Macroeconomic Dynamics, **23**, 2019, 2360–2377. Printed in the United States of America.
doi:10.1017/S1365100517000724

DEBT POLICY RULE, UTILITY-GENERATING GOVERNMENT SPENDING, AND INDETERMINACY OF THE TRANSITION PATH IN AN AK MODEL

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This study examines the effects of borrowing for public services that increase households' utility (i.e., utility-generating government services) in an AK endogenous growth model. We assume that the government has a target debt ratio. The European Union and the United Kingdom adopt such debt policy rules. We find that application of a debt policy rule into utility-generating government spending causes indeterminacy of the transition path. We point out that the level of the target debt ratio, the tax rate, and the household utility parameters are important determinants of indeterminacy when considering utility-generating government services.

Keywords: Public Expenditure, Public Debt, Debt Policy Rule, Indeterminacy

1. INTRODUCTION

Assuming that government expenditure is financed by income tax and by issuing bonds in an endogenous growth model with productive government services, as proposed by Barro (1990), Futagami et al. (2008; hereafter, FIO) investigate the

We would like to thank the editor, the associate editor, and two anonymous referees for their helpful comments on the earlier version of this paper. We also gratefully acknowledge Koichi Futagami, Tarishi Matsuoka, Akihisa Shibata, Kozo Ueda, and seminar participants at Otaru University of Commerce and The University of Kitakyushu. This study was conducted while the first author was in Aoyamagakuin University. We are responsible for any remaining errors. Address correspondence to: Noritaka Maebayashi, Faculty of Economics and Business Administration, the University of Kitakyushu, 4-2-1 Kitagata, Kokura Minami-ku, Kitakyushu, Fukuoka 802-8577, Japan; e-mail: non818mn@kitakyu-u.ac.jp.

 1365-1005/17

following government debt adjustment rule:

$$\dot{b}_t = -\phi(b_t - \bar{b}), \tag{1}$$

where b_t is defined as the government debt (B_t)-to-private capital (K_t) ratio, $\phi > 0$ is the adjustment coefficient, and $\bar{b} > 0$ is the target level of the government debt ratio.[1] Under (1), the government gradually adjusts government debt so that b_t becomes equal to the target level, $\bar{b}$, in the long run. Such debt rules are found in the Maastricht Treaty, the Stability and Growth Pact (SGP) in the European Union (EU), and the Code of Fiscal Stability (CFS) in the United Kingdom.[2]

FIO show that (i) two steady states (low growth and high growth) exist and (ii) there is indeterminacy of the transition path to the high-growth steady state (i.e., the high-growth steady state is a sink). Although FIO's results are interesting, Minea and Villieu (2013; hereafter, MV) point out that if b_t is defined as the government debt (B_t)-to-gross domestic product (GDP) (Q_t) ratio, the results change dramatically. In fact, MV show that (i) a unique steady state exists and (ii) indeterminacy of the transition path is removed (i.e., the steady state is a saddle point). As MV point out, a GDP-based government debt rule would be easier to implement in the real world. In fact, the Maastricht Treaty, the SGP, and the CFS employ GDP-based public debt rules.

MV explain why the multiplicity of the steady states disappears under their setting. The difference between FIO and MV comes from the government budget constraint in the long run. In FIO, the primary surplus is increasing in government spending when the latter is small but turns to decreasing in government spending when it is large. Such a nonmonotonic relationship between the primary surplus and government spending generates the multiple steady states. In contrast, in MV, the primary surplus is monotonically decreasing in government spending, which induces a unique steady state. MV point out that a positive relationship between the primary surplus and government spending in FIO is a very special case.

Although these controversies on the debt policy rule (1) between FIO and MV are very interesting, there remains some room for discussion. First, the roles of the debt policy rule on the properties of the transition paths in FIO and MV are limited to the case of productive expenditure. It is important to examine the case of government expenditure that directly affects household utility, such as public healthcare services, environmental protection, education, justice, police, and public broadcasting. Our first objective is to investigate the roles of the debt policy rule (1) on the properties of transition paths in the case of such utility-generating spending. Second, whereas FIO and MV do not derive policy implications of the target debt ratio or the tax rate, this study investigates how the target debt ratio and the tax rate influence the properties of transition paths.

For our purpose, we construct an endogenous growth model with simple AK production technology. We define b_t in (1) as the government debt (B_t)-to-GDP (Q_t) ratio. When utility-generating public spending is considered, AK production technology can generate a unique steady state because the primary surplus is monotonically decreasing in government spending.

We find that in the presence of utility-generating government spending, the transition path to the steady state can be either determinate or indeterminate (i.e., the steady state can be a saddle point or a sink), depending on the level of the target debt ratio, income tax rate, and preference parameters. Mechanisms behind (in)determinacy of the transition path are different from those of FIO and MV. Unlike FIO and MV, our results regarding indeterminacy may not be affected by specifications of $\bar{b}$. More precisely, we obtain the following three results. (i) There is a unique steady state. (ii) Under some preference parameter conditions, a threshold exists for the target debt ratio. If $\bar{b}$ is larger than the threshold, indeterminacy of the transition path arises. In contrast, if $\bar{b}$ is smaller than the threshold or outside the previous preference parameter conditions, the transition path is determined uniquely. (iii) The threshold of the target debt ratio increases with the income tax rate.

Although FIO point out the possibility of indeterminacy of the transition path, they do not examine how their indeterminacy condition depends on the target debt-GDP ratio and income tax rate. In MV, their determinacy condition does not depend on the target debt-GDP ratio and income tax rate. In contrast to FIO and MV, our results provide the following policy implications. In order to avoid indeterminacy of the transition path, the government has two choices. It could set the target debt ratio to be low or the income tax rate to be high.

Fernández et al. (2004) and Guo and Harrison (2008) examine the indeterminacy condition in a model with utility-generating government spending in which government spending is financed by income taxation. Both of the previous studies assume balanced-budget rules and elastic labor supply. In Fernández et al. (2004), as the income tax rate increases, indeterminacy is more likely to occur while the indeterminacy condition of Guo and Harrison (2008) is independent of income taxation. Allowing for debt financing of the government, we show that the income tax rate affects the threshold of the target debt ratio.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 derives the dynamic system of the economy. Section 4 derives the steady state. Section 5 examines the properties of transition paths to the steady state. Section 6 provides numerical examples to establish whether indeterminacy could occur in some EU member countries. Section 7 concludes the paper.

