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FUSION SYSTEMS ON BICYCLIC 2-GROUPS

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Abstract We classify all (saturated) fusion systems on bicyclic 2-groups. Here, a bicyclic group is a product of two cyclic subgroups. This extends previous work on fusion systems on metacyclic 2-groups. As an application we prove Olsson's conjecture for all blocks with bicyclic defect groups.

Keywords: fusion systems; bicyclic 2-groups; Olsson's conjecture

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1. Introduction

Fusion systems occur in many areas of mathematics; for example, group theory, representation theory and topology. This makes it interesting to classify fusion systems on a given family of finite p-groups. In particular, it is of general interest to find so-called *exotic* fusion systems, i.e. fusion systems that do not occur among finite groups (see, for example, [**24**]). On the other hand, it is often useful to know which p-groups admit only nilpotent (sometimes called trivial) fusion systems, i.e. fusion systems coming from p-groups. One family of p-groups that comes quickly to mind is the class of metacyclic p-groups. Here, for odd primes p it is known by the work of Stancu [**30**] that every fusion system is controlled. This means that one can classify these fusion systems by looking at p'-subgroups of the outer automorphism group and their action. In particular, only non-exotic fusion systems occur. The fusion systems on metacyclic 2-groups were determined in [**28**]. In this case the 2-groups of maximal class play an important role.

In order to generalize these results we consider p-groups P, which can be written in the form $P = \langle x\rangle\langle y\rangle$ for some $x, y \in P$. We call these groups *bicyclic*. For odd primes p, Huppert showed [**13**] that the class of bicyclic groups coincides with the class of metacyclic groups (see also [**14**, Satz III.11.5]). He also pointed out that this is not true for $p = 2$. A prominent counterexample is the wreath product $C_4 \wr C_2$. So the aim of this paper is to classify fusion systems on the wider class of bicyclic 2-groups.

Apart from Huppert's work, there are many other contributions to the theory of bicyclic 2-groups (see, for example, [**4**, **15**–**17**]). One of these early results is the following: let P

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be a non-metacyclic, bicyclic 2-group. Then the commutator subgroup P' is abelian of rank at most 2 and P/P' contains a cyclic maximal subgroup. Moreover, if P/P' has exponent at least 8, then P' also contains a cyclic maximal subgroup.

Recently, Janko [**18**] presented all bicyclic 2-groups in terms of generators and relations using an equivalent property (see Theorem 3.1). However, the classification of the bicyclic 2-groups is not complete, since in Janko's presentation it is not clear if some of the parameters give isomorphic groups. Even more recent results that deal with an application to bipartite graphs can be found in [**7**].

If not explicitly stated otherwise, all groups in this paper are finite and all fusion systems are saturated. In the second section we prove some general results about fusion systems on p-groups that are more or less consequences of Alperin's fusion theorem. After that we consider fusion systems on bicyclic 2-groups. Here we obtain the unexpected result that every fusion system on a bicyclic 2-group P is nilpotent unless P' is cyclic. Conversely, every bicyclic non-metacyclic 2-group with cyclic commutator subgroup provides a non-nilpotent fusion system. All these groups are cyclic extensions of (possibly abelian) dihedral or quaternion groups, and their number grows with the square of the logarithm of their order. Moreover, it turns out that no exotic fusion system shows up here (after the paper was written, Oliver [**23**] proved this for a larger class of p-groups). In fact, we construct these fusion systems as fusion systems of cyclic extension of finite groups of Lie type. The complete classification is given in Theorem 3.19. As a byproduct, we also investigate the isomorphism problem of some of the groups in Janko's paper [**18**]. At the end, as an application we prove that Olsson's conjecture of block theory holds for all blocks with bicyclic defect groups. Other conjectures for blocks with bicyclic defect groups have been investigated in a separate paper [**29**].

Most of our notation is standard. A finite p-group P is *r-generator* if $|P:\Phi(P)| = p^r$, i.e. P is generated by r elements, but not by fewer. Similarly, the *p-rank* of P is the maximal rank of an abelian subgroup of P. We denote the members of the lower central series of a p-group P by $\mathrm{K}_i(P)$; in particular, $\mathrm{K}_2(P) = P'$. Moreover, $\Omega_i(P) = \langle x \in P\colon x^{p^i} = 1\rangle$ and $\mho_i(P) := \langle x^{p^i}\colon x \in P\rangle$ for $i \geqslant 1$. For convenience we write $\Omega(P) := \Omega_1(P)$ and $\mho(P) := \mho_1(P)$. A cyclic group of order $n \in \mathbb{N}$ is denoted by C_n. Moreover, we set $\mathrm{C}_n^k := \mathrm{C}_n \times \cdots \times \mathrm{C}_n$ (k factors). In particular, groups of the form C_n^2 are called *homocyclic*. A dihedral (respectively, semi-dihedral, quaternion) group of order 2^n is denoted by D_{2^n} (respectively, SD_{2^n}, Q_{2^n}). A group G is *minimal non-abelian* if G is non-abelian, but all proper subgroups of G are abelian. We say that a p-group P is minimal non-abelian of *type* (r, s) if

$$P \cong \langle x, y \mid x^{p^r} = y^{p^s} = [x, y]^p = [x, x, y] = [y, x, y] = 1\rangle, \tag{1.1}$$

where $[x, y] := xyx^{-1}y^{-1}$ and $[x, y, z] := [x, [y, z]]$ (see [**25**]). Moreover, we set ${}^x y := xyx^{-1}$ for elements x and y of a group. A group extension with normal subgroup N is denoted by $N.H$. If the extension splits, we write $N \rtimes H$ for the semi-direct product. A central product is denoted by $N * H$, where it will be always clear which subgroup of $\mathrm{Z}(N)$ is merged with a subgroup of $\mathrm{Z}(H)$. For the language of fusion systems we refer the reader to [**5**].

2. General results

We begin with two elementary lemmas about minimal non-abelian groups.

Lemma 2.1. *A finite p-group P is minimal non-abelian if and only if P is 2-generator and $|P'| = p$.*

Proof. Assume first that P is minimal non-abelian. Choose two non-commuting elements $x, y \in P$. Then $\langle x, y\rangle$ is non-abelian and $P = \langle x, y\rangle$ is 2-generator. Every element $x \in P$ lies in a maximal subgroup $M \leqslant P$. Since M is abelian, $M \subseteq \mathrm{C}_P(x)$. In particular, all conjugacy classes of P have length at most p. By a result of Knoche (see [**14**, Aufgabe III.24 (b)]), we obtain $|P'| = p$.

Next, suppose that P is 2-generator and that $|P'| = p$. Then $P' \leqslant \mathrm{Z}(P)$. For $x, y \in P$ we have $[x^p, y] = [x, y]^p = 1$. Hence, $\Phi(P) = P'\langle x^p : x \in P\rangle \leqslant \mathrm{Z}(P)$. For any maximal subgroup $M \leqslant P$ it follows that $|M : \mathrm{Z}(P)| \leqslant |M : \Phi(P)| = p$. Therefore, M is abelian and P is minimal non-abelian. □

Lemma 2.2. *Let P be a minimal non-abelian group of type (r, s). Then the following hold:*

(i) $|P| = p^{r+s+1}$,

(ii) $\Phi(P) = \mathrm{Z}(P) = \langle x^2, y^2, [x, y]\rangle \cong \mathrm{C}_{p^{r-1}} \times \mathrm{C}_{p^{s-1}} \times \mathrm{C}_p$,

(iii) $P' = \langle [x, y]\rangle \cong \mathrm{C}_p$.

Proof. The proof is straightforward. □

By Alperin's fusion theorem, the morphisms of a fusion system $\mathcal{F}$ on a p-group P are controlled by the $\mathcal{F}$-essential subgroups of P.

Definition 2.3. A subgroup $Q \leqslant P$ is called *$\mathcal{F}$-essential* if the following properties hold:

(i) Q is *fully $\mathcal{F}$-normalized*, i.e. $|\mathrm{N}_P(R)| \leqslant |\mathrm{N}_P(Q)|$ if $R \leqslant P$ and Q are $\mathcal{F}$-isomorphic;

(ii) Q is *$\mathcal{F}$-centric*, i.e. $\mathrm{C}_P(R) = \mathrm{Z}(R)$ if $R \leqslant P$ and Q are $\mathcal{F}$-isomorphic;

(iii) $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ contains a strongly p-embedded subgroup H, i.e. $p \mid |H| < |\mathrm{Out}_{\mathcal{F}}(Q)|$ and $p \nmid |H \cap {}^xH|$ for all $x \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus H$.

Notice that in [**5**] the first property is not required. It should be pointed out that there are usually very few $\mathcal{F}$-essential subgroups. In many cases there are none. For the convenience of the reader we state a version of Alperin's fusion theorem. For this, let $\mathcal{E}$ be a set of representatives for the $\mathrm{Aut}_{\mathcal{F}}(P)$-conjugacy classes of $\mathcal{F}$-essential subgroups of P.

Theorem 2.4 (Alperin's fusion theorem). *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Then every isomorphism in $\mathcal{F}$ is a composition of finitely many isomorphisms of the form $\phi\colon S \to T$ such that $S, T \leqslant Q \in \mathcal{E} \cup \{P\}$, and there exists $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ with $\psi_{|S} = \phi$. Moreover, if $Q \neq P$, we may assume that ψ is a p-element.*

Proof. Apart from the last sentence, this is [**5**, Theorem 4.51]. Thus, for $S \in \mathcal{E}$ and $\phi \in \mathrm{Aut}_{\mathcal{F}}(S)$ we need to show that ϕ can be written as a composition of isomorphisms in the stated form. As $S < P$, also $S < \mathrm{N}_P(S)$, so by induction on $|P : S|$ we can assume that the claim is true for any $\mathcal{F}$-automorphism of $\mathrm{N}_P(S)$. Let $K := \langle f \in \mathrm{Aut}_{\mathcal{F}}(S)\ p\text{-element}\rangle \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(S)$. Since $\mathrm{Aut}_P(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, the Frattini argument implies that $\mathrm{Aut}_{\mathcal{F}}(S) = K\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_P(S))$. Hence, we can write $\phi = \alpha\beta$ such that $\alpha \in K$ and $\beta \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_P(S))$. With the notation of [**5**] we have $\mathrm{N}_\beta = \mathrm{N}_P(S)$. Then β can be extended to a morphism β' on $\mathrm{N}_P(S)$. Since $S < \mathrm{N}_P(S)$, induction shows that β' is a composition of isomorphisms of the stated form and so are $\beta = \beta'_{|S}$ and β^{-1}. Thus, after replacing ϕ by $\phi \circ \beta^{-1}$, we may assume that $\phi \in K$. Then it is obvious that ϕ is a composition of isomorphisms as desired. □

We deduce some necessary conditions for a subgroup $Q \leqslant P$ to be $\mathcal{F}$-essential. Since Q is $\mathcal{F}$-centric, we have $\mathrm{C}_P(Q) \subseteq Q$. Since $\mathrm{Out}_{\mathcal{F}}(Q)$ contains a strongly p-embedded subgroup, $\mathrm{Out}_{\mathcal{F}}(Q)$ is not a p-group and not a p'-group. Moreover, $\mathrm{N}_P(Q)/Q$ is isomorphic to a Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$. This shows that $Q < P$. We also have $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Inn}(Q)$. Consider the canonical homomorphism

$$F\colon \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Q/\Phi(Q)).$$

It is well known that $\mathrm{Ker}\, F$ is a p-group. On the other hand, $\mathrm{Inn}(Q)$ acts trivially on the abelian group $Q/\Phi(Q)$. This gives $\mathrm{Ker}\, F = \mathrm{Inn}(Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\mathcal{F}}(Q/\Phi(Q))$.

Lemma 2.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $Q \leqslant P$ be $\mathcal{F}$-essential. If $|Q| \leqslant p^2$ or if Q is non-abelian of order p^3, then P has maximal class.*

Proof. This follows from [**2**, Propositions 1.8 and 10.17]. □

Now we turn to 2-groups.

Lemma 2.6. *Let $\mathcal{F}$ be a fusion system on a finite 2-group P. If $Q \leqslant P$ is an $\mathcal{F}$-essential r-generator subgroup with $r \leqslant 3$, then $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{S}_3$ and $|\mathrm{N}_P(Q) : Q| = 2$.*

Proof. By the remark above, we have $\mathrm{Out}_{\mathcal{F}}(Q) \leqslant \mathrm{GL}(r, 2)$. Hence, we may assume that Q is 3-generator. Then $\mathrm{Out}_{\mathcal{F}}(Q) \leqslant \mathrm{GL}(3, 2)$. A computer calculation (which, of course, can be carried out by hand as well) shows that S_3 is the only subgroup of $\mathrm{GL}(3, 2)$ (up to isomorphism) with a strongly 2-embedded subgroup. □

Proposition 2.7. *Let $\mathcal{F}$ be a fusion system on a finite 2-group P. If $Q \leqslant P$ is an $\mathcal{F}$-essential 2-generator subgroup, then one of the following holds:*

(i) *$Q \cong \mathrm{C}_2^2$ and $P \in \{\mathrm{D}_{2^n}, \mathrm{SD}_{2^n}\}$ for some $n \geqslant 3$,*

(ii) *$Q \cong \mathrm{Q}_8$ and $P \in \{\mathrm{Q}_{2^n}, \mathrm{SD}_{2^n}\}$ for some $n \geqslant 3$,*

(iii) *$Q \cong \mathrm{C}_{2^r}^2$ and $P \cong \mathrm{C}_{2^r} \wr \mathrm{C}_2$ for some $r \geqslant 2$,*

(iv) *$Q/\Phi(Q')\,\mathrm{K}_3(Q)$ is minimal non-abelian of type (r, r) for some $r \geqslant 2$.*

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Proof. By Lemma 2.6, we have $|\mathrm{N}_P(Q) : Q| = 2$. If Q is metacyclic, then we have $Q \cong \mathrm{Q}_8$ or $Q \cong \mathrm{C}_{2^r}^2$ for some $r \in \mathbb{N}$ by [**22**, Lemma 1]. Then for $|Q| \leqslant 8$ the result follows from Lemma 2.5. Thus, assume that $Q \cong \mathrm{C}_{2^r}^2$ for some $r \geqslant 2$. Here we argue along the lines of [**6**, Lemma 2.4]. Let $g \in \mathrm{N}_P(Q) \setminus Q$. Since g acts non-trivially on $Q/\Phi(Q)$, we may assume that ${}^g x = y$ and ${}^g y = x$ for $Q = \langle x, y \rangle$. We can write $g^2 = (xy)^i$ for some $i \in \mathbb{Z}$, because g centralizes g^2. Then an easy calculation shows that gx^{-i} has order 2. Hence, $\mathrm{N}_P(Q) \cong \mathrm{C}_{2^r} \wr \mathrm{C}_2$. Since Q is the only abelian maximal subgroup in $\mathrm{N}_P(Q)$, we also have $Q \trianglelefteq \mathrm{N}_P(\mathrm{N}_P(Q))$, and $\mathrm{N}_P(Q) = P$ follows.

Now consider the case in which Q is non-metacyclic. Then Q is also non-abelian. By [**14**, Hilfssatz III.1.11 (c)] we know that $Q'/\mathrm{K}_3(Q)$ is cyclic. In particular, $Q'/\Phi(Q')\,\mathrm{K}_3(Q)$ has order 2. By Lemma 2.1, $\bar{Q} := Q/\mathrm{K}_3(Q)\Phi(Q')$ is minimal non-abelian. The case $\bar{Q} \cong \mathrm{Q}_8$ is impossible, because Q does not have maximal class (Taussky's theorem; see [**14**, Satz III.11.9]). Let α be an automorphism of Q of order 3. Since α acts non-trivially on $\bar{Q}$, [**27**, Lemma 2.2] implies that $\bar{Q}$ is of type (r, r) for some $r \geqslant 2$. □

The fusion systems in the first three parts of Proposition 2.7 are determined in [**6**] (see also Theorem 3.19). Notice that we have not proved that case (iv) actually occurs. However, calculations with GAP [**8**] show that there are at least small examples, and it is reasonable that many examples exist for arbitrary $r \geqslant 2$. However, we have no example of case (iv) where $Q \ntrianglelefteq P$.

