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ON MAXIMA OF STATIONARY FIELDS

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Abstract

Let $\{X_{\mathbf{n}}\colon \mathbf{n} \in \mathbb{Z}^d\}$ be a weakly dependent stationary random field with maxima $M_A := \sup\{X_{\mathbf{i}}\colon \mathbf{i} \in A\}$ for finite $A \subset \mathbb{Z}^d$ and $M_{\mathbf{n}} := \sup\{X_{\mathbf{i}}\colon \mathbf{1} \le \mathbf{i} \le \mathbf{n}\}$ for $\mathbf{n} \in \mathbb{N}^d$. In a general setting we prove that $\mathbb{P}(M_{(N_1(n),N_2(n),\ldots,N_d(n))} \le v_n) = \exp(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n, M_{A_n} \le v_n)) + \mathrm{o}(1)$ for some increasing sequence of sets A_n of size $\mathrm{o}(n^d)$, where $(N_1(n), N_2(n), \ldots, N_d(n)) \to (\infty, \infty, \ldots, \infty)$ and $N_1(n)N_2(n)\cdots N_d(n) \sim n^d$. The sets A_n are determined by a translation-invariant total order $\preccurlyeq$ on $\mathbb{Z}^d$. For a class of fields satisfying a local mixing condition, including m-dependent ones, the main theorem holds with a constant finite A replacing A_n. The above results lead to new formulas for the extremal index for random fields. The new method for calculating limiting probabilities for maxima is compared with some known results and applied to the moving maximum field.

Keywords: stationary random fields; extremes; limit theorems; extremal index; m-dependence; moving maxima

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1. Introduction

Let us consider a d-dimensional stationary random field $\{X_{\mathbf{n}}\colon \mathbf{n} \in \mathbb{Z}^d\}$ with its partial maxima

$$M_A := \sup\{X_{\mathbf{i}}\colon \mathbf{i} \in A\}$$

defined for finite $A \subset \mathbb{Z}^d$. We also put $M_{\mathbf{j},\mathbf{n}} := \sup\{X_{\mathbf{i}}\colon \mathbf{j} \le \mathbf{i} \le \mathbf{n}\}$ and $M_{\mathbf{n}} := M_{\mathbf{1},\mathbf{n}}$ for $\mathbf{j}, \mathbf{n} \in \mathbb{Z}^d$. The goal is to study the asymptotic behaviour of $\mathbb{P}(M_{\mathbf{N}(n)} \le v_n)$ as $n \to \infty$, for $\{v_n\} \subset \mathbb{R}$ and $\mathbf{N}(n) \to \boldsymbol{\infty}$ coordinate-wise.

In the case $d = 1$, when $\{X_n\colon n \in \mathbb{Z}\}$ is a stationary sequence, the well-known result of O'Brien [17, Theorem 2.1] states that under a broad class of circumstances

$$\mathbb{P}(M_n \le v_n) = \exp(-n\mathbb{P}(X_0 > v_n, M_{p(n)} \le v_n)) + \mathrm{o}(1) \tag{1.1}$$

holds for some $p(n) \to \infty$ satisfying $p(n) = \mathrm{o}(n)$. For m-dependent $\{X_n\}$ we can set $p(n) := m$ in formula (1.1), as Newell [16] shows. It follows that the extremal index θ for $\{X_n\}$, defined by Leadbetter [14], equals

$$\theta = \lim_{n\to\infty} \mathbb{P}(M_{p(n)} \le v_n \mid X_0 > v_n), \tag{1.2}$$

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where $p(n) = m$ in the m-dependent case. More generally, we can put $p(n) = m$ in (1.2) whenever condition $D^{(m+1)}(v_n)$, introduced by Chernick, Hsing, and McCormick [5], is satisfied.

We recall that the extremal index $\theta \in [0, 1]$ is interpreted as the reciprocal of the mean number of high threshold exceedances in a cluster. Formula (1.2) for θ may be treated as an answer to the question: *Asymptotically, what is the probability that a given element of a cluster of large values is its last element on the right?*

Looking for formulas analogous to (1.1) and (1.2) for arbitrary $d \in \mathbb{N}_+$, one can try to answer the properly formulated d-dimensional version of the above question. This point is realized in Sections 3 and 4. In Section 3 we prove the main result, Theorem 3.1. We establish that in a general setting the approximation

$$\mathbb{P}(M_{\mathbf{N}(n)} \leq v_n) = \exp\left(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \leq v_n)\right) + \mathrm{o}(1) \tag{1.3}$$

holds with $A(\mathbf{p}(n)) \subset \{\mathbf{j} \in \mathbb{Z}^d : -\mathbf{p}(n) \leq \mathbf{j} \leq \mathbf{p}(n)\}$ defined by (2.6), $\mathbf{N}(n)$ fulfilling (2.1), and $\mathbf{p}(n) \to \boldsymbol{\infty}$ satisfying $\mathbf{p}(n) = \mathrm{o}(\mathbf{N}(n))$ and some other rate of growth conditions. For $d = 1$ we have $A(p(n)) = \{1, 2, \ldots, p(n)\}$ and formula (1.3) simplifies to (1.1). Corollary 3 provides the local mixing condition (3.7) equivalent to the fact that (1.3) holds with $\mathbf{p}(n) := (m, m, \ldots, m)$. Section 4 is devoted to considerations concerning the notion of the extremal index for random fields. Formula (4.2), being a generalization of (1.2), and its simplified version (4.3) for fields fulfilling (3.7) are proposed there. In Section 6 the results from Sections 3 and 4 are used to describe the asymptotics of partial maxima for the moving maximum field.

In Section 5 we focus on m-dependent fields and present a corollary of the main theorem generalizing the aforementioned Newell's formula [16]. We also compare the obtained result with the limit theorem for m-dependent fields proved by Jakubowski and Soja-Kukieła [12, Theorem 2.1].

