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A GENERALIZATION OF A FIXED POINT THEOREM OF GOEBEL, KIRK AND SHIMI[1]

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1. In [7], Goebel, Kirk and Shimi proved the following:

THEOREM. *Let X be a uniformly convex Banach space, K a nonempty bounded closed and convex subset of X, and $F: K \to K$ a continuous mapping satisfying for each $x, y \in K$:*

(1) $\|Fx-Fy\| \leq a_1\|x-y\|+a_2\|x-Fx\|+a_3\|y-Fy\|+a_4\|x-Fy\|+a_5\|y-Fx\|$
where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i=1$. Then F has a fixed point in K.

In this paper we shall prove that this theorem remains true in any Banach space X, provided that K is a nonempty, weakly compact convex subset of X and has normal structure (see Definition 1 below). Moreover, F need not be continuous. It is well known (see [1], [5], [6]) that if X is a uniformly convex Banach space or if K is compact, then K has normal structure.

In [2], Belluce, Kirk and Steiner give an example of a Banach space which is reflexive, strictly convex and which possesses normal structure, but which is not isomorphic to any uniformly convex Banach space.

2. The following definitions were introduced by Brodskii and Milman [4] who also proved Lemma 1 below (see also Gossez and Lami Dozo [8]).

DEFINITION 1. A convex subset K of X has *normal structure* if in each bounded and convex subset W of K, which contains more than one point, there is a non-diametral point, i.e. a point x such that

$$\sup\{\|x-y\|; y \in W\} < \delta(W),$$

where $\delta(W)$ is the diameter of W.

DEFINITION 2. A non-constant bounded sequence $\{x_n\}$ in X is said to be *diametral* if

$$\lim_{n\to\infty} d(x_n, \operatorname{conv}\{x_1, \ldots, x_{n-1}\}) = \delta(\{x_n\})$$

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where $d(x_n, \operatorname{conv}\{x_1, \ldots, x_{n-1}\})$ is the distance between x_n and the convex hull of $\{x_1, \ldots, x_{n-1}\}$.

LEMMA 1 [4]. *A convex subset C of X has normal structure if and only if C does not contain a diametral sequence.*

DEFINITION 3. Let C be a nonempty subset of a Banach space X, and $\{W_\alpha, \alpha \in A\}$ be a decreasing net of nonempty bounded subsets of C. For each $x \in C$ and each $\alpha \in A$ define:

$$\begin{aligned} r_\alpha(x) &= \sup\{\|x-y\|, y \in W_\alpha\} \\ r(x) &= \inf\{r_\alpha(x); \alpha \in A\} \\ r &= \inf\{r(x); x \in C\} \\ M &= \{x \in C; r(x) = r\} \end{aligned} \tag{2}$$

The set M and the number r will be called the *asymptotic center* and the *asymptotic radius* of $\{W_\alpha; \alpha \in A\}$ in C, respectively.

DEFINITION 4 (Lim [11]). A convex set C of X is said to have *asymptotic normal structure* if, given any bounded convex subset W of C which contains more than one point and given any decreasing net of nonempty subsets $\{W_\alpha; \alpha \in A\}$ of W, the asymptotic center of $\{W_\alpha; \alpha \in A\}$ in W is a proper subset of W.

It is easy to see ([11]) that if W is convex and weakly compact then the asymptotic center of $\{W_\alpha; \alpha \in A\}$ in W is a nonempty closed convex subset of W.

LEMMA 2 [11]. *A convex subset C of X has normal structure if and only if it has asymptotic normal structure.*

From the two lemmas we get the following:

COROLLARY 1. *A convex subset C of X has normal structure if and only if it possesses the following property (B):*

For each non-constant bounded sequence $\{x_n\}$ in C, the function $g(x)=\lim_n\sup \|x_n-x\|$ *is not constant in* $\operatorname{conv}\{x_n\}$.

Proof. Let C have normal structure, let $\{x_n\}$ be a non-constant bounded sequence in C and let $W=\operatorname{conv}\{x_n\}$. Then, defining $W_k=\{x_n; n\geq k\}$ $(k=1,2,3,\ldots)$ we get a decreasing net $\{W_k; k=1,2,\ldots\}$ of subsets of W and, according to the notations (2), for any $x \in W$:

$$r(x) = \lim_n\sup \|x-x_n\| = g(x).$$

By Lemma 2 $g(x)$ is not constant in W.

Now, let C possess property (B). Let $\{x_n\}$ be any non-constant bounded sequence in C. By property (B) there is a point x in $W=\operatorname{conv}\{x_n\}$ such that $\lim_n\sup \|x-x_n\|<\delta(\{x_n\})$. For n sufficiently large, $x \in \operatorname{conv}\{x_1, \ldots, x_{n-1}\}$ and thus

$$\lim_n\sup d(x_n, \operatorname{conv}\{x_1, \ldots, x_{n-1}\}) \leq \lim_n\sup \|x-x_n\| < \delta(\{x_n\}).$$

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So $\{x_n\}$ is a non-diametral sequence. Hence, by Lemma 1, C has normal structure. Q.E.D.

