2}| \leqslant C\left(W(d(p_1, p_2))\right), \tag{3.94}$$

for some universal C, where $W(.)$ is a modulus function defined by (3.84).

Proof of (3.94). Let $t = d(p_1, p_2)$. We consider a 'non-tangential' point $p_3 \in \mathscr{W}_1$ at a (pseudo) distance from p_1 comparable to t, i.e. let p_3 be such that

$$d(p_3, p_1) \sim t, \;\; d(p_3, \partial\Omega) \sim t, \tag{3.95}$$

where we have assumed $d(p, \partial\Omega) = \inf_{p' \in \partial\Omega} d(p, p')$. Since $\mathscr{S}_1$ is a non-characteristic $C^{1,\mathrm{Dini}}$ portion of $\partial\Omega$, therefore, it is possible to find such a point p_3. Arguing as in the proof of [**15**, theorem 7.6], at any scale t one can find a non-tangential pseudo-ball from inside centred at p_3. In fact, there exists a universal $a > 0$ sufficiently

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small (which can be seen to depend on the Lipschitz character of $\partial\Omega$ near the non-characteristic portion $\mathscr{S}_1$) such that for some c_0 universal one has

$$d(p, \partial\Omega) \geqslant c_0 t \quad \text{for all} \quad p \in B(p_3, at).$$

This allow us to apply step (3) above and conclude that there exists a universal $C > 0$ such that for all $p \in B(p_3, at)$, we have:

$$|u(p) - L_{p_1}(p)| \leqslant C\, t\, W(t), \quad |u(p) - L_{p_2}(p)| \leqslant CtW(t). \tag{3.96}$$

Now, for $\ell = 1, 2$ we note that $v_\ell = u - L_{p_\ell}$ solves

$$\sum_{i,j=1}^{m} X_i^\star (a_{ij} X_j v_\ell) = \sum_{i=1}^{m} X_i^\star F_i^\ell + g, \tag{3.97}$$

where we have let

$$F_i^\ell \overset{def}{=} f_i - \sum_{j=1}^{m} a_{ij} X_j L_{p_\ell}.$$

Since f_i and a_{ij} are Dini-continuous, therefore, without loss of generality we can assume that F_i^ℓ, are Dini-continuous. Also, from (3.96) we see that v_ℓ satisfies

$$||v_\ell||_{L^\infty(B(p_3, at))} \leqslant CtW(t), \quad \ell = 1, 2. \tag{3.98}$$

With (3.98) in hand, we can now use the interior estimate (3.87) in corollary 3.5 in the pseudo-ball $B(p_3, at)$ to obtain the following estimate for $\ell = 1, 2$

$$\begin{aligned} |\nabla_{\mathscr{H}} v(p_3)| = |\nabla_{\mathscr{H}} u(p_3) - \nabla_{\mathscr{H}} L_{p_\ell}(p_3)| &\leqslant \frac{C}{t} ||u - L_{p_\ell}||_{L^\infty(B(p_3, t))} (1 + W(t)) \\ &\leqslant C\, W(t), \end{aligned} \tag{3.99}$$

by (3.96). From (3.99) and the triangle inequality, we obtain that the following estimate holds:

$$|\nabla_{\mathscr{H}} L_{p_1} - \nabla_{\mathscr{H}} L_{p_2}| \leqslant C\, W(t) \leqslant C\, (W(d(p_1, p_2))),$$

where we have used $t \sim d(p_1, p_2)$, which is the desired estimate (3.94). □

Step-(5) Patching the interior and boundary estimate: In this step, we prove that the horizontal gradient of a weak solution to (1.1) is Γ^1 up to the boundary. First, we observe that there is an $\epsilon > 0$ sufficiently small such that for any $p \in \mathscr{W}_\epsilon$, there exists $p_0 \in \mathscr{S}_{1/2}$ such that

$$d(p, p_0) = d(p, \partial\Omega). \tag{3.100}$$

To finish the proof of theorem 1.3, we will show that for all $p, p^\star \in \mathscr{W}_\epsilon$ we have:

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(p^\star)| \leqslant C^\star \left(W(d(p, p^\star)) \right), \tag{3.101}$$

for some universal constant $C^\star > 0$. Let $p, p^\star \in \mathscr{W}_\epsilon$ be the two given points. Let $p_0, p_0^\star$ be the corresponding points in $\mathscr{S}_{1/2}$ for which (3.100) holds. Let us write

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$\delta(p) = d(p, \partial\Omega)$ for $p \in \Omega$. Without loss of generality we may assume that

$$\delta(p) = \min\{\delta(p), \delta(p^\star)\}. \tag{3.102}$$

By step (3), there exists a first-order polynomial L_{p_0} such that for every $q \in \mathscr{W}_1$ we have

$$|u(q) - L_{p_0}(q)| \leqslant C_2 d(p_0, q) W(d(p_0, q)), \tag{3.103}$$

where p_0 is as in (3.100). Now, there are two possibilities:

(a) $d(p, p^\star) \leqslant \delta(p)/2$;

(b) $d(p, p^\star) > \delta(p)/2$.