Lemma 2.8. *Let $\mathcal{F}$ be a fusion system on a finite 2-group P. If $Q \leqslant P$ is an $\mathcal{F}$-essential 3-generator subgroup, then $\mathrm{N}_P(Q)/\Phi(Q) \cong \mathrm{D}_8 \times \mathrm{C}_2$ or $\mathrm{N}_P(Q)/\Phi(Q)$ is minimal non-abelian of type $(2, 1)$.*

Proof. By Lemma 2.6, we have $|\mathrm{N}_P(Q) : Q| = 2$. Since $\mathrm{N}_P(Q)$ acts non-trivially on $Q/\Phi(Q)$, we conclude that $\mathrm{N}_P(Q)/\Phi(Q)$ is non-abelian. One can check that there are only two non-abelian groups of order 16 with an elementary abelian subgroup of order 8. The claim follows. □

3. Bicyclic 2-groups

Janko gave the following characterization of bicyclic 2-groups (see [**18**] or alternatively [**3**, § 87]). Notice that in [**18**] Janko defines commutators differently than we do.

Theorem 3.1 (Janko [18]). *A non-metacyclic 2-group P is bicyclic if and only if P is 2-generator and contains exactly one non-metacyclic maximal subgroup.*

Using this, Janko classified all bicyclic 2-groups in terms of generators and relations. However, it is not clear if different parameters in his paper give non-isomorphic groups. In particular, the number of isomorphism types of bicyclic 2-groups is unknown.

As a corollary of Theorem 3.1, we obtain the structure of the automorphism group of a bicyclic 2-group.

Proposition 3.2. *Let P be a bicyclic 2-group such that $\mathrm{Aut}(P)$ is not a 2-group. Then P is homocyclic or a quaternion group of order 8. In particular, P is metacyclic.*

Proof. By [**22**, Lemma 1], we may assume that P is non-metacyclic. Since P is 2-generator, every non-trivial automorphism of odd order permutes the maximal subgroups of P transitively. By Theorem 3.1, such an automorphism cannot exist. □

As another corollary of Theorem 3.1 we see that every subgroup of a bicyclic 2-group contains a metacyclic maximal subgroup. Since quotients of bicyclic groups are also bicyclic, it follows that every section of a bicyclic 2-group is r-generator with $r \leqslant 3$. This will be used in the following without any explicit comment. Since here and in the following the arguments are very specific (i.e. not of general interest), we will sometimes apply computer calculations in order to handle small cases.

Proposition 3.3. *Let $\mathcal{F}$ be a fusion system on a bicyclic non-metacyclic 2-group P. Suppose that P contains an $\mathcal{F}$-essential 2-generator subgroup Q. Then $Q \cong \mathrm{C}_{2^m}^2$ and $P \cong \mathrm{C}_{2^m} \wr \mathrm{C}_2$ for some $m \geqslant 2$. Moreover, $\mathcal{F} = \mathcal{F}_P(\mathrm{C}_{2^m}^2 \rtimes \mathrm{S}_3)$ or $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(3, q))$ for some $q \equiv 1 \pmod 4$.*

Proof. By Proposition 2.7, it suffices for the first claim to show that Q is metacyclic, since minimal non-abelian groups of type (m, m) for $m \geqslant 2$ are non-metacyclic (see [**18**, Proposition 2.8]). Let $M \leqslant P$ be a metacyclic maximal subgroup of P. We may assume that $Q \nsubseteq M$. Then $M \cap Q$ is a maximal subgroup of Q. Since Q admits an automorphism of order 3, the maximal subgroups of Q are isomorphic. Now the first claim follows from [**18**, Proposition 2.2]. The fusion systems on $\mathrm{C}_{2^m} \wr \mathrm{C}_2$ are given by [**6**, Theorem 5.3]. Two of them have $\mathrm{C}_{2^m}^2$ as essential subgroup. □

It can be seen that the group $\mathrm{C}_{2^m} \wr \mathrm{C}_2$ is in fact bicyclic. Observe that [**6**, Theorem 5.3] provides another non-nilpotent fusion system on $\mathrm{C}_{2^m} \wr \mathrm{C}_2$. For the rest of this paper we consider the case in which the bicyclic non-metacyclic 2-group P has no $\mathcal{F}$-essential 2-generator subgroup.

Definition 3.4. Two fusion systems $\mathcal{F}$ and $\mathcal{F}'$ on a finite p-group P are *isomorphic* if there is an automorphism $\gamma \in \mathrm{Aut}(P)$ such that

$$\mathrm{Hom}_{\mathcal{F}'}(\gamma(S), \gamma(T)) = \gamma(\mathrm{Hom}_{\mathcal{F}}(S, T)) := \{\gamma \circ \phi \circ \gamma^{-1} \colon \phi \in \mathrm{Hom}_{\mathcal{F}}(S, T)\}$$

for all subgroups $S, T \leqslant P$.

Observe that if γ is an inner automorphism of P, then $\mathrm{Hom}_{\mathcal{F}}(\gamma(S), \gamma(T)) = \gamma(\mathrm{Hom}_{\mathcal{F}}(S, T))$ for all $S, T \leqslant P$. In the following we consider fusion systems only up to isomorphism.

Proposition 3.5. *Let $\mathcal{F}$ be a non-nilpotent fusion system on a bicyclic 2-group P. Suppose that P contains an elementary abelian normal subgroup of order 8. Then P is minimal non-abelian of type $(n, 1)$ for some $n \geqslant 2$ and $\mathrm{C}_{2^{n-1}} \times \mathrm{C}_2^2$ is the only $\mathcal{F}$-essential subgroup of P. Moreover, $\mathcal{F} = \mathcal{F}_P(\mathrm{A}_4 \rtimes \mathrm{C}_{2^n})$, where C_{2^n} acts as a transposition in $\mathrm{Aut}(\mathrm{A}_4) \cong \mathrm{S}_4$ (thus, $\mathrm{A}_4 \rtimes \mathrm{C}_{2^n}$ is unique up to isomorphism).*

Proof. By hypothesis, P is non-metacyclic. Suppose first that $|P'| = 2$. Then P is minimal non-abelian of type $(n, 1)$ for some $n \geqslant 2$ by [**18**, Theorem 4.1]. We show that P contains exactly one $\mathcal{F}$-essential subgroup Q. Since P is minimal non-abelian, every self-centralizing subgroup is maximal. Moreover, Q is 3-generator by Proposition 3.3. Hence, $Q = \langle x^2, y, z\rangle \cong \mathrm{C}_{2^{n-1}} \times \mathrm{C}_2^2$ is the unique non-metacyclic maximal subgroup of P (notation from (1.1)). We prove that $\mathcal{F}$ is unique up to isomorphism. By Alperin's fusion theorem and Proposition 3.2 it suffices to describe the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ on Q. First of all $P = \mathrm{N}_P(Q)$ acts on only two 4-subgroups $\langle y, z\rangle$ and $\langle x^{2^{n-1}}y, z\rangle$ of Q non-trivially. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ of order 3. Then α is unique up to conjugation in $\mathrm{Aut}(Q)$ since $\langle\alpha\rangle \in \mathrm{Syl}_3(\mathrm{Aut}(Q))$ and $\mathrm{Aut}(Q)$ is not 3-nilpotent. Hence, α acts on only one 4-subgroup R of Q. Let $\beta \in P/Q \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$. Then $(\alpha\beta)(R) = (\beta\alpha^{-1})(R) = \beta(R) = R$ since $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{S}_3$ by Lemma 2.6. Thus, $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts (non-trivially) on $\langle y, z\rangle$ or on $\langle x^{2^{n-1}}y, z\rangle$. It can be easily seen that the elements x and $x^{2^{n-1}}y$ satisfy the same relations as x and y. Hence, after replacing y by $x^{2^{n-1}}y$ if necessary, we may assume that $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts on $\langle y, z\rangle$. Since $\mathrm{C}_Q(\alpha) \cong \mathrm{C}_{2^{n-1}}$, we see that $x^2y \notin \mathrm{C}_Q(\alpha)$ or $x^2yz \notin \mathrm{C}_Q(\alpha)$. But then $x^2y, x^2yz \notin \mathrm{C}_Q(\alpha)$, because $\beta(x^2y) = x^2yz$. Hence, $\mathrm{C}_Q(\alpha) = \mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) \in \{\langle x^2\rangle, \langle x^2z\rangle\}$. However, xy and y fulfil the same relations as x and y. Hence, after replacing x by xy if necessary, we have $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \langle x^2\rangle$. This determines the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ on Q completely. In particular, $\mathcal{F}$ is uniquely determined up to isomorphism. The group $G = \mathrm{A}_4 \rtimes \mathrm{C}_{2^n}$ as described in the proposition has a minimal non-abelian Sylow 2-subgroup of type $(n, 1)$. Since A_4 is not 2-nilpotent, $\mathcal{F}_P(G)$ is not nilpotent. It follows that $\mathcal{F} = \mathcal{F}_P(G)$.

Now suppose that $|P'| > 2$. Then [**18**, Theorem 4.2] describes the structure of P. We use the notation of this theorem. Let $Q < P$ be $\mathcal{F}$-essential. By Proposition 3.3, Q is 3-generator. In particular, Q is contained in the unique non-metacyclic maximal subgroup $M := E\langle a^2\rangle$ of P. Since $\langle a^4, u\rangle = \mathrm{Z}(M) < Q$, it follows that $Q \in \{\langle a^4, u, v\rangle, \langle a^4, a^2v, u\rangle, M\}$. In the first two cases we have $P' = \langle u, z\rangle \subseteq Q \trianglelefteq P$, which contradicts Lemma 2.6. Hence, $Q = M$. Every automorphism of M of order 3 acts non-trivially on $M/\Phi(M)$, and thus freely on $M/\mathrm{Z}(M) \cong \mathrm{C}_2^2$. However, the subgroups $L \leqslant M$ such that $\mathrm{Z}(M) < L < M$ are non-isomorphic, which is a contradiction. □

It remains to deal with the case in which P does not contain an elementary abelian normal subgroup of order 8. In particular, [**18**, Theorem 4.3] applies.

Lemma 3.6. *Let $\mathcal{F}$ be a fusion system on a bicyclic 2-group P. If $Q \leqslant P$ is $\mathcal{F}$-essential and 3-generator, then one of the following holds:*

(i) *$Q \trianglelefteq P$ and $P/\Phi(Q)$ is minimal non-abelian of type $(2, 1)$,*

(ii) *$Q \ntrianglelefteq P$ and $P/\Phi(Q) \cong \mathrm{D}_8 \times \mathrm{C}_2$.*

Proof. By Lemma 2.8, we always have that $\mathrm{N}_P(Q)/\Phi(Q)$ is minimal non-abelian of type $(2, 1)$ or isomorphic to $\mathrm{D}_8 \times \mathrm{C}_2$. In the $\mathrm{N}_P(Q) = P$ case only the first possibility can apply, since P is 2-generator. Now assume that $Q \ntrianglelefteq P$ and $\mathrm{N}_P(Q)/\Phi(Q)$ is minimal non-abelian of type $(2, 1)$. Take $g \in \mathrm{N}_P(\mathrm{N}_P(Q)) \setminus \mathrm{N}_P(Q)$ such that $g^2 \in \mathrm{N}_P(Q)$. Then

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$Q_1 := {}^gQ \neq Q$ and $Q_1 \cap Q$ is $\langle g\rangle$-invariant. Moreover, $\Phi(Q) \subseteq \Phi(N_P(Q)) \subseteq Q_1$ and

$$\begin{aligned}|\Phi(Q) : \Phi(Q) \cap \Phi(Q_1)| &= |\Phi(Q_1) : \Phi(Q) \cap \Phi(Q_1)| \\ &= |\Phi(Q_1)\Phi(Q) : \Phi(Q)| \\ &= |\Phi(Q_1/\Phi(Q))| \\ &= 2,\end{aligned}$$

since $Q_1/\Phi(Q)$ ($\neq Q/\Phi(Q)$) is abelian of rank 2. Hence, $N_P(Q)/\Phi(Q) \cap \Phi(Q_1)$ is a group of order 32 and 2-generator with two distinct normal subgroups of order 2 such that their quotients are isomorphic to the minimal non-abelian group of type $(2,1)$. It follows that $N_P(Q)/\Phi(Q) \cap \Phi(Q_1)$ is the minimal non-abelian group of type $(2,2)$ (this can be checked by computer). However, then all maximal subgroups of $N_P(Q)/\Phi(Q) \cap \Phi(Q_1)$ are 3-generator, which contradicts Theorem 3.1. Thus, we have proved that $N_P(Q)/\Phi(Q) \cong D_8 \times C_2$. □

We are in a position to determine all $\mathcal{F}$-essential 3-generator subgroups on a bicyclic 2-group. The following is a key result for the rest of the paper.

Proposition 3.7. *Let $\mathcal{F}$ be a fusion system on a bicyclic 2-group P. If $Q \leqslant P$ is $\mathcal{F}$-essential and 3-generator, then one of the following holds:*

(i) $Q \cong C_{2^m} \times C_2^2$ *for some* $m \geqslant 1$,

(ii) $Q \cong C_{2^m} \times Q_8$ *for some* $m \geqslant 1$,

(iii) $Q \cong C_{2^m} * Q_8$ *for some* $m \geqslant 2$.

Proof. If P contains an elementary abelian normal subgroup of order 8, then the conclusion holds by Proposition 3.5. Hence, we will assume that there is no such normal subgroup. Let $\alpha \in \mathrm{Out}_{\mathcal{F}}(Q)$ be of order 3 (see Lemma 2.6). Since $|\mathrm{Aut}(Q)|$ is not divisible by 9, we can regard α as an element of $\mathrm{Aut}(Q)$ by choosing a suitable preimage. We apply [**31**, Theorem B] to the group Q (observe that the rank in [**31**] is the p-rank in our setting). Let $C := C_Q(\alpha)$. Suppose first that C has 2-rank 3, i.e. $m(C) = 3$ with the notation of [**31**]. Since $[Q, \alpha]$ is generated by at most three elements, only the first part of [**31**, Theorem B] can occur. In particular, $Q \cong Q_8 * C$. However, this implies that Q contains an r-generator subgroup with $r \geqslant 4$, which is a contradiction.

Now assume that $m(C) = 2$. Then [**31**, Theorem A] gives $Q \cong Q_8 * C$. Let $Z \leqslant Z(Q_8 \times C) = \Phi(Q_8) \times Z(C)$ such that $Q \cong (Q_8 \times C)/Z$. Then $|Z| = 2$ and C is r-generator with $r \leqslant 2$, since Q is 3-generator. Moreover, it follows that $\Omega(Z(C)) \nsubseteq \Phi(C)$ (otherwise $Z \leqslant \Phi(Q_8) \times \Phi(C) = \Phi(Q_8 \times C)$). By Burnside's basis theorem, $C \cong C_2 \times C_{2^m}$ is abelian and $Q \cong Q_8 \times C_{2^m}$ for some $m \geqslant 1$.

Finally, suppose that $m(C) \leqslant 1$, i.e. C is cyclic or quaternion. By Theorem 3.1, $\Phi(P)$ is metacyclic. Since $\Phi(Q) \subseteq \Phi(P)$ (see [**14**, Satz III.3.14]), $\Phi(Q)$ is also metacyclic. According to the action of α on $\Phi(Q)$, one of the following holds (see Proposition 3.2):

(a) $\Phi(Q) \leqslant C \trianglelefteq Q$,

(b) $\Phi(Q) \cong Q_8$,

(c) $\Phi(Q) \cap C = 1$ and $\Phi(Q) \cong C_{2^n}^2$ for some $n \geqslant 1$.

We handle these cases separately. First assume case (a). By [**20**, 8.2.2 (a)], we have $|Q : C| = 4$ and α acts freely on Q/C. On the other hand, α acts trivially on $Q/\mathrm{C}_Q(C)$ by [**20**, 8.1.2 (b)]. This shows that $Q = C\,\mathrm{C}_Q(C)$. If C is quaternion, then $Q = Q_{2^n} * \mathrm{C}_Q(C)$. In particular, $\mathrm{C}_Q(C)$ is r-generator with $r \leqslant 2$. Thus, a similar argument as above yields $Q \cong Q_{2^n} \times C_{2^m}$. However, this is impossible here, because α would act trivially on $Q/\Phi(Q)$ by the definition of C. Hence, C is cyclic and central of index 4 in Q. Since Q is 3-generator, the exponents of C and Q coincide. If Q is abelian, we must have $Q \cong C_{2^m} \times C_2^2$ for some $m \geqslant 1$. Now assume that Q is non-abelian. Write $C = \langle a \rangle$ and choose $b, c \in Q$ such that $Q/C = \langle bC, cC \rangle$. Since $\langle b \rangle C$ is abelian and non-cyclic, we may assume that $b^2 = 1$. Similarly, $c^2 = 1$. Since Q is non-abelian, ${}^{c}b \neq b$. Let $|C| = 2^m$, where $m \geqslant 2$. Then $a \in \mathrm{Z}(Q)$ implies that ${}^{c}b = a^{2^{m-1}}b$. Thus, Q is uniquely determined as

$$Q = \langle a, b, c \mid a^{2^m} = b^2 = c^2 = [a, b] = [a, c] = 1,\ {}^{c}b = a^{2^{m-1}}b \rangle.$$

Since the group $Q_8 * C_{2^m} \cong D_8 * C_{2^m}$ has the same properties, we obtain $Q \cong Q_8 * C_{2^m}$.

