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Decay spectrum and decay subspace of normal operators

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Let A be a self-adjoint operator on a Hilbert space. It is well known that A admits a unique decomposition into a direct sum of three self-adjoint operators A_{p}, A_{ac} and A_{sc} such that there exists an orthonormal basis of eigenvectors for the operator A_{p}, the operator A_{ac} has purely absolutely continuous spectrum and the operator A_{sc} has purely singular continuous spectrum. We show the existence of a natural further decomposition of the singular continuous component A_{sc} into a direct sum of two self-adjoint operators $A_{\mathrm{sc}}^{\mathrm{D}}$ and $A_{\mathrm{sc}}^{\mathrm{ND}}$. The corresponding subspaces and spectra are called decaying and purely non-decaying singular subspaces and spectra. Similar decompositions are also shown for unitary operators and for general normal operators.

1. Introduction

Based on our decomposition of measures over arbitrary non-discrete locally compact topological abelian groups as the direct sum of decaying and purely non-decaying measures [1, 2], we obtain a natural decomposition of the singular continuous subspace of a normal operator as the orthogonal direct sum of two stable subspaces: the decaying and purely non-decaying components. The decaying subspaces describe mixing and decay in classical and quantum physics [4, 7, 8].

First, we introduce the key concepts.

DEFINITION 1.1 (see [5]). An operator $A : D_A \subseteq \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is said to be *normal* if and only if

(a) the operator A is closed and densely defined;

(b) the domains of the operators $A^\dagger$, $A^\dagger A$ and $AA^\dagger$ are dense in $\mathcal{H}$, where $A^\dagger$ is the adjoint of A;

(c) $A^\dagger A = AA^\dagger$ in the sense that their domains are equal: $D_{A^\dagger A} = D_{AA^\dagger}$ and $A^\dagger Ax = AA^\dagger x$ for any $x \in D_{A^\dagger A} = D_{AA^\dagger}$.

The spectral theorem for normal operators implies that an operator A in a Hilbert space is normal if and only if A is unitarily equivalent to an operator of multiplication by a measurable function in an L_2-space over some measurable space.

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The class of normal operators contains the class of self-adjoint operators and the class of unitary operators.

DEFINITION 1.2. We say that a function f on a locally compact topological space X *vanishes at infinity* if and only if, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $f(x) < \varepsilon$ for any $x \in X \setminus K$. We denote the space of all continuous complex-valued functions on X vanishing at infinity by $C_0(X)$. This space is a Banach space with respect to the norm

$$\|f\|_c = \sup |f|. \tag{1.1}$$

DEFINITION 1.3. Let A be a normal operator in a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$. The σ-additive Borel measure μ_f is said to be a *spectral measure associated to the vector* f if and only if

$$(f, \varphi(A)f) = \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_f \tag{1.2}$$

for any $\varphi \in C_0(\mathbb{C})$. This measure exists, is unique and positive according to the Riesz–Markov theorem [9]. The scalar product $(\cdot, \cdot)$ is antilinear and linear with respect to the first and second argument, respectively.

Note that equation (1.2) actually holds for any Borel function $\varphi : \mathbb{C} \to \mathbb{C}$ and any $f \in D_{\varphi(A)}$. Moreover, $f \in \mathcal{H}$ belongs to $D_{\varphi(A)}$ if and only if

$$\int_{\mathbb{C}} |\varphi|^2 \, \mathrm{d}\mu_f < +\infty.$$

DEFINITION 1.4. Let A be a normal operator in a Hilbert space $\mathcal{H}$. We say that a closed linear subspace $\mathcal{H}_0$ of $\mathcal{H}$ is *stable with respect to* A if and only if $\varphi(A)f \in \mathcal{H}_0$ for any $f \in \mathcal{H}_0 \cap D_{\varphi(A)}$ and any Borel function $\varphi : \mathbb{C} \to \mathbb{C}$.

If $\mathcal{H}$ is a direct orthogonal sum of a sequence (finite or infinite) of closed stable (with respect to A) subspaces $\mathcal{H}_n : \mathcal{H} = \bigoplus_n \mathcal{H}_n$ (here and everywhere below we use symbol $\oplus$ for orthogonal direct sum of closed subspaces of a Hilbert space), then A is the *direct sum* of the restrictions A_n of A to $\mathcal{H}_n$ and each A_n is a normal operator in $\mathcal{H}_n$.

Recall [10] that the Fourier transform of a Borel finite σ-additive measure μ on a locally compact abelian topological group G is the function $\tilde{\mu} : G^\times \to \mathbb{C}$,

$$\tilde{\mu}(h) = \int_G \mathrm{e}^{\mathrm{i}(g|h)} \, \mathrm{d}\mu(g),$$

where $(g \mid h) = h(g)$, $G^\times$ is the dual group of G, i.e. $G^\times$ is the group of all continuous homomorphisms $h : G \to \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ from G to the circle $\mathbb{T}^1$.