3. We arrive at the main theorems. Let (X, d) be a metric space and $F:X\to X$ a mapping satisfying, for each $x, y \in X$:

$$d(Fx, Fy) \leq ad(x, y)+b[d(x, Fx)+d(y, Fy)]+c[d(x, Fy)+d(y, Fx)] \tag{3}$$

where a, b, $c\geq 0$ and $a+2b+2c=1$.

Lemma 3. *For F satisfying* (3) *and for each $x \in X$:*

$$d(Fx, F^2x) \leq d(x, Fx). \tag{4}$$

Proof. By (3):

$$d(Fx, F^2x) \leq ad(x, Fx)+b[d(x, Fx)+d(Fx, F^2x)] +c[d(x, F^2x)+d(Fx, Fx)] \leq (a+b+c)d(x, Fx)+(b+c)d(Fx, F^2x)$$

and so:

$$d(Fx, F^2x) \leq \frac{a+b+c}{1-b-c} d(x, Fx) = d(x, Fx). \qquad \text{Q.E.D.}$$

Lemma 4. *If $b>0$ in* (3) *then there is a number $\kappa<2$ such that for each $x \in X$:*

$$d(Fx, F^3x) \leq \kappa d(x, Fx) \tag{5}$$

Proof. By (3) and (4):

$$\begin{aligned} d(Fx, F^3x) \leq\ & ad(x, F^2x)+b[d(x, Fx)+d(F^2x, F^3x)] \\ & +c[d(x, F^3x)+d(F^2x, Fx)] \leq a[d(x, Fx)+d(Fx, F^2x)] \\ & +b[d(x, Fx)+d(F^2x, F^3x)]+c[d(x, Fx)+d(Fx, F^3x)+d(F^2x, Fx)] \\ & \leq (2a+2b+2c)d(x, Fx)+cd(Fx, F^3x) \end{aligned}$$

and so, since $b>0$:

$$d(Fx, F^3x) \leq (2a+2b+2c)/(1-c)d(x, Fx)$$

where $\kappa=(2a+2b+2c)/(1-c)=(2a+2b+2c)/(a+2b+c)<$

$$(2a+2b+2c)/(a+b+c)=2.$$

Q.E.D.

Lemma 5. *If $b>0$ and $c>0$ in* (3), *then there is a number $m<1$ such that for each $x \in X$:*

$$d(F^2x, F^3x) \leq md(x, Fx) \tag{6}$$

Proof. By (3), (4) and (5):

$$d(F^2x, F^3x) \leq ad(Fx, F^2x)+b[d(Fx, F^2x)+d(F^2x, F^3x)] +c[d(Fx, F^3x)+d(F^2x, F^2x)] \leq (a+2b+\kappa c)d(x, Fx)$$

where $m=a+2b+\kappa c<a+2b+2c=1$. Q.E.D.

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THEOREM 1. *Let (X, d) be a complete metric space and $F:X\to X$ a mapping satisfying* (3) *with $b>0$, $c>0$. Then F has a unique fixed point in X.*

Proof. Choose an arbitrary x in X. Then, by (6), for any positive integer $n\geq 2$:

$$d(F^n x, F^{n+1}x) \leq md(F^{n-2}x, F^{n-1}x) \qquad (m < 1).$$

Thus, if n is even, then $d(F^n x, F^{n+1}x)\leq m^{n/2}\, d(x, Fx)$ and if n is odd then $d(F^n x, F^{n+1}x)\leq m^{(n-1)/2}\, d(Fx, F^2x)\leq m^{(n-1)/2}\, d(x, Fx)$, so for any $n\geq 2$:

$$d(F^n x, F^{n+1}x) \leq (\sqrt{m})^{n-1} d(x, Fx).$$

By a standard argument the sequence $\{F^n x\}$ can be shown to be a Cauchy sequence. Thus $F^n x\to z$ for some $z\in X$.

If F is continuous then z is obviously a fixed point of F. Otherwise, by (3):

$$d(F^n x, Fz) \leq ad(F^{n-1}x, z)+b[d(F^{n-1}x, F^n x)+d(z, Fz)] + c[d(F^{n-1}x, Fz)+d(z, F^n x)].$$

Passing to the limit when $n\to\infty$ we get:

$$d(z, Fz) \leq (b+c)d(z, Fz)$$

and since $b+c<1$ we have $z=Fz$, completing the proof.

The uniqueness follows from (3) and $b>0$.

THEOREM 2. *Let K be a nonempty weakly compact convex subset of a Banach space X, and suppose K has normal structure. Let $F:K\to K$ be a mapping satisfying* (1). *Then F has a fixed point in K.*

Proof. By interchanging x and y we see that Condition (1) can be written as follows:

(7) $\|Fx-Fy\| \leq a\,\|x-y\|+b[\|x-Fx\|+\|y-Fy\|]+c[\|x-Fy\|+\|y-Fx\|]$,

where a, b, $c\geq 0$ and $a+2b+2c=1$.

By Theorem 1 it is sufficient to prove the theorem for the cases $b=0$ and $c=0$ only.