Next we will show that case (b) cannot occur for any finite group Q. On the one hand we have $Q/\mathrm{C}_Q(\Phi(Q)) \leqslant \mathrm{Aut}(Q_8) \cong S_4$. On the other hand,

$$C_2^2 \cong \Phi(Q)\,\mathrm{C}_Q(\Phi(Q))/\mathrm{C}_Q(\Phi(Q)) \leqslant \Phi(Q/\mathrm{C}_Q(\Phi(Q))),$$

which is a contradiction.

It remains to deal with case (c). Again we will derive a contradiction. By [**31**, Theorem D], $C \neq 1$ (U_{64} is 4-generator). The action of α on $Q/\Phi(Q)$ shows that $|P : C\Phi(Q)| \geqslant 4$. Now $\Phi(Q) \cap C = 1$ implies that $|C| = 2$. There exists an α-invariant maximal subgroup $N \trianglelefteq Q$. Thus, $N \cap C \subseteq N \cap C\Phi(Q) \cap C = \Phi(Q) \cap C = 1$. In particular, we can apply [**31**, Theorem D], which gives $N \cong C_{2^{n+1}}^2$. Hence, $Q \cong N \rtimes C = C_{2^{n+1}}^2 \rtimes C_2$ (here $\rtimes$ can also mean $\times$). Choose $x, y \in N$ such that $\alpha(x) = y$ and $\alpha(y) = x^{-1}y^{-1}$. Let $C = \langle c \rangle$. Since Q is 3-generator, c acts trivially on $N/\Phi(N)$. Hence, we find integers i, j such that ${}^{z}x = x^i y^j$ and $i \equiv 1 \pmod 2$ and $j \equiv 0 \pmod 2$. Then ${}^{c}y = \alpha({}^{z}x) = x^{-j}y^{i-j}$. In particular, the isomorphism type of Q only depends on i, j. Since $c^2 = 1$, we obtain $i^2 - j^2 \equiv 1 \pmod{2^{n+1}}$ and $j(2i - j) \equiv 0 \pmod{2^{n+1}}$. We will show that $j \equiv 0 \pmod{2^n}$. This is true for $n = 1$. Thus, assume that $n \geqslant 2$. Then $1 - j^2 \equiv i^2 - j^2 \equiv 1 \pmod 8$. Therefore, $j \equiv 0 \pmod 4$. Now $j(2i - j) \equiv 0 \pmod{2^{n+1}}$ implies that $j \equiv 0 \pmod{2^n}$. In particular, $i^2 \equiv i^2 - j^2 \equiv 1 \pmod{2^{n+1}}$. Hence, we have two possibilities for j and at most four possibilities for i. This gives at most eight isomorphism types for Q. Now we split the proof into cases in which $Q \trianglelefteq P$ and $Q \ntrianglelefteq P$.

Suppose that $Q \trianglelefteq P$. Then $|P : Q| = 2$ by Lemma 2.6. Moreover, $\Omega(Q) \trianglelefteq P$. Since P does not contain an elementary abelian normal subgroup of order 8, it follows that Q contains more than seven involutions. With the notation above, let $x^r y^s c$ be an involution such that $x^r y^s \notin \Omega(N)$. Then $1 = x^r y^s c x^r y^s c = x^{r+ir-js} y^{s+jr+(i-j)s}$ and $r(1+i) - js \equiv$

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$s(1+i)+jr-js \equiv 0 \pmod{2^{n+1}}$. In the $n=1$ case we have $|P|=64$. Here it can be shown by computation that P does not exist. Hence, suppose that $n \geqslant 2$ in the following. Suppose furthermore that $i \equiv 1 \pmod{2^n}$. Then we obtain $2r \equiv 2s \equiv 0 \pmod{2^n}$. Since $x^r y^s \notin \Omega(N)$ we may assume that $r \equiv \pm 2^{n-1} \pmod{2^{n+1}}$ (the $s \equiv \pm 2^{n-1} \pmod{2^{n+1}}$ case is similar). However, this leads to the contradiction $0 \equiv r(1+i)-js \equiv 2^n \pmod{2^{n+1}}$. This shows that $i \equiv -1 \pmod{2^n}$. In particular, $x^{i-1}y^i = {}^c x x^{-1} = [c,x] \in Q'$ and $x^{-j}y^{i-j-1} = [c,y] \in Q'$. This shows that $\mathrm{C}_{2^n}^2 \cong Q' = \Phi(Q)$. By Lemma 3.6, $P/\Phi(Q)$ is minimal non-abelian of type $(2,1)$. Since $Q' \subseteq P'$, we conclude that $P/P' \cong \mathrm{C}_4 \times \mathrm{C}_2$. Then P is described in [**18**, Theorem 4.11]. In particular, $\Phi(P)$ is abelian. Choose $g \in P \setminus Q$. Then g acts non-trivially on $N/\Phi(Q)$, because α does as well. This shows that $N \trianglelefteq P$ and $\mathrm{C}_2^2 \cong N/\Phi(Q) \neq \mathrm{Z}(P/\Phi(Q)) = \Phi(P/\Phi(Q))$. Hence, P/N is cyclic and $\Phi(P) \neq N$. Therefore, Q contains two abelian maximal subgroups and $N \cap \Phi(P) \subseteq \mathrm{Z}(Q)$. Now a result of Knoche (see [**14**, Aufgabe III.7.24]) gives the contradiction $|Q'|=2$.

Now assume that $Q \ntrianglelefteq P$. We will derive the contradiction that $\mathrm{N}_P(Q)$ does not contain a metacyclic maximal subgroup. By Lemma 3.6, $\mathrm{N}_P(Q)/\Phi(Q) \cong \mathrm{D}_8 \times \mathrm{C}_2$. Choose $g \in \mathrm{N}_P(Q) \setminus Q$. Then g acts non-trivially on $N/\Phi(N)$, because α does as well. In particular, $N \trianglelefteq \mathrm{N}_P(Q)$. This implies that

$$g^2\Phi(Q) \in \mho(\mathrm{N}_P(Q)/\Phi(Q)) = (\mathrm{N}_P(Q)/\Phi(Q))' \subseteq N/\Phi(Q)$$

and $g^2 \in N$. As above, we may choose $x, y \in N$ such that ${}^g x = y$ and ${}^g y = x$. Since g centralizes g^2, we can write $g^2 = (xy)^i$ for some $i \in \mathbb{Z}$. Then gx^{-i} has order 2. Hence, we may assume that $g^2 = 1$ and $\langle N, g\rangle \cong \mathrm{C}_{2^{n+1}} \wr \mathrm{C}_2$. In the $n=1$ case we have $|\mathrm{N}_P(Q)| = 64$. Here one can show by computation that $\mathrm{N}_P(Q)$ does not exist. Hence, $n \geqslant 2$. Let M be a metacyclic maximal subgroup of $\mathrm{N}_P(Q)$. Since $\langle \Phi(Q), g\rangle \cong \mathrm{C}_{2^n} \wr \mathrm{C}_2$ is not metacyclic, we conclude that $g \notin M$. Let $C = \langle c\rangle$. Then $\langle \Phi(Q), c\rangle$ is 3-generator. In particular, $c \notin M$. This leaves two possibilities for M. It is easy to see that $\langle N, gc\rangle \cong \mathrm{C}_{2^{n+1}} \wr \mathrm{C}_2$. Thus, $M = \langle \Phi(Q), xc, gc\rangle$. Assume that $(gc)^2 \in \Phi(Q)$. Then it is easy to see that $\langle \Phi(Q), gc\rangle \cong \mathrm{C}_{2^n} \wr \mathrm{C}_2$ is not metacyclic. This contradiction shows that $(gc)^2 \equiv xy \pmod{\Phi(Q)}$. Moreover, $c(gc)^2c = (cg)^2 = (gc)^{-2}$. Since $N = \langle gc, \alpha(gc)\rangle$, c acts by inversion on N. In particular, $(xc)^2 = 1$. Hence, $\langle \Omega(Q), xc\rangle \subseteq M$ is elementary abelian of order 8, which is a contradiction. □

Let Q be one of the groups in Proposition 3.7. Then it can be seen that there is an automorphism $\alpha \in \mathrm{Aut}(Q)$ of order 3. Since the kernel of the canonical map $\mathrm{Aut}(Q) \to \mathrm{Aut}(Q/\Phi(Q)) \cong \mathrm{GL}(3,2)$ is a 2-group, we have $\langle\alpha\rangle \in \mathrm{Syl}_3(\mathrm{Aut}(Q))$. If α is not conjugate to α^{-1} in $\mathrm{Aut}(Q)$, then Burnside's transfer theorem implies that $\mathrm{Aut}(Q)$ is 3-nilpotent. But then $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{S}_3$ would also be 3-nilpotent, which is not the case. Hence, α is unique up to conjugation in $\mathrm{Aut}(Q)$. In particular, the isomorphism type of $\mathrm{C}_Q(\alpha)$ is uniquely determined.

Proposition 3.8. *Let $\mathcal{F}$ be a fusion system on a bicyclic 2-group P. If $Q \trianglelefteq P$ is $\mathcal{F}$-essential and 3-generator, then one of the following holds.*

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(i) *P is minimal non-abelian of type $(n,1)$ for some $n \geqslant 2$.*

(ii) *$P \cong \mathrm{Q}_8 \rtimes \mathrm{C}_{2^n}$ for some $n \geqslant 2$. Here C_{2^n} acts as a transposition in $\mathrm{Aut}(\mathrm{Q}_8) \cong \mathrm{S}_4$.*

(iii) *$P \cong \mathrm{Q}_8.\mathrm{C}_{2^n}$ for some $n \geqslant 2$.*

In particular, P' is cyclic.

Proof. We use Proposition 3.7. If Q is abelian, then $\mathrm{C}_2^3 \cong \Omega(Q) \trianglelefteq P$. By Proposition 3.5, P is minimal non-abelian of type $(n,1)$ for some $n \geqslant 2$. Now assume that $Q \cong \mathrm{Q}_8 \times \mathrm{C}_{2^{n-1}}$ for some $n \geqslant 2$. We write $Q = \langle x, y, z\rangle$ such that $\langle x, y\rangle \cong \mathrm{Q}_8$ and $\langle z\rangle \cong \mathrm{C}_{2^{n-1}}$. Moreover, choose $g \in P \setminus Q$. Let $\alpha \in \mathrm{Out}_{\mathcal{F}}(Q)$ as usual. Then α acts non-trivially on $Q/\mathrm{Z}(Q) \cong \mathrm{C}_2^2$ and so does g. Hence, we may assume that ${}^g x = y$. Since $g^2 \in Q$, it follows that ${}^g y = {}^{g^2} x \in \{x, x^{-1}\}$. By replacing g with gx if necessary, we may assume that ${}^g y = x$. Hence, $g^2 \in \mathrm{Z}(Q)$. By Lemma 3.6, $P/\Phi(Q)$ is minimal non-abelian of type $(2,1)$. In particular, $Q/\Phi(Q) = \Omega(P/\Phi(Q))$. This gives $g^2 \notin \Phi(Q)$ and $g^2 \in z\langle x^2, z^2\rangle$. Since ${}^g(x^2) = x^2$, we obtain ${}^g z = z$. After replacing g with gz^i for a suitable integer i, it turns out that $g^2 \in \{z, zx^2\}$. In the latter case we replace z by $x^2 z$ and obtain $g^2 = z$. Hence, $P = \mathrm{Q}_8 \rtimes \mathrm{C}_{2^n}$ as stated. Moreover, g acts on $\langle x, y\rangle$ as an involution in $\mathrm{Aut}(\mathrm{Q}_8) \cong \mathrm{S}_4$. Since an involution that is a square in $\mathrm{Aut}(\mathrm{Q}_8)$ cannot act non-trivially on $\mathrm{Q}_8/\Phi(\mathrm{Q}_8)$, g must correspond to a transposition in S_4. This describes P up to isomorphism. Since $P = \langle gx\rangle\langle g\rangle$, P is bicyclic. In particular, $P' \subseteq \langle x, y\rangle$ is abelian and thus cyclic.

Finally, suppose that $Q = \mathrm{Q}_8 * \mathrm{C}_{2^n}$ for some $n \geqslant 2$. We use the same notation as before. In particular, $x^2 = z^{2^{n-1}}$. The same arguments as above give $g^2 = z$ and

$$P = \langle x, y, g \mid x^4 = 1,\ x^2 = y^2 = g^{2^n},\ {}^y x = x^{-1},\ {}^g x = y,\ {}^g y = x\rangle \cong \mathrm{Q}_8.\mathrm{C}_{2^n}.$$

Then $P = \langle gx\rangle\langle g\rangle$ is bicyclic and P' cyclic. $\square$

We will construct the groups and fusion systems in the previous proposition systematically in our main theorem, Theorem 3.19.

Let $\mathcal{F}$ be a fusion system on a 2-group P. Following [**5**, Definition 4.26], every subgroup $Q \leqslant \mathrm{Z}(P)$ gives rise to another fusion system $\mathrm{C}_{\mathcal{F}}(Q)$ on P.

Definition 3.9. The largest subgroup $Q \leqslant \mathrm{Z}(P)$ such that $\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}$ is called the *centre* $\mathrm{Z}(\mathcal{F})$ of $\mathcal{F}$. Accordingly, we say that $\mathcal{F}$ is *centre free* if $\mathrm{Z}(\mathcal{F}) = 1$.

The following result is useful to reduce the search for essential subgroups. Notice that the centre-free fusion systems on metacyclic 2-groups are determined in [**6**].

Proposition 3.10. *Let $\mathcal{F}$ be a centre-free fusion system on a bicyclic non-metacyclic 2-group P. Then there exists an abelian $\mathcal{F}$-essential subgroup $Q \leqslant P$ isomorphic to $\mathrm{C}_{2^m}^2$ or to $\mathrm{C}_{2^m} \times \mathrm{C}_2^2$ for some $m \geqslant 1$.*

Proof. By way of contradiction assume that all $\mathcal{F}$-essential subgroups are isomorphic to $\mathrm{C}_{2^m} \times \mathrm{Q}_8$ or to $\mathrm{C}_{2^m} * \mathrm{Q}_8$ (use Propositions 3.3 and 3.7). Let $z \in \mathrm{Z}(P)$ be an involution. Since $\mathrm{Z}(\mathcal{F}) = 1$, Alperin's fusion theorem in connection with Theorem 3.1 implies that

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there exists an $\mathcal{F}$-essential subgroup $Q \leqslant P$ such that $z \in \mathrm{Z}(Q)$. Moreover, there is an automorphism $\alpha \in \mathrm{Aut}(Q)$ such that $\alpha(z) \neq z$. Of course, α restricts to an automorphism of $\mathrm{Z}(Q)$. In the $Q \cong \mathrm{C}_{2^m} * \mathrm{Q}_8$ case this is not possible, since $\mathrm{Z}(Q)$ is cyclic. Now assume that $Q \cong \mathrm{C}_{2^m} \times \mathrm{Q}_8$. Observe that we can assume that α has order 3, because the automorphisms in $\mathrm{Aut}_P(Q)$ fix z anyway. But then α acts trivially on Q' and on $\Omega(Q)/Q'$, and thus also on $\Omega(Q) \ni z$, which is a contradiction. □

3.1. The P' non-cyclic case

The aim of this section is to prove that there are only nilpotent fusion systems provided that P' is non-cyclic. We do this using a case-by-case analysis corresponding to the theorems in [**18**]. By Proposition 3.8, we may assume that there are no normal $\mathcal{F}$-essential subgroups.

Let $\mathcal{F}$ be a non-nilpotent fusion system on the bicyclic 2-group P. Assume for the moment that $P' \cong \mathrm{C}_2^2$. Then P does not contain an elementary abelian subgroup of order 8 by Proposition 3.5. Hence, [**18**, Theorem 4.6] shows that P is unique of order 32. In this case we can prove with a computer that there are no candidates for $\mathcal{F}$-essential subgroups. Hence, we may assume that $\Phi(P') \neq 1$ in the following.