DEFINITION 1.5. A Borel positive finite σ-additive measure μ on a locally compact abelian topological group G is said to be *decaying* [2] if its Fourier transform $\tilde{\mu} : G^\times \to \mathbb{C}$ vanishes at infinity. The measure μ is said to be *purely non-decaying* if any non-zero measure ν absolutely continuous with respect to μ is not decaying.

DEFINITION 1.6. Let A be a normal operator on a Hilbert space $\mathcal{H}$. We say that A has *purely point spectrum* if and only if, for any $f \in \mathcal{H}$, the spectral measure μ_f has countable support, or, equivalently, there exists an orthonormal basis in $\mathcal{H}$

consisting of the eigenvectors of A. We denote by $\mathcal{H}_{\mathrm{p}}$ the closed linear hull of the eigenvectors of a normal operator A. Equivalently, $\mathcal{H}_{\mathrm{p}}$ is the set of all vectors $f \in \mathcal{H}$ for which the spectral measure μ_f has countable support.

After stating the results for self-adjoint (see § 2) unitary (see § 3) and normal (see § 4) operators, we prove the necessary lemmas (see § 5) and the main results (see § 6) and show the non-triviality of our decomposition of singular continuous subspace (see § 7).

2. The decay spectrum of self-adjoint operators

DEFINITION 2.1. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. We say that A has *purely absolutely continuous spectrum* if, for any $f \in \mathcal{H}$, the spectral measure μ_f is absolutely continuous with respect to the Lebesgue measure on the real line. We say that A has *purely singular continuous spectrum* if, for any $f \in \mathcal{H}$, the spectral measure μ_f is continuous and singular with respect to the Lebesgue measure on the real line.

The *absolutely continuous subspace* $\mathcal{H}_{\mathrm{ac}}$ for A is the set of vectors $f \in \mathcal{H}$ for which the spectral measure μ_f is absolutely continuous with respect to the Lebesgue measure on the real line. The *singular continuous subspace* $\mathcal{H}_{\mathrm{sc}}$ for A is the set of vectors $f \in \mathcal{H}$ for which the spectral measure μ_f is continuous and singular with respect to the Lebesgue measure on the real line.

The following proposition is a well-known result (see, for example, [13]).

PROPOSITION 2.2. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then the Hilbert space $\mathcal{H}$ is the orthogonal direct sum of three closed linear subspaces $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$ and $\mathcal{H}_{\mathrm{p}}$, stable with respect to A,*

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{p}}. \tag{2.1}$$

Thus A is the direct sum of three self-adjoint operators A_{ac}, A_{sc} and A_{p}, being the restrictions of A to $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$ and $\mathcal{H}_{\mathrm{p}}$, respectively. Moreover, A_{p} has purely point spectrum, A_{ac} has purely absolutely continuous spectrum and A_{sc} has purely singular continuous spectrum.

DEFINITION 2.3. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is said to be *decaying* (or $f \in \mathcal{H}^{\mathrm{D}}$) if

$$\lim_{|t|\to\infty} (f, \mathrm{e}^{\mathrm{i}tA} f) = 0, \quad t \in \mathbb{R}. \tag{2.2}$$

A vector $f \in \mathcal{H}$ is said to be *purely non-decaying* (or $f \in \mathcal{H}^{\mathrm{ND}}$) if it is orthogonal to all decaying vectors. We say that A has *purely decaying* spectrum if $\mathcal{H} = \mathcal{H}^{\mathrm{D}}$ and we say that A has *purely non-decaying* spectrum if $\mathcal{H}^{\mathrm{D}} = \{0\}$

THEOREM 2.4. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is decaying if and only if the spectral measure μ_f is decaying as a measure on $\mathbb{R}$; f is purely non-decaying if and only if the spectral measure μ_f is purely non-decaying as a measure on $\mathbb{R}$.*

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Theorem 2.5. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is the orthogonal direct sum of two closed linear subspaces $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$ stable with respect to A,*

$$\mathcal{H} = \mathcal{H}^{\mathrm{D}} \oplus \mathcal{H}^{\mathrm{ND}}. \tag{2.3}$$

Thus the operator A is the direct sum of two self-adjoint operators A^{D} and A^{ND}, being the restrictions of A to $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$, respectively. Moreover, $\mathcal{H}_{\mathrm{p}} \subseteq \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{ac}} \subseteq \mathcal{H}^{\mathrm{D}}$.

Corollary 2.6. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then*

$$\left.\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \\ \mathcal{H}_{\mathrm{sc}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}, \\ \mathcal{H}^{\mathrm{ND}} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}, \\ \mathcal{H}^{\mathrm{D}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \end{aligned}\right\} \tag{2.4}$$

where $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} = \mathcal{H}_{\mathrm{sc}} \cap \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} = \mathcal{H}^{\mathrm{D}} \cap \mathcal{H}_{\mathrm{sc}}$

Moreover, A is the direct sum of four self-adjoint operators A_{ac}, $A_{\mathrm{sc}}^{\mathrm{D}}$, $A_{\mathrm{sc}}^{\mathrm{ND}}$ and A_{p}, being the restrictions of A to $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{p}}$, respectively. The operator $A_{\mathrm{sc}}^{\mathrm{D}}$ has purely singular continuous and purely decaying spectrum and the operator $A_{\mathrm{sc}}^{\mathrm{ND}}$ has purely singular continuous and purely non-decaying spectrum.