2. MODEL

We consider a competitive economy. Time is continuous and denoted by $t \geq 0$. The representative firm produces a final good using the following AK form production function:

$$Q_t = AK_t, \tag{2}$$

where A is a positive constant. Through perfect competition and profit maximization, the interest rate, r_t, remains constant over time: $r = A$.[3]

We assume that the economy is populated by identical households. The population size is constant over time and normalized to 1. The representative household is endowed with an infinite lifetime and perfect foresight. We specify the utility of the representative household as

$$U = \int_0^\infty \frac{u(C_t, G_t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \qquad \sigma > 0, \tag{3}$$

where

$$u(C_t, G_t) = \begin{cases} (\theta_1 C_t^\eta + \theta_2 G_t^\eta)^{\frac{1}{\eta}}, & \eta < 1,\ \eta \neq 0, \\ C_t^{\theta_1} G_t^{\theta_2}, & \eta = 0. \end{cases} \tag{4}$$

In (3), $\rho > 0$ and $1/\sigma$ denote the subjective discount rate and the intertemporal elasticity of substitution (IES) of u_t, respectively.[4] In this study, we assume $\sigma > 1$.[5] In (4), G_t is the public expenditure for utility-generating government services. θ_1 and θ_2 are positive constants that satisfy $\theta_1 + \theta_2 = 1$, and $1/(1-\eta)$ is the elasticity of substitution between C_t and G_t in u_t. The budget constraint of the household is given by $\dot{W}_t = (1-\tau)rW_t - C_t$, where W_t represents the asset holdings of the household and $\tau \in [0, 1)$ is the interest income tax rate. To ensure positive growth, we assume $(1-\tau)A > \rho$.

The utility maximization of the household yields $\dot{\lambda}_t = \lambda_t[\rho - (1-\tau)r]$ and

$$(\theta_1 C_t^\eta + \theta_2 G_t^\eta)^{\frac{1-\sigma-\eta}{\eta}} \theta_1 C_t^{\eta-1} = \lambda_t, \tag{5}$$

where λ_t is the costate variable associated with W_t. The transversality condition, $\lim_{t\to\infty} \lambda_t W_t e^{-\rho t} = 0$, must be satisfied. From these two equations, we obtain

$$\frac{\dot{g}_t}{g_t} - \frac{\sigma + (1-\eta)\theta x_t^\eta}{\sigma(1+\theta x_t^\eta)} \frac{\dot{x}_t}{x_t} + \frac{\dot{Q}_t}{Q_t} = \frac{1}{\sigma}[(1-\tau)r - \rho], \tag{6}$$

where $g_t \equiv G_t/Q_t$, $x_t \equiv G_t/C_t = g_t/c_t$, $c_t \equiv C_t/Q_t$, and $\theta \equiv \theta_2/\theta_1$. Equation (6) shows that the growth rate of government spending, $\dot{G}_t/G_t (= \dot{g}_t/g_t + \dot{Q}_t/Q_t)$, affects the consumption and saving behavior of the household.

As in FIO and MV, the government finances its expenditure, G_t, by levying the interest income tax and by issuing bonds. The government is allowed to run budget deficits. The budget constraint of the government is

$$rB_t + G_t = \dot{B}_t + \tau r W_t. \tag{7}$$

We assume that the government follows the debt adjustment rule given by (1), as in FIO and MV. This means that the government does not consider the Keynes–Ramsey policy. As in the MV model, we define $b_t \equiv B_t/Q_t$. Because of the AK production function, defining the debt ratio, b_t, as B_t/Q_t or B_t/K_t does not affect the dynamics of this economy and does not change the results in this study.

3. DYNAMIC SYSTEM

This section derives the dynamic system of the economy. The asset market equilibrium implies that $W_t = K_t + B_t$. Using (2), (7), $r = A$, and $W_t = K_t + B_t$, we obtain $\dot{B}_t = (1-\tau)AB_t + G_t - \tau Q_t$. This equation is rearranged as

$$\frac{\dot{b}_t}{b_t} = (1-\tau)A + \frac{G_t}{B_t} - \frac{\tau}{b_t} - \frac{\dot{Q}_t}{Q_t}. \quad \textbf{(8)}$$

Here, we use $\dot{B}_t = \dot{b}_t Q_t + \dot{Q}_t b_t$. The goods market equilibrium condition is given by $\dot{K}_t = Q_t - C_t - G_t$. From (19), $\dot{K}_t = Q_t - C_t - G_t$, and the definitions of g_t and x_t, we obtain

$$g_t = \frac{[(1+Ab_t)\tau - \phi(b_t - \bar{b})]x_t}{(1+Ab_t)x_t + Ab_t}. \quad \textbf{(9)}$$

Differentiating both sides of (9) with respect to t yields

$$\frac{\dot{g}_t}{g_t} = \frac{Ab_t}{(1+Ab_t)x_t + Ab_t}\frac{\dot{x}_t}{x_t} - \Psi(x_t, b_t), \quad \textbf{(10)}$$

where $\Psi(x_t, b_t) \equiv [\frac{A\tau - \phi}{(1+Ab_t)\tau - \phi(b_t - \bar{b})} - \frac{A(1+x_t)}{(1+Ab_t)x_t + Ab_t}]\phi(b_t - \bar{b})$. From (2) and $\dot{K}_t = Q_t - C_t - G_t$, we have $\frac{\dot{Q}_t}{Q_t} = \frac{\dot{K}_t}{K_t} = A(1 - \frac{1+x_t}{x_t}g_t)$. Substituting this and (10) into (6) and using (9), we obtain

$$\dot{x}_t = \left[\frac{Ab_t}{(1+Ab_t)x_t + Ab_t} - \frac{\sigma + (1-\eta)\theta x_t^{\eta}}{\sigma(1+\theta x_t^{\eta})}\right]^{-1}\Gamma(x_t, b_t)x_t, \quad \textbf{(11)}$$

where

$$\Gamma(x_t, b_t) \equiv \frac{1}{\sigma}[(1-\tau)A - \rho] - A\left\{1 - \frac{[(1+Ab_t)\tau - \phi(b_t - \bar{b})](1+x_t)}{(1+Ab_t)x_t + Ab_t}\right\} + \Psi(x_t, b_t).$$

Given the initial value of b_t, the dynamic system of the economy is composed of (1) and (11). Then, the equilibrium paths are characterized by the dynamic systems of b_t (the state variable) and x_t (the jump variable). In the standard AK model, there are no transitional dynamics because the dynamic system is composed of only one jump variable C_t/K_t. In contrast, the present model has transitional dynamics because a state variable, b_t, is included in the dynamic system.

4. STEADY STATE

This section derives the steady state, in which b_t and x_t are both constant over time and B_t, G_t, C_t, K_t, and Q_t grow at the same constant rate. We omit the time index, t, from those variables that are constant in the steady state and use an asterisk to denote a steady-state variable. In the steady state, g_t becomes constant, from

(9). Then, from (6), we immediately obtain the growth rate in the steady state as follows:

$$\gamma^* = \frac{1}{\sigma}\left[(1-\tau)A - \rho\right]. \tag{12}$$

In contrast to productive government spending in FIO and MV, utility-generating government spending has no long-run effect on growth despite the potential presence of indeterminacy, as we see in the next section.

To prove the existence of a steady state, we assume the following conditions:

$$1 - \tau - \frac{\gamma^*}{A} > 0, \tag{13}$$

$$\bar{b} < \frac{\tau}{A\left(1 - \tau - \frac{\gamma^*}{A}\right)} \equiv \bar{b}_1. \tag{14}$$

The condition (13) is always satisfied when $\sigma > 1$. In addition, the condition (14) tends to be satisfied under reasonable parameter values.[6] We can prove the next proposition.