The present paper provides a d-dimensional generalization of O'Brien's formula (1.1) with a handy and immediate conclusion for m-dependent fields. Another general theorem by Turkman [19, Theorem 1] is not really applicable in the m-dependent case. A recent result obtained independently by Ling [15, Lemma 3.2] is a special case of Theorem 3.1. Other theorems on the topic were given for some subclasses of weakly dependent fields: in the two-dimensional Gaussian setting by French and Davis [9]; for two-dimensional moving maxima and moving averages by Basrak and Tafro [3]; for m-dependent and max-m-approximable fields by Jakubowski and Soja-Kukieła [12]; for regularly varying fields by Wu and Samorodnitsky [20]. The proof of Theorem 3.1 presented in the paper, although achieved independently, is similar to proofs of [9, Lemma 4] and [15, Lemma 3.2].

2. Preliminaries

An element $\mathbf{n} \in \mathbb{Z}^d$ is often denoted by $(n_1, n_2, \ldots, n_d)$ and $\|\mathbf{n}\|$ is its sup norm. We write $\mathbf{i} \leq \mathbf{j}$ and $\mathbf{n} \to \boldsymbol{\infty}$ whenever $i_l \leq j_l$ and $n_l \to \infty$, respectively, for all $l \in \{1, 2, \ldots, d\}$. We put $\mathbf{0} := (0, 0, \ldots, 0)$, $\mathbf{1} := (1, 1, \ldots, 1)$, and $\boldsymbol{\infty} := (\infty, \infty, \ldots, \infty)$.

In our considerations $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ is a d-dimensional stationary random field. We ask for the asymptotics of $\mathbb{P}(M_{\mathbf{N}(n)} \leq v_n)$ as $n \to \infty$, for $\mathbf{N} = \{\mathbf{N}(n) : n \in \mathbb{N}\} \subset \mathbb{N}^d$, such that

$$\mathbf{N}(n) \to \boldsymbol{\infty} \quad \text{and} \quad N^*(n) := N_1(n)N_2(n)\cdots N_d(n) \sim n^d \tag{2.1}$$

and $\{v_n : n \in \mathbb{N}\} \subset \mathbb{R}$.

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We are interested in *weakly dependent* fields. We assume that

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \mathbb{P}(M_{\mathbf{p}(n)} \le v_n)^{k_n^d} + \mathrm{o}(1) \tag{2.2}$$

is satisfied for some $r_n \to \infty$ and all $k_n \to \infty$ such that $k_n = \mathrm{o}(r_n)$, with

$$\mathbf{p}(n) := (\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor). \tag{2.3}$$

Applying the classical fact (see e.g. O'Brien [17])

$$(a_n)^n - \exp(-n(1-a_n)) \to 0 \quad \text{as } n \to \infty \quad \text{for } a_n \in [0, 1], \tag{2.4}$$

we obtain that (2.2) implies

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp(-k_n^d \mathbb{P}(M_{\mathbf{p}(n)} > v_n)) + \mathrm{o}(1). \tag{2.5}$$

Above, $p_l(n) = \mathrm{o}(N_l(n))$ for $l \in \{1, 2, \ldots, d\}$, which we denote by $\mathbf{p}(n) = \mathrm{o}(\mathbf{N}(n))$.

Remark 2.1. For $d = 1$, weak dependence in the sense of (2.2) is ensured by any of the following conditions: Leadbetter's $D(v_n)$, O'Brien's $\mathrm{AIM}(v_n)$, or Jakubowski's $B_1(v_n)$; see [14], [17], and [10]. For $d \in \mathbb{N}_+$ the considered property follows, for example, from condition $B_1^{\mathbf{N}}(v_n)$ introduced by Jakubowski and Soja-Kukieła [13]. In particular, m-dependent fields are weakly dependent; see Section 5. A similar notion of weak dependence was investigated by Ling [15, Lemma 3.1].

Let $\preccurlyeq$ be an arbitrary total order on $\mathbb{Z}^d$ which is *translation-invariant*, that is, $\mathbf{i} \preccurlyeq \mathbf{j}$ implies $\mathbf{i} + \mathbf{k} \preccurlyeq \mathbf{j} + \mathbf{k}$. An example of such an order is the lexicographic order:

$$\mathbf{i} \preccurlyeq \mathbf{j} \quad \text{if and only if } (\mathbf{i} = \mathbf{j} \text{ or } i_l < j_l \text{ for the first } l \text{ where } i_l \text{ and } j_l \text{ differ}).$$

We will write $\mathbf{i} \prec \mathbf{j}$ whenever $\mathbf{i} \preccurlyeq \mathbf{j}$ and $\mathbf{i} \neq \mathbf{j}$. For technical requirements in further sections, we define the set $A(\mathbf{p}) \subset \mathbb{Z}^d$ for each $\mathbf{p} \in \mathbb{N}^d$ as follows:

$$A(\mathbf{p}) := \{\mathbf{j} \in \mathbb{Z}^d : -\mathbf{p} \le \mathbf{j} \le \mathbf{p} \text{ and } \mathbf{0} \prec \mathbf{j}\}. \tag{2.6}$$

3. Main theorem

In the following the main result of the paper is presented. The asymptotic behaviour of $\mathbb{P}(M_{\mathbf{N}(n)} \le v_n)$ as $n \to \infty$, for weakly dependent $\{X_{\mathbf{n}}\}$ and for $\{\mathbf{N}(n)\}$ and $\{v_n\}$ as in Section 2, is described.

Theorem 3.1. *Let $\{X_{\mathbf{n}}\}$ satisfy (2.2) for some $r_n \to \infty$ and all $k_n \to \infty$ such that $k_n = \mathrm{o}(r_n)$. If*

$$\liminf_{n\to\infty} \mathbb{P}(M_{\mathbf{N}(n)} \le v_n) > 0, \tag{3.1}$$

then, for every $\{k_n\}$ as above, we obtain

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n)) + \mathrm{o}(1), \tag{3.2}$$

with $\mathbf{p}(n)$ and $A(\mathbf{p}(n))$ given by (2.3) and (2.6), respectively.