We introduce some notation from [**18**, Theorem 4.3] that will be used for the rest of the paper:

$$\Phi(P) = P'\langle a^2\rangle = \langle a^2\rangle\langle v\rangle, \qquad M = E\langle a^2\rangle = \langle x\rangle\langle a^2\rangle\langle v\rangle.$$

Here, M is the unique non-metacyclic maximal subgroup of P.

Proposition 3.11. *Let P be a bicyclic 2-group such that P' is non-cyclic and $P/\Phi(P')$ contains no elementary abelian normal subgroup of order 8. Then every fusion system on P is nilpotent.*

Proof. The $\Phi(P') = 1$ case was already handled, so we may assume that $\Phi(P') \neq 1$. In particular, [**18**, Theorem 4.7] applies. Let $\mathcal{F}$ be a non-nilpotent fusion system on P. Assume first that there exists an $\mathcal{F}$-essential subgroup $Q \in \{\mathrm{C}_{2^m} \times \mathrm{C}_2^2, \mathrm{C}_{2^m} * \mathrm{Q}_8 \cong \mathrm{C}_{2^m} * \mathrm{D}_8\}$ (the letter m is not used in [**18**, Theorem 4.7]). Theorem 4.7 of [**18**] also shows that $\Phi(P)$ is metacyclic and abelian. Since Q contains more than three involutions, there is an involution $\beta \in M \setminus \Phi(P)$. Hence, we can write $\beta = xa^{2i}v^j$ for some $i, j \in \mathbb{Z}$. Now, using [**18**, Theorem 4.7, case (a)] we derive the following contradiction:

$$\beta^2 = xa^{2i}v^jxa^{2i}v^j = xa^{2i}xa^{2i} = x^2(av)^{2i}a^{2i} = x^2a^{2i}u^iz^{\xi i}a^{2i} = uz^{(\eta+\xi)i} \neq 1.$$

Similarly, in case (b) we obtain

$$\begin{aligned}\beta^2 = xa^{2i}v^jxa^{2i}v^j = xa^{2i}xz^ja^{2i} = x^2(av)^{2i}z^ja^{2i} &= x^2a^{2i}u^iv^{2^{n-2}i}z^{\xi i}z^ja^{2i} \\ &= x^2v^{2^{n-2}i}z^{\eta i}v^{2^{n-2}i}z^{\xi i}z^j \\ &= uz^{i(1+\eta+\xi)+j} \\ &\neq 1.\end{aligned}$$

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Next assume that there is an $\mathcal{F}$-essential subgroup $\mathrm{C}_{2^m} \times \mathrm{Q}_8 \cong Q \leqslant P$ for some $m \geqslant 1$. Suppose that $m \geqslant 3$ for the moment. Since $Q \subseteq M$, it is easy to see that $M \setminus \Phi(P)$ contains an element of order at least 8. However, we have seen above that this is impossible. Hence, $m \leqslant 2$. By Proposition 3.8, Q is not normal in P. Since $Q < \mathrm{N}_M(Q) \leqslant \mathrm{N}_P(Q)$, we have $\mathrm{N}_P(Q) \leqslant M = \mathrm{N}_P(Q)\Phi(P)$. A computer calculation shows that $\mathrm{N}_P(Q) \cong \mathrm{Q}_{16} \times \mathrm{C}_{2^m}$. Thus, $\mathrm{N}_P(Q) \cap \Phi(P) \cong \mathrm{C}_8 \times \mathrm{C}_{2^m}$, because $\Phi(P)$ is abelian. Hence, there exist $\beta = xa^{2^i}y^j \in \mathrm{N}_P(Q)\setminus\Phi(P) \subseteq M\setminus\Phi(P)$ and $\delta \in \mathrm{N}_P(Q)\cap\Phi(P)$ such that $\beta^2 = \delta^4$. As above, we always have $\beta^2 \in u\langle z\rangle$. However, in both case (a) and case (b) we have $\delta^4 \in \mho_2(\Phi(P)) \cap \Omega(\Phi(P)) = \langle a^8\rangle\langle v^{2^{n-1}}\rangle = \langle z\rangle$, which is a contradiction. $\square$

If P' is cyclic, $P/\Phi(P')$ is minimal non-abelian, and thus contains an elementary abelian normal subgroup of order 8. Hence, it remains to deal with the case in which $P/\Phi(P')$ has a normal subgroup isomorphic to C_2^3.

Our next goal is to show that P' requires a cyclic maximal subgroup in order to admit a non-nilpotent fusion system.

Proposition 3.12. *Let P be a bicyclic 2-group such that $P' \cong \mathrm{C}_{2^r} \times \mathrm{C}_{2^{r+s}}$ for some $r \geqslant 2$ and $s \in \{1, 2\}$. Then every fusion system on P is nilpotent.*

Proof. We apply [**18**, Theorems 4.11 and 4.12] simultaneously. As usual, assume first that P contains an $\mathcal{F}$-essential subgroup $Q \cong \mathrm{C}_{2^m} \times \mathrm{C}_2^2$ for some $m \geqslant 1$ (m is not used in the statement of [**18**, Theorem 4.11]). Then $Q \cap \Phi(P) \cong \mathrm{C}_{2^m} \times \mathrm{C}_2$, since $\Phi(P)$ is abelian and metacyclic. We choose $\beta := xa^{2i}v^j \in Q \setminus \Phi(P)$. In the $m \geqslant 2$ case, β fixes an element of order 4 in $Q \cap \Phi(P)$. Since $\Phi(P)$ is abelian, all elements of $\Phi(P)$ of order 4 are contained in

$$\Omega_2(\Phi(P)) = \begin{cases} \langle b^{2^{r-2}}, v^{2^{r-1}}\rangle & \text{if Theorem 4.11 applies,} \\ \langle b^{2^{r-1}}, v^{2^{r-1}}\rangle & \text{if Theorem 4.12 applies.} \end{cases}$$

However, the relations in [**18**, Theorem 4.11 and 4.12] show that x, and thus β, acts by inversion on $\Omega_2(\Phi(P))$. Hence, $m = 1$. Then $\mathrm{N}_P(Q) \cap \Phi(P) \cong \mathrm{C}_4 \times \mathrm{C}_2$ by Lemma 3.6. In particular, there exists an element $\rho \in \Omega_2(\Phi(P))\setminus(\mathrm{N}_P(Q)\cap\Phi(P))$. Then ${}^{\rho}\beta = \beta\rho^{-2} \in Q$. Since $Q = \langle\beta\rangle(Q \cap \Phi(P))$, we derive the contradiction $\rho \in \mathrm{N}_P(Q)$.

Next suppose that $Q \cong \mathrm{C}_{2^m} \times \mathrm{Q}_8$ for some $m \geqslant 1$. Here we can repeat the argument word for word. Finally, the case $Q \cong \mathrm{C}_{2^m} * \mathrm{Q}_8$ cannot occur, since $\mathrm{Z}(P)$ is non-cyclic. $\square$

The next lemma is useful in a more general context.

Lemma 3.13. *Let P be a metacyclic 2-group which does not have maximal class. Then every homocyclic subgroup of P is given by $\Omega_i(P)$ for some $i \geqslant 0$.*

Proof. Let $\mathrm{C}_{2^k}^2 \cong Q \leqslant P$ with $k \in \mathbb{N}$. We argue by induction on k. By [**2**, Exercise 1.85], $\mathrm{C}_2^2 \cong \Omega(P)$. Hence, we may assume that $k \geqslant 2$. By induction it suffices to show that $P/\Omega(P)$ does not have maximal class. Let us assume the contrary. Since $P/\Omega(P)$ contains more than one involution, $P/\Omega(P)$ is a dihedral group or a semi-dihedral group. Let $\langle x\rangle \trianglelefteq P$ such that $P/\langle x\rangle$ is cyclic. Then $\langle x\rangle\Omega(P)/\Omega(P)$ and $(P/\Omega(P))/(\langle x\rangle\Omega(P)/\Omega(P)) \cong P/\langle x\rangle\Omega(P)$ are also cyclic. This yields $|P/\langle x\rangle\Omega(P)| = 2$ and $|P/\langle x\rangle| = 4$. Since $P/\Omega(P)$ is a dihedral group or a semi-dihedral group, there exists

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an element $y \in P$ such that the following hold:

(i) $P/\Omega(P) = \langle x\Omega(P), y\Omega(P)\rangle$,

(ii) $y^2 \in \Omega(P)$,

(iii) ${}^y x \equiv x^{-1} \pmod{\Omega(P)}$ or ${}^y x \equiv x^{-1+2^{n-2}} \pmod{\Omega(P)}$ with $|P/\Omega(P)| = 2^n$ and without loss of generality, $n \geqslant 4$.

Since $P = \langle x, y\rangle\Omega(P) \subseteq \langle x, y\rangle\Phi(P) = \langle x, y\rangle$, we have shown that P is the semi-direct product of $\langle x\rangle$ with $\langle y\rangle$. Moreover,

$${}^y x \in \{x^{-1}, x^{-1+2^{n-1}}, x^{-1+2^{n-2}}, x^{-1-2^{n-2}}\}.$$

Since $Q \cap \langle x\rangle$ and $Q/Q \cap \langle x\rangle \cong Q\langle x\rangle/\langle x\rangle$ are cyclic, we obtain $k = 2$ and $x^{2^{n-2}} \in Q$. But then Q cannot be abelian, since $n \geqslant 4$, which is a contradiction. □

Note that, in general, for a metacyclic 2-group P that does not have maximal class it can happen that $P/\Omega(P)$ has maximal class (e.g. $P \cong \mathrm{C}_8 \rtimes \mathrm{C}_4$, where C_4 acts by inversion on C_8).

Proposition 3.14. *Let P be a bicyclic 2-group such that $P' \cong \mathrm{C}_{2^r}^2$ for some $r \geqslant 2$. Then every fusion system on P is nilpotent.*

Proof. We apply [**18**, Theorem 4.9]. The general argument is quite similar to Proposition 3.12, but we need more details. Assume first that $Q \cong \mathrm{C}_{2^m} \times \mathrm{C}_2^2$ for some $m \geqslant 1$ is $\mathcal{F}$-essential in P (m is not used in the statement of [**18**, Theorem 4.9]). Since $\Phi(P)$ is 2-generator, we obtain $Q \cap \Phi(P) \cong \mathrm{C}_{2^m} \times \mathrm{C}_2$. We choose $\beta := xa^{2i}v^j \in Q \setminus \Phi(P)$. Suppose first that $m \geqslant 2$. Then β fixes an element $\delta \in Q \cap \Phi(P)$ of order 4. Now $\Phi(P)$ is a metacyclic group with $\Omega(\Phi(P)) \cong \mathrm{C}_2^2$ and $\mathrm{C}_4^2 \cong \Omega_2(P') \leqslant \Phi(P)$. So Lemma 3.13 implies that $\Omega_2(\Phi(P)) = \langle v^{2^{r-2}}, b^{2^{r-2}}\rangle \cong \mathrm{C}_4^2$. In the $r = 2$ case we have $|P| = 2^7$, and the claim follows by computational verification. Thus, we may assume that $r \geqslant 3$. Then $x^{-1}v^{2^{r-2}}x = v^{-2^{r-2}}$. Moreover, $\Omega_2(\Phi(P)) \subseteq \mho(\Phi(P)) = \Phi(\Phi(P)) \subseteq \mathrm{Z}(\Phi(P))$, since $\Phi(P)$ is abelian or minimal non-abelian depending on η. This shows that β acts by inversion on $\Omega_2(\Phi(P))$, and thus cannot fix δ. It follows that $m = 1$. Then $|\mathrm{N}_P(Q) \cap \Phi(P)| \leqslant 8$. In particular, there exists an element $\rho \in \Omega_2(\Phi(P)) \setminus (\mathrm{N}_P(Q) \cap \Phi(P))$. Then ${}^\rho\beta = \beta\rho^{-2} \in Q$. Since $Q = \langle\beta\rangle(Q \cap \Phi(P))$, we derive the contradiction $\rho \in \mathrm{N}_P(Q)$.

Now assume that $Q \cong \mathrm{C}_{2^m} \times \mathrm{Q}_8$ for some $m \geqslant 1$. We choose again $\beta := xa^{2i}v^j \in Q \setminus \Phi(P)$. If $\Phi(P)$ contains a subgroup isomorphic to Q_8, then $\Omega_2(\Phi(P))$ cannot be abelian. So, in the $m = 1$ case we have $\mathrm{N}_P(Q) \cap \Phi(P) \cong \mathrm{C}_8 \times \mathrm{C}_2$. Then the argument above reveals a contradiction (using $r \geqslant 3$). Now let $m \geqslant 2$. We write $Q = \langle q_1\rangle \times \langle q_2, q_3\rangle$, where $\langle q_1\rangle \cong \mathrm{C}_{2^m}$ and $\langle q_2, q_3\rangle \cong \mathrm{Q}_8$. In the $q_1 \notin \Phi(P)$ case we can choose $\beta = q_1$. In any case, it follows that β fixes an element of order 4 in $Q \cap \Phi(P)$. This leads to a contradiction as above.

Finally, suppose that $Q \cong \mathrm{C}_{2^m} * \mathrm{Q}_8 \cong \mathrm{C}_{2^m} * \mathrm{D}_8$ for some $m \geqslant 2$. Here we can choose $\beta \in Q \setminus \Phi(P)$ as an involution. Then there is always an element of order 4 in $Q \cap \Phi(P)$ that is fixed by β. The contradiction follows as before. □

Proposition 3.15. *Let P be a bicyclic 2-group such that $P' \cong C_{2^r} \times C_{2^{r+s+1}}$ for some $r, s \geqslant 2$. Then every fusion system on P is nilpotent.*

Proof. Here [**18**, Theorem 4.13] applies. The proof is a combination of the proofs of Propositions 3.12 and 3.14. In fact, for Theorem 4.13 (a) we can copy the proof of Proposition 3.12. Similarly, the arguments of Proposition 3.14 remain correct for case (b). Here observe that there is no need to discuss the $r = 2$ case separately, since $x^{-1}v^{2^{r+s-1}}x = v^{-2^{r+s-1}}$. □

Now it suffices to consider the case in which P' contains a cyclic maximal subgroup. If P' is non-cyclic, [**18**, Theorem 4.8] applies. This case is more complicated, since $|P/P'|$ is not bounded anymore.

Proposition 3.16. *Let P be a bicyclic 2-group such that $P' \cong C_{2^n} \times C_2$ for some $n \geqslant 2$, and $P/\Phi(P')$ has a normal elementary abelian subgroup of order 8. Then every fusion system on P is nilpotent.*

Proof. There are two possibilities for P according to if $Z(P)$ is cyclic or not. We handle them separately.