PROPOSITION 1. *Suppose that* (13) *and* (14) *are satisfied. A unique steady-state equilibrium exists in which $b^* = \bar{b}$ holds, c^* and g^* are strictly positive, and B_t, G_t, C_t, K_t, and Q_t grow at the rate of γ^*.*

Proof. We set $\dot{b}_t = 0$ in (1) to obtain $b^* = \bar{b}$. Substituting $\dot{x}_t = 0$ and $b^* = \bar{b}$ into (11) yields

$$x^* = \frac{(1 + A\bar{b})\tau - A\bar{b}\left(1 - \frac{\gamma^*}{A}\right)}{(1 + A\bar{b})\left(1 - \tau - \frac{\gamma^*}{A}\right)} = \frac{A(\bar{b}_1 - \bar{b})}{1 + A\bar{b}}, \tag{15}$$

where $\bar{b}_1$ is defined in (14).[7] Inserting (15) and $b^* = \bar{b}$ into (9) and $c^* = g^*/x^*$ yields

$$g^* = (1 + A\bar{b})\tau - A\bar{b}\left(1 - \frac{\gamma^*}{A}\right) = A\left(1 - \tau - \frac{\gamma^*}{A}\right)(\bar{b}_1 - \bar{b}),$$

$$c^* = (1 + A\bar{b})\left(1 - \tau - \frac{\gamma^*}{A}\right).$$

Assumptions (13) and (14) ensure $\bar{b}_1$, c^*, g^*, and $x^* > 0$. ■

In the steady state, the government budget constraint (8) can be expressed as the following solvency constraint for the government by using (12):

$$\begin{aligned} \tau - g^* &= [(1-\tau)A - \gamma^*]\bar{b} \\ &= \left[\frac{\sigma - 1}{\sigma}(1-\tau)A + \frac{\rho}{\sigma}\right]\bar{b}. \end{aligned} \tag{16}$$

The left-hand side (LHS), $\tau - g^*$, is the primary surplus expressed as a share of GDP.[8] The primary surplus expressed as a share of GDP is linked negatively to the ratio of public expenditure, g^*. As pointed by MV [p. 949, Figure 1(b)], the negative and monotonic relationship between primary surplus and government spending ensures the uniqueness of the steady state.

5. TRANSITION PATHS

This section focuses on the properties of transition paths to a unique steady state and compares them with those of FIO and MV.

5.1. Stability of the Steady State

To examine the local stability of the unique steady state, we define

$$\bar{b}_2 \equiv \frac{(1-\eta)\tau}{\sigma A\left(1-\tau-\frac{\gamma^*}{A}\right)} = \frac{(1-\eta)\bar{b}_1}{\sigma}(>0). \tag{17}$$

Appendix A proves the next proposition.

PROPOSITION 2. *Suppose that* (13) *and* (14) *hold.*

1. *If $\sigma + \eta < 1$, the steady state is a saddle point. The transition path is determinate.*
2. *If $\sigma + \eta > 1$, a unique $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$ exists.*
   (a) *If $\bar{b} < \bar{b}_3$, the steady state is a saddle point. The transition path is determinate.*
   (b) *If $\bar{b} > \bar{b}_3$, the steady state is a sink. The transition path is indeterminate.*

Proposition 2 shows that whether indeterminacy of the transition path arises depends on the following factors: household preferences and the target debt ratio, $\bar{b}$. When $\sigma + \eta > 1$, if the target debt ratio, $\bar{b}$, is large enough, indeterminacy of the transition path occurs. In contrast, if $\bar{b}$ is small enough, indeterminacy of the transition path does not occur. This result has important policy implications not considered in FIO and MV.

To understand the mechanisms behind Proposition 2, we rewrite the Euler equation (6) and the government budget constraint (8) as

$$\frac{\sigma + (1-\eta)\theta x_t^\eta}{1+\theta x_t^\eta}\frac{\dot{C}_t}{C_t} = (1-\tau)r_t - \rho + \frac{(1-\sigma-\eta)\theta x_t^\eta}{1+\theta x_t^\eta}\frac{\dot{G}_t}{G_t}, \tag{18}$$

$$G_t = -\phi(b_t - \bar{b})Q_t + \dot{Q}_t b_t - (1-\tau)r_t B_t + \tau Q_t, \tag{19}$$

where $r_t = r = A$. Here, we use the definition of $x_t \equiv G_t/C_t$ and $g_t \equiv G_t/Q_t$ to derive (18). As seen in (12), utility-generating government spending does not affect household savings in the long run but does affect them in the transition through $x_t \equiv G_t/C_t$ and $\dot{G}_t/G_t$. The latter is the crucial source of indeterminacy in our model.

Suppose that the economy is initially in the steady state and households expect an increase in the growth rate of government spending, $\dot{G}/G$. Assume $\sigma + \eta > 1$. The Euler equation (18) shows that an increase in $\dot{G}/G$ makes current consumption, C_t, more attractive and then households reduce savings for future consumption. Capital accumulation is depressed and GDP growth slows down, which leads to the following three effects on $\dot{G}_t/G_t$.

The first effect is represented in the first and second terms of the right-hand side (RHS) of the government budget constraint (19). Because the economy is initially in the steady state and $b_t = \bar{b}$ holds over time, these terms can be written as $-\phi(b_t - \bar{b})Q_t + \dot{Q}_t b_t = \bar{b}\dot{Q}_t = \dot{B}_t$.[9] The second equality of this equation shows that when GDP growth decreases, the government must reduce its bond issuance today to follow the debt policy rule (1). Given B_t and Q_t, the government must reduce current spending, G_t, to satisfy its budget constraint (19), which positively affects $\dot{G}_t/G_t$. We call this positive effect the *current adjustment effect*.

The second effect is represented by the term, $-(1-\tau)r_t B_t$, in the government budget constraint (19). Because the government must reduce its bond issuance today, $\dot{B}_t$, as just discussed, future outstanding government debt, B_{t+dt}, is reduced. Consequently, the government's future interest payments, $-(1-\tau)r_t B_{t+dt}$, also decrease. This leads to a positive effect on future government spending G_{t+dt}, which positively affects $\dot{G}_t/G_t$. We call this positive effect the *net interest payment effect*.

The third effect is represented by the term, τQ_t, in the government budget constraint (19). Depressed capital accumulation reduces the future tax revenue, τQ_{t+dt}, which has a negative effect on future government spending, G_{t+dt}. Therefore, $\dot{G}_t/G_t$ is negatively affected. This negative effect is called the *tax base effect*.

The relation $\bar{b}\dot{Q}_t = \dot{B}_t$ shows that reductions in the government's current bond issuance are proportional to $\bar{b}$. This suggests that when $\bar{b}$ is large (small), the first and second positive effects tend to be strong (weak) relative to the third. Then, the increase in the growth of government spending, $\dot{G}_t/G_t$, does (not) become self-fulfilling and indeterminacy (determinacy) of the transition path occurs when $\bar{b}$ is sufficiently large (small).

Next, we turn to the case of $\sigma + \eta < 1$. Here, the expectation of an increase in the growth of government spending, $\dot{G}_t/G_t$, makes future consumption more attractive. As a result, households increase savings, which accelerates capital accumulation. In this case, the abovementioned three effects work in the opposite directions to those described earlier. Since a rise in current bond issuance crowds out some capital accumulation, the third positive effect is weak relative to the other two negative ones when $\sigma + \eta < 1$. Then, household expectations cannot be self-fulfilling.