Remark 3.1. If (2.2) holds for some $k_n \to \infty$, then (3.1) is implied by the condition

$$\limsup_{n\to\infty} n^d \mathbb{P}(X_{\mathbf{0}} > v_n) < \infty. \tag{3.3}$$

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This follows from (2.5) and the inequality

$$k_n^d \mathbb{P}(M_{\mathbf{p}(n)} > v_n) \le N^*(n)\mathbb{P}(X_{\mathbf{0}} > v_n) \sim n^d \mathbb{P}(X_{\mathbf{0}} > v_n).$$

The proof of the theorem generalizes the reasoning proposed by O'Brien [17, Theorem 2.1] for sequences. A way of dividing the event $\{M_{\mathbf{p}(n)} > v_n\}$ into $p^*(n) := p_1(n)p_2(n)\cdots p_d(n)$ mutually exclusive events determined by $\preccurlyeq$ (which are similar in some sense) plays a key role in the proof. An analogous technique was used by French and Davis [9, Lemma 4] in the two-dimensional Gaussian case. Recently, Ling [15, Lemma 3.1] expanded their result to some non-Gaussian fields. In both papers the authors restrict themselves to the lexicographic order on $\mathbb{Z}^2$.

Proof of Theorem 3.1. Let the assumptions of the theorem be satisfied. Then (2.5) holds. Dividing the set $\{M_{\mathbf{p}(n)} > v_n\}$ into $p^*(n) = p_1(n)p_2(n)\cdots p_d(n)$ disjoint sets and applying monotonicity and stationarity, we obtain

$$\begin{aligned}
\mathbb{P}(M_{\mathbf{p}(n)} > v_n) &= \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{p}(n)} \mathbb{P}(X_{\mathbf{j}} > v_n, X_{\mathbf{i}} \le v_n \text{ for all } \mathbf{i} \succ \mathbf{j} \text{ such that } \mathbf{1} \le \mathbf{i} \le \mathbf{p}(n)) \\
&\ge \sum_{\mathbf{1}\le\mathbf{j}\le\mathbf{p}(n)} \mathbb{P}(X_{\mathbf{j}} > v_n, X_{\mathbf{i}} \le v_n \ \text{ for all } \ \mathbf{i} \in A(\mathbf{p}(n)) + \mathbf{j}) \\
&= p^*(n)\mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n),
\end{aligned}$$

which, combined with (2.5) and the fact that $k_n^d p^*(n) \sim n^d$, gives

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) \le \exp(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n)) + \mathrm{o}(1). \tag{3.4}$$

In the second step of the proof we will show that the reverse inequality to (3.4) also holds. It is sufficient to consider the case $\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) \to \gamma$ for $\gamma \in [0, 1]$, and we do so. Since $\gamma = 0$ is excluded by assumption (3.1) and for $\gamma = 1$ the proven inequality is obvious, we focus on $\gamma \in (0, 1)$. Let us choose $\{t_n\} \subset \mathbb{N}_+$ so that $t_n \to \infty$ and $t_n = \mathrm{o}(k_n)$. Put

$$\mathbf{s}(n) := (\lfloor N_1(n)/t_n \rfloor, \lfloor N_2(n)/t_n \rfloor, \ldots, \lfloor N_d(n)/t_n \rfloor) \quad \text{and} \quad s^*(n) := s_1(n)s_2(n)\cdots s_d(n).$$

Since $t_n = \mathrm{o}(r_n)$, (2.5) holds with k_n replaced by t_n and $\mathbf{p}(n)$ replaced by $\mathbf{s}(n)$. Also, $\mathbf{p}(n) = \mathrm{o}(\mathbf{s}(n))$ and $\mathbf{s}(n) = \mathrm{o}(\mathbf{N}(n))$. Moreover, for the sets

$$C(\mathbf{p}(n), \mathbf{s}(n)) := \{\mathbf{j} \in \mathbb{Z}^d : \mathbf{p}(n) + \mathbf{1} \le \mathbf{j} \le \mathbf{s}(n) - \mathbf{p}(n)\}$$

and

$$B(\mathbf{p}(n), \mathbf{s}(n)) := \{\mathbf{j} \in \mathbb{Z}^d : \mathbf{1} \le \mathbf{j} \le \mathbf{s}(n)\} \setminus C(\mathbf{p}(n), \mathbf{s}(n)),$$

we obtain

$$\begin{aligned}
\frac{\mathbb{P}(M_{\mathbf{s}(n)} > v_n, M_{B(\mathbf{p}(n),\mathbf{s}(n))} \le v_n)}{\mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)} &= \frac{\mathbb{P}(M_{\mathbf{s}(n)} > v_n) - \mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)}{\mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)} \\
&= \frac{\mathbb{P}(M_{\mathbf{s}(n)} > v_n)}{\mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)} - 1 \\
&= \frac{\mathbb{P}(M_{\mathbf{s}(n)} > v_n)}{\mathrm{o}(s^*(n)/p^*(n)) \cdot \mathbb{P}(M_{\mathbf{p}(n)} > v_n)} - 1 \\
&= \frac{1 + \mathrm{o}(1)}{\mathrm{o}(1)} \cdot \frac{t_n^d \mathbb{P}(M_{\mathbf{s}(n)} > v_n)}{k_n^d \mathbb{P}(M_{\mathbf{p}(n)} > v_n)} - 1.
\end{aligned}$$

