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ON POLY-EULER NUMBERS

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Abstract

Poly-Euler numbers are introduced as a generalization of the Euler numbers in a manner similar to the introduction of the poly-Bernoulli numbers. In this paper, some number-theoretic properties of poly-Euler numbers, for example, explicit formulas, a Clausen–von Staudt type formula, congruence relations and duality formulas, are given together with their combinatorial properties.

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1. Introduction

For every integer k, we define *poly-Euler numbers* $E_n^{(k)}$ $(n = 0, 1, 2, \ldots)$, which are introduced as a generalization of the Euler number, by

$$\frac{\mathrm{Li}_k(1 - e^{-4t})}{4t(\cosh t)} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n. \tag{1.1}$$

Here,

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, k \in \mathbb{Z})$$

is the kth polylogarithm. When $k = 1$, $E_n^{(1)}$ is the Euler number E_n defined by

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n. \tag{1.2}$$

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Poly-Euler numbers are special values of a certain L-function as in the case of poly-Bernoulli numbers [2]. We shall see this in Section 2.

In this paper, we show how poly-Euler numbers have nice properties similar to the poly-Bernoulli numbers. In fact, we show explicit formulas, a Clausen–von Staudt type formula, congruence relations, combinatorial interpretations and duality formulas of poly-Euler numbers. Other properties are found in [10, 11].

In Section 3 we give two kinds of explicit formulas for poly-Euler numbers. In Section 4 the sign change of di-Euler numbers is determined completely. In Section 5 we show a Clausen–von Staudt type formula for poly-Euler numbers. From Section 6 we focus on the negative index case, and in Section 6 we show explicit formulas and see some combinatorial properties. In Section 7 we give some congruence formulas for poly-Euler numbers. In Section 8 we consider the 2-orders of poly-Euler numbers. In Section 9 we give two kinds of duality formula. Finally, in Section 10, we consider the positivity of poly-Euler numbers.

2. Related L-function

For every integer k, Kaneko [6] introduced the poly-Bernoulli number $\mathbb{B}_n^{(k)}$ given by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(k)}}{n!} t^n.$$

When $k = 1$, $\mathbb{B}_n^{(1)}$ is the classical Bernoulli number. Many interesting properties of the poly-Bernoulli numbers have been pointed out by Kaneko [6], Arakawa and Kaneko [3], Sánchez-Peregrino [12] and others. Furthermore, the combinatorial interpretations of $\mathbb{B}_n^{(-k)}$ were given by Brewbaker [4] and Launois [9]. Arakawa and Kaneko [2] introduced a zeta function that is called the *Arakawa–Kaneko zeta function*,

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1}\, dt \quad (k \geq 1),$$

which is a kind of generalization of the Riemann zeta function (in fact, $\xi_1(s) = s\zeta(s+1)$), and showed that

$$\xi_k(-n) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \mathbb{B}_l^{(k)} \quad (n \in \mathbb{Z}_{\geq 0}).$$

From this viewpoint, it is natural to define poly-Euler numbers as special values of a suitable L-function. The Euler number E_n is the generalized Bernoulli number associated with the Dirichlet character of conductor 4, and the corresponding L-function for E_n is

$$L(s) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{e^t + e^{-t}}\, dt$$

because $L(-n) = E_n/2$ is satisfied by all nonnegative integers n. The second author gave in [13] a general method for defining L-functions that have similar properties to the Arakawa–Kaneko zeta function. By using the method, a natural generalization of $L(s)$ is

$$L_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{\mathrm{Li}_k(1-e^{-4t})}{4(e^t+e^{-t})}\, dt \quad (k \geq 1).$$

Then $L_1(s) = sL(s+1)$ and

$$L_k(-n) = \frac{(-1)^n n E_{n-1}^{(k)}}{2} \quad (n \in \mathbb{N})$$

under the definition of poly-Euler numbers by (1.1).

3. Explicit formulas

In this section we present two kinds of explicit formula for $E_n^{(k)}$ which are essential tools to investigate properties of poly-Euler numbers. The first explicit formula is a standard type, while the second explicit formula is appropriate for investigating the relations between poly-Euler numbers and ordinary Euler numbers. For example, as we see in Corollary 3.3, a di-Euler number with odd n equals an ordinary Euler number up to some elementary factor, which is naturally derived from the second explicit formula. Thus we use both formulas as the situation demands.

The following theorem gives the first explicit formula.

Theorem 3.1. *For any nonnegative integer n and integer k,*

$$E_n^{(k)} = \frac{1}{2(n+1)} \sum_{m=0}^{n+1} \binom{n+1}{m} \mathbb{B}_{n-m+1}^{(k)} 4^{n-m+1}((-1)^m - (-3)^m). \tag{3.1}$$

Proof. Comparing with the Taylor coefficients of both sides of

$$e^{-t} \frac{\mathrm{Li}_k(1-e^{-4t})}{1-e^{-4t}} - e^{-3t} \frac{\mathrm{Li}_k(1-e^{-4t})}{1-e^{-4t}} = \frac{\mathrm{Li}_k(1-e^{-4t})}{e^t+e^{-t}}, \tag{3.2}$$

we immediately obtain Theorem 3.1. □

Next we state the second explicit formula for $E_n^{(k)}$, which is the finite sum of the product of the modified poly-Bernoulli numbers and the Euler numbers. For every integer k, the modified poly-Bernoulli number $C_n^{(k)}$ is defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} \frac{C_n^{(k)}}{n!} t^n.$$

When $k = 1$, $C_n^{(1)}$ is the classical Bernoulli number unless $n = 1$, and $C_1^{(1)} = 1/2$. We remark that the following relations hold for the two kinds of poly-Bernoulli numbers:

$$C_n^{(k)} = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \mathbb{B}_m^{(k)}, \quad \mathbb{B}_n^{(k)} = \sum_{m=0}^{n} \binom{n}{m} C_m^{(k)}.$$

Then we have the following result.