Comparison of our results with those of FIO and MV. FIO and MV focus on multiplicity or the uniqueness of the steady state. Here, we attempt to provide some interpretations of their results regarding indeterminacy using the Euler equation.

In both FIO and MV, where there is productive government spending, the Euler equation is given by $\dot{C}_t/C_t = (1-\tau)r_t - \rho$. To simplify the interpretation, we approximate this Euler equation by $(C_{t+dt}-C_t)/(C_t dt) = [(1-\tau)r_{t+dt}-\rho]/(1+\rho dt)$, where $dt > 0$ is sufficiently small. This approximation can be rearranged as

$$\frac{C_{t+dt}}{C_t} = \frac{1+(1-\tau)r_{t+dt}dt}{1+\rho dt}, \tag{20}$$

where $r_{t+dt} = aA(G_{t+dt}/K_{t+dt})^{1-a}$ [$a \in (0,1)$] because government spending is productive ($Q_t = AK_t{}^a G_t{}^{1-a}$) and the interest rate is equal to the marginal product of capital.

Suppose that households in FIO and MV expect an increase in r_{t+dt}. Then, they increase savings because future consumption becomes more attractive. Capital accumulation is stimulated, which negatively affects $r_{t+dt} = aA(G_{t+dt}/K_{t+dt})^{1-a}$. If future government spending, G_{t+dt}, increases more than capital, K_{t+dt}, the expectation of households is satisfied. In FIO's setting, such increases in G_{t+dt} are possible under some conditions, and hence, indeterminacy of the transition path might occur.

In MV's setting, such increases in G_{t+dt} seem to be impossible under any conditions, as suggested by the following observation. With productive government spending ($Q_t = AK_t{}^a G_t{}^{1-a}$), MV defines the target debt ratio as $b_t = B_t/Q_t = B_t/(AK_t{}^a G_t^{1-a})$. Faced with increases in K_{t+dt}, the government reduces G_{t+dt} and increases B_{t+dt} to satisfy the debt policy rule (1). G_{t+dt} cannot increase more than K_{t+dt}. Hence, indeterminacy of the transition path never occurs in MV.

We now compare our results with those of FIO and MV. In the models with *productive* government spending, the government spending-to-capital ratio, G_t/K_t, affects the dynamics of C_t through *the interest rate*, $r_t = aA(G_t/K_t)^{1-a}$. In FIO and MV, this channel affects the uniqueness and indeterminacy of the transition path. In our model, this channel is absent because the interest rate is constant ($r_t = A$) and is therefore independent of G_t and K_t. Instead, with *utility-generating* government spending, *the dynamics of* G_t, $\dot{G}_t$ (not G_t/K_t), directly affect those of C_t [see (18)], which generates (in)determinacy in the transition path, as discussed after Proposition 2. Therefore, the mechanisms behind the (in)determinacy of the transition path in our model are different from those of FIO and MV.

Furthermore, the specification of the target debt ratio does not seem to affect our results. To confirm this, let us redefine the debt target ratio as $b_t = B_t/(K_t{}^\xi G_t^{1-\xi})$, where $\xi \in (0,1)$. Although this specification of b_t conveys no practical meaning in our setting, it allows for a comparison between MV's results and ours. We assume that $\sigma + \eta > 1$ in our setting. Suppose that households expect an increase in $\dot{G}_t/G_t$. As discussed after Proposition 2, capital accumulation is depressed.[10] Faced with a decrease in K_{t+dt}, the government raises G_{t+dt} and decreases B_{t+dt} to satisfy the debt policy rule (1) under $b_t = B_t/(K_t{}^\xi G_t^{1-\xi})$. In addition, decreases in B_{t+dt} cause the net payment and current adjustment effects (see the discussion after Proposition 2). Because of these positive effects on $\dot{G}_t/G_t$, the expectation of households could be self-filling. Hence, even under $b_t = B_t/(K_t{}^{\xi}G_t^{1-\xi})$, indeterminacy in the transition path may arise in our model.

5.2. The Role of Income Tax

The objective in this subsection is to examine the effects of the income tax rate on the (in)determinacy of the transition path. Focusing on balanced-budget rules, Fernández et al. (2004) and Guo and Harrison (2008) also examine the effects of income tax rate on the indeterminacy condition in a model with utility-generating government spending. Fernández et al. (2004) show that as income tax rate increases, indeterminacy is more likely to occur in the transition path, while Guo and Harrison (2008) show that it is independent of income tax. In contrast, we show that an increase in income tax rate reduces the likelihood of indeterminacy in the transition path.

We assume $\sigma + \eta > 1$ because indeterminacy of the transition path occurs only in this case. Appendix B proves the next proposition.

PROPOSITION 3. *Suppose that* (13) *and* (14) *are satisfied. When* $\sigma + \eta > 1$ *holds, the debt target ratio threshold,* $\bar{b}_3$*, increases with the income tax rate* τ.

If the income tax rate is larger, indeterminacy is less likely to occur. Such a policy implication is not pointed out in FIO and MV. The intuition behind Proposition 3 is as follows: When τ is large, the net interest payment effect becomes smaller while the tax base effect becomes larger. Then, the likelihood of indeterminacy of the transition path is reduced.

6. NUMERICAL EXAMPLES

Here, we provide numerical examples to establish whether indeterminacy could occur in some countries. We find the debt adjustment rule given by (1) in the Maastricht Treaty, the SGP in the EU, and the CFS in the United Kingdom (see footnote 2). Therefore, we discuss mainly European countries.

To make the model more realistic and to provide interesting numerical examples, we modify it so that the production of the final good requires labor input and we consider capital depreciation. The capital depreciation rate is $\delta > 0$. As far as possible, we retain the notation used and the assumptions in the original benchmark model. Following Romer (1986), we specify the production function of firm i as $Q_{i,t} = A(K_{i,t})^{\alpha}(K_t L_{i,t})^{1-\alpha}$, where $\alpha \in (0, 1)$ and $L_{i,t}$ are the labor inputs and $K_{i,t}$ is the capital input of firm i. The aggregate capital stock, $K_t = \sum_i K_{i,t}$, yields external effects. The budget constraints of the household and government are given by $\dot{W}_t = (1-\tau)r_t(W_t + w_t) - C_t$ and $r_t B_t + G_t = \dot{B}_t + \tau(r_t W_t + w_t)$, respectively, where w_t is the wage rate. We can prove the following results in this modified model similarly to the benchmark (see Appendix C).

