COMPOSITIO MATHEMATICA

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Compositio Math. **145** (2009), 1147–1162.

doi:10.1112/S0010437X08003734

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

On Shimura varieties of orthogonal type over totally real fields, we prove a product formula and the modularity of Kudla's generating series of special cycles in Chow groups.

1. Introduction

On a modular curve $X_0(N)$, Gross *et al.* prove [GKZ87] that certain generating series of Heegner points are modular forms of weight 3/2 with values in the Jacobian as a consequence of their formula for Néron–Tate height pairing of Heegner points. Such a result is the analogue of an earlier result of Hirzebruch and Zagier [HZ76] on intersection numbers of Shimura curves on Hilbert modular surfaces, and has been extended to orthogonal Shimura varieties in various settings:

- cohomological cycles over totally real fields by Kudla and Millson [KM90] using their theory of cohomological theta lifting;
- divisor classes in the Picard group over $\mathbb{Q}$ by Borcherds [Bor99] as an application of his construction of singular theta lifting;
- high-codimensional Chow cycles over $\mathbb{Q}$ by one of us, Wei Zhang [Zha09], as a consequence of his modularity criterion by induction on the codimension.

The main result of this paper is a further extension of the modularity to Chow cycles on orthogonal Shimura varieties over totally real fields. For applications of our result, we would like to mention our ongoing work on the Gross–Zagier formula [GZ86] and the Gross–Kudla conjecture on triple product L-series [GK92] over totally real fields. Our result is also necessary for extending the work of Kudla *et al.* [KRY06] to totally real fields.

Different from the work of Gross *et al.* and Borcherds, our main ingredients in the proof are some product formulae and the modularity of Kudla and Millson. In the codimension-one case, our result is new only in the case of Shimura curves and their products, as the Kudla–Millson result already implies the modularity in Chow groups where the first Betti numbers of ambient Shimura varieties vanish. Both the modularity and product formulae for certain special cycles were proposed by Kudla [Kud97, Kud04]. In the following, we give details of his definitions and our results.

Let F be a totally real field of degree $d=[F:\mathbb{Q}]$ with a fixed real embedding ι. Let V be a vector space over F with an inner product $\langle\cdot,\cdot\rangle$ which is non-degenerate with signature $(n,2)$

Received 5 November 2007, accepted in final form 28 May 2008, published online 10 August 2009.
2000 Mathematics Subject Classification 11G18, 14G35.
Keywords: Shimura varieties, Kudla cycles, Siegel modular forms.

on $V_{\iota,\mathbb{R}}$ and signature $(n+2, 0)$ at all other real places. Let G denote the reductive group $\mathrm{Res}_{F/\mathbb{Q}} G\mathrm{Spin}(V)$.

Let $D \subset \mathbb{P}(V_{\mathbb{C}})$ be the Hermitian symmetric domain for $G(\mathbb{R})$ as follows:

$$D = \{v \in V_{\iota,\mathbb{C}} : \langle v, v\rangle = 0,\ \langle v, \bar{v}\rangle < 0\}/\mathbb{C}^{\times},$$

where the quadratic form extends by $\mathbb{C}$-linearity, and $v \mapsto \bar{v}$ is the involution on $V_{\mathbb{C}} = V \otimes_F \mathbb{C}$ induced by complex conjugation on $\mathbb{C}$. Then, for any open compact subgroup K of $G(\widehat{\mathbb{Q}})$, we have a Shimura variety with $\mathbb{C}$-points

$$M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\widehat{\mathbb{Q}})/K.$$

It is known that, for K sufficiently small, $M_K(\mathbb{C})$ has a canonical model M_K over F as a quasi-projective variety. In our case, M_K is actually complete if $F \neq \mathbb{Q}$. Let $\mathcal{L}_D$ be the bundle of lines corresponding to points on D. Then $\mathcal{L}_D$ descends to an ample line bundle $\mathcal{L}_K \in \mathrm{Pic}(M_K) \otimes \mathbb{Q}$ with $\mathbb{Q}$ coefficients.

For an F-subspace W of V with positive definite inner product at all real places of F, and an element $g \in G(\widehat{\mathbb{Q}})$, we define a *Kudla cycle* $Z(W, g)_K$ represented by points $(z, hg) \in D \times G(\widehat{\mathbb{Q}})$, where $z \in D_W$ is in the subset of lines in D perpendicular to W, and $h \in G_W(\widehat{\mathbb{Q}})$ is in the subgroup of elements in $G(\widehat{\mathbb{Q}})$ fixing every point in $\widehat{W} = W \otimes \widehat{\mathbb{Q}}$. The cycle $Z(W, g)$ depends only on the K-class of the F-subspace $g^{-1}W$ of $\widehat{V} := V \otimes_F \widehat{F}$.

For a positive number r, an element $x = (x_1, \ldots, x_r) \in K\backslash V(F)^r$, and an element $g \in G(\widehat{\mathbb{Q}})$, we define a *Kudla Chow cycle* $Z(x, g)_K$ in M_K as follows: let W be the subspace of V generated by the components x_i of x, then

$$Z(x, g)_K := \begin{cases} Z(W, g)_K c_1(\mathcal{L}_K^{\vee})^{r-\dim W} & \text{if } W \text{ is positive definite,} \\ 0 & \text{otherwise.} \end{cases}$$

For any Bruhat–Schwartz function $\phi \in \mathcal{S}(V(\widehat{F})^r)^K$, we define *Kudla's generating function* of cycles in the Chow group $\mathrm{Ch}(M_K, \mathbb{C})$ with complex coefficients as follows:

$$Z_\phi(\tau) = \sum_{x \in G(\mathbb{Q})\backslash V^r} \sum_{g \in G_x(\widehat{\mathbb{Q}})\backslash G(\widehat{\mathbb{Q}})/K} \phi(g^{-1}x) Z(x, g)_K q^{T(x)}, \quad \tau = (\tau_k) \in (\mathcal{H}_r)^d,$$

where $\mathcal{H}_r$ is the Siegel upper-half plane of genus r, and $T(x) = \frac{1}{2}(\langle x_i, x_j\rangle)$ is the intersection matrix, and

$$q^{T(x)} = \exp\left(2\pi i \sum_{k=1}^{d} \mathrm{tr}\, \tau_k \iota_k T(x)\right),$$

where $\iota_1 := \iota, \ldots, \iota_d$ are all real embeddings of F. Note that $Z_\phi(\tau)$ does not depend on the choice of K when we consider the sum in the direct limit $\mathrm{Ch}(M)_{\mathbb{C}}$ of $\mathrm{Ch}(M_K)_{\mathbb{C}}$ via pull-back maps of cycles. Since the natural map $\mathrm{Ch}(M_K)_{\mathbb{C}} \longrightarrow \mathrm{Ch}(M)_{\mathbb{C}}$ is injective with image being the subspace of K-invariants of $\mathrm{Ch}(M)$, we see that the identities among Z_ϕ as generating series of $\mathrm{Ch}(M)$ are equivalent to identities as generating series of $\mathrm{Ch}(M_K)$ once ϕ are invariant under K.