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THEOREM 3.2. *For any nonnegative integer n and integer k,*

$$E_n^{(k)} = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m}4^{n-m}C_{n-m}^{(k-1)}E_m.$$

PROOF. Since $(d/dt)\,\mathrm{Li}_k(1-e^{-t}) = \mathrm{Li}_{k-1}(1-e^{-t})/(e^t-1)$,

$$\mathrm{Li}_k(1-e^{-t}) = \sum_{n=1}^{\infty}\frac{C_{n-1}^{(k-1)}}{n!}t^n. \tag{3.3}$$

Therefore we obtain Theorem 3.2 by combining (3.3) with (1.1). □

Note that $C_n^{(1)} = B_n$ unless $n = 1$. Therefore we have the following corollary.

COROLLARY 3.3. *For any nonnegative integer n,*

$$E_{2n+1}^{(2)} = -(2n+1)E_{2n}.$$

4. The sign change of di-Euler numbers

It is known that the sign change of the classical Euler number is determined by

$$(-1)^n E_{2n} > 0 \quad (n \ge 0). \tag{4.1}$$

We determine the sign change of di-Euler numbers completely as follows.

THEOREM 4.1. *For any positive integer n,*

$$(-1)^n E_{2n-1}^{(2)} > 0 \tag{4.2}$$

and

$$(-1)^{n-1}E_{2n}^{(2)} > 0 \quad (n \ge 2).$$

First, we show an algorithm to calculate the alternative Euler numbers which plays important role in proving Theorem 4.1. Further, it gives an alternative proof of (4.1).

PROPOSITION 4.2 (Algorithm for the Euler numbers). *We define a sequence $a_{n,m}$, which denotes the mth $(m = 0, 1, 2, \dots)$ number in the nth $(n = 0, 1, 2, \dots)$ row, by*

$$a_{n,0} = a_{n-1,1}, \quad a_{n,m} = ma_{n-1,m-1} + (m+1)a_{n-1,m+1} \quad (m \ge 1) \tag{4.3}$$

with the initial values

$$a_{0,0} = 1, \quad a_{0,m} = 0 \quad (m > 0). \tag{4.4}$$

Then

$$a_{2n,0} = (-1)^n E_{2n}.$$

REMARK 4.3. From (4.3) and (4.4) we can easily see that all $a_{2n,0}$ are positive, which implies that $(-1)^n E_{2n} = |E_{2n}|$.

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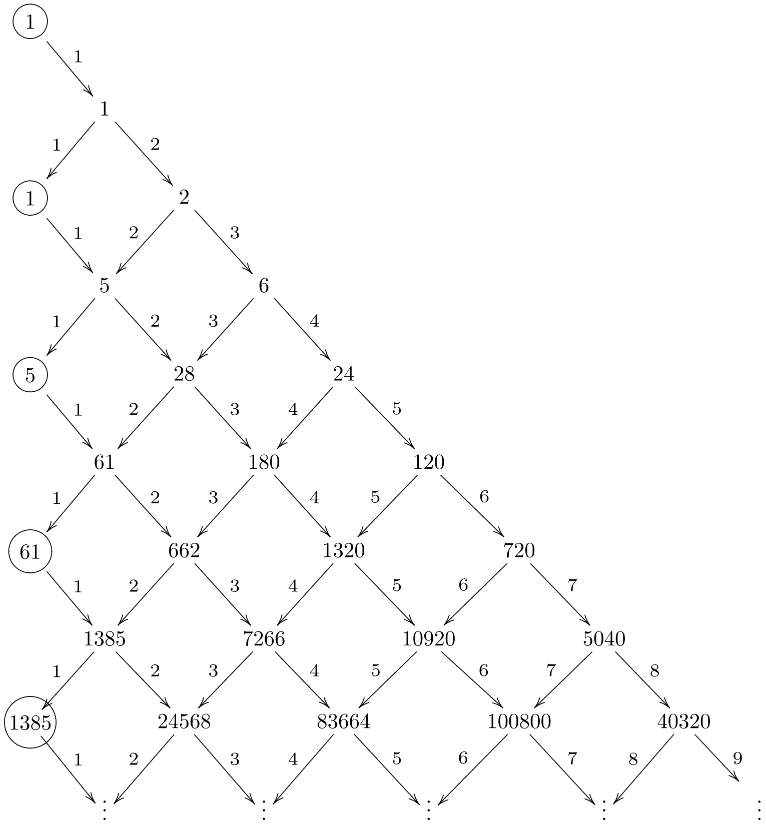

FIGURE 1. Algorithm for the Euler numbers.

REMARK 4.4. Proposition 4.2 is illustrated by Figure 1. In Figure 1, entries which are equal to 0 are not shown, and numbers on each arrow denote the multipliers for the previous values. We see that Figure 1 indicates that the alternative Euler number $(-1)^n E_{2n}$ is given as a sum of contributions come from all possible routes from $a_{0,0}$ to $a_{2n,0}$. For example, $a_{4,0} = 5 = 2!^2 + 1^4$. The first term $2!^2$ of the equation corresponds to the route $a_{0,0} \to a_{1,1} \to a_{2,2} \to a_{3,1} \to a_{4,0}$ in Figure 1, and the second term 1^4 corresponds to the route $a_{0,0} \to a_{1,1} \to a_{2,0} \to a_{3,1} \to a_{4,0}$. In general, the route given by going n times lower right on the first n steps and n times lower left on the second n steps gives the contribution $n!^2$, and hence we have the inequality

$$|E_{2n}| \geq n!^2 \quad (n \in \mathbb{Z}_{\geq 0}).$$

The above inequality is a weak one. However, this kind of inequality will be needed later.