Case 1 ($Z(P)$ non-cyclic). Then $a^{2^m} = uz^\eta$. Moreover,

$$a^{-2}va^2 = a^{-1}vuv^{2+4s}a = a^{-1}uv^{3+4s}a = u(uv^{3+4s})^{3+4s} = v^{(3+4s)^2} \in v\langle v^8\rangle. \qquad (3.1)$$

Using this we see that $\langle a^{2^{m-1}}, v^{2^{n-2}}\rangle \cong C_4^2$. Thus, Lemma 3.13 implies that $\Omega_2(\Phi(P)) = \langle a^{2^{m-1}}, v^{2^{n-2}}\rangle$. As usual we assume that there is an $\mathcal{F}$-essential subgroup $Q \cong C_{2^t} \times C_2^2$ for some $t \geqslant 1$. Then $Q \cap \Phi(P) \cong C_{2^t} \times C_2$, since $\Phi(P)$ is 2-generator. For $t = 1$ we obtain $Q \cap \Phi(P) = \Omega(\Phi(P)) \subseteq Z(P)$. Write $\bar{P} := P/\Omega(\Phi(P))$, $\bar{Q} := Q/\Omega(\Phi(P))$ and so on. Then $C_{\bar{P}}(\bar{Q}) \subseteq \overline{N_P(Q)}$. So, by [**14**, Satz III.14.23], $\bar{P}$ has maximal class. Hence, $P' = \Phi(P)$ and $m = 1$, which is a contradiction. Thus, we may assume that $t \geqslant 2$. Then as usual we can find an element $\delta \in Q \cap \Phi(P)$ of order 4 that is fixed by some involution $\beta \in Q \setminus \Phi(P)$. We write $\delta = a^{2^{m-1}d_1}v^{2^{n-2}d_2}$ and $\beta = xv^ja^{2i}$. Assume first that $2 \mid d_1$. Then $2 \nmid d_2$. Since $a^{2^m}v^{2^{n-2}} \in Z(\Phi(P))$, it follows that $\delta = {}^\beta\delta = {}^x\delta = \delta^{-1}$. This contradiction shows that $2 \nmid d_1$. After replacing δ with its inverse if necessary, we can assume that $d_1 = 1$. Now we consider β. We have

$$1 = \beta^2 = (xv^ja^{2i})^2 \equiv x^2v^{2j}a^{4i} \equiv a^{4i} \pmod{P'}.$$

Since

$$2^{n+m} = |\Phi(P)| = \frac{|\langle a^2\rangle|\,|P'|}{|\langle a^2\rangle \cap P'|} = \frac{2^{n+m+1}}{|\langle a^2\rangle \cap P'|},$$

we get $2^{m-2} \mid i$. In the $i = 2^{m-2}$ case we obtain the contradiction

$$\langle z\rangle \ni x^2 = xv^{j-2^{n-2}d_2}xv^{j-2^{n-2}d_2} = (\beta\delta^{-1})^2 = \delta^2 \in u\langle z\rangle.$$

Hence, $2^{m-1} \mid i$. So, after multiplying β by δ^2 if necessary, we may assume that $i = 0$, i.e. $\beta = xv^j$. Then $1 = xv^jxv^j = x^2$. Conjugation with a^{-1} gives $\beta = a^{-1}xv^ja =$

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$xv^{-1}a^{-1}v^ja = xu^jv^{(3+4s)j-1}$. Since $u \in Q$, we may assume that $\beta = xv^{2j}$. After we conjugate Q with v^j, we even obtain $\beta = x$. Since $x(a^2v^i)x^{-1} = a^2uv^{4(1+s)-i}$, no element of the form a^2v^i is fixed by x. On the other hand,

$$x(a^4v^i)x^{-1} = (a^2uv^{4(1+s)})^2v^{-i} = a^4v^{4(1+s)(3+4s)^2+4(1+s)-i}.$$

This shows that there is an i such that $a^4v^i =: \lambda$ is fixed by x. Assume that there is another element $\lambda_1 := a^4v^j$ that is also fixed by x. Then $\lambda^{-1}\lambda_1 = v^{j-i} \in \langle z\rangle$. This holds in a similar way for elements containing higher powers of a. In particular, $u = a^{2^m}z^\eta \in \langle\lambda, z\rangle$. Recall that $\Phi(P) = \langle v\rangle \rtimes \langle a^2\rangle$. This shows that $\mathrm{C}_{\Phi(P)}(x) = \langle\lambda\rangle \times \langle z\rangle \cong \mathrm{C}_{2^{m-1}} \times \mathrm{C}_2$. Since $Q \cap \Phi(P) \subseteq \mathrm{C}_{\Phi(P)}(x)$ and $Q = (Q \cap \Phi(P))\langle x\rangle$, we deduce that $\mathrm{C}_{\Phi(P)}(x) \subseteq \mathrm{C}_P(Q) \subseteq Q$. Moreover, $Q \cap \Phi(P) = \mathrm{C}_{\Phi(P)}(x)$ and $t = m - 1$. Therefore, $Q = \langle\lambda, x, z\rangle$. The calculation above shows that there is an element $\mu := a^2v^j$ such that ${}^\mu x = ux \in Q$. Now $\mu^2 \in \mathrm{C}_{\Phi(P)}(x)$ implies that $\mathrm{C}_{\Phi(P)}(x) = \langle\mu^2, z\rangle$ and $\mu \in \mathrm{N}_P(Q) = Q\langle v^{2^{n-2}}\rangle$, which is a contradiction.

Now assume that $Q \cong \mathrm{C}_{2^t} \times \mathrm{Q}_8$ for some $t \geqslant 1$. Since $\Phi(P)$ does not contain a subgroup isomorphic to Q_8, we see that $\Omega(\Phi(P)) \subseteq Q$. First assume that $t = 1$. Then we look again at the quotients $\bar{P} := P/\Omega(\Phi(P))$ and $\bar{Q} := Q/\Omega(\Phi(P)) \cong \mathrm{C}_2^2$. Since $\mathrm{N}_P(Q)$ acts non-trivially on $\bar{Q}$, we obtain $\mathrm{C}_{\bar{P}}(\bar{Q}) \subseteq \bar{Q}$. In particular, [**2**, Proposition 1.8] implies that $\bar{P}$ has maximal class. This leads to a contradiction, as in the first part of the proof. Thus, we may assume that $t \geqslant 2$ from now on. Then $\Omega_2(\Phi(P)) \subseteq Q$. Since Q contains more elements of order 4 than $\Phi(P)$, we can choose $\beta \in Q \setminus \Phi(P)$ of order 4. Write $\beta = xa^{2i}v^j$. Then $\beta^2 \in \Omega(\Phi(P)) \subseteq P'$. So the same discussion as above shows that we can assume that $\beta = x$. In particular, $|\langle x\rangle| = 4$. Since $\mathrm{C}_{\Phi(P)}(x)$ is abelian, λ centralizes $(\mathrm{C}_Q(x) \cap \Phi(P))\langle x\rangle\langle v^{2^{n-2}}\rangle = \mathrm{C}_Q(x)\langle v^{2^{n-2}}\rangle = Q$. This shows that $\lambda \in Q$ and $t = m - 1$ again. More precisely we have $Q = \langle\lambda\rangle \times \langle v^{2^{n-2}}, x\rangle$. Equation (3.1) shows that $v^{2^{n-3}}$ still lies in the centre of $\Phi(P)$. It follows easily that $\mathrm{N}_P(Q) = Q\langle v^{2^{n-3}}\rangle$. However, as above we also have that $\mu \in \mathrm{N}_P(Q)$, which is a contradiction.

Finally, the $Q \cong \mathrm{C}_{2^t} * \mathrm{Q}_8$ case cannot occur, since $\mathrm{Z}(P)$ is non-cyclic.

Case 2 ($\mathbf{Z}(\boldsymbol{P})$ cyclic). Here we have $a^{2^m} = uv^{2^{n-2}}z^\eta$, $n \geqslant m + 2 \geqslant 4$ and $1 + s \not\equiv 0 \pmod{2^{n-3}}$. Again we begin with $Q \cong \mathrm{C}_{2^t} \times \mathrm{C}_2^2$ for some $t \geqslant 1$. By [**18**, Theorem 4.3 (b)] we still have $\langle u, z\rangle = \Omega(\mathrm{Z}(\Phi(P)))$. Since $\Phi(P)$ does not have maximal class, $\langle u, z\rangle = \Omega(\Phi(P))$ also holds. In particular, $\Omega(\Phi(P)) \subseteq Q$. In the $t = 1$ case we see that $P/\Omega(\Phi(P))$ has maximal class, which leads to a contradiction as before. Thus, $t \geqslant 2$. Since $u \in \mathrm{Z}(\Phi(P))$, (3.1) is still true. Hence, $\Omega_2(\Phi(P)) = \langle a^{2^{m-1}}v^{2^{n-3}}, v^{2^{n-2}}\rangle \cong \mathrm{C}_4^2$. We choose an involution $\beta = xv^ja^{2i} \in Q \setminus \Phi(P)$. Then, as usual, $v^{2^{n-2}} \in \mathrm{N}_P(Q) \setminus Q$. Since $a^{2^m} \in \langle u\rangle \times \langle v^{2^{n-2}}\rangle$, we find an element $\delta = a^{2^{m-1}}v^{d_1} \in Q \cap \Omega_2(\Phi(P))$ of order 4 fixed by β. Now exactly the same argument as in Case 1 shows that $\beta = x$ after changing the representative of β and the conjugation of Q if necessary. Similarly, we obtain $\lambda := a^4v^j \in \mathrm{C}_{\Phi(P)}(x)$. Moreover, $u = a^{2^m}v^{-2^{n-2}}z^\eta \in \{\lambda^{2^{m-2}}, \lambda^{2^{m-2}}z\}$. Therefore, $\mathrm{C}_{\Phi(P)}(x) = \langle\lambda\rangle \times \langle z\rangle \cong \mathrm{C}_{2^{m-1}} \times \mathrm{C}_2$. The contradiction follows as before.

Now assume that $Q \cong \mathrm{C}_{2^t} \times \mathrm{Q}_8$ or $Q \cong \mathrm{C}_{2^{t+1}} * \mathrm{Q}_8$ for some $t \geqslant 1$. Proposition 3.10 shows that $\mathcal{F} = \mathrm{C}_{\mathcal{F}}(\langle z\rangle)$. Theorem 5.60 in [**5**] implies that $\bar{Q} := Q/\langle z\rangle$ is an $\mathcal{F}/\langle z\rangle$-essential subgroup of $\bar{P} := P/\langle z\rangle$. Now $\bar{P}$ is bicyclic and has commutator subgroup isomorphic to $C_{2^{n-1}} \times C_2$. Hence, the result follows by induction on t. □

Combining these propositions we deduce one of the main results of this paper.

Theorem 3.17. *Every fusion system on a bicyclic 2-group P is nilpotent unless P' is cyclic.*

It seems that there is no general reason for Theorem 3.17. For example, there are non-nilpotent fusion systems on 2-generator 2-groups with non-cyclic commutator subgroup.

For the convenience of the reader we state a consequence for finite groups.

Corollary 3.18. *Let G be a finite group with bicyclic Sylow 2-subgroup P. If P' is non-cyclic, then P has a normal complement in G.*

3.2. The P' cyclic case

In this section we consider the remaining case, where the bicyclic 2-group P has cyclic commutator subgroup. Here [**18**, Theorem 4.4] plays an important role. The following theorem classifies all fusion systems on bicyclic 2-groups together with some more information.

Theorem 3.19. *Let $\mathcal{F}$ be a fusion system on a bicyclic 2-group P. Then one of the following holds.*

(1) *$\mathcal{F}$ is nilpotent, i.e. $\mathcal{F} = \mathcal{F}_P(P)$.*

(2) *$P \cong C_{2^n}^2$ and $\mathcal{F} = \mathcal{F}_P(P \rtimes C_3)$ for some $n \geqslant 1$.*

(3) *$P \cong D_{2^n}$ for some $n \geqslant 3$, and $\mathcal{F} = \mathcal{F}_P(\mathrm{PGL}(2, 5^{2^{n-3}}))$ or $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(2, 5^{2^{n-2}}))$. Moreover, $\mathcal{F}$ possesses one (in the first case) or two (in the second case) essential subgroups isomorphic to C_2^2 up to conjugation.*

(4) *$P \cong Q_8$ and $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2,3))$ is controlled, i.e. there are no $\mathcal{F}$-essential subgroups.*

(5) *$P \cong Q_{2^n}$ for some $n \geqslant 4$, and $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-4}}).C_2)$ or $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-3}}))$. Moreover, $\mathcal{F}$ possesses one (in the first case) or two (in the second case) essential subgroups isomorphic to Q_8 up to conjugation.*

(6) *$P \cong SD_{2^n}$ for some $n \geqslant 4$, and $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(2, 5^{2^{n-3}}) \rtimes C_2)$, $\mathcal{F} = \mathcal{F}_P(\mathrm{GL}(2,q))$ or $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(3,q))$, where in the last two cases q is a suitable prime power such that $q \equiv 3 \pmod 4$. Moreover, in the first (respectively, second) case, C_2^2 (respectively, Q_8) is the only $\mathcal{F}$-essential subgroup up to conjugation, and in the last case both are $\mathcal{F}$-essential and these are the only ones up to conjugation.*

(7) *$P \cong C_{2^n} \wr C_2$ for some $n \geqslant 2$ and $\mathcal{F} = \mathcal{F}_P(C_{2^n}^2 \rtimes S_3)$, $\mathcal{F} = \mathcal{F}_P(\mathrm{GL}(2,q))$ or $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(3,q))$, where in the last two cases $q \equiv 1 \pmod 4$. Moreover, in the first (respectively, second) case $C_{2^n}^2$ (respectively, $C_{2^n} * Q_8$) is the only $\mathcal{F}$-essential subgroup up to conjugation, and in the last case both are $\mathcal{F}$-essential and these are the only ones up to conjugation.*

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(8) *$P \cong C_2^2 \rtimes C_{2^n}$ is minimal non-abelian of type $(n,1)$ for some $n \geqslant 2$ and $\mathcal{F} = \mathcal{F}_P(A_4 \rtimes C_{2^n})$. Moreover, $C_{2^{n-1}} \times C_2^2$ is the only $\mathcal{F}$-essential subgroup of P.*

(9) *$P \cong \langle v, x, a \mid v^{2^n} = x^2 = 1,\ {}^x v = v^{-1},\ a^{2^m} = v^{2^{n-1}},\ {}^a v = v^{-1+2^{n-m+1}},\ {}^a x = vx\rangle \cong D_{2^{n+1}}.C_{2^m}$ for $n > m > 1$ and $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(2, 5^{2^{n-1}}).C_{2^m})$. Moreover, $C_{2^{m-1}} \times C_2^2$ is the only $\mathcal{F}$-essential subgroup up to conjugation.*

(10) *$P \cong \langle v, x, a \mid v^{2^n} = x^2 = a^{2^m} = 1,\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^i},\ {}^a x = vx\rangle \cong D_{2^{n+1}} \rtimes C_{2^m}$ for $\max(2, n-m+2) \leqslant i \leqslant n$ and $n, m \geqslant 2$. Moreover, $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}(2, 5^{2^{n-1}}) \rtimes C_{2^m})$ and $C_{2^{m-1}} \times C_2^2$ is the only $\mathcal{F}$-essential subgroup up to conjugation. In the $i = n$ case there are two possibilities for $\mathcal{F}$, which differ by $Z(\mathcal{F}) \in \{\langle a^2\rangle, \langle a^2 v^{2^{n-1}}\rangle\}$.*

(11) *$P \cong \langle v, x, a \mid v^{2^n} = 1,\ x^2 = a^{2^m} = v^{2^{n-1}},\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^{n-m+1}},\ {}^a x = vx\rangle \cong Q_{2^{n+1}}.C_{2^m}$ for $n > m > 1$ and $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-2}}).C_{2^m})$. Moreover, $C_{2^{m-1}} \times Q_8$ is the only $\mathcal{F}$-essential subgroup up to conjugation.*

(12) *$P \cong \langle v, x, a \mid v^{2^n} = a^{2^m} = 1,\ x^2 = v^{2^{n-1}},\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^i},\ {}^a x = vx\rangle \cong Q_{2^{n+1}} \rtimes C_{2^m}$ for $\max(2, n-m+2) \leqslant i \leqslant n$ and $n, m \geqslant 2$. Moreover, $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-2}}) \rtimes C_{2^m})$ and $C_{2^{m-1}} \times Q_8$ is the only $\mathcal{F}$-essential subgroup up to conjugation.*

(13) *$P \cong \langle v, x, a \mid v^{2^n} = a^{2^m} = 1,\ x^2 = v^{2^{n-1}},\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^{n-m+1}},\ {}^a x = vx\rangle \cong Q_{2^{n+1}} \rtimes C_{2^m}$ for $n > m > 1$ and $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-2}}) \rtimes C_{2^m})$. Moreover, $C_{2^m} * Q_8$ is the only $\mathcal{F}$-essential subgroup up to conjugation.*

(14) *$P \cong \langle v, x, a \mid v^{2^n} = 1,\ x^2 = a^{2^m} = v^{2^{n-1}},\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^i},\ {}^a x = vx\rangle \cong Q_{2^{n+1}}.C_{2^m}$ for $\max(2, n-m+2) \leqslant i \leqslant n$ and $n, m \geqslant 2$. In the $m = n$ case we have $i \neq n$. Moreover, $\mathcal{F} = \mathcal{F}_P(\mathrm{SL}(2, 5^{2^{n-2}}).C_{2^m})$ and $C_{2^m} * Q_8$ is the only $\mathcal{F}$-essential subgroup up to conjugation.*

In particular, $\mathcal{F}$ is non-exotic. Conversely, for every group described in these cases there exists a fusion system with the given properties. Moreover, different parameters give non-isomorphic groups.

Proof. Assume that $\mathcal{F}$ is non-nilpotent. By Theorem 3.17, P' is cyclic. The $P \cong Q_8$ case is easy. For the other metacyclic cases and the $P \cong C_{2^n} \wr C_2$ case we refer the reader to [**6**, Theorem 5.3]. Here we add a few additional details. An induction on $i \geqslant 2$ shows that $5^{2^{i-2}} \equiv 1 + 2^i \pmod{2^{i+1}}$. This implies that the Sylow 2-subgroups of $\mathrm{SL}(2, 5^{2^{n-3}})$, $\mathrm{PSL}(2, 5^{2^{n-2}})$ and so on have the right order. For the groups SD_{2^n} and $C_{2^n} \wr C_2$ it is *a priori* not clear if for every n an odd prime power q can be found. However, this can be shown using Dirichlet's prime number theorem (cf. [**32**, Theorem 6.2]). Hence, for a given n all these fusion systems can be constructed.