THEOREM 1.1 (Product formula). *Let $\phi_1 \in \mathcal{S}(V(\widehat{F})^{r_1})$, and $\phi_2 \in \mathcal{S}(V(\widehat{F})^{r_2})$ be two Bruhat–Schwartz functions. Then, in the Chow group,*

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = Z_{\phi_1 \otimes \phi_2}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right).$$

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For a linear functional ℓ on $\mathrm{Ch}^r(M_K)_{\mathbb{C}}$, we define a series with complex coefficients:

$$\ell(Z_\phi)(\tau) = \sum_{x \in G(\mathbb{Q})\backslash V^r} \sum_{g \in G_x(\widehat{\mathbb{Q}})\backslash G(\widehat{\mathbb{Q}})/K} \phi(g^{-1}x)\ell(Z(x,g)_K)q^{T(x)}.$$

THEOREM 1.2 (Modularity). *Let ℓ be a linear functional on $\mathrm{Ch}^r(M_K)_{\mathbb{C}}$ such that the generating function $\ell(Z_\phi(\tau))$ is convergent. Then $\ell(Z_\phi(\tau))$ is a Siegel modular form of weight $n/2+1$ for $\mathrm{Sp}_r(F)$ with respect to a Weil representation on $\mathcal{S}(\widehat{V}^r)$.*

Let us explain the meaning of the last sentence in Theorem 1.2 in more detail. For each place v, we have a double cover $\mathrm{Mp}_r(F_v)$ of $\mathrm{Sp}_r(F_v)$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_r(F_v) \longrightarrow \mathrm{Sp}_r(F_v) \longrightarrow 1.$$

If v is non-archimedean, $\mathrm{Mp}_r(F_v)$ has a Weil representation on $\mathcal{S}(V_v^r)$ with respect to the standard additive character $\psi_{\widehat{F}}$ of $\widehat{F}$. The subgroup $\{\pm 1\}$ acts as the scalar multiplication. If v is real and induces an isomorphism $F_v \simeq \mathbb{R}$, then $\mathrm{Mp}_r(F_v)$ consists of pairs $(g, J(g,\tau))$ where $g \in \mathrm{Sp}_r(F_v)$ and $J(g,\tau)$ is an analytic function of $\tau \in \mathcal{H}_r$ such that

$$J(g,\tau)^2 = \det(c\tau + d), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For any half integer k, the group $\mathrm{Mp}_r(F_v)$ has an action of weight k on the space of functions on $\mathcal{H}_r$ by

$$(f|_k g)(\tau) = f(g\tau)J(g,\tau)^{-2k}.$$

The global double cover $\mathrm{Mp}_r(\mathbb{A})$ of $\mathrm{Sp}_r(\mathbb{A})$ is defined as the quotient of the restricted product of $\mathrm{Mp}_r(F_v)$ modulo the subgroup of $\oplus_v\{\pm\}$ consisting of an even number of components -1:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_r(\mathbb{A}) \longrightarrow \mathrm{Sp}_r(\mathbb{A}) \longrightarrow 1.$$

Then the preimage $\mathrm{Mp}_r(F)$ of $\mathrm{Sp}_r(F)$ in $\mathrm{Mp}(\mathbb{A})$ has a unique splitting:

$$\mathrm{Mp}_r(F) = \{\pm 1\} \times \mathrm{Sp}_r(F).$$

The modularity in the theorem means the following identity for each $\gamma \in \mathrm{Sp}_r(F)$:

$$\ell(Z_{\omega(\gamma_f)\phi})|_{1+n/2}\gamma_\infty = \ell(Z_\phi), \quad \gamma \in \mathrm{Sp}_r(F),$$

where $(\gamma_\infty, \gamma_f) \in \prod_v \mathrm{Mp}_r(F_v)$ is one representative of γ.

Remarks. (1) We conjecture that the series $\ell(Z_\phi)$ is convergent for all ℓ. A good example is the functional derived from a cohomological class as follows. For a cohomological cycle $\alpha \in H^{2r}(M_k, \mathbb{Q})$, we may define a functional ℓ_α by taking the intersection pairing between the cohomological class $[Z]$ of $Z \in \mathrm{Ch}^r(M_K)$ and α:

$$\ell_\alpha(Z) := [Z] \cdot \alpha.$$

In this case, the generating series $\ell_\alpha(Z_\phi)$ is convergent and modular by a fundamental result of Kudla and Millson [KM90].

(2) Let $N^r(M_K)_{\mathbb{Q}}$ and $\mathrm{Ch}^r(M_K)^0_{\mathbb{Q}}$ be the image and kernel, respectively, of the class map $\mathrm{Ch}^r(M_K)_{\mathbb{Q}} \longrightarrow H^{2r}(M_K)$. We expect that there is a canonical decomposition of modules over the Hecke algebra of M_K:

$$\mathrm{Ch}^r(M_K)_{\mathbb{Q}} \simeq \mathrm{Ch}^r(M_K)^0_{\mathbb{Q}} \oplus N^r(M_K)_{\mathbb{Q}}.$$

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In this way, for any $\alpha \in \mathrm{Ch}^r(M_K)^0_{\mathbb{Q}}$, we can define a functional ℓ_α by taking (a conjectured) Beilinson–Bloch height pairing between the projection Z_0 of $Z \in \mathrm{Ch}^r(M_K)$ and α:

$$\ell_\alpha(Z) := Z_0 \cdot \alpha.$$

The convergence problem is reduced to estimating the height pairing.

(3) Beilison and Bloch have conjectured that the cohomologically trivial cycles $\mathrm{Ch}^r(M_K)^0 \otimes \mathbb{Q}$ will map injectively to the rth intermediate Jacobian. Thus, when $H^{2r-1}(M_K, \mathbb{Q}) = 0$, a combination of this conjecture with Kudla and Millson's work implies the modularity of the generating series $\ell(Z_\phi)$.

For Kudla divisors, we have an unconditional result.

THEOREM 1.3 (Modularity in codimension one). *For any $\phi \in \mathcal{S}(\widehat{V})$, the generating function $Z_\phi(\tau)$ of Kudla divisor classes is convergent and defines a modular form of weight $n/2+1$.*

Remark. The Shimura variety M_K has vanishing Betti number $h^1(M_K)$ unless M_K is a Shimura curve or the product of two Shimura curves. In this case, $\mathrm{Ch}^1(M_K)^0_{\mathbb{Q}} = 0$ and the modularity in the Chow group $\mathrm{Ch}^1(M_K) \otimes \mathbb{Q}$ follows from Kudla–Millson modularity for the cohomology group $H^2(M_K, \mathbb{Q})$.