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Proof of Proposition 4.2. Let us put

$$f(x) := \sum_{m\geq 0} a_{0,m}(\sin x)^m(\cos x)^{-(m+1)}$$

and consider the differentiation of $f(x)$. Then

$$\begin{aligned} f'(x) &= a_{0,1}(\cos x)^{-1} + \sum_{m\geq 1}(ma_{0,m-1} + (m+1)a_{0,m+1})(\sin x)^m(\cos x)^{-(m+1)} \\ &=: \sum_{m\geq 0} a_{1,m}(\sin x)^m(\cos x)^{-(m+1)} \end{aligned}$$

and the coefficients $a_{1,m}$ satisfy (4.3) with $n = 1$. Repeating this argument, we see that coefficients of $f^{(n)}(x)$ also satisfy (4.3). Since

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{x^{2n}}{(2n)!},$$

which is given by substituting $t = ix$ into (1.2), we have Proposition 4.2 by setting the initial values as (4.4).

Remark 4.5. The algorithm in Proposition 4.2 can also be found in [8]. However, our proof is completely different from the proof described in [8] and applicable to the case of tangent numbers. It is needed to prove Theorem 4.1.

By using the method as in the case of Proposition 4.2, we have the following lemma needed to prove Theorem 4.1.

Lemma 4.6. *For any positive integer n,*

$$\frac{4^{2n}|B_{2n}|}{2n+1} > \frac{T_n}{2} \geq \frac{n!(n-1)!}{2},$$

where $T_n := (2^{2n} - 1)2^{2n}|B_{2n}|/(2n)$ *and* B_n *are the nth tangent number and the nth Bernoulli number, respectively.*

Proof. The first inequality is trivial. The second inequality is obtained by considering the algorithm for tangent numbers similar to that for Euler numbers as above (see Remark 4.4). □

Proof of Theorem 4.1. By combining Corollary 3.3 and (4.1), we immediately obtain (4.2).

Next, we treat the case of di-Euler numbers with even n. From Theorem 3.2,

$$\begin{aligned} (-1)^{n-1}E_{2n}^{(2)} &= \frac{1}{2n+1}\sum_{m=0}^{n}\binom{2n+1}{2m}4^{2(n-m)}(-1)^{n-m-1}B_{2(n-m)}(-1)^m E_{2m} \\ &= \sum_{m=0}^{n-1}\binom{2n}{2m}\frac{4^{2(n-m)}|B_{2(n-m)}|}{2(n-m)+1}|E_{2m}| - |E_{2n}| \\ &= \sum_{m=0}^{n-1}\binom{2n}{2m}|E_{2m}|\left\{\frac{4^{2(n-m)}|B_{2(n-m)}|}{2(n-m)+1} + (-1)^{n-m}\right\}. \end{aligned} \tag{4.5}$$

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Here, we have used the formula

$$E_{2n} = -\sum_{j=0}^{n-1}\binom{2n}{2j}E_{2j} \quad (n \geq 1). \tag{4.6}$$

From Lemma 4.6, we see that values in the inner brackets in the above formula are positive except for the case $m = n - 1$. In fact, the value in the inner brackets for $m = n - 1$ is negative. Therefore, we have to treat such term appropriately. By applying (4.6) to (4.5) again,

$$\begin{aligned}(-1)^{n-1}E_{2n}^{(2)} &= \sum_{m=0}^{n-2}\binom{2n}{2m}|E_{2m}|\left\{\frac{4^{2(n-m)}|B_{2(n-m)}|}{2(n-m)+1} + (-1)^{n-m}\right\}\\ &\quad - \frac{1}{9}\binom{2n}{2}|E_{2n-2}|\\ &= \sum_{m=0}^{n-2}\binom{2n}{2m}|E_{2m}|\left\{\frac{4^{2(n-m)}|B_{2(n-m)}|}{2(n-m)+1} + (-1)^{n-m}\right.\\ &\quad \left. - \frac{(-1)^{n-m}}{9}(n-m)(2(n-m)-1)\right\}.\end{aligned}$$

When $n - m$ is odd, such terms are all positive from Lemma 4.6. On the other hand, when $n - m$ is even, we just have to show that

$$\frac{4^{2m}|B_{2m}|}{2m+1} - \frac{m(2m-1)}{9} \geq 0 \tag{4.7}$$

for any $m \geq 2$. We can confirm the above inequality by using Lemma 4.6. For $m \geq 2$,

$$\begin{aligned}\frac{4^{2m}|B_{2m}|}{2m+1} - \frac{m(2m-1)}{9} &> m\left(\frac{(m-1)!^2}{2} - \frac{2m-1}{9}\right)\\ &> m\left(\frac{(m-1)^2}{2} - \frac{m}{4}\right) = \frac{m}{4}(2m-1)(m-2) \geq 0,\end{aligned}$$

which provides (4.7). Therefore the proof of Theorem 4.1 is now complete. □

5. The Clausen–von Staudt type formula

Since every Euler number E_n is an integer, the Clausen–von Staudt type formula for them does not exist. However, our poly-Euler numbers $E_n^{(k)}$ are rational numbers in general. Therefore it is possible to consider an analogue of the Clausen–von Staudt formula for $E_n^{(k)}$. We first present the following theorem which describes the p-order of pth poly-Euler numbers can be determined completely.