3. The decay spectrum of unitary operators

The definitions of unitary operators with purely absolutely continuous and with purely singular continuous spectrum, as well as the definition of absolutely continuous and singular continuous subspaces for unitary operators, can be obtained from the corresponding definitions for self-adjoint operators (see definition 2.1) by replacing the Lebesgue measure on the real line with the Lebesgue measure on the unit circle.

The following proposition can be obtained from proposition 2.2 by the Cayley transform.

Proposition 3.1. *Let U be a unitary operator in a Hilbert space $\mathcal{H}$. Then the Hilbert space $\mathcal{H}$ is the orthogonal direct sum of three closed linear subspaces $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$ and $\mathcal{H}_{\mathrm{p}}$, stable with respect to U,*

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{p}}. \tag{3.1}$$

Thus U is the direct sum of three unitary operators U_{ac}, U_{sc} and U_{p}, being the restrictions of U to $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}$ and $\mathcal{H}_{\mathrm{p}}$, respectively. Moreover, U_{p} has purely point spectrum, U_{ac} has purely absolutely continuous spectrum and U_{sc} has purely singular continuous spectrum.

Definition 3.2. Let U be a unitary operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is said to be *decaying* (or $f \in \mathcal{H}^{\mathrm{D}}$) if

$$\lim_{|n|\to\infty} (f, U^n f) = 0, \quad n \in \mathbb{Z}. \tag{3.2}$$

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A vector $f \in \mathcal{H}$ is said to be *purely non-decaying* (or $f \in \mathcal{H}^{\mathrm{ND}}$) if it is orthogonal to all decaying vectors. We say that A has *purely decaying* spectrum if $\mathcal{H} = \mathcal{H}^{\mathrm{D}}$ and we say that A has *purely non-decaying* spectrum if $\mathcal{H}^{\mathrm{D}} = \{0\}$.

THEOREM 3.3. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is decaying if and only if the spectral measure μ_f is decaying as a measure on the unit circle; f is purely non-decaying if and only if the spectral measure μ_f is purely non-decaying as a measure on the unit circle.*

THEOREM 3.4. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is the orthogonal direct sum of two closed linear subspaces $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$, stable with respect to U,*

$$\mathcal{H} = \mathcal{H}^{\mathrm{D}} \oplus \mathcal{H}^{\mathrm{ND}}. \tag{3.3}$$

Thus the operator U is the direct sum of two unitary operators U^{D} and U^{ND}, being the restrictions of U to $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$, respectively. Moreover, $\mathcal{H}_{\mathrm{p}} \subseteq \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{ac}} \subseteq \mathcal{H}^{\mathrm{D}}$.

COROLLARY 3.5. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Then*

$$\left.\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \\ \mathcal{H}_{\mathrm{sc}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}, \\ \mathcal{H}^{\mathrm{ND}} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}, \\ \mathcal{H}^{\mathrm{D}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \end{aligned}\right\} \tag{3.4}$$

where $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} = \mathcal{H}_{\mathrm{sc}} \cap \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} = \mathcal{H}^{\mathrm{D}} \cap \mathcal{H}_{\mathrm{sc}}$. Thus U is the direct sum of four unitary operators U_{ac}, $U_{\mathrm{sc}}^{\mathrm{D}}$, $U_{\mathrm{sc}}^{\mathrm{ND}}$ and U_{p}, being the restrictions of U to $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{p}}$, respectively. The operator $U_{\mathrm{sc}}^{\mathrm{D}}$ has purely singular continuous and purely decaying spectrum and the operator $U_{\mathrm{sc}}^{\mathrm{ND}}$ has purely singular continuous and purely non-decaying spectrum.

4. The decay spectrum of normal operators

In this section we deal with normal operators whose spectrum is not concentrated in the real line or unit circle. Therefore, they are neither self-adjoint nor unitary. The definitions of normal operators with purely absolutely continuous and with purely singular continuous spectrum, as well as the definition of absolutely continuous and singular continuous subspaces for normal operators, can be obtained from the corresponding definitions for self-adjoint operators (see definition 2.1) by replacing the Lebesgue measure on the real line with the Lebesgue measure on the complex plane.

The following proposition is the well-known analogue of proposition 2.2 for normal operators.