Now we would like to describe the contents of this paper. In § 2 we prove some intersection formulae for Kudla cycles in Chow groups and then some product formulae for generating series. The modularity Theorems 1.2 and 1.3 will be proved in § 3. For modularity of divisors (Theorem 1.3), we use Kudla–Millson modularity for generating functions of cohomological classes and an embedding trick that relies on the vanishing of the first Betti number of our Shimura varieties by results of Kumaresan and Vogan and Zuckerman [VZ84]. For modularity of high-codimensional cycles, we use an induction method described in [Zha09].

2. Intersection formulae

Our aim in this section is to study the intersections of Kudla cycles $Z(W, g)_K$ in the Chow group $\mathrm{Ch}^*(M_K)$ of cycles modulo the rational equivalence. We first prove some scheme-theoretic formulae and then some intersection formulae in Chow groups.

First we need a more intrinsic definition of Kudla cycles. We say that an F-vector subspace W of $\widehat{V}$ is *admissible* if the inner product on W takes F-rational values and is positive definite.

LEMMA 2.1. *An F-vector subspace W of $\widehat{V}$ is admissible if and only if $W = gW'$ where W' is a positive-definite subspace of V and $g \in G(\widehat{\mathbb{Q}})$.*

Proof. Indeed, for any element $w \in W$ with non-zero norm, the F-rational number $\|w\|^2$ is locally a norm of vectors in V_v for every place v of F. Thus, it is a norm of $v \in V$ by the Hasse–Minkowski theorem (see [Ser73, p. 41, Theorem 8]). Now we apply Witt's theorem (see [Ser73, p. 31, Theorem 3]) to obtain an element $g \in G(\widehat{\mathbb{Q}})$ such that $gw = v$. Replacing W by gW we may assume that $v = w$. Let V_1 be the orthogonal complement of v in V and W_1 be the orthogonal complement of v in W. Then we may use induction to embed W_1 into V_1. This induces an obvious embedding from W to V. $\square$

For an admissible subspace $W = g^{-1}W'$, $W' \subset V$ and $g \in G(\widehat{\mathbb{Q}})$, we have a well-defined Kudla cycle $Z(W)_K := Z(W', g)_K$. For an open subgroup K' of K, the pull-back of the cycle $Z(W)_K$

on $M_{K'}$ has a decomposition

$$Z(W)_K = \sum_k Z(k^{-1}W)_{K'}, \tag{2.1}$$

where k runs though a set of representatives of the coset $K_W\backslash K/K'$ with K_W the stabilizer of W.

PROPOSITION 2.2. *Let $Z(W_1)_K$ and $Z(W_2)_K$ be two Kudla cycles. The scheme-theoretic intersection is the union of $Z(W)$ indexed by admissible classes W in $K\backslash(KW_1 + KW_2)$.*

Proof. Assume that $W_i = g_i^{-1}V_i$ with $V_i \subset V$. Then the scheme-theoretic intersection is represented by $(z, g) \in D \times G(\widehat{\mathbb{Q}})$ such that for some $\gamma \in G(\mathbb{Q})$, $k \in K$,

$$z \in D_{V_1} \cap \gamma D_{V_2}, \quad g \in G_{V_1}(\widehat{\mathbb{Q}})g_1 \cap \gamma G_{V_2}(\widehat{\mathbb{Q}})g_2k.$$

It is easy to see that

$$\gamma D_{V_2} = D_{\gamma V_2}, \quad \gamma G_{V_2} = G_{\gamma V_2} \cdot \gamma.$$

Thus, we can rewrite the above condition as

$$z \in D_{V_1} \cap D_{\gamma V_2} = D_{V_1+\gamma V_2}, \quad g \in G_{V_1}(\widehat{\mathbb{Q}})g_1 \cap G_{\gamma V_2}(\widehat{\mathbb{Q}})\gamma g_2k.$$

It follows that the intersection is a union of $Z(V_1 + \gamma V_2, g)_K$ indexed by $\gamma \in G(\mathbb{Q})$ and $g \in G(\widehat{\mathbb{Q}})$ such that

$$g \in G_{V_1+\gamma V_2}(\widehat{\mathbb{Q}})\backslash(G_{V_1}(\widehat{\mathbb{Q}})g_1K \cap G_{\gamma V_2}(\widehat{\mathbb{Q}})\gamma g_2K)/K.$$

For such a g, we may write

$$g = h_1g_1k_1 = h_2\gamma g_2k_2$$

with elements in the corresponding components. Then

$$g^{-1}(V_1 + \gamma V_2) = k_1^{-1}g_1^{-1}V_1 + k_2^{-1}g_2^{-1}V_2 = k_1^{-1}W_1 + k_2^{-1}W_2.$$

Thus, the intersection is parameterized by admissible classes in

$$K\backslash(KW_1 + KW_2). \qquad \square$$

The following lemma gives the uniqueness of the admissible class with fixed generators when K is sufficiently small.

PROPOSITION 2.3. *Let $x_1, \ldots, x_r$ be a basis of an admissible subspace W of $\widehat{V}$. Then there is an open normal subgroup K' in K such that, for any $k \in K$, the only possible admissible class in*

$$K'\Big\backslash \sum K'k^{-1}(x_i)$$

is $\sum_i k^{-1}(x_i)$, where (x_i) denotes the subspace Fx_i of V.

Proof. We proceed with the proof in several steps.

Step 0. Let us reduce to the case $k = 1$. Assume that K' is a normal subgroup. Then we have a bijection of classes:

$$K'\Big\backslash \sum K'k^{-1}(x_i) \longrightarrow K'\Big\backslash \sum_i K'(x_i), \quad t \mapsto kt.$$

Thus, we *may assume $k = 1$ to prove the proposition.*

Step 1: work on a congruence group for a fixed lattice. Choose an $\mathcal{O}_F$-lattice $V_{\mathcal{O}_F}$ stable under K and taking integer-valued inner products. Then, for each positive integer N, we have an open subgroup $K(N)$ of $G(\widehat{\mathbb{Q}})$ consisting of elements h such that

$$hx - x \in NV_{\mathcal{O}_F} \quad \text{for all } x \in V_{\mathcal{O}_F}.$$

Now we take $K' = K(N)$ for large N so that K' is a normal subgroup of K. Assume that, for some $h_i \in K(N)$, the class $\sum_i Fh_i x_i$ is admissible. We are reduced to showing that *there is a $k \in K'$ such that $kx_i = h_i x_i$ when N is sufficiently large.*

Step 2: reduce to a problem of extending maps. Without loss of generality, assume that $x_1, \ldots, x_r \in V_{\mathcal{O}_F}$ and generate $W_{\mathcal{O}_F} := W \cap V_{\mathbb{Z}}$. Then we have the following properties for the inner product of $h_i x_i$:

- for some $t_{i,j} \in \mathcal{O}_F$,

$$(h_i x_i, h_j x_j) = (x_i, x_j) + Nt_{i,j};$$

- the Schwartz inequality (as the pairing on $\sum_i Fh_i x_i$ is positive definite),

$$|(h_i x_i, h_j x_j)| \leqslant \|h_i x_i\| \|h_j x_j\| = \|x_i\| \|y_j\|.$$

It follows that for large N, $t_{i,j} = 0$. In other words, there is an isometric embedding

$$\xi : W \longrightarrow \widehat{V}, \quad x_i \mapsto h_i x_i.$$

Thus we reduce to *extending this embedding to an isomorphism $k : \widehat{V} \longrightarrow \widehat{V}$ by a $k \in K$ for N sufficiently large.*

Step 3: work with an orthogonal basis. Write $W = g^{-1}W'$ and take an orthogonal basis $f_1, \ldots, f_{n+2}$ of V over F such that $f_1, \ldots, f_r$ is a basis of W'. Then we can take $V_{\mathcal{O}_F}$ in Step 1 to be generated by f_i over $\mathcal{O}_F$. Write $e_i = g^{-1} f_i$ and $e'_i = \xi(e_i)$ for $1 \leqslant i \leqslant r$. Note that e_i is an integral combination of x_i; thus $e_i - e'_i \in N' \widehat{V}_{\mathcal{O}_F}$ for an integer N' which can be arbitrarily large as N goes to infinity. Thus, we are in a situation of finding an element $k \in K$ such that $ke_i = \xi e_i = e'_i$ for i between 1 and m. We reduce this problem to finding a *local component of $k_\wp$ for each finite prime $\wp$ of $\mathcal{O}_F$.*

Step 4: work with good primes. Let S be a finite set of primes in $\mathcal{O}_F$ consisting of the factors of $2N$, and the norms of e_i. If $\wp$ is not in S, we claim that one of $(e_1 \pm e'_1)/2$ has invertible norm. Otherwise, the sum of their norms, $(\|e_1\|^2 + \|e_2\|^2)/2$, is in $\wp\mathcal{O}_\wp$. This is a contradiction because e_1 and e'_1 have the same norm. Thus, we may have a decomposition $V_{\mathcal{O}_\wp}$ into a sum of $\mathcal{O}_\wp(e_1 \pm e'_1)/2$ and its complement V'. We may take k_1 which is ± 1 on the first term and ∓ 1 on the second term. Then $k_1 \in G\mathrm{Spin}(V_\wp)$ such that $k_1 e'_1 = e_1$. Now we may replace e'_i by $k_1 e_i$ and then reduce to the case where $e_1 = e'_1$. We may continue this process for $\mathcal{O}_\wp e_1^\perp$ and so on until all $e_m = e'_m$. In other words, we find a $k_\wp \in G\mathrm{Spin}(V_\wp)$ such that $k_\wp e_i = e'_i$.

Step 5: work with bad primes. If $\wp \in S$, we may replace N by Np^ℓ so that the order α of p in N is arbitrarily large. We define e'_i for $i > m$ by induction such that $\langle e'_i, e'_j \rangle = \langle e_i, e_j \rangle$ for all $j \leqslant i$, and such that e_i is close to e'_i. This is done by applying the Schmidt process for the elements

$$e'_1, \ldots, e'_m, e_{m+1}, \ldots, e_{n+2}.$$

More precisely, assume that $e'_1, \dots, e'_{i-1}$ are defined, and define e'_i by

$$e''_i := e_i - \sum_{j=1}^{i-1} \frac{(e_i, e'_j)}{(e'_j, e'_j)} e'_j,$$
$$e'_i := \sqrt{\frac{(e_i, e_i)}{(e''_i, e''_i)}} \cdot e''_i.$$

Note that when the order of p in N is sufficiently large, e''_i is arbitrarily close to e_i and $(e_i, e_i)/(e''_i, e''_i)$ is arbitrarily close to one. Thus, the square root is well defined. In summary, we find a $k_\wp \in K(\wp^\beta)$ for β arbitrarily large when $\operatorname{ord}_\wp(N)$ is arbitrarily large. □

As an application, we want to decompose the cycle $Z(W, g)_K$ as a complete intersection after raising the level.

Proposition 2.4. *Let $x_1, \dots, x_r$ be a basis of W over F. Then there is an open normal subgroup K' in K such that the pull-back of $Z(W)_K$ is a (rational) multiple of unions of the complete intersection*

$$\sum_{k \in K' \backslash K} \prod_i Z(k^{-1} x_i)_{K'}.$$

Proof. For an open subgroup K' of K, the cycle $Z(W)_K$ has a decomposition

$$Z(W)_K = \sum_{k \in K_W \backslash K / K'} Z(k^{-1} W)_{K'}.$$

Here K_W is the subgroup of K consisting of elements fixing every element in W.

We want to compare the right-hand side with $\sum_{k \in K/K'} \prod_i Z(k^{-1} x_i)_{K'}$. By Proposition 2.2, the components of $\prod_i Z(k^{-1} x_i)_{K'}$ correspond to the admissible classes in

$$K' \Big\backslash \sum_i K' k^{-1}(x_i).$$

By Proposition 2.3, when K' is small, the only admissible class in the above coset is $\sum k^{-1}(x_i) = k^{-1} W$. Thus,

$$\prod_i Z(x_i)_{K'} = \sum_j Z(k_j^{-1} W)_{K'}$$

for some $k_j \in K$. Now we translate both sides by $k \in K/K'$ to obtain

$$\sum_{k \in K/K'} \prod_i Z(k^{-1} x_i)_{K'} = c_1 \sum_{k \in K/K'} Z(k^{-1} W)_{K'} = c_2 Z(W)_K,$$

where c_1 and c_2 are some positive rational numbers. □

By comparing the codimensions, we conclude that the intersection of $Z(W_1)_K$ and $Z(W_2)_K$ in Proposition 2.2 is proper if and only if $k_1 W_1 \cap k_2 W_2$ is zero for all admissible classes $k_1 W_1 + k_2 W_2$. In this case, the set-theoretic intersection gives the intersection in the Chow group. In the following we want to study what happens if the intersection is not proper. First, we need to express the canonical bundle of M_K in terms of $\mathcal{L}_K$.