THEOREM 5.1. *Let k be a positive integer greater than* 1. *For any odd prime p, $p^k E_{p-1}^{(k)} \in \mathbb{Z}_{(p)}$ and $p^k E_{p-1}^{(k)} \equiv -1 \bmod p$.*

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The following lemma is needed to prove theorems in this section.

LEMMA 5.2 [6, Theorem 1]. *For any nonnegative integer n and integer k,*

$$\mathbb{B}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}, \tag{5.1}$$

where the symbol $\left\{ {k \atop l} \right\}$ *denotes Stirling numbers of the second kind.*

PROOF. From Theorem 3.1,

$$p^k E_{p-1}^{(k)} = \frac{1}{2} \sum_{m=1}^{p} \binom{p}{m} p^{k-1} \mathbb{B}_{p-m}^{(k)} 4^{p-m}((-1)^m - (-3)^m).$$

From the viewpoint of Lemma 5.2, we can easily see that terms except for the case $m = 1$ belong to $\mathbb{Z}_{(p)}$, and are equivalent to 0 modulo p. Therefore we just have to treat the case $m = 1$. From Lemma 5.2, $p^k \mathbb{B}_{p-1} \in \mathbb{Z}_{(p)}$ and $p^k \mathbb{B}_{p-1}^{(k)} \equiv -1 \bmod p$. □

Next, we treat a more general case. Then the p-order of poly-Bernoulli numbers plays an important role. Arakawa and Kaneko [3] showed the Clausen–von Staudt type formula for the poly-Bernoulli numbers. First, we review the argument due to Arakawa and Kaneko. They denoted each term on the right-hand side of (5.1) by

$$b_n^{(k)}(m) := \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}$$

and estimated $b_n^{(k)}(m)$. In particular, if we rewrite (5.1) as

$$(-1)^n \mathbb{B}_n^{(k)} = \{b_n^{(k)}(p-1) + b_n^{(k)}(2p-1) + b_n^{(k)}(p^2-1)\} + \mathcal{R}_n^{(k)} \tag{5.2}$$

for prime p satisfying $k + 2 \leq p \leq n + 1$, Arakawa and Kaneko showed that $p^{k-1}\mathcal{R}_n^{(k)} \in \mathbb{Z}_{(p)}$ and $p^{k-1}\mathcal{R}_n^{(k)} \equiv 0 \bmod p$. Here, $\mathcal{R}_n^{(k)}$ denotes the remainder terms. Thus we consider contributions from the former three terms on the left-hand side of (5.2) and obtain the following result.

THEOREM 5.3. *Let k be a positive integer greater than* 1. *For any prime number p satisfying* $k + 2 \leq p \leq n + 1$, $p^{k-1}(n+1)E_n^{(k)} \in \mathbb{Z}_{(p)}$ *and*

$$\begin{aligned} p^{k-1}(n+1)E_n^{(k)} \equiv \frac{(-1)^n}{2} \bigg\{ &-\frac{1}{p} \sum_{m=0}^{n+1} \binom{n+1}{m} \left\{ {m \atop p-1} \right\} 4^m (3^{n-m+1} - 1) \\ &+ \frac{3^n - 1}{2^{k-2}} \sum_{\substack{2p-1 \leq m \leq n+1 \\ m \equiv 1 \bmod p-1}} \binom{n+1}{m} m + \mathcal{S}_{n,p}^{(k)} \bigg\} \bmod p, \end{aligned}$$

where

$$\mathcal{S}_{n,p}^{(k)} = \begin{cases} (3^{n+1} - 1) \sum_{1 \leq l} \binom{n+1}{(p-1)(lp+1)} & \text{if } p = k+2, \\ 0 & \text{if } p > k+2. \end{cases}$$

Proof. From Theorem 3.1 and (5.2),

$$p^{k-1}(n+1)E_n^{(k)} = \frac{(-1)^n}{2}\sum_{j=1,2,p} S_{n,j}^{(k)} + \frac{1}{2}\sum_{m=0}^{n+1}\binom{n+1}{m}p^{k-1}\mathcal{R}_m^{(k)}4^m(3^{n-m+1}-1), \tag{5.3}$$

where

$$S_{n,j}^{(k)} := \sum_{m=0}^{n+1}\binom{n+1}{m}p^{k-1}b_m^{(k)}(jp-1)4^m(3^{n-m+1}-1) \quad (j=1,2,p).$$

We estimate $S_{n,j}^{(k)}$.

(i) The case $j=1$. From the definition,

$$S_{n,1}^{(k)} = (-1)^{p-1}(p-1)!\sum_{m=1}^{n+1}\frac{1}{p}\binom{n+1}{m}\left\{ {m \atop p-1} \right\}4^m(3^{n-m+1}-1).$$

Since

$$\left\{ {m \atop p-1} \right\} \equiv \begin{cases} 1 \bmod p & \text{if } (p-1)|m, \\ 0 \bmod p & \text{otherwise,} \end{cases}$$

terms which satisfy $(p-1)\nmid m$ belong to $\mathbb{Z}_{(p)}$. We will show that the other terms also belong to $\mathbb{Z}_{(p)}$. For $(p-1)|m$,

$$\sum_{\substack{1\le m\le n+1,\\ (p-1)|m}}\binom{n+1}{m}4^m(3^{n-m+1}-1) \equiv (3^{n+1}-1)\sum_{\substack{1\le m\le n+1,\\ (p-1)|m}}\binom{n+1}{m} \bmod p.$$

From Fermat's theorem, when $(p-1)|(n+1)$, the previous factor on the right-hand side of the above formula is equivalent to 0 modulo p. On the other hand, we see that the latter sum is equivalent to 0 modulo p by Lemma 5.4. Hence $S_{n,1}^{(k)} \in \mathbb{Z}_{(p)}$ and

$$S_{n,1}^{(k)} \equiv -\frac{1}{p}\sum_{m=1}^{n+1}\binom{n+1}{m}\left\{ {m \atop p-1} \right\}4^m(3^{n-m+1}-1) \bmod p. \tag{5.4}$$