Using Proposition 3.3 we can assume that every $\mathcal{F}$-essential subgroup is 3-generator. Finally, by Proposition 3.5 it remains to consider $|P'| > 2$. Hence, let P be as in [**18**, Theorem 4.4]. We adapt our notation slightly as follows. We replace a by a^{-1} in order to

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write ${}^a v$ instead of v^a. Then we have ${}^a x = vx$. After replacing v by a suitable power, we may assume that i is a 2-power (accordingly, we need to change x to $v^\eta x$ for a suitable number η). Then we can also replace i by $2 + \log i$. This gives

$$P \cong \langle v, x, a \mid v^{2^n} = 1,\ x^2,\ a^{2^m} \in \langle v^{2^{n-1}} \rangle,\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^i},\ {}^a x = vx \rangle. \quad (3.2)$$

Since [**18**, Theorem 4.4] also states that v and $a^{2^{m-1}}$ commute, we obtain $i \in \{\max(n - m + 1, 2), \dots, n\}$. We set $z := v^{2^{n-1}}$ as in [**18**]. Moreover, let $\lambda := v^{-2^{i-1}} a^2$. Then

$$x \lambda x^{-1} = v^{2^{i-1}} (v^{-1} a)^2 = v^{-2^{i-1}} a^2 = \lambda$$

and $\lambda \in \mathrm{C}_{\Phi(P)}(x)$. Assume also that $v^j a^2 \in \mathrm{C}_{\Phi(P)}(x)$. We then obtain $v^j a^2 \in \{\lambda, \lambda z\}$. Hence, $\mathrm{C}_{\Phi(P)}(x) \in \{\langle \lambda \rangle, \langle \lambda \rangle \times \langle z \rangle\}$. It should be pointed out that it was not shown in [**18**] that these presentations really give groups of order 2^{n+m+1} (although some evidence by way of computational results is stated). However, we assume in the first part of the proof that these groups with the 'right' order exist. Later we construct $\mathcal{F}$ as a fusion of a finite group and it will be clear that P shows up as a Sylow 2-subgroup of order 2^{n+m+1}. Now we distinguish between the three different types of essential subgroups.

Case 1 ($Q \cong \mathrm{C}_{2^t} \times \mathrm{C}_2^2$ is $\mathcal{F}$-essential in P for some $t \geqslant 1$). As usual, $Q \leqslant M = E\langle a^2 \rangle$. Since $Q \cap E$ is abelian and $Q/Q \cap E \cong QE/E \leqslant P/E$ is cyclic, it follows that E is dihedral and $Q \cap E \cong \mathrm{C}_2^2$. After conjugation of Q we may assume that $Q \cap E \in \{\langle z, x \rangle, \langle z, vx \rangle\}$. Further conjugation with a gives $Q \cap E = \langle z, x \rangle$. Since $\mathrm{C}_Q(x) \cap \Phi(P)$ is non-cyclic, it follows that $\mathrm{C}_{\Phi(P)}(x) = \langle \lambda \rangle \times \langle z \rangle \cong \mathrm{C}_{2^{m-1}} \times \mathrm{C}_2$. As usual we obtain $Q = \langle \lambda, z, x \rangle$ and $t = m - 1$. Moreover, $a^2 v a^{-2} \equiv v \pmod{\langle v^8 \rangle}$. Hence, $\mathrm{N}_P(Q) = \langle \lambda, v^{2^{n-2}}, x \rangle$.

We prove that Q is the only $\mathcal{F}$-essential subgroup of P up to conjugation. If there is an $\mathcal{F}$-essential 2-generator subgroup, then Proposition 3.3 implies that P is a wreath product. However, by the proof of Theorem 5.3 in [**6**], all the other $\mathcal{F}$-essential subgroups are of type $\mathrm{C}_{2^r} * \mathrm{Q}_8$. Hence, this case cannot occur. Thus, by construction it is clear that Q is the only abelian $\mathcal{F}$-essential subgroup up to conjugation. Now assume that $\mathrm{Q}_1 \cong \mathrm{C}_{2^s} \times \mathrm{Q}_8$ is also $\mathcal{F}$-essential. Since Q_1 has three involutions, $\mathrm{Q}_1 \cap E$ is cyclic or isomorphic to C_2^2. In either case $Q/Q \cap Q \cong QE/E \leqslant P/E$ cannot be cyclic, which is a contradiction. Suppose now that $\mathrm{Q}_1 \cong \mathrm{C}_{2^s} * \mathrm{Q}_8 \cong \mathrm{C}_{2^s} * \mathrm{D}_8$ for some $s \geqslant 2$. Then $\mathrm{Q}_1 \cap E$ cannot be cyclic, since Q_1 is 3-generator. Suppose that $\mathrm{Q}_1 \cap E \cong \mathrm{C}_2^2$. Then $\Omega(\mathrm{Z}(\mathrm{Q}_1)) \subseteq \mathrm{Q}_1 \cap E$ and $\exp \mathrm{Q}_1/\mathrm{Q}_1 \cap E \leqslant 2^{s-1}$. On the other hand, $|\mathrm{Q}_1/\mathrm{Q}_1 \cap E| = 2^s$. In particular, $\mathrm{Q}_1/\mathrm{Q}_1 \cap E \cong \mathrm{Q}_1 E/E \leqslant P/E$ cannot be cyclic. It follows that $\mathrm{Q}_1 \cap E$ must be a (non-abelian) dihedral group. Hence, $2^{s-1}|\mathrm{Q}_1 \cap E| = |(\mathrm{Q}_1 \cap E)\,\mathrm{Z}(\mathrm{Q}_1)| \leqslant |\mathrm{Q}_1| = 2^{s+2}$ and $\mathrm{Q}_1 \cap E \cong \mathrm{D}_8$. After conjugation of Q_1 we have $\mathrm{Q}_1 \cap E = \langle v^{2^{n-2}}, x \rangle$. Let $\lambda_1 \in \mathrm{Z}(\mathrm{Q}_1) \setminus E$ be an element of order 2^s such that $\lambda_1^{2^{s-1}} = z$. Since $x \in \mathrm{Q}_1$, we have $\lambda_1^2 \in \mathrm{C}_{\Phi(P)}(x) = \langle \lambda \rangle \times \langle z \rangle$. This implies that $s = 2$ and $\lambda_1 \notin \Phi(P)$. Since $\mathrm{Q}_1 = (\mathrm{Q}_1 \cap \Phi(P))\langle x \rangle$, we obtain $\lambda_1 x \in \mathrm{C}_{\Phi(P)}(x)$. But this contradicts $z = \lambda_1^2 = (\lambda_1 x)^2$. Hence, we have proved that Q is in fact the only $\mathcal{F}$-essential subgroup of P up to conjugation.

Now we try to pin down the structure of P more precisely. We show by induction on $j \geqslant 0$ that $\lambda^{2^j} = v^{2^{i+j-1}\nu} a^{2^{j+1}}$ for an odd number ν. This is clear for $j = 0$. For arbitrary

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$j \geqslant 1$ we have

$$\lambda^{2^j} = \lambda^{2^{j-1}}\lambda^{2^{j-1}} = v^{2^{i+j-2}\nu}a^{2^j}v^{2^{i+j-2}\nu}a^{2^j} = v^{2^{i+j-2}\nu(-1+2^i)^{2^j}+2^{i+j-2}\nu}a^{2^{j+1}}$$
$$= v^{2^{i+j-2}\nu((-1+2^i)^{2^j}+1)}a^{2^{j+1}},$$

and the claim follows. In particular, we obtain

$$1 = \lambda^{2^{m-1}} = v^{2^{i+m-2}\nu}a^{2^m}. \tag{3.3}$$

We distinguish between whether P splits or not.

Case 1 (a) ($a^{2^m} = z$). Here, (3.3) shows that $i = n - m + 1$. Then $n > m > 1$, and the isomorphism type of P is completely determined by m and n. We show next that $\mathcal{F}$ is uniquely determined. For this we need to describe the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ in order to apply Alperin's fusion theorem. As in the proof of Proposition 3.5, $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts on $\langle x, z\rangle$ or on $\langle x\lambda^{2^{m-2}}, z\rangle$ non-trivially (recall that $\mathrm{N}_P(Q) \cong \mathrm{D}_8 \times \mathrm{C}_{2^{m-1}}$). Set $\tilde{x} := x\lambda^{2^{m-2}}$ and $\tilde{a} := av^{2^{n-2}}$. Then, as above, $\tilde{x} = xv^{\pm 2^{n-2}}a^{2^{m-1}}$. Hence, $\tilde{x}^2 = 1$ and ${}^{\tilde{x}}v = v^{-1}$. Moreover, $\tilde{a}^2 = a^2$, and thus $\tilde{a}^{2^m} = z$. Finally, ${}^{\tilde{a}}v = {}^{a}v$ and ${}^{\tilde{a}}\tilde{x} = {}^{a}(xzv^{\pm 2^{n-2}}a^{2^{m-1}}) = vxzv^{\mp 2^{n-2}}a^{2^{m-1}} = v\tilde{x}$. Hence, v, $\tilde{x}$ and $\tilde{a}$ satisfy the same relations as v, x and a. Obviously, $P = \langle v, \tilde{x}, \tilde{a}\rangle$. Therefore, we may replace x by $\tilde{x}$ and a by $\tilde{a}$. After doing this if necessary, we see that $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts non-trivially on $\langle x, z\rangle$ (observe that Q remains fixed under this transformation). As usual, it follows that $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) \in \{\langle\lambda\rangle, \langle\lambda z\rangle\}$ (cf. the proof of Proposition 3.5). Define $\tilde{a} := a^{1+2^{m-1}}$ and $\tilde{v} := v^{1+2^{n-1}} = vz$. Then $\tilde{a}^2 = a^2 z$, $\tilde{a}^{2^m} = z$, $\tilde{v}^{2^n} = 1$, ${}^{x}\tilde{v} = \tilde{v}^{-1}$ and ${}^{\tilde{a}}\tilde{v} = \tilde{v}^{-1+2^{n-m+1}}$. Now we show by induction on $j \geqslant 1$ that $a^{2^j}xa^{-2^j} = v^{2^{n-m+j}\nu}x$ for an odd integer ν. For $j = 1$ we have $a^2xa^{-2} = {}^{a}(vx) = v^{2^{n-m+1}}x$. For arbitrary $j \geqslant 1$ induction gives

$$a^{2^{j+1}}xa^{-2^{j+1}} = a^{2^j}(a^{2^j}xa^{-2^j})a^{-2^j} = a^{2^j}(v^{2^{n-m+j}\nu}x)a^{-2^j}$$
$$= v^{2^{n-m+j}\nu((-1+2^{n-m+1})^{2^j}+1)}x,$$

and the claim follows. In particular, $a^{2^{m-1}}xa^{-2^{m-1}} = zvx$ and ${}^{\tilde{a}}x = \tilde{v}x$. Obviously, $P = \langle\tilde{v}, \tilde{a}, x\rangle$. Hence, we may replace v, a by $\tilde{v}$, $\tilde{a}$ if necessary. Under this transformation Q and $\langle x, z\rangle$ remain fixed as sets and λ goes to λz. So, we may assume that $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \langle\lambda\rangle$. Then the action on $\mathrm{Aut}_{\mathcal{F}}(Q)$ on Q is completely described. In particular, $\mathcal{F}$ is uniquely determined.

It remains to prove that P and $\mathcal{F}$ really exist. Let $q := 5^{2^{n-1}}$. It is not hard to verify that $H := \mathrm{PSL}(2, q)$ has Sylow 2-subgroup $E \cong \mathrm{D}_{2^{n+1}}$. More precisely, E can be generated by the matrices

$$v := \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad x := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\omega \in \mathbb{F}_q^\times$ has order 2^{n+1}. Moreover, we regard these matrices modulo $\mathrm{Z}(\mathrm{SL}(2, q)) = \langle -1_2\rangle$. Now consider the matrix $a_1 := \left(\begin{smallmatrix} 0 & \omega \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}(2, q)/\mathrm{Z}(\mathrm{SL}(2, q))$. Then a_1 acts on H and a calculation shows that ${}^{a_1}v = v^{-1}$ and ${}^{a_1}x = vx$. Let γ_1 be the Frobenius automorphism of $\mathbb{F}_q$ with respect to $\mathbb{F}_5$, i.e. $\gamma_1(\tau) = \tau^5$ for $\tau \in \mathbb{F}_q$. As usual we may regard γ_1 as an automorphism of H. Let $\gamma := \gamma_1^{2^{n-m-1}}$ so that $|\langle\gamma\rangle| = 2^m$. Recall that $(\mathbb{Z}/2^{n+1}\mathbb{Z})^\times = \langle 5 + 2^{n+1}\mathbb{Z}\rangle \times \langle -1 + 2^{n+1}\mathbb{Z}\rangle \cong \mathrm{C}_{2^{n-1}} \times \mathrm{C}_2$. It is easy to show that $\langle 5^{2^{n-m-1}} + 2^{n+1}\mathbb{Z}\rangle$ and $\langle 1 - 2^{n-m+1} + 2^{n+1}\mathbb{Z}\rangle$ are subgroups of $(\mathbb{Z}/2^{n+1}\mathbb{Z})^\times$ of order 2^m. Since

$$5^{2^{n-m-1}} \equiv 1 - 2^{n-m+1} \pmod 8,$$

it follows that

$$\langle 5^{2^{n-m-1}} + 2^{n+1}\mathbb{Z}\rangle = \langle 1 - 2^{n-m+1} + 2^{n+1}\mathbb{Z}\rangle.$$

In particular, we can find an odd integer ν such that $5^{2^{n-m-1}\nu} \equiv 1 - 2^{n-m+1} \pmod{2^{n+1}}$. Now we set

$$a := a_1\gamma^\nu.$$

Since γ_1 fixes x, we obtain ${}^{a}v = v^{-1+2^{n-m+1}}$ and ${}^{a}x = vx$. It remains to show that $a^{2^m} = v^{2^{n-1}} =: z$. Here we identify elements of H with the corresponding inner automorphisms in $\mathrm{Inn}(H) \cong H$. For an element $u \in H$ we have

$$a^2(u) = (a_1\gamma^\nu a_1\gamma^\nu)(u) = (a_1\gamma^\nu(a_1))\gamma^{2\nu}(u)(a_1\gamma^\nu(a_1))^{-1} = \left(\begin{pmatrix}\omega & 0\\ 0 & \omega^{5^{2^{n-m-1}\nu}}\end{pmatrix}\gamma^{2\nu}\right)(u).$$

After multiplying the matrix in the last equation by $\left(\begin{smallmatrix}\omega & 0\\ 0 & \omega\end{smallmatrix}\right)^h \in \mathrm{Z}(\mathrm{GL}(2,q))$ for $h := -(5^{2^{n-m-1}\nu} + 1)/2$, we obtain

$$a^2(u) = \left(\begin{pmatrix}\omega^{2^{n-m}} & 0\\ 0 & \omega^{-2^{n-m}}\end{pmatrix}\gamma^{2\nu}\right)(u),$$

since $(1 - 5^{2^{n-m-1}\nu})/2 \equiv 2^{n-m} \pmod{2^n}$. Using induction and the same argument we obtain

$$a^{2^j} = \begin{pmatrix}\omega^{h_j} & 0\\ 0 & \omega^{-h_j}\end{pmatrix}\gamma^{2^j\nu},$$

where $2^{n-m+j-1} \mid h_j$ and $2^{n-m+j} \nmid h_j$ for $j \geqslant 1$. In particular, $a^{2^m} = z$ as claimed. Now [**10**, Theorem 15.3.1] shows that the following non-split extension exists:

$$G := H\langle a\rangle \cong \mathrm{PSL}(2, 5^{2^{n-1}}).\mathrm{C}_{2^m}.$$

Moreover, the construction shows that G has Sylow 2-subgroup P. Since H is non-abelian simple, $\mathcal{F}_P(G)$ is non-nilpotent. Hence, $\mathcal{F} = \mathcal{F}_P(G)$.