TABLE 1. Values of $\bar{b}_3$ with $(\bar{b}_2, \bar{b}_1)$

	(1) $\sigma = 2$			
	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.3$	$\tau = 0.4$
$\eta = -1$	–	–	–	–
$\eta = -0.5$	0.6695	1.8052	3.5694	6.3362
	(0.4917, 1.2325)	(1.2741, 2.8712)	(2.5468, 5.1564)	(4.6672, 8.5649)
$\eta = 0$	0.7052	1.7997	3.4857	6.1458
	(0.2059, 1.2325)	(0.5114, 2.8712)	(0.9991, 5.1564)	(1.8552, 8.5649)
$\eta = 0.5$	1.0831	2.4593	4.4123	7.3816
	(0.0736, 1.2325)	(0.1733, 2.8712)	(0.3156, 5.1564)	(0.5348, 8.5649)

	(2) $\sigma = 4$			
	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0.3$	$\tau = 0.4$
$\eta = -1$	0.2000	0.6010	1.3205	2.5365
	(0.1516, 0.8068)	(0.3713, 1.8557)	(0.7098, 3.2750)	(1.2718, 5.3032)
$\eta = -0.5$	0.2292	0.6763	1.4349	2.6804
	(0.0961, 0.8068)	(0.2300, 1.8557)	(0.4276, 3.2750)	(0.7434, 5.3032)
$\eta = 0$	0.3772	0.9489	1.8246	3.1892
	(0.0553, 0.8068)	(0.1296, 1.8557)	(0.2341, 3.2750)	(0.3919, 5.3032)
$\eta = 0.5$	0.6990	1.5481	2.7083	4.3987
	(0.0243, 0.8068)	(0.0558, 1.8557)	(0.0984, 3.2750)	(0.1590, 5.3032)

1. If the two conditions in (C.3) are satisfied, a unique steady state exists.
2. If $\sigma + \eta < 1$, the transition path is unique, and hence, determinate.
3. If $\sigma + \eta > 1$, a unique $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$ exists. If $\bar{b} < (>)\bar{b}_3$, the transition path is (in)determinate.

The definitions of $\bar{b}_1$, $\bar{b}_2$, and $\bar{b}_3$ are presented in Appendix C.

Assuming the two conditions in (C.3) and $\sigma + \eta > 1$, we calculate the value of $\bar{b}_3$. We set $\rho = 0.05$, $\alpha = 0.3$, and $\delta = 0.05$. These parameter values are conventional. We consider several values of σ, η, and τ (see Table 1). Because the estimation of Ni (1995) suggests that η is close to 0, we include $\eta = 0$. When $\eta = 0$, the estimation of Ni (1995) suggests that θ_1 ranges from 0.64 to 0.75. Then, we set $\theta_1 = 0.7$ and $\theta_2 = 0.3$. The value of A is chosen so that for each value of σ, the long-run growth rate, $\gamma^* = (1/\sigma)[(1-\tau)(\alpha A - \delta) - \rho]$, becomes equal to 0.02 when $\tau = 0.131$.[11] Table 1 presents the values of $\bar{b}_3$ as well as $(\bar{b}_2, \bar{b}_1)$ intervals. Naturally, we have $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$ in all cases we consider. The value of $\bar{b}_3$ increases with τ. Furthermore, we draw phase diagrams assuming $\sigma = 4$, $\rho = 0.05$, $\alpha = 0.3$, $\delta = 0.05$ $\eta = 0$, $\phi = 0.05$, and $\tau = 0.1$ (see Figure 1). In this case, we have $\bar{b}_3 = 0.3772$. If $\bar{b} = 0.3 < \bar{b}_3$, the transition path is determined uniquely, as shown in the upper panel of Figure 1. The lower panel of Figure 1

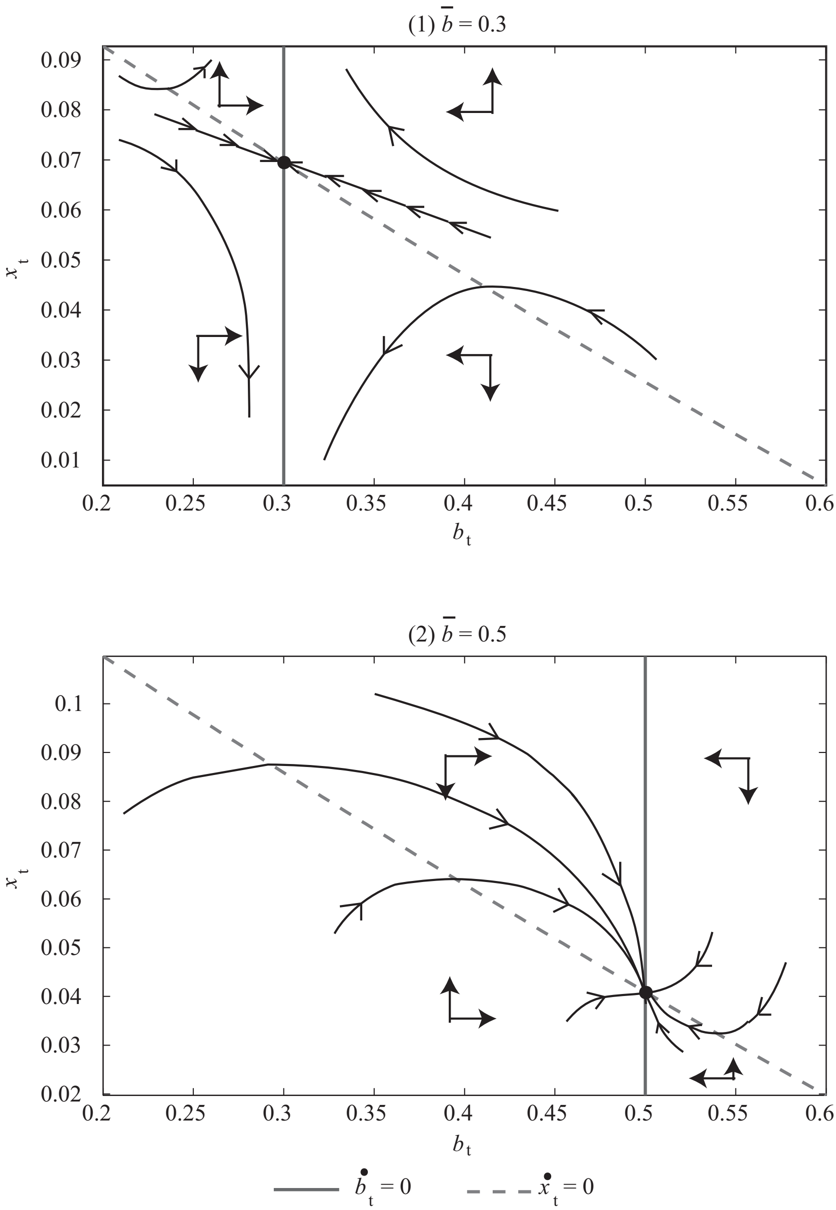

FIGURE 1. Phase diagrams.

https://doi.org/10.1017/S1365100517000724 Published online by Cambridge University Press

assumes $\bar{b} = 0.5 > \bar{b}_3$. The steady state is a sink. Hence, indeterminacy of the transition path arises.

Although our model is rather simple, we seem to obtain the following economic interpretation from our numerical examples. If countries in the EU obey the Maastricht Treaty or the SGP ($\bar{b}$ is less than 60%), indeterminacy of the transition path seems to be avoided unless the tax rate is very small.

However, the debt-to-GDP ratios in many member countries exceed the target levels in the Maastricht Treaty or the SGP. According to the Eurostat database, the debt-to-GDP ratio in Greece steadily increased from 103.4% to 174.9% in 2006–2013. Many other countries in the EU show similar situations in 2006–2013; debt-GDP ratios increased from 90.8% to 104.5% in Belgium, 23.8% to 123.3% in Ireland, 38.9% to 92.1% in Spain, 64.2% to 92.2% in France, 102.5% to 127.9% in Italy, 58.9% to 102.2% in Cyprus, 69.2% to 128% in Portugal, and 42.5% to 87.2% in the United Kingdom.