Lemma 2.5. *Let $\omega_K = \det \Omega^1_K$ denote the canonical bundle on M_K. Then for K small, there is a canonical isomorphism*

$$\omega_K \simeq \mathcal{L}_K^n \otimes \det V^\vee.$$

Proof. We need only to prove the statements in the lemma for the bundle $\mathcal{L}_D$ on D with an isomorphism

$$\omega_D \simeq \mathcal{L}_D^n \otimes \det V^\vee,$$

which is equivariant under the action of $G(\mathbb{R})$. Fix one point p on D corresponding to one point $v \in V_{\iota,\mathbb{C}}$; then $V_{\iota,\mathbb{R}}$ has an orthogonal basis given by $\mathrm{Re}(v), \mathrm{Im}(v), e_1, \ldots, e_n$ and $V_{\iota,\mathbb{C}}$ has a basis $v, \bar{v}, e_1, \ldots, e_n$. After rescaling, we may assume that $\langle v, \bar{v}\rangle = -2$, and $\langle e_i, e_i\rangle = 1$. Then we can define local coordinates $z = (z_1, \ldots, z_n)$ near p such that the vector v extends to a section of $\mathcal{L}_D$ in a neighborhood of p:

$$v_z := v + \frac{1}{2}\sum_i z_i^2 \bar{v} + \sum_i z_i e_i.$$

For a point $p \in D$ corresponding to a line ℓ in $V_{\mathbb{C}}$, the tangent space of D at p is canonically isomorphic to $\mathrm{Hom}(\ell, \ell^\perp/\ell)$. In terms of coordinates z for $\ell = \mathbb{C}v$, this isomorphism takes $\partial/\partial z_i \otimes v$ to the class of e_i in $\ell^\perp/\ell$. In terms of bundles, one has an equivariant isomorphism

$$T_D \simeq \underline{\mathrm{Hom}}(\mathcal{L}_D, \mathcal{L}_D^\perp/\mathcal{L}_D) = \mathcal{L}_D^\perp \otimes \mathcal{L}_D^\vee/\mathcal{O}_D.$$

Since $\omega_D = \det T_D^\vee$, we have an equivariant isomorphism

$$\omega_D = (\det \mathcal{L}_D^\perp)^\vee \otimes \mathcal{L}_D^{1+n}.$$

In terms of coordinates z, this isomorphism is given by

$$dz_1 \cdots dz_n \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_n \wedge v) \longmapsto v^{\otimes(n+1)}.$$

Note that the pairing $\langle\cdot,\cdot\rangle$ induces an equivariant isomorphism between $\mathcal{L}_D^\vee$ and $V_D/\mathcal{L}_D^\perp$ which is represented by $\mathbb{C}\bar{v}$ in our base of $V_{\mathbb{C}}$. This defines an isomorphism $\det \mathcal{L}_D^\perp \simeq \mathcal{L}_D \otimes \det V$ which is given by

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n \wedge v \longmapsto v \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_n \wedge v \wedge \bar{v}).$$

Thus, we have a canonical isomorphism

$$\omega_D \simeq \mathcal{L}_D^n \otimes \det V^\vee,$$

which is given by

$$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \longmapsto v^n \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_n \wedge v \wedge \bar{v}).$$

This completes the proof of the lemma. □

Now we have a version of Proposition 2.2 in the Chow group. For a positive number r and an element $x = (x_1, \ldots, x_r) \in K\backslash \widehat{V}^r$, recall that the Kudla cycle $Z(x)_K$ in M_K is defined as follows: let W be the subspace of $V(\widehat{F})$ generated by the components x_i of x. Then we define

$$Z(x)_K = \begin{cases} Z(W)_K c_1(\mathcal{L}^\vee)^{r-\dim W} & \text{if } W \text{ is admissible,}\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 2.6. *Let $Z(W_1)_K$ and $Z(W_2)_K$ be two Kudla cycles. Their intersection in the Chow group is given as a sum of $Z(W)_K$ indexed by admissible classes W in*

$$K\backslash(KW_1 + KW_2).$$

Proof. First we treat the case where W_2 is one dimensional. Then the set-theoretic intersection is indexed by admissible classes $k_1W_1 + k_2W_2$ in $K\backslash KW_1 + KW_2$. There is nothing we need to prove if this intersection is proper. Otherwise, we may assume $k_2W_2 \subset k_1W_1$ and $Z_1 \subset Z_2$ for some components Z_1 of $Z(W_1)$ and Z_2 of $Z(W_2)$. Let Z be a connected component of M_K containing Z_2. Let i denote the embedding $i : Z_2 \longrightarrow Z$. Then the intersection in the Chow group has the expression

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot i^* c_1(\mathcal{O}(Z_2))).$$

Let I be the ideal sheaf of Z_2 on Z. Then $\mathcal{O}(Z_2) = I^{-1}$, and $i^*c_1(Z(W_2)) = -c_1(i^*I)$ is the first Chern class of the bundle i^*I^{-1}. From the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_Z|_{Z_2} \longrightarrow \Omega_{Z_2} \longrightarrow 0$$

we obtain the following isomorphism from the determinant:

$$i^*(I) \otimes \omega_{Z_2} \simeq i^*\omega_Z.$$

Thus, we have shown the following equality in the Chow group:

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot c_1(\omega_{Z_2} \otimes i^*\omega_Z^{-1})). \tag{2.2}$$

Now we use the canonical isomorphism in Lemma 2.5,

$$\omega_{Z_i} \simeq \mathcal{L}_{Z_i}^{\dim Z_i}, \quad \mathcal{L}_Z|_{Z_i} = \mathcal{L}_{Z_i}$$

to conclude that

$$i^*\mathcal{L}^\vee \simeq \omega_{Z_2} \otimes i^*\omega_Z^{-1}. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot i^*c_1(\mathcal{L}^\vee)) = Z_1 \cdot c_1(\mathcal{L}^\vee).$$

Thus we have proved the proposition when W_2 is one dimensional. Now we want to prove the proposition for the general case. We use Proposition 2.4 to write $Z(W_2)_K$ as a sum:

$$Z(W_2)_K = c \sum_{k \in K/K'} \prod_i Z(k^{-1}x_i)_{K'}.$$

Working on the intersections of $Z(k^{-1}x_i)$ with scheme-theoretic components of

$$Z(W_1) \prod_{j<i} Z(k^{-1}x_i),$$

we find that the intersection of $Z(W_1)$ and $Z(W_2)$ in the Chow group in level K' is simply the sum of terms of the form

$$Z(W)c_1(\mathcal{L}^\vee)^{\dim W_1 + \dim W_2 - \dim W}$$

where W runs through admissible classes in

$$\coprod_{k \in K'\backslash K} K' \bigg\backslash \bigg(KW_1 + \sum_i K'k'(x_i) \bigg).$$

In other words, in level K', the Chow intersection is the Zariski intersection corrected by powers of the first Chern class of $c_1(\mathcal{L}^\vee)$. As $\mathcal{L}$ is invariant under pull-back, and the dimension does not change under push-forward, we have the same conclusion in level K. $\square$

Proposition 2.4 still holds for Chow cycles.

Proposition 2.7. *Let $x = (x_1, \dots, x_r) \in K\backslash \widehat{V}^r$. Then there is an open normal subgroup K' in K such that in the Chow group the pull-back of $Z(x)_K$ is a (rational) multiple of a sum of complete intersections*

$$\sum_{k \in K'\backslash K} \prod_i Z(k^{-1}x_i)_{K'}.$$

Proof. By Proposition 2.3, we may choose K' such that, for any $k \in K$, the only admissible class in

$$K' \Big\backslash \sum K'k^{-1}(x_i)$$

is $\sum Fk^{-1}x_i$. By Proposition 2.6, the product $\prod_i Z(k^{-1}x_i)_{K'}$ is simply $Z(k^{-1}x)_{K'}$. Their sum is simply $Z(x)_K$. □

3. Product formulae

In this section, we want to apply the formulae in the previous section to obtain some product formulae for Kudla's generating series. The first is the product formula in Theorem 1.1 which has been conjectured by Kudla [Kud04], and the second is the pull-back formula for embedding of Shimura varieties.