(a) In view of (3.102), it is clear that $B(p, \delta(p)) \subset \Omega$. Now, let us consider the function $v := u - L_{p_0}$, where $p_0 \in \mathscr{S}_{1/2}$ is the point corresponding to p discussed above and L_{p_0} is the polynomial from step-(3). Again it is easy to see that v satisfies an equation of the type (3.97) in $B(p, \delta(p)) \subset \Omega$. Now, we can apply corollary 3.5 (interior estimate) along with (3.103) to get the following estimate:

$$||v||_{L^\infty(B(p,\delta(p)))} \leqslant \tilde{C}_2 \delta(p) W(\delta(p)), \tag{3.104}$$

for some $\tilde{C}_2 > 0$. Since $p^\star \in B(p, \delta(p)/2)$, so by using the interior estimate (3.88) (corollary 3.5) and (3.104), we find that for some $\tilde{C}$ depending also on $\tilde{C}_2$ the following estimates hold:

$$\begin{aligned} &|\nabla_{\mathscr{H}} v(p) - \nabla_{\mathscr{H}} v(p^\star)| = |\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(p^\star)| \\ &\leqslant C\left(W(d(p,p^\star)) \left[||u - L_{p_0}||_{L^\infty(B(p,\delta(p)))} \right] + \frac{|d(p,p^\star)|^\alpha}{\delta(p)^{1+\alpha}} \left[||u - L_{p_0}||_{L^\infty(B(p,\delta(p)))} \right] \right) \\ &\leqslant C\left(W(d(p,p^\star)) \left[\delta(p)\, W(\delta(p)) \right] + \frac{|d(p,p^\star)|^\alpha}{\delta(p)^{\alpha}} \left[W(\delta(p)) \right] \right). \end{aligned} \tag{3.105}$$

Now, α-decreasing property of $W(\cdot)$ implies

$$\frac{|d(p,p^\star)|^\alpha}{\delta(p)^\alpha} [W(\delta(p))] \leqslant W(d(p,p^\star)). \tag{3.106}$$

With the help of (3.106), (3.105) can be rewritten as follows:

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(p^\star)| \leqslant C(W(d(p,p^\star))),$$

which gives (3.101). **(b)** In this case, we have $d(p, p^\star) > \delta(p)/2$ and from (3.100) we get

$$d(p, p_0) = d(p, \partial\Omega) = \delta(p) < 2d(p, p^\star). \tag{3.107}$$

Let us recall the following pseudo-triangle inequality for d

$$d(p, p') \leqslant C_0(d(p, p'') + d(p'', p')), \tag{3.108}$$

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for all $p, p', p'' \in \mathbb{G}$, and a universal $C_0 > 0$. From (3.107) and (3.108), we get

$$d(p^\star, p_0) \leqslant C_0(d(p^\star, p) + d(p, p_0)) \leqslant C_0(d(p^\star, p) + 2d(p^\star, p)) = 3C_0 d(p, p^\star). \tag{3.109}$$

Since, we also have $d(p^\star, p_0) \geqslant d(p^\star, \partial\Omega) = \delta(p^\star)$, therefore, in view of (3.109), we get

$$\delta(p^\star) \leqslant 3C_0 d(p, p^\star). \tag{3.110}$$

So by combining (3.108)–(3.110) we finally obtain

$$d(p_0, p_0^\star) \leqslant C_0(d(p_0, p^\star) + d(p^\star, p_0^\star)) = C_0(d(p_0, p^\star) + \delta(p^\star)) \leqslant 6C_0^2 d(p, p^\star). \tag{3.111}$$

Let b be the universal constant in the existence of a non-tangential (pseudo)-ball in the previous step-(4). Therefore, from step (3), we have the following estimates:

$$\begin{aligned} &||u - L_{p_0}||_{L^\infty(B(p, b\delta(p)))} \leqslant \tilde{K}_0 \delta(p) W(\delta(p)), \\ &||u - L_{p_0^\star}||_{L^\infty(B(p, b\delta(p^\star)))} \leqslant \tilde{K}_0 \delta(p^\star) W(\delta(p^\star)). \end{aligned} \tag{3.112}$$

Let us define $v = u - L_{p_0}$, and observe that v satisfies an equation of the type (3.97). Therefore, arguing as in (3.96)–(3.99) and using the former estimate (3.112) in $B(p, b\delta(p))$ along with the interior estimate in corollary 3.5, we obtain that for some universal constant $C > 0$, we have

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} L_{p_0}| = |\nabla_{\mathscr{H}} v(p)| \leqslant C\, W(\delta(p)) \leqslant C\, W(d(p, p^\star)), \tag{3.113}$$

where in the last inequality we have used $\delta(p) \leqslant 2d(p, p^\star)$. Arguing as before (3.113), we obtain

$$|\nabla_{\mathscr{H}} u(p^\star) - \nabla_{\mathscr{H}} L_{p_0^\star}| \leqslant CW(\delta(p^\star)) \leqslant C\, W(d(p, p^\star)) \tag{3.114}$$

by (3.110). Now, from (3.94) and (3.111) we have

$$|\nabla_{\mathscr{H}} L_{p_0} - \nabla_{\mathscr{H}} L_{p_0^\star}| \leqslant C\, W(d(p_0, p_0^\star)) \leqslant C\, W(d(p, p^\star)). \tag{3.115}$$

Applying the triangle inequality along with the estimates (3.113)–(3.115), we get

$$|\nabla_{\mathscr{H}} u(p) - \nabla_{\mathscr{H}} u(p^\star)| \leqslant C^\star \left(W(d(p, p^\star))\right).$$

This completes the proof of theorem 1.3. □

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