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Applying (2.5) twice, we get

$$\frac{t_n^d \mathbb{P}(M_{\mathbf{s}(n)} > v_n)}{k_n^d \mathbb{P}(M_{\mathbf{p}(n)} > v_n)} \to \frac{-\log\gamma}{-\log\gamma} = 1 \quad \text{as } n \to \infty,$$

and consequently

$$\frac{\mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)}{\mathbb{P}(M_{\mathbf{s}(n)} > v_n,\, M_{B(\mathbf{p}(n),\mathbf{s}(n))} \le v_n)} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Now, observe that

$$\begin{aligned}\mathbb{P}(M_{\mathbf{s}(n)} > v_n) &= \mathbb{P}(M_{\mathbf{s}(n)} > v_n,\, M_{B(\mathbf{p}(n),\mathbf{s}(n))} \le v_n) + \mathbb{P}(M_{B(\mathbf{p}(n),\mathbf{s}(n))} > v_n)\\ &= \mathbb{P}(M_{\mathbf{s}(n)} > v_n,\, M_{B(\mathbf{p}(n),\mathbf{s}(n))} \le v_n)(1 + \mathrm{o}(1))\\ &\le \sum_{\mathbf{j} \in C(\mathbf{p}(n),\mathbf{s}(n))} \mathbb{P}(X_{\mathbf{j}} > v_n,\, M_{A(\mathbf{p}(n))+\mathbf{j}} \le v_n) \cdot (1 + \mathrm{o}(1))\\ &\le s^*(n)\mathbb{P}(X_{\mathbf{0}} > v_n,\, M_{A(\mathbf{p}(n))} \le v_n)(1 + \mathrm{o}(1)),\end{aligned}$$

by property (3.5), subadditivity and monotonicity of probability, and by stationarity of the field $\{X_{\mathbf{n}}\}$. Applying (2.5) with $(k_n, \mathbf{p}(n))$ replaced by $(t_n, \mathbf{s}(n))$ and the fact that $t_n^d s^*(n) \sim n^d$, we conclude that

$$\begin{aligned}\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) &\ge \exp\big(-t_n^d s^*(n)\mathbb{P}(X_{\mathbf{0}} > v_n,\, M_{A(\mathbf{p}(n))} \le v_n)(1 + \mathrm{o}(1))\big) + \mathrm{o}(1)\\ &= \exp\big(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n,\, M_{A(\mathbf{p}(n))} \le v_n)\big) + \mathrm{o}(1).\end{aligned} \tag{3.6}$$

Since inequalities (3.4) and (3.6) are both satisfied, the proof is complete. □

Theorem 3.1 immediately yields the following generalization of the result established by Chernick, Hsing, and McCormick [5, Proposition 1.1] for $d = 1$. Assumption (3.7) given below is a multidimensional counterpart of the local mixing condition $D^{(m+1)}(v_n)$ defined in [5] for sequences and it is satisfied by m-dependent fields, for example (see Section 5.1).

Corollary 3.1. *Let the assumptions of Theorem 3.1 be satisfied. Then*

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp\big(-n^d \mathbb{P}(X_{\mathbf{0}} > v_n,\, M_{A((m,m,\dots,m))} \le v_n)\big) + \mathrm{o}(1)$$

if and only if

$$n^d \mathbb{P}(X_{\mathbf{0}} > v_n \ge M_{A((m,m,\dots,m))},\, M_{A(\mathbf{p}(n))\setminus A((m,m,\dots,m))} > v_n) \xrightarrow[n\to\infty]{} 0, \tag{3.7}$$

where $k_n \to \infty$ is such that $k_n = \mathrm{o}(r_n)$.

We point out that Corollary 3.1 reforms a faulty formula for m-dependent fields proposed by Ferreira and Pereira [8, Proposition 2.1]; see [12, Example 5.5]. We also suggest comparing the above condition (3.7) with assumption $D''(v_n, \mathcal{B}_n, \mathcal{V})$ proposed by Pereira, Martins, and Ferreira [18, Definition 3.1].

Remark 3.2. There exists a close relationship between Theorem 3.1 and compound Poisson approximations in the spirit of Arratia, Goldstein, and Gordon [1, Section 4.2.1]. The random variable

$$\Lambda_n^{(1)} := \sum_{\mathbf{1} \le \mathbf{k} \le \mathbf{N}(n)} \mathbb{I}_{\{X_{\mathbf{k}} > v_n,\, M_{\mathbf{k}+A(\mathbf{p}(n))} \le v_n\}},$$

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with the expectation $\lambda_n^{(1)} := N^*(n)\mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n)$, estimates the number of clusters of exceedances over v_n in the set $\{\mathbf{k} \colon \mathbf{1} \le \mathbf{k} \le \mathbf{N}(n)\}$, and we have

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp(-\lambda_n^{(1)}) + \mathrm{o}(1).$$

Remark 3.3. It is worth noting that translation-invariant linear orders on the set of indices $\mathbb{Z}^d$ play a significant role in considerations (by Basrak and Planinić [4], Wu and Samorodnitsky [20]) on the extremes of regularly varying fields.

4. Extremal index

In this part we use the results given in Section 3 to establish formulas (4.2) and (4.3) for the extremal index θ for random fields. We refer to Choi [6] or Jakubowski and Soja-Kukieła [12] for definitions and some considerations on the extremal index in the multidimensional setting.

Here we present a method for calculating the number $\theta \in [0, 1]$ satisfying

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) - \mathbb{P}(X_{\mathbf{0}} \le v_n)^{\theta n^d} \to 0 \quad \text{as } n \to \infty, \tag{4.1}$$

whenever such a θ exists. Let us observe that according to (2.4) we have

$$\mathbb{P}(X_{\mathbf{0}} \le v_n)^{n^d} = \exp(-n^d\mathbb{P}(X_{\mathbf{0}} > v_n)) + \mathrm{o}(1)$$

and, moreover, Theorem 3.1 yields

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp(-n^d\mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n)) + \mathrm{o}(1)$$

for $\mathbf{N}(n)$, v_n and $\mathbf{p}(n)$ satisfying appropriate assumptions. Hence, provided that

$$0 < \liminf_{n\to\infty} n^d\mathbb{P}(X_{\mathbf{0}} > v_n) \le \limsup_{n\to\infty} n^d\mathbb{P}(X_{\mathbf{0}} > v_n) < \infty,$$

condition (4.1) is satisfied if and only if

$$\theta = \lim_{n\to\infty} \mathbb{P}(M_{A(\mathbf{p}(n))} \le v_n \mid X_{\mathbf{0}} > v_n). \tag{4.2}$$