3.1 Proof of the product formula

First of all, using Lemma 2.1, we see that the cycle $Z(y, g)_K$ depends only on $x := g^{-1}y \in \widehat{V}$ which is *admissible* in the sense that it generates an admissible subspace of $\widehat{V}$. Thus, we may write such a cycle as $Z(x)_K$. We extend this definition by setting $Z(x)_K = 0$ if $x \in K\backslash \widehat{V}^r$ is not admissible. In this way, we can rewrite the generating series in the introduction as follows:

$$Z_\phi(\tau) = \sum_{x \in K\backslash \widehat{V}^r} \phi(x) Z(x)_K q^{T(x)}.$$

Now we return to the proof of the product formula. By the above formula,

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = \sum_{(x_1,x_2)} Z(x_1)_K Z(x_2)_K \phi_1(x_1)\phi_2(x_2) q^{T(x_1)} q^{T(x_2)}.$$

By Proposition 2.6,

$$Z(x_1)_K Z(x_2)_K = \sum_W Z(W),$$

where W runs through the admissible classes in

$$K\backslash K(x_1) + K(x_2),$$

and (x_i) denote the subspaces of $\widehat{V}$ generated by the components x_{ij} of x_i. It is clear that such W are generated by αx_{1i} and βx_{2i} for some α, β in K. Thus, we write $x = (\alpha x_1, \beta x_2)$. On the other hand, it is easy to see that for such an x,

$$\phi_1(x_1)\phi_2(x_2) q^{T(x_1)} q^{T(x_2)} = (\phi_1 \otimes \phi_2)(x) q^{T(x)}.$$

Thus, we have shown the following:

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = \sum_x Z(x)_K (\phi_1 \otimes \phi_2)(x) q^{T(x)} = Z_{\phi_1 \otimes \phi_2}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right).$$

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3.2 A pull-back formula

The rest of this section is devoted to proving a pull-back formula for generating functions for Kudla cycles with respect to an embedding of Shimura varieties. First let us describe the generating series as a function on adelic points. Recall that $\mathrm{Mp}_r(\mathbb{R})$ is a double cover of $\mathrm{Sp}_r(\mathbb{R})$. Let K' denote the preimage of the compact subgroup

$$\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a+ib \in U(r)\right\}$$

of $\mathrm{Sp}_r(\mathbb{R})$. The group K' has a character $\det^{1/2} := j(\cdot, i_r)^{1/2}$ whose square descends to the determinant character of $U(n)$, where $i_r := \sqrt{-1}\cdot I_r \in \mathcal{H}_r$. Write

$$Z_\phi(g') = Z_{\omega(g'_f)\phi}(g' i_r) j(g', i_r)^{-n/2-1}.$$

Then $Z_\phi(g')$ has a Fourier expansion

$$Z_\phi(g') = \sum_{x \in K\backslash \widehat{V}^r} (\omega_f(g'_f)\phi)(x) Z(x)_K W_{T(x)}(g'_\infty),$$

where $W_t(g'_\infty)$ is the tth 'holomorphic' Whittaker function on $\mathrm{Mp}_r(\mathbb{R})$ of weight $r/2+1$; for each $g' \in \mathrm{Mp}_r(\mathbb{R})$ with Iwasawa decomposition

$$g = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} k, \quad a \in \mathrm{GL}_r(\mathbb{R})^+, \quad k \in K'$$

we have

$$W_t(g) = |\det(a)|^{n/2+1} e(\mathrm{tr}\, t\tau) \det(k)^{n/2+1}.$$

Here

$$\tau = u + ia \cdot {}^t a.$$

Now the modularity of Z_ϕ is equivalent to the following identity:

$$Z_\phi(g') = Z_\phi(\gamma g') \quad \text{for all } \gamma \in \mathrm{Sp}_r(F).$$

Let $W \subseteq V$ be a positive-definite F-subspace of dimension d and let W' be its orthogonal complement. Then we have a decomposition $\mathcal{S}(V(\mathbb{A})^r) = \mathcal{S}(W'(\mathbb{A})^r) \otimes \mathcal{S}(W(\mathbb{A})^r)$. Consider the embedding map

$$i : M_{K,W} = G_W(\mathbb{Q})\backslash D_W \times G_W(\widehat{\mathbb{Q}})/K_W \to M_K = G(\mathbb{Q})\backslash D \times G(\widehat{\mathbb{Q}})/K,$$

where $K_W = G_W(\widehat{Q}) \cap K$. Then we have a pull-back map

$$i^* : \mathrm{Ch}^r(M_K) \to \mathrm{Ch}^r(M_{K,W}).$$

Now we want to prove a pull-back formula for

$$i^*(Z_\phi)(g') = \sum_{x \in K\backslash V(\widehat{F})^r} \omega(g'_f)\phi_f(x)\, i^*(Z(x)_K)\, W_{T(x)}(g'_\infty).$$

Proposition 3.1. *Let $\phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(V(\mathbb{A})^r) = \mathcal{S}(W'(\mathbb{A})^r) \otimes \mathcal{S}(W(\mathbb{A})^r)$ and suppose that ϕ_1, ϕ_2 are K-invariant. Then, we have an equality in the Chow group:*

$$i^*(Z_\phi)(g') = Z_{\phi_1}(g')\theta_{\phi_2}(g'), \tag{3.1}$$

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where $\theta_{\phi_1}(g')$ *is the generating function with coefficients in* $\mathrm{Ch}^r(M_{K,W},\mathbb{C})$,

$$Z_{\phi_1}(g') = \sum_{y\in K_W\backslash W'(\widehat{F})^r} \omega_f(g'_f)\phi(y)Z(y)_K W_{T(y)}(g'_\infty),$$

and

$$\theta_{\phi_2}(g') = \sum_{z\in W(F)^r} \omega(g')(\phi_2\otimes\phi_{2\infty})(z)$$

is the usual theta function, where $\phi_{2\infty}$ *is the standard spherical function on* $W'(\mathbb{R})$.

Proof. Let $x\in K\setminus V(\widehat{F})^r$. By Proposition 2.6, the intersection components of $Z(x)_K$ and $Z(W)_K$ are indexed by admissible classes in $K\setminus KW+Kx$. For an admissible class (W,kx), the projection of kx to $\widehat{W}$, denoted by z, must lie in W by the definition of admissibility. Thus, $y:=kx-z\in W'(\widehat{F})^r$. Conversely, for $y+z\in W'(\widehat{F})^{r,ad}\oplus W(F)^r$, $(W,y+z)$ must be admissible.