PROPOSITION 4.1. *Let A be a normal operator on a Hilbert space $\mathcal{H}$. Then the Hilbert space $\mathcal{H}$ is the orthogonal direct sum of three closed linear subspaces $\mathcal{H}_{\mathrm{p}}$, $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{\mathrm{sc}}$, stable with respect to A,*

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}. \tag{4.1}$$

Thus A is the direct sum of three normal operators A_{p}, A_{ac} and A_{sc}, being the restrictions of A to $\mathcal{H}_{\mathrm{p}}$, $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{\mathrm{sc}}$, respectively. Moreover, A_{p} has purely point spectrum, A_{ac} has purely absolutely continuous spectrum and A_{sc} has purely singular continuous spectrum.

We would like to point out here that any non-zero self-adjoint or unitary operator considered as a normal operator has purely singular spectrum, since any measure, concentrated on the real line or on the unit circle, is singular with respect to the two-dimensional Lebesgue measure.

Definition 4.2. Let A be a normal operator on a Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is said to be *decaying* (with respect to the two-dimensional Lebesgue measure) (or $f \in \mathcal{H}^{\mathrm{D}}$) if and only if

$$\lim_{|t|+|s|\to\infty} (f, \mathrm{e}^{\mathrm{i}tA_{\mathrm{R}}+\mathrm{i}sA_{\mathrm{I}}} f) = 0, \quad t, s \in \mathbb{R}, \tag{4.2}$$

where $A_{\mathrm{R}} = \frac{1}{2}(A + A^{\dagger})$ is the real part of A and $A_{\mathrm{I}} = (A - A^{\dagger})/2\mathrm{i}$ is the imaginary part of A. A vector $f \in \mathcal{H}$ is said to be *purely non-decaying* (or $f \in \mathcal{H}^{\mathrm{ND}}$) if it is orthogonal to all decaying vectors. We say that A has *purely decaying* spectrum if $\mathcal{H} = \mathcal{H}^{\mathrm{D}}$ and we say that A has *purely non-decaying* spectrum if $\mathcal{H}^{\mathrm{D}} = \{0\}$.

Theorem 4.3. *Let A be a normal operator on a Hilbert space $\mathcal{H}$, $f \in \mathcal{H}$. Then f is decaying if and only if the spectral measure μ_f is decaying as a measure on the complex plane; f is purely non-decaying if and only if the spectral measure μ_f is purely non-decaying as a measure on the complex plane.*

Theorem 4.4. *Let A be a normal operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is the orthogonal direct sum of two closed linear subspaces $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$, stable with respect to A,*

$$\mathcal{H} = \mathcal{H}^{\mathrm{D}} \oplus \mathcal{H}^{\mathrm{ND}}. \tag{4.3}$$

Thus the operator A is the direct sum of two normal operators A^{D} and A^{ND}, being the restrictions of A to $\mathcal{H}^{\mathrm{D}}$ and $\mathcal{H}^{\mathrm{ND}}$, respectively. Moreover, $\mathcal{H}_{\mathrm{p}} \subseteq \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{ac}} \subseteq \mathcal{H}^{\mathrm{D}}$.

Corollary 4.5. *Let A be a normal operator on a Hilbert space $\mathcal{H}$. Then*

$$\left.\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \\ \mathcal{H}_{\mathrm{sc}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}, \\ \mathcal{H}^{\mathrm{ND}} &= \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}, \\ \mathcal{H}^{\mathrm{D}} &= \mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathcal{H}_{\mathrm{ac}}, \end{aligned}\right\} \tag{4.4}$$

where $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}} = \mathcal{H}_{\mathrm{sc}} \cap \mathcal{H}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}} = \mathcal{H}^{\mathrm{D}} \cap \mathcal{H}_{\mathrm{sc}}$ and A is the direct sum of four normal operators A_{ac}, $A_{\mathrm{sc}}^{\mathrm{D}}$, $A_{\mathrm{sc}}^{\mathrm{ND}}$ and A_{p}, being the restrictions of A to $\mathcal{H}_{\mathrm{ac}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{D}}$, $\mathcal{H}_{\mathrm{sc}}^{\mathrm{ND}}$ and $\mathcal{H}_{\mathrm{p}}$, respectively. The operator $A_{\mathrm{sc}}^{\mathrm{D}}$ has purely singular continuous and purely decaying spectrum and the operator $A_{\mathrm{sc}}^{\mathrm{ND}}$ has purely singular continuous and purely non-decaying spectrum.

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5. Basic lemmas

Let μ and ν be positive σ-additive measures on some measurable space $(X, \mathcal{F})$. We write $\mu \prec \nu$ if μ is absolutely continuous with respect to the measure ν and we write $\mu \perp \nu$ if the measure μ is singular with respect to ν.

THEOREM 5.1 (Lebesgue theorem [11]). *Let μ and ν be positive σ-additive measures on some measurable space $(X, \mathcal{F})$. Then there exist unique measures μ_1 and μ_2 such that $\mu_1 \prec \nu$, $\mu_2 \perp \nu$ and $\mu = \mu_1 + \mu_2$.*

DEFINITION 5.2. Let A be a normal operator in a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$. The *cyclic subspace* of $\mathcal{H}$ generated by f is

$$\mathcal{H}_f = \{\varphi(A)f | \varphi : \mathbb{C} \to \mathbb{C} \text{ is a Borel function and } f \in D_{\varphi(A)}\}. \tag{5.1}$$

Evidently, $\mathcal{H}_f$ is a closed linear subspace of $\mathcal{H}$, stable with respect to A. Note that a closed linear subspace $\mathcal{H}^0$ of $\mathcal{H}$ is stable with respect to A if and only if $\mathcal{H}_f \subseteq \mathcal{H}^0$ for any $f \in \mathcal{H}^0$.