Lemma 5.4. *For any odd prime number p and positive integer $n \ge p-2$,*

$$\sum_{\substack{1\le m\le n+1,\\ (p-1)|m}}\binom{n+1}{m} \equiv \begin{cases} 1 \bmod p & \textit{if } p-1|n+1, \\ 0 \bmod p & \textit{otherwise.} \end{cases} \tag{5.5}$$

Proof. The generating function of the left-hand side of (5.5) is

$$\frac{x^{p-2}}{(1-x)^p - x^{p-1} + x^p} = \sum_{n=0}^{\infty}\sum_{\substack{1\le m\le n+1,\\ (p-1)|m}}\binom{n+1}{m}x^n. \tag{5.6}$$

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Then we can easily see that

$$\frac{x^{p-2}}{(1-x)^p - x^{p-1} + x^p} \equiv \frac{x^{p-2}}{1-x^{p-1}} = \sum_{l=1}^{\infty} x^{(p-1)l-1} \bmod p. \tag{5.7}$$

Comparing the coefficients of (5.6) and (5.7), we have Lemma 5.4. □

(ii) The case $j = 2$. Arakawa and Kaneko [3] showed that

$$p^{k-1} b_m^{(k)}(2p-1) \equiv \begin{cases} \dfrac{m}{2^k} \bmod p & \text{if } m \geq 2p-1 \text{ and } m \equiv 1 \bmod p-1, \\ 0 \bmod p & \text{otherwise.} \end{cases}$$

Therefore

$$S_{n,2}^{(k)} \equiv \frac{3^n - 1}{2^{k-2}} \sum_{\substack{2p-1 \leq m \leq n+1, \\ m \equiv 1 \bmod p-1}} \binom{n+1}{m} m \bmod p. \tag{5.8}$$

(iii) The case $j = p$. From

$$\frac{(p^2-1)!}{p^{k+1}} \equiv \begin{cases} 1 \bmod p & \text{when } p = k+2, \\ 0 \bmod p & \text{when } p > k+2, \end{cases}$$

we just have to estimate the contribution from $p^{k-1} b_m^{(k)}(p^2-1) \bmod p$ for $p = k+2$. From the generating function for Stirling numbers of the second kind, the following congruence relations hold:

$$\left\{ {m \atop p^2-1} \right\} \equiv \begin{cases} 1 \bmod p & \text{if } m = (p-1)(pl+1)\ (l \geq 1), \\ 0 \bmod p & \text{otherwise.} \end{cases}$$

Hence

$$S_{n,p}^{(p-2)} \equiv (3^{n+1} - 1) \sum_{\substack{p^2-1 \leq m \leq n+1, \\ m \equiv p-1 \bmod p(p-1)}} \binom{n+1}{m} \bmod p. \tag{5.9}$$

Thus, by combining (5.3), (5.4), (5.8) and (5.9), we have Theorem 5.3. □

REMARK 5.5. In numerical experimentation, when n is even, the p-order of the denominator of $(n+1)E_n^{(k)}$ is $k-1$ for almost all prime p subject to $k+2 \leq p \leq n+1$. On the other hand, when n is odd, the p-order of the denominator of $(n+1)E_n^{(k)}$ is expected to be exactly $k-2$ for almost all prime p subject to the same condition as above.

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6. Explicit formulas for the case of negative index

In the case of negative index, an explicit formula (3.1) can be rewritten as the following form.

THEOREM 6.1. *For any nonnegative integers k and n,*

$$(n+1)E_n^{(-k)} = \frac{(-1)^k}{2}\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\{(4l+3)^{n+1}-(4l+1)^{n+1}\}. \tag{6.1}$$

PROOF. By using the duality formula $\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}$ given by Kaneko [6], (5.1) can be rewritten as

$$\mathbb{B}_n^{(-k)} = (-1)^k\sum_{m=0}^{k}(-1)^m m!\left\{ {k \atop m} \right\}(m+1)^n.$$

Applying the above formula to Theorem 3.1,

$$\begin{aligned}(n+1)E_n^{(-k)} &= \frac{(-1)^k}{2}\sum_{m=0}^{n+1}\binom{n+1}{m}\\ &\quad\times\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}(4(l+1))^{n+1-m}((-1)^m-(-3)^m)\\ &= \frac{(-1)^k}{2}\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\\ &\quad\times\sum_{m=0}^{n+1}\binom{n+1}{m}(4(l+1))^{n+1-m}((-1)^m-(-3)^m)\\ &= \frac{(-1)^k}{2}\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\{(4l+3)^{n+1}-(4l+1)^{n+1}\}.\end{aligned}$$

□

REMARK 6.2. We can easily see that the right-hand side of (6.1) is an integer. This indicates that the denominator of poly-Euler number $E_n^{(-k)}$ is at most $n+1$.

We can easily obtain the following corollaries by specializing Theorem 6.1.

COROLLARY 6.3. *For any nonnegative integer k,*

$$E_0^{(-k)} = 1, \quad E_1^{(-k)} = 2^{k+2}-2.$$

COROLLARY 6.4. *For any nonnegative integer n,*

$$(n+1)E_n^{(0)} = \frac{3^{n+1}-1}{2}, \quad (n+1)E_n^{(-1)} = \frac{7^{n+1}-5^{n+1}}{2}.$$

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Numerically, the $nE_{n-1}^{(0)}$ are equal to the number of different nonparallel lines determined by pairs of vertices in the n-dimensional hypercube (see [7]). Furthermore, the $nE_{n-1}^{(-1)}$ are equal to the number of collinear 5-tuples of points in a $\underbrace{5\times\cdots\times 5}_{n}$ n-dimensional cubic grid (see [5]).