Therefore, we have in the Chow group of M_W the following identity:

$$i^*Z(x)_K = \sum_{(y,z)} Z(y)_{K_W}, \tag{3.2}$$

where the sum is over all admissible $y\in K_W\setminus W'(\widehat{F})^r$ and all $z\in W(F)^r$ such that

$$K_W(y+z)=K_Wy+z\supseteq Kx.$$

By the discussion above, we have

$$\begin{aligned} i^*Z_\phi(g') &= \sum_{x\in K\backslash V(\widehat{F})^r} \omega(g'_f)\phi(x)i^*Z(x)_K W_{T(x)}(g'_\infty)\\ &= \sum_{y\in K_W\backslash W'(\widehat{F})^r} \omega(g'_f)\phi_1(y)Z_{W'}(y)W_{T(y)}(g'_\infty)\sum_{z\in W(F)^r}\omega(g')\phi_2(z)W_{T(z)}(g'_\infty)\\ &= Z_{\phi_1}(g')\theta_{\phi_2}(g'). \end{aligned}$$

This completes the proof of the proposition. □

4. Modularity in Chow groups

In this section, we want to prove the modularity (Theorems 1.2 and 1.3) of generating series for a linear functional on Chow groups. We first treat the case of codimension one. Quite different from Borcherds' proof in [Bor99], our proof does not use Borcherds' 'singular theta lifting', which is actually unavailable on a totally real field except for $F=\mathbb{Q}$. Roughly speaking, the modularity for large n follows from the Kudla–Millson modularity for a cohomological class and the vanishing of the first Betti number of our Shimura varieties. For small n, we use a pull-back trick to deduce the desired modularity from that of large n.

4.1 Proof of Theorem 1.3

Suppose that ϕ is K-invariant. The group of cohomologically trivial line bundles, up to torsion, is parameterized by the connected component of the Picard variety of M_K. The tangent of the Picard is $H^1(M_K,\mathcal{O})$. For $n\geqslant 3$, $\dim_{\mathbb{C}} H^1(M_K,\mathcal{O})=0$ since it is half of the first Betti number of M_K, which is zero by Kumaresan's vanishing theorem and Vogan's and Zuckerman's explicit computation in [VZ84, Theorem 8.1]. Thus, the cycle class map is injective up to torsion and the theorem follows from the modularity of Kudla and Millson [KM90, Theorem 2] where the statement extends obviously to the adelic setting by their proof.

Now we assume $n \leqslant 2$. We can embed V into a higher dimension quadratic space $V' = V \oplus W$ such that $\dim_F V' \geqslant 5$ and with the desired signature at archimedean places. By Proposition 3.1, for any $\phi' \in \mathcal{S}(\widehat{W})$ we have

$$i^* Z_{\phi\otimes\phi'}(g') = Z_\phi(g')\theta_{\phi'}(g').$$

Since both $Z_{\phi\otimes\phi'}(g')$ and the usual theta function $\theta_{\phi'}(g')$ are convergent and $\mathrm{SL}_2(F)$-invariant, we deduce the convergence of $Z_\phi(g')$ and invariance under $\mathrm{SL}_2(F)$, provided that, for each g', we can choose ϕ' such that $\theta_{\phi'}(g') \neq 0$. However, we can make such a choice because, otherwise, for some g', $\theta_{\phi'}(g') = 0$ for all choices of ϕ'. This would imply that $\theta_{\phi'}(g'g_f) = \theta_{\omega(g_f)\phi'}(g') = 0$ for any $g_f \in \mathrm{Mp}(\widehat{F})$, which is a contradiction. This completes the proof of the theorem.

Remark. When $\dim_F V = 3$, the theorem above generalizes the Gross–Kohnen–Zagier theorem regarding Heegner points on modular curves to CM points on Shimura curves. The pull-back trick has already been used, as explained in Zagier's paper [Zag85] and the introduction to [GKZ87], to deduce the Gross–Kohnen–Zagier theorem in a special case from the theorem of Hirzebruch and Zagier [HZ76]. There, a key ingredient is the simple connectedness of Hilbert modular surfaces.

Combining their computation of the Néron–Tate pairing and a result of Waldspurger [Wal81], Gross *et al.* also proved in [GKZ87] that eigen-components of Heegner divisors on $X_0(N)$ are co-linear in the Mordell–Weil group. This can be viewed as a 'multiplicity one' result. We can give a representation-theoretic proof of this result along the same lines as in [Zha09]. Let B be a quaternion algebra over F such that it splits at exactly one archimedean place of F. Let V be the trace-free subspace of B. Together with the reduced norm, we obtain a three-dimensional quadratic space, and $G = G\mathrm{Spin}(V) = B^\times$. Let M_K be the Shimura curve for an open compact subgroup $K \subseteq G(\mathbb{A}_f)$. Let ξ be the Hodge class defined as in [Zha01]. Consider the subspace $\mathcal{M}_K$ of $\mathrm{Jac}(M_K)(F)$ generated by CM-divisors $Z(x)_K - \deg(Z(x)_K)\xi$ for all $x \in K \setminus \widehat{V}$. Consider the direct limit $\mathcal{M}$ of $\mathcal{M}_K$ for all K and consider it as a $G(\mathbb{A}_f)$-module. For a $G(\mathbb{A}_f)$-module π_f, let σ_f be the representation of $\mathrm{GL}_2(\mathbb{A}_f)$ associated by Jacquet–Langlands correspondence. Let $\sigma_{\infty,(2,2,\ldots,2)}$ be the homomorphic discrete series of $\mathrm{GL}_2(F_\infty)$ of parallel weight $(2, 2, \ldots, 2)$.

THEOREM 4.1. *For a $G(\mathbb{A}_f)$-module π_f with trivial central character,*

$$\dim \mathrm{Hom}_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f) \leqslant 1.$$

If $\mathrm{Hom}_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ is non-trivial, the product $\sigma = \sigma_f \otimes \sigma_{\infty,(2,2,\ldots,2)}$ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$.

Proof. For the first assertion, we sketch the proof and the complete details can be found in [Zha09, § 6]. Let ρ_f be the representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_f)$ defined by the local Howe duality for the pair $(\mathrm{SO}(V), \widetilde{\mathrm{SL}}_2)$ as in the work of Waldspurger. Let $\rho_{\infty,(3/2,\ldots,3/2)}$ be the homomorphic discrete series of $\widetilde{\mathrm{SL}}_2(F_\infty)$ of parallel weight $(3/2, \ldots, 3/2)$. Note that we have an equivariance of Hecke actions on the space $\mathcal{M}$ and the space $\mathcal{S}(V(\mathbb{A}_f))$. In our case, Theorem 1.3 actually implies that generating functions valued in $\mathcal{M}$ are all cuspidal forms. Then, $\mathrm{Hom}_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ vanishes unless $\rho = \rho_f \otimes \rho_{\infty,(3/2,\ldots,3/2)}$ is a cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, and the dimension of $\mathrm{Hom}_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ is bounded by the multiplicity of ρ in the space of cuspidal automorphic forms on $\widetilde{\mathrm{SL}}(2)$. The 'multiplicity one' result for cuspidal automorphic representations on $\widetilde{\mathrm{SL}}(2)$

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holds by Waldspurger's work. For the second assertion, the automorphy of ρ implies that of σ, again by Waldspurger's work. □

4.2 High-codimensional cycles

In the following we want to prove the modularity in Theorem 1.2 along the same lines as in [Zha09]. Now that we have already assumed the convergence of the generating series, we only need to verify the automorphy.