LEMMA 5.3. *For any normal operator A on a Hilbert space $\mathcal{H}$ and any $f, g \in \mathcal{H}$, we have*

(i) *if $g \perp \mathcal{H}_f$, then $\mathcal{H}_g \perp \mathcal{H}_f$;*

(ii) *if $\mu_g \perp \mu_f$, then $\mathcal{H}_g \perp \mathcal{H}_f$;*

(iii) *if $g \in \mathcal{H}_f$, then $\mu_g \prec \mu_f$.*

Proof. If $g \in \mathcal{H}_f$, then there exists a Borel function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $g = \varphi(A)f$. Then, for any $\psi \in C_0(\mathbb{C})$,

$$\int_{\mathbb{C}} \psi \, \mathrm{d}\mu_g = (g, \psi(A)g) = (\varphi(A)f, \psi(A)\varphi(A)f) = (f, (\psi|\varphi|^2)(A)f) = \int_{\mathbb{C}} \psi|\varphi|^2 \, \mathrm{d}\mu_f.$$

Therefore, $\mu_g \prec \mu_f$ (the density of μ_g with respect to μ_f is $|\varphi|^2$). Condition (iii) is proved.

Let $g \perp \mathcal{H}_f$. To prove (i), it suffices to check that, for any bounded Borel functions $\varphi, \psi : \mathbb{C} \to \mathbb{C}$, the equality $(\varphi(A)g, \psi(A)f) = 0$ is valid. But $(\varphi(A)g, \psi(A)f) = (g, (\bar{\varphi}\psi)(A)f) = 0$, since $(\bar{\varphi}\psi)(A)f \in \mathcal{H}_f$. Condition (i) is proved.

Suppose now that $\mu_g \perp \mu_f$. Let $h \in \mathcal{H}_f$ and $p \in \mathcal{H}_g$. According to the already proven condition (iii), $\mu_h \prec \mu_f$ and $\mu_p \prec \mu_g$. Therefore, $\mu_h \perp \mu_p$, i.e. there exists a Borel set $B \subseteq \mathbb{C}$ such that $\mu_h(B) = \mu_p(\mathbb{C} \setminus B) = 0$. Let $\varphi(z) = 1$ for $z \in B$ and $\varphi(B) = 0$ for $z \in \mathbb{C} \setminus B$. Evidently, the operator $P = \varphi(A)$ is an orthoprojection, $Ph = 0$ and $Pp = p$. Therefore, $h \perp p$. Hence $\mathcal{H}_f \perp \mathcal{H}_g$. Lemma 5.3 is proved. □

LEMMA 5.4. *Let A be a normal operator in a Hilbert space $\mathcal{H}$ and $\mathcal{H}^0$ be a closed linear subspace of $\mathcal{H}$, stable with respect to A. Then the orthocomplement $\mathcal{H}^1$ of $\mathcal{H}^0$ is also stable with respect to A.*

Proof. Let $f \in \mathcal{H}^1$, $g \in \mathcal{H}^0$. Since $\mathcal{H}^0$ is stable with respect to A, $\mathcal{H}_g \subseteq \mathcal{H}^0$. Therefore, $f \perp \mathcal{H}_g$. Lemma 5.3 implies that $\mathcal{H}_g \perp \mathcal{H}_f$ and, in particular, $g \perp \mathcal{H}_f$. Since g is an arbitrary element of $\mathcal{H}^0$, $\mathcal{H}_f \perp \mathcal{H}^0$. Therefore, $\mathcal{H}_f \subseteq \mathcal{H}^1$. Hence $\mathcal{H}^1$ is stable with respect to A. Lemma 5.4 is proved. □

The following two lemmas are proved in [2].

Lemma 5.5. *Let μ be a decaying measure on a non-discrete locally compact abelian topological group and $\nu \in \mathfrak{M}$ be a measure absolutely continuous with respect to μ. Then the measure ν is also decaying.*

Lemma 5.6. *Let μ be a decaying measure on a non-discrete locally compact abelian topological group G and ν be a purely non-decaying measure on G. Then μ is singular with respect to ν.*

6. Proof of the theorems

Proof of theorem 2.5. Suppose that the conditions of theorem 2.5 are satisfied. First, let us check that the set $\mathcal{H}^{\mathrm{D}}$ is a closed linear subspace of $\mathcal{H}$. Let $f \in \mathcal{H}^{\mathrm{D}}$ and $c \in \mathbb{C}$. Then

$$\lim_{t\to\infty}(cf, \mathrm{e}^{\mathrm{i}tA}(cf)) = |c|^2 \lim_{t\to\infty}(f, \mathrm{e}^{\mathrm{i}tA} f) = 0.$$