Case 1 (b) ($a^{2^m} = 1$). Here $P \cong \mathrm{D}_{2^{n+1}} \rtimes \mathrm{C}_{2^m}$. Moreover, by (3.3) we have $n-m+2 \leqslant i$. As in Case 1 (a) we may assume that $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts on $\langle x, z\rangle$ using the following automorphism of P if necessary:

$$v \mapsto v, \qquad x \mapsto x\lambda^{2^{m-2}}, \qquad a \mapsto av^{2^{n-2}}.$$

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Now assume that $i < n$ (and thus $m, n \geqslant 3$). Here we consider the map

$$v \mapsto v^{1+2^{n-1}} = vz =: \tilde{v}, \qquad x \mapsto x, \qquad a \mapsto a^{1+2^{n-i}} =: \tilde{a}.$$

It can be seen that $\tilde{v}$, x and $\tilde{a}$ generate P and satisfy the same relations as v, x and a. Moreover, as above, we have $\lambda^{2^{n-i}} = za^{2^{n-i+1}}$. This shows that

$$\lambda \mapsto \tilde{v}^{-2^{i-1}}\tilde{a}^2 = v^{-2^{i-1}}a^{2+2^{n-i+1}} = \lambda^{1+2^{n-i}}z = (\lambda z)^{1+2^{n-i}}.$$

Hence, we obtain $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \langle\lambda\rangle$ after applying this automorphism if necessary. This determines $\mathcal{F}$ completely, and we will construct $\mathcal{F}$ later.

We continue by looking at the case in which $i = n$. Here we show that $\lambda = za^2$ is not a square in P. Assume the contrary, i.e. $za^2 = (v^jx^ka^l)^2$ for some $j, k, l \in \mathbb{Z}$. Of course l must be odd. In the $k = 0$ case we obtain the contradiction $(v^ja^l)^2 = a^{2l}$. Thus, $k = 1$. Then $[v, xa^l] = 1$ and $(v^jxa^l)^2 = v^{2j}(xaxa^{-1})a^{2l} = v^{2j-1}a^{2l}$. Again a contradiction. Hence, λ is in fact a non-square. However, $\lambda z = a^2$ is a square and so is every power. As a consequence, it turns out that the two possibilities $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Z}(\mathcal{F}) = \langle\lambda\rangle$ or $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Z}(\mathcal{F}) = \langle a^2\rangle$ give in fact *non-isomorphic* fusion systems (in the sense of Definition 3.4). We denote the latter possibility by $\mathcal{F}'$, i.e. $\mathrm{Z}(\mathcal{F}') = \langle a^2\rangle$.

Now, for every $i \in \{\max(2, n-m+2), \ldots, n\}$ we construct P and $\mathcal{F}$. After that we explain how to obtain $\mathcal{F}'$ for $i = n$. This works similarly to Case 1 (a). Let q, H, v, x, a_1 and γ_1 be as there. It is easy to see that $\langle 1 - 2^i + 2^{n+1}\mathbb{Z}\rangle$ has order 2^{n+1-i} as a subgroup of $(\mathbb{Z}/2^{n+1}\mathbb{Z})^\times$. Set $\gamma := \gamma_1^{2^{i-2}}$. Then $\gamma^{2^m} = 1$, since $m + i - 2 \geqslant n$. Again we can find an odd integer ν such that $5^{2^{i-2}}\nu \equiv 1 - 2^i \pmod{2^{n+1}}$. Setting $a := a_1\gamma^\nu \in \mathrm{Aut}(H)$ we obtain ${}^av = v^{-1+2^i}$ and ${}^ax = vx$. It remains to prove $a^{2^m} = 1$. As above we obtain

$$a^2 = \begin{pmatrix} \omega^{2^{i-1}} & 0 \\ 0 & \omega^{-2^{i-1}} \end{pmatrix} \gamma^{2\nu}.$$

This leads to $a^{2^m} = 1$. Now we can define $G := H \rtimes \langle a\rangle$ (notice that the action of $\langle a\rangle$ on H is usually not faithful). It is easy to see in fact that $P \in \mathrm{Syl}_2(G)$ and $\mathcal{F}_P(G)$ is non-nilpotent. Hence, for $i < n$ we obtain $\mathcal{F} = \mathcal{F}_P(G)$ immediately. Now assume that $i = n$. Since $\omega^{2^n} = -1 \in \mathbb{F}_q$, we can choose ω such that $\omega^{2^{n-1}} = 2 \in \mathbb{F}_5 \subseteq \mathbb{F}_q$. Define

$$\alpha := \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \in H.$$

A calculation shows that α has order 3 and acts on $\langle x, z\rangle$ non-trivially. Moreover, $\gamma^{2\nu} = 1$ and a^2 is the inner automorphism induced by z. In particular, a^2 does not fix α. We can view α as an element of $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$. Then $\mathrm{C}_Q(\mathrm{Aut}_{\mathcal{F}_P(G)}(Q)) = \langle\lambda\rangle = \mathrm{Z}(\mathcal{F})$ is generated by a non-square in P. This shows again that $\mathcal{F} = \mathcal{F}_P(G)$. It remains to construct $\mathcal{F}'$. Observe that γ acts trivially on $\langle v, x\rangle$, since $5^{2^{n-2}} \equiv 1 \pmod{2^n}$. Hence, we can replace the automorphism a by $a_1 = \left(\begin{smallmatrix} 0 & \omega \\ -1 & 0 \end{smallmatrix}\right)$ without changing the isomorphism type of P. Again we define $G := H \rtimes \langle a_1\rangle$. Then it turns out that $a_1^2 = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathrm{Z}(\mathrm{GL}(2, q))$. In particular, a_1^2 is fixed by the element $\alpha \in \mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$ above. So here $\mathrm{Z}(\mathcal{F}) = \langle a_1^2\rangle$ is generated by a square in P. Thus, we obtain $\mathcal{F}' = \mathcal{F}_P(G)$.

Case 2 ($Q \cong C_{2^t} \times Q_8$ is $\mathcal{F}$-essential in P for some $t \geqslant 1$). We have seen above that E cannot be dihedral. Hence, E is (generalized) quaternion, i.e. $x^2 = z$. Now, $|Q : Z(Q)| = 4$ implies that $Q \cap E \cong Q_8$. After conjugation of Q we may assume that $Q \cap E = \langle v^{2^{n-2}}, x\rangle$. Proposition 3.10 implies that $z \in Z(\mathcal{F})$. In particular, $Q/\langle z\rangle \cong C_{2^t} \times C_2^2$ is an $\mathcal{F}/\langle z\rangle$-essential subgroup of $P/\langle z\rangle$ (see [**5**, Theorem 5.60]). So by the first part of the proof and Proposition 3.5 (for $n = 2$) we obtain $t = m - 1$, and Q is the only $\mathcal{F}$-essential subgroup up to conjugation. Since $C_Q(x) \cap \Phi(P)$ is still non-cyclic, we have $C_{\Phi(P)}(x) = \langle\lambda\rangle \times \langle z\rangle \cong C_{2^{m-1}} \times C_2$, as in Case 1. Moreover, a^2 fixes $v^{2^{n-2}}$, and it follows that $Q = \langle v^{2^{n-2}}, x, \lambda\rangle$.

Here we can handle the uniqueness of $\mathcal{F}$ uniformly without discussing the split and non-split cases separately. Since $\mathrm{Inn}(Q) \cong C_2^2$, $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a group of order 24 that is generated by $N_P(Q)/Z(Q)$ and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ of order 3. Hence, in order to describe the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ on Q (up to automorphisms from $\mathrm{Aut}(P)$), it suffices to know how α acts on Q. First of all, α acts on only one subgroup $Q_8 \cong R \leqslant Q$. It is not hard to see that $Q' = \langle z\rangle \subseteq R$, and thus $R \trianglelefteq Q$. In particular, R is invariant under inner automorphisms of Q. Now let β be an automorphism of Q coming from $N_P(Q)/Q \leqslant \mathrm{Out}_{\mathcal{F}}(Q)$. Then $\beta\alpha \equiv \alpha^{-1}\beta \pmod{\mathrm{Inn}(Q)}$. In particular, $\beta(R) = \alpha^{-1}(\beta(R)) = R$. Looking at the action of $N_P(Q)$, we see that $R \in \{\langle v^{2^{n-2}}, x\rangle, \langle v^{2^{n-2}}, x\lambda^{2^{m-2}}\rangle\}$. Again, the automorphism

$$v \mapsto v, \qquad x \mapsto x\lambda^{2^{m-2}}, \qquad a \mapsto av^{2^{n-2}}$$

leads to $R = \langle v^{2^{n-2}}, x\rangle$. The action of α on R is not quite unique. However, after inverting α if necessary, we have $\alpha(x) \in \{v^{2^{n-2}}, v^{-2^{n-2}}\}$. If we conjugate α with the inner automorphism induced by x if necessary, we end up with $\alpha(x) = v^{2^{n-2}}$. Since α has order 3, it follows that $\alpha(v^{2^{n-2}}) = xv^{2^{n-2}}$. So we know precisely how α acts on R. Since α is unique up to conjugation in $\mathrm{Aut}(Q)$, we have $C_Q(\alpha) = Z(Q) = \langle\lambda, z\rangle$. Hence, the action of $\mathrm{Aut}_{\mathcal{F}}(Q)$ on Q is uniquely determined. By Alperin's fusion theorem, $\mathcal{F}$ is unique up to isomorphism. For the construction of $\mathcal{F}$ we split up the proof again.

Case 2 (a) ($a^{2^m} = z$). Again, $n > m > 1$ and $i = n - m + 1$ by (3.3). So the isomorphism type of P is determined by m and n. We construct P and $\mathcal{F}$ in a similar manner as above. For this, set $q := 5^{2^{n-2}}$ and $H := \mathrm{SL}(2, q)$. Then a Sylow 2-subgroup H is given by $E := \langle v, x\rangle \cong Q_{2^{n+1}}$, where v and x are defined similarly to in Case 1 (a). The only difference is that $\omega \in \mathbb{F}_q^\times$ now has order 2^n and the matrices are not considered modulo $Z(\mathrm{SL}(2, q))$ anymore. Also, the element a_1 as above still satisfies ${}^{a_1}v = v^{-1}$ and ${}^{a_1}x = vx$. Now we can repeat the calculations in Case 1 (a) word for word. Doing so, we obtain $G := H\langle a\rangle \cong \mathrm{SL}(2, q).C_{2^m}$ and $\mathcal{F} = \mathcal{F}_P(G)$.

Case 2 (b) ($a^{2^m} = 1$). Here (3.3) gives $\max(n + m + 2, 2) \leqslant i \leqslant n$. For every i in this interval we can again construct P and $\mathcal{F}$ in the same manner as before. We omit the details.

Case 3 ($Q \cong C_{2^t} * Q_8$ is $\mathcal{F}$-essential in P for some $t \geqslant 2$). Again the arguments above reveals that E is a quaternion group and $x^2 = z$. Moreover, $Q \cap E = \langle v^{2^{n-2}}, x\rangle \cong Q_8$ after conjugation if necessary. Going over to $P/\langle z\rangle$, it follows that $t = m$. Assume that $n = m = i$ and $a^{2^m} = z$ for a moment. Then $(ax)^2 = vza^2$ and $F_1 := \langle v, ax\rangle \cong C_{2^n}^2$ is

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maximal in P. Since $P/\Phi(F_1)$ is non-abelian, we obtain $P \cong C_{2^n} \wr C_2$ (cf. the proof of Proposition 2.7). Thus, in the case in which $n = m$ and $a^{2^m} = z$ we assume that $i < n$ in the following. We will see later that other parameters cannot lead to a wreath product. After excluding this special case, it follows as before that Q is the only $\mathcal{F}$-essential subgroup up to conjugation. Since $C_Q(x)$ contains an element of order 2^m, we have $C_{\Phi(P)}(x) = \langle\lambda\rangle$. Hence, we have to replace (3.3) by

$$z = \lambda^{2^{m-1}} = v^{2^{m+i-2}\nu} a^{2^m},$$

where ν is an odd number. Moreover, $Q = \langle v^{2^{n-2}}, x, \lambda\rangle$. If $a^{2^m} = z$, then $\max(n - m + 2, 2) \leqslant i \leqslant n$. On the other hand, if $a^{2^m} = 1$, then $n > m > 1$ and $i = n - m + 1$. Hence, these cases complement exactly Case 2.

The uniqueness of $\mathcal{F}$ is a bit easier than for the other types of essential subgroups. Again $\mathrm{Aut}_{\mathcal{F}}(Q)$ has order 24 and is generated by $N_P(Q)/Z(Q)$ and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ of order 3. It suffices to describe the action of α on Q up to automorphisms from $\mathrm{Aut}(P)$. By considering $Q/Q' \cong C_{2^{m-1}} \times C_2^2$ we see that $R := \langle v^{2^{n-2}}, x\rangle$ is the only subgroup of Q isomorphic to Q_8. In particular, α must act on R. Here we can also describe the action precisely by changing α slightly. Moreover, $C_Q(\alpha) = Z(Q) = \langle\lambda\rangle$, since α is unique up to conjugation in $\mathrm{Aut}(Q)$. This shows that $\mathcal{F}$ is uniquely determined up to isomorphism. Now we distinguish the split and non-split case in order to construct P and $\mathcal{F}$.

Case 3 (a) ($a^{2^m} = 1$). At first glance one might think that the construction in Case 2 should not work here. However, it does. We denote q, H and so on as in Case 2 (a). Then a^{2^m} is the inner automorphism on H induced by z. But since $z \in Z(H)$, a^{2^m} is in fact the trivial automorphism. Hence, we can construct the semi-direct product $G = H \rtimes \langle a\rangle$, which does the job.

Case 3 (b) ($a^{2^m} = z$). Here we do the opposite to Case 3 (a). With the notation of Case 3 (a), a is an automorphism of H such that $a^{2^m} = 1$ and a fixes $z \in Z(H)$. Using [**10**, Theorem 15.3.1] we can build a non-split extension $G := H\langle a\rangle$ such that $a^{2^m} = z$. This group fulfils our conditions.

Finally, we show that different parameters in all these group presentations give non-isomorphic groups. Obviously, the metacyclic groups are pairwise non-isomorphic and not isomorphic to non-metacyclic groups. Hence, it suffices to look at the groups coming from [**18**, Theorem 4.4]. So let P be as in (3.2) together with additional dependence between x^2 and the choice of i as in the statement of our theorem (this restriction is important). Assume that P is isomorphic to a similar group P_1, where we attach an index 1 to all elements and parameters of P_1. Then we have $2^{n+m+1} = |P| = |P_1| = 2^{n_1+m_1+1}$ and $2^n = |P'| = |P_1'| = 2^{n_1}$. This already shows that $n = n_1$ and $m = m_1$. As proved above, P admits a non-nilpotent fusion system with essential subgroup $C_{2^{m-1}} \times C_2^2$ if and only if $x^2 = 1$. Hence, $x^2 = 1$ if and only if $x_1^2 = 1$. Now we show that $i = i_1$. For this we consider $\Phi(P) = \langle v, a^2\rangle$. Since $\Phi(P)$ is metacyclic, it follows that $\Phi(P)' = \langle[v, a^2]\rangle = \langle v^{2^{i+1}}\rangle \cong C_{2^\eta}$, where $\eta := \max(n - i - 1, 0)$. Since $i, i_1 \leqslant n$, we may assume that $i, i_1 \in \{n - 1, n\}$. In the $i = n$ case the subgroup $C := \langle v, ax\rangle$ is abelian. By [**18**, Theorem 4.3 (f)], C is a metacyclic maximal subgroup of P. However, in the $i = n - 1$ case it is easy to see that the two metacyclic maximal subgroups $\langle v, a\rangle$ and $\langle v, ax\rangle$ of P are both non-abelian. This gives $i = i_1$. It remains to show that $a^{2^m} = 1 \iff a_1^{2^{m_1}} = 1$. For this we may assume that $x^2 = z$ and $x_1^2 = z_1$. In the $i = n - m + 1$ (and $n > m > 1$) case we have $a^{2^m} = 1$ if and only if P provides a fusion system with essential subgroup $C_{2^m} * Q_8$. A similar equivalence holds for $\max(n - m + 2, 2) \leqslant i \leqslant n$ (even in the case in which $n = m = i$). This completes the proof. □

We present an example to shed more light on the alternative in part (10) of Theorem 3.19. Let us consider the smallest case, $n = m = i = 2$. The group $N := A_6 \cong PSL(2, 3^2)$ has Sylow 2-subgroup D_8. Let $H := \langle h\rangle \cong C_4$. It is well known that $\mathrm{Aut}(N)/N \cong C_2^2$, and the three subgroups of $\mathrm{Aut}(N)$ of index 2 are isomorphic to S_6, $PGL(2, 9)$ and the Mathieu group M_{10} of degree 10. We choose two homomorphisms $\phi_j : H \to \mathrm{Aut}(N)$ for $j = 1, 2$ such that $\phi_1(h) \in PGL(2, 9) \setminus N$ is an involution and $\phi_2(h) \in M_{10} \setminus N$ has order 4 (we do not define ϕ_j precisely). Then it turns out that the groups $G_j := N \rtimes_{\phi_j} H$ for $j = 1, 2$ have Sylow 2-subgroup P as in part (10). Moreover, one can show that $\mathcal{F}_1 := \mathcal{F}_P(G_1) \neq \mathcal{F}_P(G_2) =: \mathcal{F}_2$. More precisely, $Z(\mathcal{F}_1) = Z(G_1) = \langle \phi_1(h)^2\rangle$ is generated by a square in P and $Z(\mathcal{F}_2)$ is not. The indices of G_j in the 'small group library' are `[1440,4592]` and `[1440,4595]`, respectively. It should be clarified that this phenomenon is not connected to the special behaviour of A_6, since it occurs for all n with $PSL(2, 5^{2^{n-1}})$.