Remark 6.5. In [5], it is conjectured that the sequence $(7^{n+1}-5^{n+1})/2$ is equal to the number of collinear 5-tuples of points in a $\underbrace{5\times\cdots\times 5}_{n}$ n-dimensional cubic grid, and this is verified for $n\le 9$. However, Shuji Yamamoto kindly gave us a proof that it holds in general.

The explicit formula (6.1) is useful to evaluate $E_n^{(-k)}$. However, the following form is more often useful than Theorem 6.1 in investigating some properties of $E_n^{(-k)}$ numerically.

Corollary 6.6. *For any nonnegative integers k and n,*

$$(n+1)E_n^{(-k)} = (-1)^k\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\sum_{j=0}^{[n/2]}\binom{n+1}{2j+1}(4l+2)^{n-2j}. \tag{6.2}$$

In the following section, we often use Corollary 6.6. For convenience, we denote the inner sum on the right-hand side of (6.2) by $A(n,l)$, that is,

$$A(n,l) := \sum_{j=0}^{[n/2]}\binom{n+1}{2j+1}(4l+2)^{n-2j}. \tag{6.3}$$

7. Congruence relations

Here, we treat several congruence relations of $(n+1)E_n^{(-k)}$ to investigate some numerical information about them.

First, we show Kummer-type congruences for poly-Euler numbers with negative index.

Proposition 7.1. *Let p be an odd prime and k be any nonnegative integer. For any nonnegative integers n and m such that $n\equiv m \bmod p-1$,*

$$(n+1)E_n^{(-k)} \equiv (m+1)E_m^{(-k)} \bmod p.$$

Proof. Let us put $q(l,n)=(4l+3)^{n+1}-(4l+1)^{n+1}$. Then, from Fermat's theorem, $q(l,n)\equiv q(l,m) \bmod p$ for any integers n and m satisfying $n\equiv m \bmod p-1$ and any integer l. Applying this to Theorem 6.1, we have the theorem. □

Next, we consider the property of

$$(n+1)E_n^{(-k)} \bmod n+1,$$

which allows us to evaluate whether $E_n^{(-k)}$ is an integer. We begin with the case where $n+1$ is an odd prime.

Theorem 7.2. *For any odd prime p,*

$$pE_{p-1}^{(-k)} \equiv 1 \bmod p.$$

Proof. Putting $n = p-1$ on both sides of (3.1),

$$\begin{aligned} pE_{p-1}^{(-k)} &= \frac{1}{2}\sum_{m=0}^{p}\binom{p}{m}\mathbb{B}_{p-m}^{(-k)}4^{(p-m)}(-1)^m(1-3^m) \\ &\equiv 2^{-1}\cdot(3^p-1) \equiv 2^{-1}\cdot 2 \equiv 1 \bmod p. \end{aligned}$$

Thus we obtain the formula. □

Secondly, we treat the modulo p congruence, where p is an odd prime number. From the following congruence relations, we see that some poly-Euler numbers $E_n^{(-k)}$ are indeed integers.

Theorem 7.3. *Let p be an odd prime. For any positive odd integers n and k satisfying* $k \equiv p-2 \bmod p-1$,

$$(n+1)E_n^{(-k)} \equiv 0 \bmod p.$$

Combining the above theorem with the parity result for poly-Euler numbers (8.1) below, we obtain the following congruence relation.

Corollary 7.4. *We put* p_j $(j = 1, \ldots, \nu)$ *distinct primes. For a positive integer k satisfying* $k \equiv p_j - 2 \bmod p_j - 1$ *for each* p_j, *and a positive odd integer n,*

$$(n+1)E_n^{(-k)} \equiv 0 \bmod p_1 \cdots p_\nu.$$

Remark 7.5. If $n+1 = p_1 \cdots p_\nu$ in the above corollary, then we see that the poly-Euler number $E_n^{(-k)}$ is an integer.

To prove Theorem 7.3, we first show the following lemmas concerning the properties of Stirling numbers of the second kind.

Lemma 7.6. *Let p be a prime number and put l a positive integer less than p. Then, for any positive integers k and k′ satisfying* $k \equiv k' \bmod p-1$,

$$\left\{ {k \atop l} \right\} \equiv \left\{ {k' \atop l} \right\} \bmod p.$$

Proof. Note that an explicit formula for Stirling numbers of the second kind is

$$(-1)^l l! \left\{ {k \atop l} \right\} = \sum_{j=0}^{l}(-1)^j\binom{l}{j}j^k.$$

From the above formula, we obtain Lemma 7.6, since the congruence $j^k \equiv j^{k'} \bmod p$ holds for $k \equiv k' \bmod p-1$. □

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LEMMA 7.7. *Let p be an odd prime. For any positive integer l less than $p-1$,*

$$(-1)^l l! \left\{ {p-2 \atop l} \right\} \equiv (-1)^{p-l-1}(p-l-1)! \left\{ {p-2 \atop p-l-1} \right\} \bmod p.$$

PROOF. By using the congruence relation (see [1, Lemma 14.19])

$$(-1)^m m! \left\{ {p-2 \atop m} \right\} \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{m} \bmod p \quad (m = 1, \ldots, p-2),$$

we have

$$\begin{aligned} &(-1)^l l! \left\{ {p-2 \atop l} \right\} - (-1)^{p-l-1}(p-l-1)! \left\{ {p-2 \atop p-l-1} \right\} \\ &\quad \equiv \frac{1}{l+1} + \cdots + \frac{1}{p-1-l} \bmod p. \end{aligned} \tag{7.1}$$