In addition, the income tax rate in these countries seems to be relatively small. We obtain the data of direct taxes (Personal income + Corporate income + Other) as a percentage of GDP from the 2014 edition of *Taxation trends in the European Union* published by the EU. The direct income tax rate in 2012 was 10.2% in Greece, 17.4% in Belgium, 13.1% in Ireland, 10.6% in Spain, 12.4% in France, 15.2% in Italy, 11.1% in Cyprus, 9.4% in Portugal, and 15.1% in the United Kingdom. These countries might face indeterminacy of the transition path unless (i) the income tax rate is increased or (ii) the target debt ratio is reduced significantly.

In contrast, Denmark might be on a determinacy path because its debt-to-GDP ratio is relatively low (45.6 % in 2012) and direct income tax is relatively high (30.6% in 2012).

If $\sigma = 4$ and $\eta = 0$ [here, $\eta = 0$ is based on Ni (1995)], Table 1 suggests that, to avoid indeterminacy in the transitional path, (i) if the income tax rate is around 10% (EU 28), the target debt ratio is reduced to around 30%, and (ii) if the target debt ratio remains at 60% as in the SGP, the income tax rate is increased to around 20%.

7. CONCLUSION

In this study, we examine the effect of financing utility-generating government services through public debt in an AK endogenous growth model. In this model, the government obeys the debt policy rule examined by FIO and MV, in which the government has a target debt ratio level.

In an endogenous growth model of productive government service, FIO and MV provide the following controversies on the debt policy rule: The rule generates indeterminacy of the transition path if the target debt ratio is defined in terms of the debt-to-capital ratio (FIO), whereas it induces the transition path to be determinate if the target debt ratio is defined in terms of the debt-to-GDP ratio (MV).

Our model shows that the transition path can be either determinate or indeterminate under the different sources and mechanisms from FIO and MV. In contrast to

FIO and MV, the level of the target debt ratio, income tax rate, and the preference parameters of household utility can play important roles in determining whether indeterminacy occurs. More precisely, we find that under some conditions for the preference parameters, a threshold of the target debt ratio exists beyond which the steady state exhibits indeterminacy. In addition, the threshold of the target debt ratio increases with the income tax rate.

The abovementioned results provide the following policy implications: In order to avoid indeterminacy of the transition path, the governments in the EU may have to reduce the target debt ratio or increase the income tax rate. Specifically, our results suggest that if the income tax rate is around 10% (EU 28), the target debt ratio is reduced to around 30%, while if the target debt ratio remains at 60% as in the SGP, the income tax rate is increased to around 20%.

The Maastricht Treaty and the SGP have fiscal rules on the deficit-to-GDP (deficit/GDP) rules, in addition to the debt-to-GDP (debt/GDP) rules. They state that EU member states must keep their government deficit-to-GDP ratio below 3%. Minea and Villieu (2012) introduce a constant deficit/GDP rule into a model with productive government spending that is similar to that of FIO and MV. It is shown that their constant deficit/GDP rule generates multiple steady states. Interestingly, the steady state that the economy reaches in the long run is indeterminate. The results of Minea and Villieu (2012) suggest that in models with productive government spending, constant deficit/GDP rules generate results that are quite different from those generated by debt/GDP rules. Therefore, it is of interest to examine the effects of constant deficit/GDP rules in a model of utility-generating government spending in future studies.

NOTES

1. In addition, Maebayashi et al. (2015) and Morimoto et al. (2017) examine this debt policy rule in models in which the *stock* of public capital positively affects labor productivity.
2. The Maastricht Treaty requires that EU member countries keep their government debt-to-GDP ratios below 60%. In the debt reduction benchmark (rule) introduced by the reform of the SGP in December 2011, the so-called Six-Pack, EU member countries with current debt-to-GDP ratios more than 60%, must reduce their distances to 60% by an average rate of 1/20th per year. The CFS in the United Kingdom requires the government to keep its debt–to-GDP ratio less than 30%.
3. We ignore the depreciation of capital for simplicity. However, Section 6 shows that if we include capital depreciation, our main results are not affected.
4. When $\sigma = 1$, the instantaneous utility function takes a logarithmic form, $\ln u_t$.
5. Most empirical estimates find that the IES, $1/\sigma$, is less than unity.
6. In Section 6, we modify the model so that production requires labor input and capital depreciates. The modified model needs conditions similar to (13) and (14). We observe that those conditions hold under reasonable parameters.
7. Although $\dot{x}_t = 0$ in (11) also yields $x^* = 0$, we pay no attention to such a trivial steady state.
8. When $\sigma > 1$, the no-Ponzi game condition for the government is always satisfied in the long run [i.e., the growth of public debt $\gamma^*\bar{b}$ is inferior to the net real interest payment, $(1-\tau)A\bar{b}$].
9. The second equality holds because $\dot{b}_t/b_t = \dot{B}_t/B_t - \dot{Q}_t/Q_t = 0$ holds when $b_t = \bar{b}$.
10. In MV, capital accumulation is stimulated.

11. According to the 2014 edition of *Taxation trends in the European Union* published by the EU, the weighted average of direct taxes (Personal income + Corporate income + Other) as the percentage of GDP from 2002 to 2012 in the EU 28 is around 0.131. Estimates of IES, $1/\sigma$ are variable throughout the literature. It is conventional to set the value of A so that the steady-state per capita growth rate is 0.02 in the literature of endogenous growth models, such as Barro and Sala-i-Martin (2004), Chapter 5. In addition, the average real GDP growth rate in the EU 28 from 2003 to 2014 was around 2% if we exclude the years of financial crisis between 2008 and 2009.

12. $\pi'(\bar{b}) = (1-\eta)A^{\eta-1}\theta(1+A\bar{b}_1)(1+A\bar{b})^{-\eta}(\bar{b}_1-\bar{b})^{\eta-2} > 0$ holds for $\bar{b} < \bar{b}_1$ because $-\infty \leq \eta \leq 1$.

13. $\mu'(\bar{b}) = -(1+\bar{b}_2)/[A(\bar{b}-\bar{b}_2)^2] < 0$ for $\bar{b}_2 < \bar{b}$.