As another comment, we observe that the 2-groups in parts (11)–(14) have 2-rank 2. Hence, these are new examples in the classification of all fusion systems on 2-groups of 2-rank 2, which was started in [**6**]. It is natural to ask what happens if we interchange the restrictions on i in case (9) and case (10) in Theorem 3.19. We will see in the next theorem that this does not result in new groups.

Theorem 3.20. *Let P be a bicyclic non-metacyclic 2-group. Then P admits a non-nilpotent fusion system if and only if P' is cyclic.*

Proof. By Theorem 3.17 it suffices to prove only one direction. Let us assume that P' is cyclic. Since P is non-metacyclic, it follows that $P' \neq 1$. In the $|P'| = 2$ case, [**18**, Theorem 4.1] implies that P is minimal non-abelian of type $(n, 1)$ for some $n \geqslant 2$. We have already shown that there is a non-nilpotent fusion system on this group. Thus, we may assume that $|P'| > 2$. Then we are again in [**18**, Theorem 4.4]. After adapting notation, P is given as in (3.2). In the $x^2 = z$ case there is always a non-nilpotent fusion system on P, by Theorem 3.19. Hence, let $x^2 = 1$. Then it remains to deal with two different pairs of parameters.

Case 1 ($\boldsymbol{a^{2^m} = 1}$ and $\boldsymbol{i = n - m + 1 \geqslant 2}$). Set $\tilde{x} := xa^{2^{m-1}}$. Then

$$\tilde{x}^2 = xa^{2^{m-1}}xa^{2^{m-1}} = (v^{-1}a)^{2^{m-1}}a^{2^{m-1}} = v^{2^{i+m-2}\nu}a^{2^m} = z$$

for an odd integer ν. Moreover, ${}^{\tilde{x}}v = v^{-1}$, ${}^{a}\tilde{x} = vxa^{2^{m-1}} = v\tilde{x}$. This shows that P is isomorphic to a group with parameters $x^2 = z$, $a^{2^m} = 1$ and $i = n - m + 1 \geqslant 2$. In particular, Theorem 3.19 provides a non-nilpotent fusion system on P.

Table 1. *Number of non-nilpotent fusion systems.*

N	1	2	3	$\geqslant 4$ even	$\geqslant 5$ odd
$f(N)$	0	1	2	$\frac{3}{4}N^2-3N+5$	$(3N^2+1)/4-3N+3$
$g(N)$	0	1	3	$\frac{3}{4}N^2-2N+5$	$(3N^2+1)/4-2N+5$

Case 2 ($a^{2^m}=z$ and $\max(2,n-m+2)\leqslant i\leqslant n$). Again, let $\tilde{x}:=xa^{2^{m-1}}$. Then

$$\tilde{x}^2=v^{2^{i+m-2}\nu}a^{2^m}=z.$$

Hence, P is isomorphic to a group with parameters $x^2=a^{2^m}=z$ and $\max(2,n-m+2)\leqslant i\leqslant n$. The claim follows as before. □

Now we count how many interesting fusion systems we have found.

Proposition 3.21. *Let $f(N)$ be the number of isomorphism classes of bicyclic 2-groups of order 2^N that admit a non-nilpotent fusion system. Moreover, let $g(N)$ be the number of non-nilpotent fusion systems on all bicyclic 2-groups of order 2^N. Then the results of Table 1 hold.*

Proof. Without loss of generality, $N\geqslant 4$. We have to distinguish between the cases in which N is even and those in which N is odd. Assume first that N is even. Then we obtain the following five groups: $\mathrm{C}^2_{2^{N/2}}$, D_{2^N}, Q_{2^N}, SD_{2^N} and the minimal non-abelian group of type $(N-2,1)$. From case (9) of Theorem 3.19 we obtain exactly $\frac{1}{2}N-2$ groups. In case (10) the number of groups is

$$\begin{aligned}\sum_{n=2}^{N-3}(n-\max(2,2n-N+3)+1)&=\sum_{n=2}^{N/2-1}(n-1)+\sum_{n=N/2}^{N-3}(N-n-2)\\&=2\sum_{n=1}^{N/2-2}n\\&=(\tfrac{1}{2}N-2)(\tfrac{1}{2}N-1)\\&=\tfrac{1}{4}N^2-\tfrac{3}{2}N+2.\end{aligned}$$

The other cases are similar (observe that the wreath product cannot occur, since N is even). All together we obtain

$$5+3(\tfrac{1}{2}N-2)+3(\tfrac{1}{4}N^2-\tfrac{3}{2}N+2)=\tfrac{3}{4}N^2-3N+5$$

bicyclic 2-groups of order 2^N with non-nilpotent fusion system.

Now, if N is odd, we have the following four examples: D_{2^N}, Q_{2^N}, SD_{2^N} and the minimal non-abelian group of type $(N-2,1)$. From case (9) of Theorem 3.19 we obtain exactly $(N-5)/2$ groups. In case (10) the number of groups is

$$\begin{aligned}\sum_{n=2}^{N-3}(n-\max(2,2n-N+3)+1) &= \sum_{n=2}^{(N-1)/2}(n-1)+\sum_{n=(N+1)/2}^{N-3}(N-n-2)\\ &= 2\sum_{n=1}^{(N-5)/2}n+\frac{N-3}{2}\\ &= \frac{(N-5)(N-3)}{4}+\frac{N-3}{2}\\ &= \frac{N^2-6N+9}{4}.\end{aligned}$$

Adding the numbers from the other cases (this time including the wreath product), we obtain

$$4+3\frac{N^2-4N-1}{4}=\frac{3N^2+1}{4}-3N+3.$$

In order to obtain $g(N)$ from $f(N)$ we have to add one fusion system on D_{2^N}, one on Q_{2^N} and two on SD_{2^N}. If N is odd, we get two more fusion systems on the wreath product. For all $N \geqslant 5$ we have to add $N-4$ fusion systems coming from part (10) in Theorem 3.19. □

4. Applications

We present an application to finite simple groups. For this we introduce a general lemma.

Lemma 4.1. *Let G be a perfect group and let $1 \neq P \in \mathrm{Syl}_p(G)$ such that $\mathrm{N}_G(P) = P\,\mathrm{C}_G(P)$. Then there are at least two conjugacy classes of $\mathcal{F}_P(G)$-essential subgroups in P.*

Proof. Let $\mathcal{F} := \mathcal{F}_P(G)$. If there is no $\mathcal{F}$-essential subgroup, then $\mathcal{F}$ is nilpotent and G is p-nilpotent, since $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{N}_G(P)/P\,\mathrm{C}_G(P) = 1$. Then $G' \leqslant P'\,\mathrm{O}_{p'}(G) < G$, because $P \neq 1$, which is a contradiction. Now suppose that there is exactly one $\mathcal{F}$-essential subgroup $Q \leqslant P$ up to conjugation. Then Q lies in a maximal subgroup $M < P$. Moreover, $P' \subseteq \Phi(P) \subseteq M$. Now the focal subgroup theorem (see [**9**, Theorem 7.3.4]) gives the following contradiction:

$$P = P\cap G = P\cap G' = \langle x^{-1}\alpha(x)\colon x\in P,\ \alpha \text{ a morphism in } \mathcal{F}\rangle \subseteq P'Q \subseteq M.$$

□

We remark that the number of conjugacy classes of essential subgroups is sometimes called the *essential rank* of the fusion system (see, for example, [**11**]).

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Theorem 4.2. *Let G be a simple group with bicyclic Sylow 2-subgroup. Then G is one of the following groups:* C_2, $\mathrm{PSL}(i,q)$, $\mathrm{PSU}(3,q)$, A_7 *or* M_{11} *for* $i \in \{2,3\}$ *and* q *odd.*

Proof. By the Alperin–Brauer–Gorenstein theorem [**1**] on simple groups of 2-rank 2, we may assume that G has 2-rank 3 (observe that a Sylow 2-subgroup of $\mathrm{PSU}(3,4)$ is not bicyclic, since it is 4-generator). Let $P \in \mathrm{Syl}_2(G)$ and let $\mathcal{F} := \mathcal{F}_P(G)$. By Theorem 3.19, there is only one $\mathcal{F}$-essential subgroup Q in P up to conjugation. But this contradicts Lemma 4.1. □

Now we consider fusion systems coming from block theory. Let B be a p-block of a finite group G. We denote the number of irreducible ordinary characters of B by $k(B)$ and the number of irreducible Brauer characters of B by $l(B)$. Moreover, let $k_0(B)$ be the number of irreducible characters of *height* 0, i.e. the p-part of the degree of these characters is as small as possible. Let D be a defect group of B. Then for every element $u \in D$ we have a *subsection* (u, b_u), where b_u is a Brauer correspondent of B in $\mathrm{C}_G(u)$.

Theorem 4.3. *Olsson's conjecture holds for all blocks of finite groups with bicyclic defect groups.*

Proof. Let B be a p-block of a finite group with bicyclic defect group D. Since all bicyclic p-groups for an odd prime are metacyclic, we may assume that $p = 2$. If D is metacyclic, minimal non-abelian or a wreath product, then Olsson's conjecture holds by the results in [**19**, **27**, **28**]. Let $\mathcal{F}$ be the fusion system of B. Without loss of generality, $\mathcal{F}$ is non-nilpotent. Hence, we may assume that D is given by

$$D \cong \langle v, x, a \mid v^{2^n} = 1,\ x^2,\ a^{2^m} \in \langle v^{2^{n-1}} \rangle,\ {}^x v = v^{-1},\ {}^a v = v^{-1+2^i},\ {}^a x = vx \rangle,$$

where $\max(2, n-m+1) \leqslant i \leqslant n$ as in Theorem 3.19. Moreover, there is only one conjugacy class of $\mathcal{F}$-essential subgroups of D. We use [**12**, Proposition 2.5 (i)]. For this let us consider the subsection (a, b_a). Since $\langle a, v \rangle$ is a metacyclic maximal subgroup of P, a does not lie in any $\mathcal{F}$-essential subgroup of P. In particular, $\langle a \rangle$ is fully $\mathcal{F}$-centralized. Thus, [**21**, Theorem 2.4 (ii)] implies that b_a has defect group $\mathrm{C}_D(a)$. Obviously, $\mathrm{C}_{\langle v \rangle}(a) = \langle z \rangle$. Now let $v^j x \in \mathrm{C}_D(a)$ for some $j \in \mathbb{Z}$. Then $v^j x = {}^a(v^j x) = v^{1-j+2^i j} x$ and $v^{2j} = v^{1+2^i j}$, a contradiction. This shows that $\mathrm{C}_D(a) = \langle a, z \rangle$. Now, by [**12**, Proposition 2.5 (i)] we obtain $k_0(B) \leqslant |\mathrm{C}_D(a)| = 2^{m+1} = |D : D'|$, i.e. Olsson's conjecture holds. □

Using [**26**, Theorem 3.4], it is not hard to see that Brauer's $k(B)$-conjecture also holds if for the fusion system of B one of the cases (1)–(10) in Theorem 3.19 occurs.

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References

1. J. L. Alperin, R. Brauer and D. Gorenstein, Finite simple groups of 2-rank two, *Scripta Math.* **29** (1973), 191–214.
2. Y. Berkovich, *Groups of prime power order*, Volume 1, De Gruyter Expositions in Mathematics, Volume 46 (Walter de Gruyter, Berlin, 2008).
3. Y. Berkovich and Z. Janko, *Groups of prime power order*, Volume 2, De Gruyter Expositions in Mathematics, Volume 47 (Walter de Gruyter, Berlin, 2008).
4. N. Blackburn, Über das Produkt von zwei zyklischen 2-Gruppen, *Math. Z.* **68** (1958), 422–427.
5. D. A. Craven, *The theory of fusion systems*, Cambridge Studies in Advanced Mathematics, Volume 131 (Cambridge University Press, 2011).
6. D. A. Craven and A. Glesser, Fusion systems on small p-groups, *Trans. Am. Math. Soc.* **364** (2012), 5945–5967.
7. S. Du, G. Jones, J. H. Kwak, R. Nedela and M. Škoviera, 2-groups that factorise as products of cyclic groups, and regular embeddings of complete bipartite graphs, *Ars Math. Contemp.* **6** (2013), 155–170.
8. GAP Group, GAP, Groups, Algorithms, and Programming, Version 4.4.12 (2008) (available at www.gap-system.org).
9. D. Gorenstein, *Finite groups* (Harper and Row, New York, 1968).
10. M. Hall, *The theory of groups* (Macmillan, London, 1959).
11. E. Henke, Recognizing $\mathrm{SL}_2(q)$ in fusion systems, *J. Group Theory* **13** (2010), 679–702.
12. L. Héthelyi, B. Külshammer and B. Sambale, A note on Olsson's conjecture, *J. Alg.* **398** (2014), 364–385.
13. B. Huppert, Über das Produkt von paarweise vertauschbaren zyklischen Gruppen, *Math. Z.* **58** (1953), 243–264.
14. B. Huppert, *Endliche Gruppen, I*, Die Grundlehren der Mathematischen Wissenschaften, Volume 134 (Springer, 1967).
15. N. Itô, Über das Produkt von zwei zyklischen 2-Gruppen, *Publ. Math. Debrecen* **4** (1956), 517–520.
16. N. Itô and A. Ôhara, Sur les groupes factorisables par deux 2-groupes cycliques, I, Cas où leur groupe des commutateurs est cyclique, *Proc. Jpn Acad.* A **32** (1956), 736–740.
17. N. Itô and A. Ôhara, Sur les groupes factorisables par deux 2-groupes cycliques, II, Cas où leur groupe des commutateurs n'est pas cyclique, *Proc. Jpn Acad.* A **32** (1956), 741–743.
18. Z. Janko, Finite 2-groups with exactly one nonmetacyclic maximal subgroup, *Israel J. Math.* **166** (2008), 313–347.
19. B. Külshammer, On 2-blocks with wreathed defect groups, *J. Alg.* **64** (1980), 529–555.
20. H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Universitext (Springer, 2004).
21. M. Linckelmann, Simple fusion systems and the Solomon 2-local groups, *J. Alg.* **296** (2006), 385–401.
22. V. D. Mazurov, Finite groups with metacyclic Sylow 2-subgroups, *Sibirsk. Mat. Zh.* **8** (1967), 966–982.
23. B. Oliver, *Reduced fusion systems over* 2*-groups of sectional rank at most* 4, Memoirs of the American Mathematical Society, Volume 239 (American Mathematical Society, Providence, RI, 2015).
24. B. Oliver and J. Ventura, Saturated fusion systems over 2-groups, *Trans. Am. Math. Soc.* **361** (2009), 6661–6728.
25. L. Rédei, Das 'schiefe Produkt' in der Gruppentheorie, *Comment. Math. Helv.* **20** (1947), 225–264.

26. G. R. Robinson, On the number of characters in a block, *J. Alg.* **138** (1991), 515–521.
27. B. Sambale, 2-Blocks with minimal nonabelian defect groups, *J. Alg.* **337** (2011), 261–284.
28. B. Sambale, Fusion systems on metacyclic 2-groups, *Osaka J. Math.* **49** (2012), 325–329.
29. B. Sambale, Further evidence for conjectures in block theory, *Alg. Number Theory* **7** (2013), 2241–2273.
30. R. Stancu, Control of fusion in fusion systems, *J. Alg. Appl.* **5** (2006), 817–837.
31. R. M. Thomas, On 2-groups of small rank admitting an automorphism of order 3, *J. Alg.* **125** (1989), 27–35.
32. R. W. van der Waall, On p-nilpotent forcing groups, *Indagationes Math.* **2** (1991), 367–384.