Formula (4.2), allowing computation of extremal indices θ for random fields, is a multidimensional generalization of (1.2). In the special case when assumption (3.7) is satisfied, it is easy to show that formula (4.2) simplifies to the following:

$$\theta = \lim_{n\to\infty} \mathbb{P}(M_{A((m,m,\ldots,m))} \le v_n \mid X_{\mathbf{0}} > v_n). \tag{4.3}$$

The above formulas are in line with the interpretation of θ as the reciprocal of the mean number of high threshold exceedances in a cluster. Indeed, they answer the question: *What is the asymptotic probability that a given element of a cluster is the distinguished element of the cluster?*, where the distinguished element in a cluster is the greatest one with respect to the order $\preccurlyeq$. This identification of a unique representative for each cluster is called *declustering*, *declumping*, or *anchoring*, and has much in common with compound Poisson approximations (see e.g. [1], [2], and [4]).

Remark 4.1. Formula (4.2) justifies the following definition of the runs estimator $\hat{\theta}^R_{\mathbf{N}(n)}$ for the extremal index θ:

$$\hat{\theta}^R_{\mathbf{N}(n)} := S_n^{-1} \sum_{\mathbf{1}+\mathbf{p}(n) \le \mathbf{k} \le \mathbf{N}(n)-\mathbf{p}(n)} \mathbb{I}_{\{X_{\mathbf{k}} > v_n, M_{\mathbf{k}+A(\mathbf{p}(n))} \le v_n\}},$$

where S_n is the number of exceedances over v_n in the set $\{\mathbf{k} \in \mathbb{Z}^d \colon \mathbf{1} \le \mathbf{k} \le \mathbf{N}(n)\}$.

5. Maxima of m-dependent fields

In this section we focus on m-dependent fields. We recall that $\{X_{\mathbf{n}}\}$ is *m-dependent* for some $m\in\mathbb{N}$ if the families $\{X_{\mathbf{i}}:\mathbf{i}\in U\}$ and $\{X_{\mathbf{j}}:\mathbf{j}\in V\}$ are independent for all pairs of finite sets $U, V\subset\mathbb{Z}^d$ satisfying $\min\{\|\mathbf{i}-\mathbf{j}\|:\mathbf{i}\in U,\ \mathbf{j}\in V\}>m$.

Let us assume that $\{X_{\mathbf{n}}\}$ is m-dependent and satisfies (3.1) for some sequence $\{v_n\}\subset\mathbb{R}$. Then it is easy to show that condition (3.3) holds too (see [12, Remark 4.2]). Below, we present two methods that can be used to calculate the limit of $\mathbb{P}(M_{\mathbf{N}(n)}\le v_n)$. A direct connection between them can be given and we illustrate it in the case $d=2$.

5.1. First method

The first of the methods is a consequence of the main result presented in the paper. Since the field $\{X_{\mathbf{n}}\}$ is m-dependent, it satisfies (2.2) for each $k_n\to\infty$ such that $k_n=\mathrm{o}(r_n)$, for some $r_n\to\infty$ (see e.g. [12]). Moreover, the inequality

$$\begin{aligned} &n^d\mathbb{P}(X_{\mathbf{0}}>v_n\ge M_{A((m,m,\dots,m))},\, M_{A(\mathbf{p}(n))\setminus A((m,m,\dots,m))}>v_n)\\ &\quad\le n^d\sum_{\mathbf{i}\in A(\mathbf{p}(n)),\,\|\mathbf{i}\|>m}\mathbb{P}(X_{\mathbf{0}}>v_n,\, X_{\mathbf{i}}>v_n)\\ &\quad= n^d\sum_{\mathbf{i}\in A(\mathbf{p}(n)),\,\|\mathbf{i}\|>m}\mathbb{P}(X_{\mathbf{0}}>v_n)\mathbb{P}(X_{\mathbf{i}}>v_n)\\ &\quad\le n^d\cdot\frac{n^d}{k_n^d}\cdot\mathbb{P}(X_{\mathbf{0}}>v_n)^2(1+\mathrm{o}(1)) \end{aligned}$$

holds with the right-hand side tending to zero by (3.3). From Corollary 3.1, we obtain

$$\mathbb{P}(M_{\mathbf{N}(n)}\le v_n)=\exp\big(-n^d\mathbb{P}(X_{\mathbf{0}}>v_n,\, M_{A((m,m,\dots,m))}\le v_n\big)+\mathrm{o}(1). \tag{5.1}$$

5.2. Second method

The second formula comes from Jakubowski and Soja-Kukieła [12, Theorem 2.1]. It states that we have

$$\mathbb{P}(M_{\mathbf{N}(n)}\le v_n)=\exp\bigg(-n^d\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d}(-1)^{\varepsilon_1+\varepsilon_2+\dots+\varepsilon_d}\mathbb{P}(M_{\boldsymbol{\varepsilon},(m,m,\dots,m)}>v_n)\bigg)+\mathrm{o}(1) \tag{5.2}$$

under the above assumptions on $\{X_{\mathbf{n}}\}$. This result is a consequence of the Bonferroni-type inequality from Jakubowski and Rosiński [11, Theorem 2.1].