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APPENDIX A: PROOF OF PROPOSITION 2

To examine the local stability of the unique steady state, we linearly approximate the dynamic system around the steady state:

$$\begin{pmatrix} \dot{b}_t \\ \dot{x}_t \end{pmatrix} = \begin{pmatrix} -\phi & 0 \\ \omega_1 x^* & \omega_2 x^* \end{pmatrix} \begin{pmatrix} b_t - \bar{b} \\ x_t - x^* \end{pmatrix},$$

https://doi.org/10.1017/S1365100517000724 Published online by Cambridge University Press

where

$$\omega_1 \equiv f(\bar{b})^{-1}\Gamma_{b_t}(x_t, b_t)|_{(x_t,b_t)=(x^*,\bar{b})}, \quad \text{and} \quad \omega_2 \equiv -f(\bar{b})^{-1}\frac{A(1+A\bar{b})\left(1-\tau-\frac{\gamma^*}{A}\right)^2}{\tau}.$$

Here, $\Gamma_{b_t}(x_t, b_t)$ is the derivative of $\Gamma(x_t, b_t)$ with respect to b_t and $f(\bar{b})$ is defined as

$$f(\bar{b}) = \frac{A\left(1-\tau-\frac{\gamma^*}{A}\right)}{\tau\,[1+\theta(x^*)^\eta]}\left[(\bar{b}-\bar{b}_2)\theta(x^*)^\eta - (\bar{b}_1-\bar{b})\right],$$

where $\bar{b}_2$ is defined by (17). One of the two eigenvalues of the Jacobian matrix, $-\phi < 0$, is inevitably negative. The other, $\omega_2 x^*$, has an opposite sign to that of $f(\bar{b})$. Note that b_t is a state variable, whereas x_t is a jump variable. Then, if $f(\bar{b}) > 0$ holds, the steady state is a sink and indeterminacy of the transition path to the steady state occurs. In contrast, if $f(\bar{b}) < 0$, the steady state equilibrium is saddle stable and the transition path is determined uniquely. Because (13) is assumed, the transition path to the steady state is indeterminate (determinate) if and only if

$$(\bar{b}-\bar{b}_2)\theta(x^*)^\eta > (<)(\bar{b}_1-\bar{b}). \qquad \textbf{(A.1)}$$

When $\sigma+\eta < 1$, $\bar{b}_1 < \bar{b}_2$ holds by the definition of $\bar{b}_2$. Note that $\bar{b} < \bar{b}_1$ is assumed to ensure $g^* > 0$. Then, when $\sigma+\eta < 1$, the LHS of (A.1) is negative and the RHS is positive. The transition path to the steady sate is locally determinate.

When $\sigma+\eta > 1$, we have $\bar{b}_1 > \bar{b}_2$. If $\bar{b} \leq \bar{b}_2$, the LHS of (A.1) is negative and the RHS is positive. The transition path to the steady state is determinate. Next, we consider the case in which $\bar{b}_2 < \bar{b} < \bar{b}_1$ holds. Using (15), we can rewrite (A.1) as

$$\theta\left[\frac{1+A\bar{b}}{A(\bar{b}_1-\bar{b})}\right]^{1-\eta} > (<)\frac{1+A\bar{b}}{A(\bar{b}-\bar{b}_2)}. \qquad \textbf{(A.2)}$$

Let us denote the LHS and RHS of (A.2) as $\pi(\bar{b})$ and $\mu(\bar{b})$, respectively. As $\bar{b}$ increases from $\bar{b}_2$ to $\bar{b}_1$, $\pi(\bar{b})$ monotonically increases from $\pi(\bar{b}_2)(< +\infty)$ to $+\infty$ (see Figure B.1).[12] In contrast, as $\bar{b}$ increases from $\bar{b}_2$ to $\bar{b}_1$, $\mu(\bar{b})$ monotonically decreases from $+\infty$ to $\mu(\bar{b}_1)(< +\infty)$.[13] Therefore, a unique value $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$ exists that satisfies $\pi(\bar{b}_3) = \mu(\bar{b}_3)$. For $\bar{b} \in (\bar{b}_2, \bar{b}_3)$, $\pi(\bar{b}) < \mu(\bar{b})$ holds. Then, the transition path is determinate. When $\bar{b} \in (\bar{b}_3, \bar{b}_1)$, $\pi(\bar{b}) > \mu(\bar{b})$ holds, and then, indeterminacy of the transition path occurs. ■

APPENDIX B: PROOF OF PROPOSITION 3

From the definitions of $\pi(\bar{b}) \equiv \theta[\frac{1+A\bar{b}}{A(\bar{b}_1-\bar{b})}]^{1-\eta}$ and $\mu(\bar{b}) \equiv \frac{1+A\bar{b}}{A(\bar{b}-\bar{b}_2)}$, we have

$$\pi(\bar{b}_3) = \mu(\bar{b}_3) \Leftrightarrow \theta\left[\frac{1+A\bar{b}_3}{A(\bar{b}_1-\bar{b}_3)}\right]^{1-\eta} - \frac{1+A\bar{b}_3}{A(\bar{b}_3-\bar{b}_2)} = 0,$$

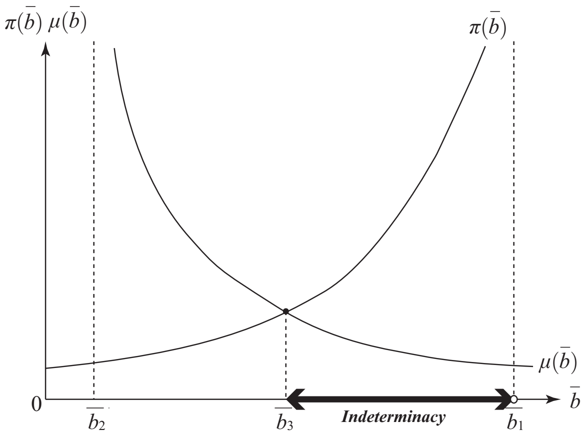

FIGURE B.1. Indeterminacy or determinacy when $\sigma + \eta > 1$.

where both $\bar{b}_1$ and $\bar{b}_2$ are functions of τ because we have defined $\bar{b}_1 \equiv \frac{\tau}{A(1-\tau-\gamma^*/A)} = \frac{\sigma\tau}{A[(\sigma-1)(1-\tau)+\rho/A]}$ and $\bar{b}_2 \equiv \frac{(1-\eta)\bar{b}_1}{\sigma}$. The total differential of $\pi(\bar{b}_3) = \mu(\bar{b}_3)$ can be written as

$$[\pi'(\bar{b}_3) - \mu'(\bar{b}_3)]d\bar{b}_3 + \left[\frac{d\pi(\bar{b}_3)}{d\bar{b}_1} - \frac{d\mu(\bar{b}_3)}{d\bar{b}_2}\frac{d\bar{b}_2}{d\bar{b}_1}\right]\frac{d\bar{b}_1}{d\tau}d\tau = 0. \qquad \textbf{(B.1)}$$

Substituting $\frac{d\pi(\bar{b}_3)}{d\bar{b}_1} = -\frac{(1-\eta)\pi(\bar{b}_3)}{\bar{b}_1-\bar{b}_3}$, $\frac{d\mu(\bar{b}_3)}{d\bar{b}_2} = \frac{\mu(\bar{b}_3)}{\bar{b}_3-\bar{b}_2} = \frac{\pi(\bar{b}_3)}{\bar{b}_3-\bar{b}_2}$ from $\pi(\bar{b}_3) = \mu(\bar{b}_3)$, and $\frac{d\bar{b}_2}{d\bar{b}_1} = \frac{1-\eta}{\sigma}$ into (B.1), we obtain

$$[\pi'(\bar{b}_3) - \mu'(\bar{b}_3)]d\bar{b}_3 = \frac{1-\eta}{\sigma}\pi(\bar{b}_3)\frac{d\bar{b}_1}{d\tau}\left(\frac{1}{\bar{b}_3-\bar{b}_2} + \frac{\sigma}{\bar{b}_1-\bar{b}_3}\right)d\tau. \qquad \textbf{(B.2)}$$