Since

$$\frac{1}{m+1} + \frac{1}{p-1-m} = \frac{p}{(m-1)(p-1-m)} \equiv 0 \bmod p$$

for $m = 1, \ldots, p-2$, the right-hand side of (7.1) is equivalent to 0 modulo p. □

PROOF OF THEOREM 7.3. Since $l! \equiv 0 \bmod p$ for $l \geq p$ and $\left\{ {k \atop p-1} \right\} \equiv 0 \bmod p$ for $(p-1) \nmid k$,

$$(n+1)E_n^{(-k)} \equiv (-1)^k \sum_{l=0}^{p-2} (-1)^l l! \left\{ {k \atop l} \right\} A(n,l) \bmod p \tag{7.2}$$

from Corollary 6.6 and (6.3). Furthermore, from the point of view of Lemma 7.6, we may replace k with $p-2$. Since $4(p-l-1)+2 \equiv -(4l+2) \bmod p$ and n is an odd integer, we see that

$$A(n,l) \equiv \begin{cases} 0 \bmod p & \text{for } l = (p-1)/2, \\ -A(n, p-l-1) \bmod p & \text{for } l = 1, \ldots, (p-3)/2. \end{cases}$$

Therefore, by combining Lemma 7.7 and the above congruence relation, (7.2) becomes

$$\begin{aligned} (n+1)E_n^{(-k)} &\equiv -\sum_{l=1}^{p-2} (-1)^l l! \left\{ {p-2 \atop l} \right\} A(n,l) \bmod p \\ &\equiv -\sum_{l=1}^{(p-3)/2} (-1)^l l! \left\{ {p-2 \atop l} \right\} (A(n,l) + A(n, p-l-1)) \bmod p \\ &\equiv 0 \bmod p. \end{aligned}$$

Thus we complete the proof. □

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8. The 2-orders of poly-Euler numbers with negative index

In [10], we showed the parity of poly-Euler numbers, that is,

$$(n+1)E_n^{(-k)} \equiv \begin{cases} 1 & n \equiv 0 \bmod 2, \\ 0 & n \equiv 1 \bmod 2. \end{cases} \tag{8.1}$$

The second equation of the above formula gives rise to our interest in computing the 2-orders of the poly-Euler numbers $E_{2n-1}^{(-k)}$. Numerically,it is expected that $\mathrm{ord}_2 E_{2n-1}^{(-k)} = 1$ for any positive integer n. In this section we show the following theorem which implies a partial answer regarding this expectation.

Theorem 8.1. *For any nonnegative integers k and positive integer s,*

$$\mathrm{ord}_2 E_{2^s-1}^{(-k)} = 1.$$

To prove Theorem 8.1, the following congruence for $A(n,l)$ is needed.

Lemma 8.2. *For any nonnegative integer l and positive integer s, the congruence $A(2^s-1,l) \equiv 2^{s+1} \bmod 2^{s+2}$ holds.*

We first prove Lemma 8.2, and then prove Theorem 8.1.

Proof of Lemma 8.2. We denote each term of $A(2^s-1,l)$ by $a_j(s,l)$, namely $a_j(s,l) := \binom{2^s}{2j+1}(4l+2)^{2^s-2j-1}$. Since $\mathrm{ord}_2(2^s-2j-1)! = \mathrm{ord}_2(2^s-2)! - \mathrm{ord}_2(2j)!$, we see that $\mathrm{ord}_2 \binom{2^s}{2j+1} = s$ for $j+1 \le 2^{s-1}$. Therefore, for any nonnegative integer l and positive integer s,

$$\mathrm{ord}_2\, a_j(s,l) \begin{cases} \ge s+2 & \text{if } j = 0, \dots, 2^{s-1}-2, \\ = s+1 & \text{if } j = 2^{s-1}-1. \end{cases}$$

Thus it follows that $A(2^s-1,l) \equiv 2^{s+1} \bmod 2^{s+2}$. □

The above lemma can be used to deduce Theorem 8.1 as follows.

Proof of Theorem 8.1. We prove that $2^s E_{2^s-1}^{(-k)} \equiv 2^{s+1} \bmod 2^{s+2}$. From Corollary 6.6 and (6.3) with $n+1 = 2^s$,

$$2^s E_{2^s-1}^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} A(2^s-1,l). \tag{8.2}$$

If $k = 0$ then, from (8.2), $2^s E_{2^s-1}^{(0)} = A(2^s-1,0)$. Hence the theorem is derived from Lemma 8.2 for this case. If $k \ge 1$, note that $l!$ is even for $l \ge 2$ and $\left\{ {k \atop 0} \right\} = 0$, and then

$$2^s E_{2^s-1}^{(-k)} \equiv (-1)^{k+1} \left\{ {k \atop 1} \right\} A(2^s-1,1) \equiv 2^{s+1} \bmod 2^{s+2}. \qquad \square$$

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9. Duality formulas for poly-Euler numbers

For the poly-Bernoulli numbers, the duality formula $\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}$ $(n, k \in \mathbb{Z}_{\geq 0})$ was given by Kaneko [6]. In this section, we present two kinds of duality formula for poly-Euler numbers.

Theorem 9.1 (Duality formula). *For any nonnegative integers k and n,*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{E_{n-m}^{(-k)}}{4^n} (B_m(1 - 2^{m-1}) + m) = \sum_{m=0}^{k} \binom{k}{m} \frac{E_{k-m}^{(-n)}}{4^k} (B_m(1 - 2^{m-1}) + m).$$

The above theorem will be shown by considering its generating function. From the generating function we also obtain the following explicit formula for poly-Bernoulli numbers.