Therefore, $cf \in \mathcal{H}^{\mathrm{D}}$. It is clear that the function $(f, \mathrm{e}^{\mathrm{i}tA} f)$ coincides with the Fourier transform of μ_f,

$$(f, \mathrm{e}^{\mathrm{i}tA} f) = \tilde{\mu}_f(t). \tag{6.1}$$

Now let $f, g \in \mathcal{H}^{\mathrm{D}}$. Then

$$\begin{aligned}\lim_{t\to\infty}(f+g, \mathrm{e}^{\mathrm{i}tA}(f+g)) &= \lim_{t\to\infty}((f, \mathrm{e}^{\mathrm{i}tA} f) + (g, \mathrm{e}^{\mathrm{i}tA} g) + (f, \mathrm{e}^{\mathrm{i}tA} g) + (g, \mathrm{e}^{\mathrm{i}tA} f)) \\ &= \lim_{t\to\infty}((f, \mathrm{e}^{\mathrm{i}tA} g) + (g, \mathrm{e}^{\mathrm{i}tA} f)). \end{aligned} \tag{6.2}$$

It is clear that the function $(f, \mathrm{e}^{\mathrm{i}tA} g)+(g, \mathrm{e}^{\mathrm{i}tA} f)$ coincides with the Fourier transform of the complex-valued Borel measure ν on $\mathbb{C}$ defined by the equality

$$\nu(B) = (f, \varphi_B(A)g) + (g, \varphi_B(A)f),$$

where $\varphi_B(z) = 1$ if $z \in B$ and $\varphi_B(z) = 0$ if $z \notin B$. Clearly, for any Borel set $B \subseteq \mathbb{C}$,

$$\begin{aligned}|\nu(B)| &\leqslant |(f, \varphi_B(A)g) + (g, \varphi_B(A)f)| \\ &= |(f, \varphi_B^2(A)g) + (g, \varphi_B^2(A)f)| \\ &\leqslant 2|(\varphi_B(A)f, \varphi_B(A)g)| \\ &\leqslant (\varphi_B(A)f, \varphi_B(A)f) + (\varphi_B(A)g, \varphi_B(A)g) \\ &= (f, \varphi_B^2(A)f) + (g, \varphi_B^2(A)g) \\ &= (f, \varphi_B(A)f) + (g, \varphi_B(A)g) \\ &= \mu_f(B) + \mu_g(B). \end{aligned} \tag{6.3}$$

Therefore, $\nu \prec \mu_f + \mu_g$. Since $\tilde{\mu}_f, \tilde{\mu}_g \in C_0(\mathbb{R})$, lemma 5.5 implies that $\tilde{\nu}$ vanishes at infinity. This, together with (6.2) and (6.1), implies that $f + g \in \mathcal{H}^{\mathrm{D}}$. Therefore, $\mathcal{H}^{\mathrm{D}}$ is a linear subspace of $\mathcal{H}$. Suppose now that f_n is a sequence of elements of $\mathcal{H}^{\mathrm{D}}$ converging to $f \in \mathcal{H}$ with respect to the norm of the Hilbert space $\mathcal{H}$. Then, obviously, the sequence of functions $g_n(t) = (f_n, \mathrm{e}^{\mathrm{i}tA} f_n)$ converges uniformly to the

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function $g(t) = (f, \mathrm{e}^{\mathrm{i}tA} f)$ $(|g_n(t) - g(t)| \leqslant 2\|f\|\|f_n - f\|)$. Since a uniform limit of a sequence of elements of $C_0(\mathbb{R})$ is an element of $C_0(\mathbb{R})$,

$$\lim_{t\to\infty} (f, \mathrm{e}^{\mathrm{i}tA} f) = 0.$$

Therefore, $f \in \mathcal{H}^{\mathrm{D}}$. Thus the space $\mathcal{H}^{\mathrm{D}}$ is closed. The inclusion $\mathcal{H}_{\mathrm{ac}} \subseteq \mathcal{H}^{\mathrm{D}}$ follows from the Riemann–Lebesgue lemma and (6.1). Since the convergence of the Fourier transform to zero implies continuity of a measure on real line, we have that $\mathcal{H}^{\mathrm{D}} \subseteq \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$. Therefore, $\mathcal{H}_{\mathrm{p}} \subseteq \mathcal{H}^{\mathrm{ND}}$ ($\mathcal{H}^{\mathrm{ND}}$ is the orthocomplement of $\mathcal{H}^{\mathrm{D}}$).