As is clear from Figure B.1, we have $\pi'(\bar{b}_3) > \mu'(\bar{b}_3)$ and $\pi(\bar{b}_3) > 0$. The definition of $\bar{b}_1$ implies $d\bar{b}_1/d\tau > 0$ when $\sigma > 1$. Since $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$, we have $d\bar{b}_3/d\tau > 0$. ■

APPENDIX C: AK MODEL WITH LABOR INPUT AND CAPITAL DEPRECIATION

The profit maximization for firms yields wage rate, $w_t = (1-\alpha)AK_t$, and interest rate, $r = \alpha A - \delta$, respectively, where $\delta > 0$ is the depreciation rate. The aggregate production function is written as $Q_t = AK_t$ because $\sum_i L_{i,t} = 1$. The resource constraint is given by $\dot{K}_t = Q_t - C_t - G_t - \delta K_t$. The first-order condition for the optimization of households remains unchanged. Equation (9) is modified as

$$g_t = \frac{\tau\left(1-\frac{\delta}{A}\right) + [A - \delta - (1-\tau)(\alpha A - \delta)]b_t - \phi(b_t - \bar{b})}{(1+Ab_t)x_t + Ab_t}x_t. \qquad \textbf{(C.1)}$$

https://doi.org/10.1017/S1365100517000724 Published online by Cambridge University Press

Equation (11) is modified as

$$\dot{x}_t = \left[\frac{Ab_t}{(1+Ab_t)x_t + Ab_t} - \frac{\sigma + (1-\eta)\theta x_t^\eta}{\sigma(1+\theta x_t^\eta)}\right]^{-1} \tilde{\Gamma}(x_t, b_t)x_t, \qquad \textbf{(C.2)}$$

where

$$\tilde{\Gamma}(x_t, b_t) \equiv \frac{1}{\sigma}[(1-\tau)(\alpha A - \delta) - \rho] + \tilde{\Psi}(x_t, b_t) - A\left(1 - \frac{\delta}{A} - \frac{1+x_t}{x_t}g_t\right),$$

$$\tilde{\Psi} \equiv \left\{\frac{A[1-\frac{\delta}{A} - (1-\tau)(\alpha - \frac{\delta}{A})] - \phi}{(1-\frac{\delta}{A})\tau + A[1-\frac{\delta}{A} - (1-\tau)(\alpha - \frac{\delta}{A})]b_t - \phi(b_t - \bar{b})} - \frac{A(1+x_t)}{(1+Ab_t)x_t + Ab_t}\right\} \times \phi(b_t - \bar{b}).$$

In the steady state, the long-run growth rate is given by $\gamma^* = (1/\sigma)[(1-\tau)(\alpha A - \delta) - \rho]$, and we have $x^* = A(\bar{b}_1 - \bar{b})/(\Omega + A\bar{b})$, where $\bar{b}_1 = (1-\delta/A)\tau/[(1-\tau)(\alpha A - \delta) - \gamma^*]$ and $\Omega = [(1-\tau)(1-\delta/A) - \gamma^*/A]/[(1-\tau)(\alpha - \delta/A) - \gamma^*/A]$. The conditions (13) and (14) are modified as

$$(1-\tau)(\alpha - \delta/A) - \gamma^*/A > 0, \quad \text{and } \bar{b} < \bar{b}_1. \qquad \textbf{(C.3)}$$

These conditions ensure $x^* > 0$, $\bar{b}_1 > 0$, and $\Omega > 1$.

The transition path to the steady state is locally indeterminate (determinate) if and only if

$$\left[\kappa(\bar{b}) - \zeta(\bar{b})\right]\theta(x^*)^\eta > (<) \frac{\sigma(\bar{b}_1 - \bar{b})}{\Omega + A\bar{b}}, \qquad \textbf{(C.4)}$$

where $\kappa(\bar{b}) \equiv (\frac{\sigma+\eta-1}{1+A\bar{b}} + \frac{1-\eta}{\Omega+A\bar{b}})\bar{b} > 0$ and $\zeta(\bar{b}) \equiv \frac{(1-\eta)\bar{b}_1}{\Omega+A\bar{b}}(> 0)$. When $\sigma + \eta < 1$, we have $\kappa(\bar{b}) < \frac{1-\eta}{\Omega+A\bar{b}}\bar{b} < \frac{1-\eta}{\Omega+A\bar{b}}\bar{b}_1 = \zeta(\bar{b})$. Since the LHS is negative and the RHS is positive, indeterminacy never arises.

Next, we consider the case in which $\sigma + \eta > 1$. In this case, $\kappa(\bar{b}) \geq 0$ holds and it is monotonically increasing with $\bar{b}$. In contrast, $\zeta(\bar{b})(> 0)$ is monotonically decreasing with $\bar{b}$. We have $\kappa(0) = 0 < \zeta(0)$ and $\kappa(\bar{b}_1) > \zeta(\bar{b}_1)$. As in the benchmark model, a unique value $\bar{b}_2$ exists such that if $\bar{b} < (>)\bar{b}_2$, the LHS is negative (positive) and the RHS is positive. Thus, if $\bar{b} < \bar{b}_2$, indeterminacy does not occur.

Next, we consider the case in which $\bar{b}_2 < \bar{b} < \bar{b}_1$ holds. Using $x^* = A(\bar{b}_1 - \bar{b})/(\Omega + A\bar{b})$, we can rewrite (C.4) as

$$\theta\left[\frac{\Omega + A\bar{b}}{A(\bar{b}_1 - \bar{b})}\right]^{1-\eta} > (<) \frac{\sigma}{A}\left[\kappa(\bar{b}) - \zeta(\bar{b})\right]^{-1}. \qquad \textbf{(C.5)}$$

Let us define the LHS and RHS of (C.5) as $\tilde{\pi}(\bar{b})$ and $\tilde{\mu}(\bar{b})$, respectively. Because of $\eta < 1$, $\tilde{\pi}(\bar{b})$ monotonically increases with $\bar{b}$. As $\bar{b}$ increases from $\bar{b}_2$ to $\bar{b}_1$, $\tilde{\pi}(\bar{b})$ increases from $\tilde{\pi}(\bar{b}_2)(< +\infty)$ to $+\infty$. Because $\kappa(\bar{b})$ monotonically increases with $\bar{b}$ and $\zeta(\bar{b})(> 0)$ monotonically decreases with $\bar{b}$, $\tilde{\mu}(\bar{b})$ monotonically decreases with $\bar{b}$. As $\bar{b}$ increases from $\bar{b}_2$ to $\bar{b}_1$, $\tilde{\mu}(\bar{b})$ decreases from $+\infty$ to $\tilde{\mu}(\bar{b}_1)(< +\infty)$. Again, a unique value $\bar{b}_3 \in (\bar{b}_2, \bar{b}_1)$ exists, which satisfies $\tilde{\pi}(\bar{b}_3) = \tilde{\mu}(\bar{b}_3)$. For $\bar{b} \in (\bar{b}_2, \bar{b}_3)$, $\tilde{\pi}(\bar{b}) < \tilde{\mu}(\bar{b})$ holds. Then, the transition path is determinate. When $\bar{b} \in (\bar{b}_3, \bar{b}_1)$, $\tilde{\pi}(\bar{b}) > \tilde{\mu}(\bar{b})$ holds, and then, indeterminacy of the transition path occurs.