Corollary 9.2. *For any nonnegative integers k and n,*

$$\mathbb{B}_n^{(-k)} = \sum_{m=0}^{n} \binom{n}{m} \frac{E_{n-m}^{(-k)}}{4^n} (2B_m(1 - 2^{m-1}) + 2m).$$

Proof. Kaneko [6] showed that

$$\sum_{k=0}^{\infty} \frac{\mathrm{Li}_{-k}(1 - e^{-x})}{1 - e^{-x}} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \tag{9.1}$$

By combining this and (3.2),

$$\sum_{k=0}^{\infty} \frac{2xe^{3x}}{e^{2x} - 1} \frac{\mathrm{Li}_{-k}(1 - e^{-4x})}{4x \cosh x} \frac{(4y)^k}{k!} = \frac{e^{4x+4y}}{e^{4x} + e^{4y} - e^{4x+4y}}. \tag{9.2}$$

The inner function on the left-hand side of the above formula has the following expansion:

$$\frac{2xe^{3x}}{e^{2x} - 1} \frac{\mathrm{Li}_{-k}(1 - e^{-4x})}{4x \cosh x} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(-k)} (2B_m(1 - 2^{m-1}) + 2m) x^n.$$

Here, we have used the decomposition

$$\frac{2xe^{3x}}{e^{2x} - 1} = 2\frac{xe^x}{e^x - 1} - \frac{2xe^{2x}}{e^{2x} - 1} + 2xe^x.$$

Since the right-hand side of (9.2) is invariant with respect to the change of variables x and y and is just the generating function of the poly-Bernoulli numbers (9.1), we obtain Theorem 9.1 and its corollary by comparing with the coefficients of both sides of (9.2) with respect to x. □

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We have another duality formula which is similar to the duality formula for poly-Bernoulli numbers. Let us consider

$$F(x,y) := \frac{1}{2xy}\frac{e^{4x+4y}}{e^{4x}+e^{4y}-e^{4x+4y}}(e^{-x}-e^{-3x})(e^{-y}-e^{-3y})$$

and denote its expansion by

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{E}(n,k)\frac{x^n}{(n+1)!}\frac{y^k}{(k+1)!},$$

where

$$\mathcal{E}(n,k) := \sum_{m=0}^{k+1}\binom{k+1}{m}(-1)^{k+1-m}(1-3^{k+1-m})(n+1)E_n^{(-m)}4^m.$$

Since $F(x,y)$ is invariant with respect to the change of variables x and y, we obtain the following result.

Theorem 9.3 (Duality formula). *For any nonnegative integers k and n,*

$$\mathcal{E}(n,k) = \mathcal{E}(k,n).$$

10. Positivity

Here, we describe the positivity of poly-Euler numbers of negative index. In Section 6 we mentioned that, in some cases, poly-Euler numbers have combinatorial interpretations. The positivity is the essential evidence suggesting that poly-Euler numbers have a combinatorial interpretation in general. We obtain the positivity of poly-Euler number of negative index from the following recursive formula.

Theorem 10.1. *For any nonnegative integer k and positive integer n,*

$$nE_{n-1}^{(-k)} = \sum_{j=0}^{\min(n-1,k)} j!^2\Bigg(\sum_{m=1}^{n-j}\binom{n}{m}mE_{m-1}^{(0)}\left\{{n-m \atop j}\right\}4^{n-m}\Bigg)\left\{{k+1 \atop j+1}\right\}.$$

Proof. From (9.2),

$$\sum_{k=0}^{\infty}\frac{\mathrm{Li}_{-k}(1-e^{-4x})}{\cosh x}\frac{(4y)^k}{k!} = 2(e^{3x}-e^{x})\frac{e^{4y}}{e^{4x}+e^{4y}-e^{4(x+y)}}. \tag{10.1}$$

The first factor on the right-hand side of (10.1) can be expressed as

$$2(e^{3x}-e^{x}) = \frac{\mathrm{Li}_0(1-e^{-4x})}{\cosh x}. \tag{10.2}$$

Furthermore, by the same argument in [1, page 227], the second factor of the right-hand side of (10.1) can be rewritten as

$$\begin{aligned}\frac{e^{4y}}{e^{4x}+e^{4y}-e^{4(x+y)}} &= e^{4y}\sum_{j=0}^{\infty}(e^{4x}-1)^j(e^{4y}-1)^j \\ &= \sum_{j=0}^{\infty}(4(j+1))^{-1}(e^{4x}-1)^j\frac{d}{dy}(e^{4y}-1)^{j+1}. \end{aligned} \tag{10.3}$$

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By combining (10.2), (10.3) and

$$\frac{(e^x-1)^m}{m!} = \sum_{k=m}^{\infty} \left\{ {k \atop m} \right\} \frac{x^k}{k!},$$

we see that the right-hand side of (10.1) can be expanded as follows:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\min\{k,n-1\}} j!^2 \left(\sum_{m=1}^{n-j} \binom{n}{m} m E_{m-1}^{(0)} \left\{ {n-m \atop j} \right\} 4^{n-m}\right) \left\{ {k+1 \atop j+1} \right\} \frac{x^n}{n!}\frac{(4y)^k}{k!}. \tag{10.4}$$

Hence we obtain Theorem 10.1 by comparing the corresponding coefficients of (10.1) and (10.4). □

We obtain the following recurrence relation from Theorem 10.1 with $k = 1$.

Corollary 10.2. *For any positive integer n,*

$$nE_{n-1}^{(-1)} = \sum_{m=1}^{n} \binom{n}{m} m E_{m-1}^{(0)} 4^{n-m}.$$

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