Let us verify that $\mathcal{H}^{\mathrm{D}}$ is stable with respect to A. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a Borel function, $f \in \mathcal{H}^{\mathrm{D}} \cap D_{\varphi(A)}$ and $g = \varphi(A)f$. We have to prove that $g \in \mathcal{H}^{\mathrm{D}}$. Lemma 5.3 implies that $g \in \mathcal{H}_f$ and therefore $\mu_g \prec \mu_f$. The measure μ_f is decaying since $f \in \mathcal{H}^{\mathrm{D}}$. Lemma 5.5 now implies that μ_g is decaying and therefore $g \in \mathcal{H}^{\mathrm{D}}$. Thus $\mathcal{H}^{\mathrm{D}}$ is stable with respect to A and so is $\mathcal{H}^{\mathrm{ND}}$ according to lemma 5.4. The spaces $\mathcal{H}^{\mathrm{D}}_{\mathrm{sc}}$ and $\mathcal{H}^{\mathrm{ND}}_{\mathrm{sc}}$ are stable with respect to A, as intersection of stable subspaces. All other statements of theorem 2.5 follow easily from the proven ones. □

Proof of theorem 3.4. This is identical to the proof of theorem 2.5; the only difference is that we deal with measures on a unit circle instead of measures on $\mathbb{R}$. Alternatively, it follows from theorem 2.5 using the Cayley transform. □

Proof of theorem 4.4. This is identical to the proof of theorem 2.5; the only difference is that we deal with measures on $\mathbb{C}$ instead of measures on $\mathbb{R}$. □

Proof of theorem 2.4. Suppose that the conditions of theorem 2.4 are satisfied. Formula (6.1) implies that $f \in \mathcal{H}^{\mathrm{D}}$ if and only if μ_f is decaying. Suppose that $f \in \mathcal{H}$ and μ_f is purely non-decaying. Also let $g \in \mathcal{H}^{\mathrm{D}}$, i.e. μ_g is decaying. According to lemma 5.6, $\mu_g \perp \mu_f$. Therefore, lemma 5.3 implies that $g \perp f$. Hence $f \perp \mathcal{H}^{\mathrm{D}}$, or, equivalently, $f \in \mathcal{H}^{\mathrm{ND}}$.

Suppose now that $f \in \mathcal{H}^{\mathrm{ND}}$. We have to prove that μ_f is purely non-decaying. If not, there exists a non-zero Borel measure ν on $\mathbb{R}$ such that ν is decaying and $\nu \prec \mu_f$. Lemma 5.5 implies that ν is decaying. Let $\varphi : \mathbb{C} \to [0, +\infty)$ be the density of ν with respect to μ_f. Consider $g = \sqrt{\varphi}(A)f$. Clearly,

$$\|g\|^2 = (\sqrt{\varphi}(A)f, \sqrt{\varphi}(A)f) = (f, \varphi(A)f) = \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_f = |\nu|(\mathbb{C}) < \infty.$$

Thus $f \in D_{\sqrt{\varphi}(A)}$ and vector $g \in \mathcal{H}_f \subseteq \mathcal{H}$ is well defined. Let $\psi \in C_0(\mathbb{C})$. Then

$$\int_{\mathbb{C}} \psi \, \mathrm{d}\mu_g = (g, \psi(A)g) = (\psi\sqrt{\varphi}(A)f, \sqrt{\varphi}(A)f) = (\varphi\psi(A)f, f) = \int_{\mathbb{C}} \psi\varphi \, \mathrm{d}\mu_f.$$

Therefore, φ is the density of μ_g with respect to μ_f. Thus $\mu_g = \nu$ is decaying. Hence $g \in \mathcal{H}^{\mathrm{D}} \cap \mathcal{H}_f$ for $f \in \mathcal{H}^{\mathrm{ND}}$, which contradicts the invariance of $\mathcal{H}^{\mathrm{ND}}$ with respect to A proved in the previous subsection. Theorem 2.4 is proved. □

Proof of theorem 3.3. This is identical to the proof of theorem 2.4; the only difference is that we deal with measures on the unit circle instead of measures on $\mathbb{R}$. □

Proof of theorem 4.3. This is identical to the proof of theorem 2.4; the only difference is that we deal with measures on $\mathbb{C}$ instead of measures on $\mathbb{R}$. □

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7. Non-triviality of the decomposition of the singular continuous subspace

The following examples show the non-triviality of the decomposition of the subspace $\mathcal{H}_{\text{sc}}$ in terms of decay and non-decay parts.

For any $\alpha > 0$, let ν_α be the measure on the real line concentrated in the two point set $\{-\alpha, \alpha\}$ such that $\nu_\alpha(\{-\alpha\}) = \nu_\alpha(\{\alpha\}) = \frac{1}{2}$. For any $\theta > 1$, let μ_θ be the weak limit of the sequence of measures $\nu_{\theta^{-1}} * \cdots * \nu_{\theta^{-n}}$, where $*$ is the convolution. Note that the standard Cantor measure (up to a linear change of variables) coincides with μ_3 and μ_2 is the normalized Lebesgue measure on the segment $[-1, 1]$. It is straightforward to verify that the Fourier transform $\tilde{\mu}_\theta$ of μ_θ is given by the formula

$$\tilde{\mu}_\theta(x) = \prod_{n=1}^{\infty} \cos\left(\frac{x}{\theta^n}\right). \tag{7.1}$$

In [2], it is shown that μ_θ is purely non-decaying if θ is an integer and $\theta \geqslant 3$. Erdös [6] showed that μ_θ, for rational non-integer theta greater than 2, is decaying (in both cases, we treat μ_θ as a measure on the real line).