By the weak compactness of K and by Zorn's Lemma there exists a subset C of K which is minimal in the family of all nonempty, closed and convex subsets of K which are invariant under F. It is sufficient to show that C contains exactly one point.

Case 1. $b=0$. Let x_0 be an arbitrary point in C and $x_n=F^n x_0$, $(n=1, 2, \ldots)$. Then, either $\{x_n\}$ is constant, implying $\{x_0\}=C$, or $\{x_n\}$ is not constant and hence, by Corollary 1, there is a number $r>0$ such that the set $M=\{y\in C;\ \lim_n \sup \|x_n-y\|\leq r\}$ is a nonempty closed and convex subset of C, and $M\neq C$. If $y\in M$ then

$$\|Fy-F^n x_0\| \leq a\,\|y-F^{n-1}x\|+c[\|y-F^n x_0\|+\|F^{n-1}x_0-Fy\|]$$

and thus

$$\lim_n\sup \|Fy-x_n\| \leq \frac{a+c}{1-c}\lim_n\sup \|y-x_n\| \leq r.$$

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So $F(M)\subseteq M$ and this contradicts the minimality of C. Hence $C=\{x_0\}$.

Case 2. $c=0$. In view of the results of Browder-Kirk ([5], [10]) and the results of Bianrchini, Soardi and Reich ([3], [12], [13], see also Kannan [9]), the cases $c=b=0$ and $c=a=0$ need not be considered. So we assume $c=0$, $a>0$, $b>0$.

Let $d=\inf\{\|x-Fx\|;\ x\in C\}$, and $\varepsilon>0$. Then there is a point x in C such that $\|x-Fx\|<d+\varepsilon$.

Define $y=(1/2)\,(F^2x+F^3x)$. Then $y\in C$ and by (7):

$$\begin{aligned}\|y-Fy\| &\leq \tfrac{1}{2}[\|F^2x-Fy\|+\|F^3x-Fy\|]\\ &\leq \tfrac{1}{2}\{a\,\|Fx-y\|+b[\|Fx-F^2x\|+\|y-Fy\|]\}\\ &\quad+\tfrac{1}{2}\{a\,\|F^2x-y\|+b[\|F^2x-F^3x\|+\|y-Fy\|]\}.\end{aligned}$$

This implies, by (4) and (5):

$$\begin{aligned}(1-b)\,\|y-Fy\| &\leq \frac{1}{2}\left\{\frac{a}{2}\,[\|Fx-F^2x\|+\|Fx-F^3x\|]+b\,\|Fx-F^2x\|\right\}\\ &\quad+\frac{1}{2}\left\{\frac{a}{2}\,\|F^2x-F^3x\|+b\,\|F^2x-F^3x\|\right\}\leq\left(\frac{a}{2}+\kappa\,\frac{a}{4}+b\right)\|x-Fx\|.\end{aligned}$$

So,

$$(1-b)d\leq(1-b)\,\|y-Fy\|\leq\left(\frac{a}{2}+\kappa\,\frac{a}{4}+b\right)(d+\varepsilon)$$

Letting $\varepsilon\to 0$ and assuming $d>0$ we get, since $a>0$: $(1-b)d<(a+b)d$ and this contradicts $a+2b=1$.

Thus $d=0$, and hence there is a sequence $\{x_n\}$ in C such that $\|x_n-Fx_n\|\to 0$. If $\{x_n\}$ is constant it defines a fixed point of F. Else, by Corollary 1, there is a number $r>0$ such that the set $M=\{y\in C;\ \lim_n\sup\|x_n-y\|\leq r\}$ is a nonempty closed and convex subset of C and $M\neq C$. For each $y\in M$ and each n:

$$\begin{aligned}\|Fy-x_n\| &\leq \|Fy-Fx_n\|+\|Fx_n-x_n\|\leq a\,\|y-x_n\|\\ &\quad+b[\|y-Fy\|+\|x_n-Fx_n\|]+\|Fx_n-x_n\|\\ &\leq a\,\|y-x_n\|+b[\|y-x_n\|+\|x_n-Fy\|]+(1+b)\,\|Fx_n-x_n\|\end{aligned}$$

and so, since $\lim_n\|x_n-Fx_n\|=0$, we have:

$$\lim_n\sup\,\|Fy-x_n\|\leq\frac{a+b}{1-b}\lim_n\sup\,\|y-x_n\|\leq r.$$

So $F(M)\subseteq M$, in a contradiction to the minimality of C. Hence $C=\{x_0\}$. The proof is now complete.

Remark. Even though the mappings of the type we consider are not assumed to be continuous, an easy argument shows that their fixed point sets are closed. Once this is accomplished it is possible to show in the usual way that if the norm of X is strictly convex then these fixed point sets are also convex. From this it

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readily follows from properties of weak compactness that if K satisfies the assumptions of Theorem 2 with X strictly convex, then every commuting family of mappings of K into K, each of which satisfies (7) (with the constants a, b, c depending on the mapping) has a common fixed point. (See Browder [5], Roux and Soardi [14]).

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