Step 0: modularity when $r=1$. When $r=1$, the assertion is implied by Theorem 1.3. We actually know that generating functions converge for all linear functionals ℓ.

Step 1: invariance under the Siegel parabolic subgroup. It is easy to see that the series $Z_\phi(g)$ is invariant under the Siegel parabolic subgroup of $\mathrm{Sp}_r(F)$. Indeed, it suffices to consider γ of the form

$$n(u) := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}.$$

By definition, we have

$$\omega(n(u)g'_f)\phi(x)W_{T(x)}(n(u)g'_\infty) = \omega(g')\phi(x)W_{T(x)}(g'_\infty).$$

Thus, every term in $Z_\phi(g)$ is invariant under $n(u)$. Also, by definition,

$$\omega(m(a)g'_f)\phi(x)W_{T(x)}(m(a)g'_\infty) = \omega(g'_f)\phi(xa)W_{T(xa)}(g'_\infty).$$

Since $Z(x)_K = Z(xa)_K$, the sum does not change after a substitution $x \to xa$.

Step 2: invariance under w_1. We want to show that $Z_\phi(g)$ is invariant under w_1, the image of $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ under the embedding of SL_2 into Sp_{2r}. This is the key step of the proof.

First, we can rewrite the sum as

$$Z_\phi(\tau) = \sum_{y \in K\backslash \widehat{V}^{r-1}} \sum_{x \in K_y \backslash \widehat{V}} \phi(x,y) Z(x,y)_K q^{T(x,y)},$$

where K_y is the stabilizer of y. One can write

$$Z(x,y)_K = \sum_{x_1,x_2} i_{y\,*} Z(x_1)_{K_y},$$

where

$$i_y : M_{K,y} \to M_K$$

and the sum is over all

$$x_1 \in y^\perp := \{z \in \widehat{V} : \langle z, y_i\rangle = 0, i = 1, 2, \dots, r-1\}$$

and $x_2 \in Fy := \sum_{i=1}^{r-1} Fy_i$ satisfying $K_y(x_1+x_2) = K_y x$ (see also (3.2)).

Thus, the sum becomes

$$Z_\phi(\tau) = \sum_{y \in K\backslash \widehat{V}^{r-1,ad}} \sum_{x_1 \in K_y \backslash y^\perp} \sum_{x_2 \in Fy} \phi(x_1+x_2, y) i_{y*}(Z(x_1)_{K_y}) q_1^{T(x_1)} \xi^{\langle x_2, y\rangle} q_2^{T(y)},$$

where

$$\xi^{\langle x_2,y\rangle} = \exp\bigg(2\pi i \sum_{k=1}^{d} \sum_{i=1}^{r-1} (z_{k,i}\langle x_2, y_i\rangle)\bigg),$$

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and we have a natural decomposition

$$q^{T(x)} = q_1^{T(x_1)} \xi^{\langle x_2, y\rangle} q_2^{T(y)}, \quad \tau_k = \begin{pmatrix} \tau_{k,1} & z_k \\ {}^t z_k & \tau_{k,2} \end{pmatrix}, \quad k = 1, \ldots, d$$

where $\tau_{k,1} \in \mathcal{H}_1,\ z_k \in \mathbb{C}^{r-1},\ \tau_{k,2} \in \mathcal{H}_{r-1}$. Here, we simply write $Z(x, y)$, $Z(x_1)$, etc., to mean cycle classes shifted by appropriate powers of tautological line bundles on ambient Shimura varieties. However, all of these tautological bundles are compatible with pull-backs (with respect to various compact subgroups K, K_y), or restrictions (from M_K to $M_{K,y}$). We therefore suppress them in the following exposition to avoid messing up the notation.

For fixed y, applying the modularity for divisors (proved in Step 0) to $\phi(x_1 + x_2, y)$ as a function of x_1, we know that under the substitution $\tau \mapsto w_1^{-1}\tau$,

$$\sum_{x_1 \in K_y \backslash y^\perp} \phi(x_1 + x_2, y) Z(x_1)_{K_y} q_1^{T(x_1)}$$

becomes

$$\sum_{x_1 \in K_y \backslash y^\perp} \widehat{\phi}^{\,1}(x_1 + x_2, y) Z(x_1)_{K_y} q_1^{T(x_1)}$$

where $\widehat{\phi}^1(x_1 + x_2, y)$ is the partial Fourier transformation with respect to x_1. Note that we here implicitly used the convergence of this partial series (by Theorem 1.3). By the Poisson summation formula, we also know that, under the same substitution,

$$\sum_{x_2 \in Fy} \widehat{\phi}^{\,1}(x_1 + x_2, y) Z(x_1)_{K_y} \xi^{\langle x_2, y\rangle} q_2^{T(y)}$$

becomes

$$\sum_{x_2 \in Fy} \widehat{\phi}^{\,1,2}(x_1 + x_2, y) Z(x_1)_{K_y} \xi^{\langle x_2, y\rangle} q_2^{T(y)}.$$

Note that $\omega(w_1)\phi(x, y) = \widehat{\phi}^x(x, y)$ is the partial Fourier transformation with respect to the first coordinate x. It is easy to see that for $x = x_1 + x_2$, also $\widehat{\phi}^{1,2}(x_1 + x_2, y) = \widehat{\phi}^x(x, y)$. This proves that

$$Z_\phi(w_1^{-1}\tau) = Z_{\omega(w_1)\phi}(\tau).$$

This proves that $Z_\phi(g')$ is invariant under w_1.

Step 3: invariance under $\mathrm{Sp}_{2r}(F)$. We claim that the Siegel parabolic subgroup and w_1 generate $\mathrm{Sp}_{2r}(F)$. In fact, $\mathrm{SL}_2(F)^r$ and the Siegel parabolic subgroup generate $\mathrm{Sp}_{2r}(F)$. Obviously, one needs just one copy of $\mathrm{SL}_2(F)$ since others can be obtained by permutations which are in the Siegel parabolic subgroup. Furthermore, one copy of $\mathrm{SL}_2(F)$ can be generated by w_1 and the Siegel parabolic subgroup. This proves the claim. Thus, we have finished the proof of Theorem 1.2 by Steps 0, 1 and 2.

Acknowledgements

This research was supported by a Focus Research Group grant from the National Science Foundation. The authors would like to thank Steve Kudla, Ken Ono, and Tonghai Yang for their help. The authors are grateful to the referee for his very helpful comments.

https://doi.org/10.1112/S0010437X08003734 Published online by Cambridge University Press

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https://doi.org/10.1112/S0010437X08003734 Published online by Cambridge University Press