5.3. Comparison

For $d=1$ both of the formulas simplify to the well-known result of Newell [16]:

$$\mathbb{P}(M_n\le v_n)=\exp(-n\mathbb{P}(X_0>v_n,\, M_{1,m}\le v_n))+\mathrm{o}(1).$$

Each of them allows us to describe the asymptotic behaviour of maxima on the base of tail properties of joint distribution of a fixed finite dimension. To apply the first method, one uses the distribution of the $(1+((2m+1)^d-1)/2)$-element family

$$\{X_{\mathbf{n}}:\mathbf{n}\in\{\mathbf{0}\}\cup A((m,m,\dots,m))\}.$$

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To involve the second method, we use the distribution of the $(m+1)^d$-element family $\{X_{\mathbf{n}}\colon \mathbf{0}\le \mathbf{n}\le (m, m, \ldots, m)\}$.

Below, a link between the two formulas is described in two ways: a more conceptual one, and one that is shorter but perhaps less intuitive. To avoid annoying technicalities, we focus on $d=2$.

5.3.1 *First approach: counting clusters.* Our aim is to calculate the number of clusters of exceedances over v_n in the window $W := \{\mathbf{k}\in\mathbb{Z}^2\colon \mathbf{1}\le\mathbf{k}\le\mathbf{N}(n)\}$ in two different ways and obtain, as a consequence, the equivalence of (5.1) and (5.2).

Let the random set J_n be given as $J_n := \{\mathbf{k}\in W\colon X_{\mathbf{k}} > v_n\}$ and let $\leftrightarrow$ be the equivalence relation on J_n, defined as follows:

$$\mathbf{i}\leftrightarrow\mathbf{j} \text{ whenever there exist } l\in\mathbb{N} \text{ and } \mathbf{k}_0, \mathbf{k}_1, \ldots, \mathbf{k}_l, \mathbf{k}_{l+1}\in J_n,\ \mathbf{k}_0=\mathbf{i},\ \mathbf{k}_{l+1}=\mathbf{j}$$
$$\text{such that } \max_{0\le h\le l}\|\mathbf{k}_{h+1}-\mathbf{k}_h\|\le m,$$

for $\mathbf{i}, \mathbf{j}\in J_n$. We define a *cluster* as an equivalence class of $\leftrightarrow$ and obtain the partition $\mathcal{C}_n := J_n/_{\leftrightarrow}$ of J_n into $\Lambda_n := \#\mathcal{C}_n$ clusters. We put

$$\lambda_n := \mathbb{E}(\Lambda_n),\qquad \mathcal{C}'_n := \Big\{C\in\mathcal{C}_n\colon \max_{\mathbf{i},\mathbf{j}\in C}\|\mathbf{i}-\mathbf{j}\|\le m\Big\},$$
$$\mathcal{C}''_n := \mathcal{C}_n\backslash\mathcal{C}'_n,\qquad \Lambda'_n := \#\mathcal{C}'_n,\qquad \lambda'_n := \mathbb{E}(\Lambda'_n).$$

Let $\Lambda_n^{(1)}$ and $\lambda_n^{(1)}$, associated with the method presented in Section 5.1, be defined as in Remark 3.2 with $\mathbf{p}(n) := (m, m)$. Recall that we have

$$A(m, m) = \{\mathbf{j}\in\mathbb{Z}^2\colon (-m, -m)\le\mathbf{j}\le(m, m) \text{ and } (0, 0)\prec\mathbf{j}\}.$$

Analogously (see [12, Remark 2.2]) we define $\Lambda_n^{(2)}$ and $\lambda_n^{(2)}$ related to the method from Section 5.2 as follows:

$$\Lambda_n^{(2)} := \sum_{\mathbf{k}\in W}\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{I}_{\{M_{\mathbf{k}+\boldsymbol{\varepsilon},\mathbf{k}+(m,m)}>v_n\}},$$
$$\lambda_n^{(2)} := \mathbb{E}(\Lambda_n^{(2)}) = N^*(n)\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{P}(M_{\boldsymbol{\varepsilon},(m,m)}>v_n).$$

Assume that $C\in\mathcal{C}'_n$. Let

$$B(C, m) := \{\mathbf{j}\in\mathbb{Z}^2\colon \|\mathbf{j}-\mathbf{k}\|\le m \text{ for some } \mathbf{k}\in C\}$$

and suppose we have $M_{B(C,m)\backslash C}\le v_n$ (which is obviously satisfied in the typical case $B(C, m)\subset W$). Observe that for such C we have

$$\sum_{\mathbf{k}\in C}\mathbb{I}_{\{M_{A(m,m)+\mathbf{k}}\le v_n\}} = \sum_{\mathbf{k}\in C}\mathbb{I}_{\{\mathbf{k}\text{ is the largest element of } C \text{ with respect to } \preccurlyeq\}} = 1 \tag{5.3}$$

and

$$\sum_{\mathbf{k}\in\bar{C}}\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{I}_{\{M_{\mathbf{k}+\boldsymbol{\varepsilon},\mathbf{k}+(m,m)}>v_n\}} = \mathbb{I}_{\{M_{\mathbf{k}(C),\mathbf{k}(C)+(m,m)}>v_n\}} = 1, \tag{5.4}$$

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where

$$\bar{C} := \{\mathbf{k} \in \mathbb{Z}^2 : \mathbf{k} + \mathbf{i} \in C \text{ for some } \mathbf{0} \le \mathbf{i} \le (m, m)\}$$

and $\mathbf{k}(C)$ satisfies the condition

$$C \subset \{\mathbf{k} \in \mathbb{Z}^2 : \mathbf{k}(C) \le \mathbf{k} \le \mathbf{k}(C) + (m, m)\}.$$

We will show that each of the following equalities holds:

$$\delta_n := \mathbb{E}|\Lambda_n - \Lambda'_n| = \mathbb{E}(\Lambda_n - \Lambda'_n) = \mathrm{o}(1), \tag{5.5}$$

$$\delta_n^{(1)} := \mathbb{E}|\Lambda_n^{(1)} - \Lambda'_n| = \mathrm{o}(1), \tag{5.6}$$

$$\delta_n^{(2)} := \mathbb{E}|\Lambda_n^{(2)} - \Lambda'_n| = \mathrm{o}(1). \tag{5.7}$$

This will entail the condition $\lambda_n = \lambda'_n + \mathrm{o}(1) = \lambda_n^{(1)} + \mathrm{o}(1) = \lambda_n^{(2)} + \mathrm{o}(1)$ and complete this section.