Example 7.1. *Self-adjoint operator with non-trivial decomposition of $\mathcal{H}_{\text{sc}}$.* Consider the measure $\mu = \mu_3 + \mu_{5/2}$ on the real line, where μ_3 and $\mu_{5/2}$ are measures defined by (7.1). Then μ is a singular continuous measure, which is neither decaying nor purely non-decaying.

Let $\mathcal{H} = L_2(\mathbb{R}, \mu)$ and let the operator A on $\mathcal{H}$ be defined by the formula $Af(x) = xf(x)$. Then A is a self-adjoint operator on $\mathcal{H}$, $\mathcal{H}_{\text{p}} = \mathcal{H}_{\text{ac}} = \{0\}$ and both spaces $\mathcal{H}_{\text{sc}}^{\text{D}}$ and $\mathcal{H}_{\text{sc}}^{\text{ND}}$ are infinite dimensional. This example shows that our decomposition of the space $\mathcal{H}_{\text{sc}}$ is non-trivial.

The analogous example for unitary operator can be obtained from the previous one using Cayley transform.

Let μ be the measure from example 7.1 and η be an arbitrary positive finite measure on $\mathbb{R}$, absolutely continuous with respect to the Lebesgue measure. Also let ν be the measure on the complex plane such that

$$\nu(A + \mathrm{i}B) = \mu(A)\eta(B) \tag{7.2}$$

for any Borel subsets A and B of $\mathbb{R}$. Then ν is an example of a singular continuous measure on the complex plane, which is neither decaying nor purely non-decaying.

Example 7.2. *Normal operator with non-trivial decomposition of $\mathcal{H}_{\text{sc}}$.* Let $\mathcal{H} = L_2(\mathbb{C}, \nu)$ and let the operator A on $\mathcal{H}$ be defined by the formula $Af(z) = zf(z)$. Then A is a normal operator on $\mathcal{H}$, $\mathcal{H}_{\text{p}} = \mathcal{H}_{\text{ac}} = \{0\}$ and the spaces $\mathcal{H}_{\text{sc}}^{\text{D}}$ and $\mathcal{H}_{\text{sc}}^{\text{ND}}$ are infinite dimensional. This example shows that our decomposition of the space $\mathcal{H}_{\text{sc}}$ for normal operators is non-trivial.

8. Concluding remarks

The motivation of the decomposition into decaying and purely non-decaying spectral subspaces is the complete spectral characterization [3] of mixing and decay in classical and quantum systems [4, 7, 8]. In the case of quantum systems, the spectral measure of the Liouville–von Neumann operator generating the statistical evolution, is the convolution of the spectral measure of the Hamiltonian with its reflection [3]. In [3], we corrected some erroneous statements [12] on the spectrum of the Liouville–von Neumann operator in the Hilbert–Schmidt space.

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References

1 I. Antoniou and S. A. Shkarin. Decomposition of singular measures. *Dokl. Math.* **61** (2000), 24–27.
2 I. Antoniou and S. A. Shkarin. Decaying measures on locally compact abelian topological groups. *J. Edinb. R. Soc.* A **131** (2001), 1257–1273. (Following paper.)
3 I. Antoniou, S. A. Shkarin and Z. Suchanetcki. The spectrum of the Liouville–von Neumann operator in the Hilbert–Schmidt space. *J. Math. Phys.* **40** (1999), 4106–4118
4 I. Cornfeld, S. Fomin and Ya. Sinai. *Ergodic theory* (Springer, 1982).
5 N. Dunford and T. Schwartz. *Linear operators*, vols I, II, III (Wiley, 1988).
6 P. Erdös. On a family of symmetric Bernoulli convolutions. *Am. J. Math.* **61** (1939), 974–976.
7 P. Exner. *Open quantum systems and Feynman integrals* (Dordrecht, The Netherlands: Reidel, 1985).
8 L. Fonda, G. Girardi and A. Rimini. Decay theory of unstable quantum systems. *Rep. Prog. Phys.* **41** (1978), 587–631.
9 M. Reed and B. Simon. Functional analysis. In *Methods of modern mathematical physics*, vol. 1 (Academic, 1972).
10 W. Rudin. *Fourier analysis on groups* (Wiley, 1990).
11 G. E. Shilov and B. L. Gurevich. *Integral, measure and derivative: a unified approach* (New York: Dover, 1977).
12 H. Spohn. The spectrum of the Liouville–von Neumann operator. *J. Math. Phys.* **17** (1976), 57–60.
13 J. Weidmann. *Linear operators in Hilbert space* (Springer, 1980).

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