To show (5.5), observe that the event $\{\#\mathcal{C}''_n = l\}$, for $l \in \mathbb{N}_+$, implies that there exist pairs $\mathbf{j}(i, a), \mathbf{j}(i, b) \in J_n$, for $i \in \{1, 2, \dots, l\}$, such that $m < \|\mathbf{j}(i, a) - \mathbf{j}(i, b)\| \le 2m$ holds for each i and $\|\mathbf{j}(i_1, c_1) - \mathbf{j}(i_2, c_2)\| > m$ is satisfied for $i_1 \ne i_2$ and $c_1, c_2 \in \{a, b\}$. Thus we have

$$\delta_n = \sum_{l=1}^{\infty} l\mathbb{P}(\#\mathcal{C}''_n = l) \le \sum_{l=1}^{\infty} l(N^*(n)((4m+1)^2 - (2m+1)^2)\mathbb{P}(X_{\mathbf{0}} > v_n)^2)^l.$$

Since

$$q_n := N^*(n)((4m+1)^2 - (2m+1)^2)\mathbb{P}(X_{\mathbf{0}} > v_n)^2 = \mathrm{o}(1)$$

by (3.3), we obtain

$$\delta_n \le \sum_{l=1}^{\infty} l(q_n)^l = q_n(1 - q_n)^{-2} \quad \text{for all large } n$$

and finally $\delta_n = \mathrm{o}(1)$.

Before we establish (5.6) and (5.7), we will give an upper bound for the probability that a fixed $\mathbf{k} \in W$ belongs to a large cluster. Note that we have

$$\begin{aligned}
&\mathbb{P}(\mathbf{k} \in C \text{ for some } C \in \mathcal{C}''_n) \\
&\quad = \mathbb{P}(\mathbf{k} \in C \text{ and } \|\mathbf{j} - \mathbf{k}\| \le m \text{ for all } \mathbf{j} \in C, \text{ for some } C \in \mathcal{C}''_n) \\
&\qquad + \mathbb{P}(\mathbf{k} \in C \text{ and } \|\mathbf{j} - \mathbf{k}\| > m \text{ for some } \mathbf{j} \in C, \text{ for some } C \in \mathcal{C}''_n) \\
&\quad \le \mathbb{P}(\|\mathbf{i} - \mathbf{k}\| \le m \text{ and } \|\mathbf{j} - \mathbf{k}\| \le m \text{ and } \|\mathbf{i} - \mathbf{j}\| > m, \text{ for some } \mathbf{i}, \mathbf{j} \in J_n) \\
&\qquad + \mathbb{P}(\mathbf{k} \in J_n \text{ and } m < \|\mathbf{k} - \mathbf{j}\| \le 2m \text{ for some } \mathbf{j} \in J_n) \\
&\quad \le ((2m+1)^4 + ((4m+1)^2 - (2m+1)^2))\mathbb{P}(X_{\mathbf{0}} > v_n)^2 = a(m)\mathbb{P}(X_{\mathbf{0}} > v_n)^2
\end{aligned} \tag{5.8}$$

with $a(m) := (2m+1)^4 + 4m(3m+1)$.

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Applying observation (5.3), property (5.8) and taking into account estimation errors for clusters situated near the edges of the window W, we obtain

$$\begin{aligned}\delta_n^{(1)} &= \mathbb{E}\left|\sum_{\mathbf{k}\in W}\left(\mathbb{I}_{\{\mathbf{k}\in\bigcup\mathcal{C}_n', M_{\mathbf{k}+A(m,m)}\le v_n\}} + \mathbb{I}_{\{\mathbf{k}\in\bigcup\mathcal{C}_n'', M_{\mathbf{k}+A(m,m)}\le v_n\}}\right) - \Lambda_n'\right| \\ &\le \mathbb{E}\left|\sum_{C\in\mathcal{C}_n'}\sum_{\mathbf{k}\in C}\mathbb{I}_{\{M_{\mathbf{k}+A(m,m)}\le v_n\}} - \Lambda_n'\right| + \mathbb{E}\left(\sum_{\mathbf{k}\in W}\mathbb{I}_{\{\mathbf{k}\in\bigcup\mathcal{C}_n''\}}\right) \\ &\le 2m(N_1(n)+N_2(n))\mathbb{P}(X_{\mathbf{0}}>v_n) + a(m)N^*(n)\mathbb{P}(X_{\mathbf{0}}>v_n)^2,\end{aligned}$$

which combined with assumption (3.3) implies (5.6). Quite similarly, using (5.4) instead of (5.3) and writing $\bar{W} := \{\mathbf{k}\in\mathbb{Z}^2 : \mathbf{1}-(m,m)\le\mathbf{k}\le\mathbf{N}(n)\}$, we conclude that

$$\begin{aligned}\delta_n^{(2)} &\le \mathbb{E}\left|\sum_{\mathbf{k}\in\bar{W}}\mathbb{I}_{\{\mathbf{k}\in\bar{C}\text{ for some }C\in\mathcal{C}_n'\}}\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{I}_{\{M_{\mathbf{k}+\boldsymbol{\varepsilon},\mathbf{k}+(m,m)}>v_n\}} - \Lambda_n'\right| \\ &\quad + \mathbb{E}\left(\sum_{\mathbf{k}\in\bar{W}\setminus W}\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}\left|(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{I}_{\{M_{\mathbf{k}+\boldsymbol{\varepsilon},\mathbf{k}+(m,m)}>v_n\}}\right|\right) \\ &\quad + 2\mathbb{E}\left(\sum_{\mathbf{k}\in W}\mathbb{I}_{\{\mathbf{k}\in\bar{C}\text{ for some }C\in\mathcal{C}_n''\}}\right) \\ &\le \mathbb{E}\left|\sum_{C\in\mathcal{C}_n'}\sum_{\mathbf{k}\in\bar{C}}\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2}\mathbb{I}_{\{M_{\mathbf{k}+\boldsymbol{\varepsilon},\mathbf{k}+(m,m)}>v_n\}} - \Lambda_n'\right| \\ &\quad + \sum_{\mathbf{k}\in\bar{W}\setminus W}(2m+1)\mathbb{P}(X_{\mathbf{0}}>v_n) + 2N^*(n)\mathbb{P}(\mathbf{k}\in\bar{C}\text{ for some }C\in\mathcal{C}_n'') \\ &\le 2m(2m+1)(N_1(n)+N_2(n))\mathbb{P}(X_{\mathbf{0}}>v_n) \\ &\quad + m(2m+1)(N_1(n)+N_2(n)+m)\mathbb{P}(X_{\mathbf{0}}>v_n) + 2(m+1)^2a(m)N^*(n)\mathbb{P}(X_{\mathbf{0}}>v_n)^2.\end{aligned}$$

Since the right-hand side tends to zero by (3.3), property (5.7) follows.

5.3.2. *Second approach: direct verification.* In this part we assume that $\preccurlyeq$ is the lexicographic order on $\mathbb{Z}^2$. Let us note that

$$\begin{aligned}&\mathbb{P}(M_{(0,0),(m,m)}>v_n) - \mathbb{P}(M_{(1,0),(m,m)}>v_n) - \mathbb{P}(M_{(0,1),(m,m)}>v_n) + \mathbb{P}(M_{(1,1),(m,m)}>v_n) \\ &\quad = \mathbb{P}(X_{(0,0)}>v_n, M_{R((m,m))}\le v_n) \\ &\quad\quad - \mathbb{P}(M_{(0,1),(0,m)}>v_n, M_{(1,0),(m,0)}>v_n, M_{(1,1),(m,m)}\le v_n)\end{aligned}$$

is true with $R((p_1,p_2)) := A((p_1,p_2))\cap\mathbb{N}^2$, where on the left-hand side of the equation the sum of probabilities from (5.2) for $d=2$ appears. Next, let us look at the exponent in (5.1) and observe that

$$\begin{aligned}&\mathbb{P}(X_{(0,0)}>v_n, M_{A((m,m))}\le v_n) \\ &\quad = \mathbb{P}(X_{(0,0)}>v_n, M_{R((m,m))}\le v_n) \\ &\quad\quad - \mathbb{P}(X_{(0,0)}>v_n, M_{R((m,m))}\le v_n, M_{(1,-m),(m,-1)}>v_n)\end{aligned}$$

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and, moreover, the second summand of the right-hand side satisfies

$$\begin{aligned}
&\mathbb{P}(X_{(0,0)} > v_n,\, M_{R((m,m))} \le v_n,\, M_{(1,-m),(m,-1)} > v_n) \\
&\quad = \sum_{l=1}^{m} \mathbb{P}(X_{(0,0)} > v_n,\, M_{R((m,m))} \le v_n,\, M_{(1,-l),(m,-l)} > v_n,\, M_{(1,-l+1),(m,-1)} \le v_n) \\
&\quad = \sum_{l=1}^{m} \mathbb{P}(X_{(0,0)} > v_n,\, M_{R((m,m-l))} \le v_n,\, M_{(1,-l),(m,-l)} > v_n,\, M_{(1,-l+1),(m,-1)} \le v_n) + \mathrm{o}(n^{-2}) \\
&\quad = \sum_{l=1}^{m} \mathbb{P}(X_{(0,l)} > v_n,\, M_{(0,l)+R((m,m-l))} \le v_n,\, M_{(1,0),(m,0)} > v_n,\, M_{(1,1),(m,l-1)} \le v_n) + \mathrm{o}(n^{-2}) \\
&\quad = \mathbb{P}(M_{(0,1),(0,m)} > v_n,\, M_{(1,0),(m,0)} > v_n,\, M_{(1,1),(m,m)} \le v_n) + \mathrm{o}(n^{-2}).
\end{aligned}$$

In the above statement, probabilities of mutually exclusive events are summed up, and m-dependence, assumption (3.3), and stationarity are applied. Finally, we obtain

$$\begin{aligned}
&\mathbb{P}(M_{(0,0),(m,m)} > v_n) - \mathbb{P}(M_{(1,0),(m,m)} > v_n) - \mathbb{P}(M_{(0,1),(m,m)} > v_n) + \mathbb{P}(M_{(1,1),(m,m)} > v_n) \\
&\quad = \mathbb{P}(X_{(0,0)} > v_n,\, M_{A((m,m))} \le v_n) + \mathrm{o}(n^{-2}).
\end{aligned}$$

Summarizing, we have confirmed that both presented methods lead to the same result.

Remark 5.1. The above reasoning for m-dependent fields can also be applied in the general setting. Suppose that formula (1.3), with $\preccurlyeq$ the lexicographic order on $\mathbb{Z}^2$, describes the asymptotics of partial maxima of the stationary field $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$. Then

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) = \exp\Bigg(-n^2 \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^2} (-1)^{\varepsilon_1+\varepsilon_2} \mathbb{P}(M_{\boldsymbol{\varepsilon},\mathbf{p}(n)} > v_n)\Bigg) + \mathrm{o}(1) \tag{5.9}$$

holds if and only if $\{X_{\mathbf{n}}\}$ satisfies the following condition:

$$\sum_{l=1}^{p_2(n)} \mathbb{P}(X_{\mathbf{0}} > v_n,\, M_{U_l(\mathbf{p}(n))} > v_n,\, M_{V_l(\mathbf{p}(n))} > v_n,\, M_{W_l(\mathbf{p}(n))} \le v_n) = \mathrm{o}(n^{-2}), \tag{5.10}$$

where

$$\begin{aligned}
U_l(\mathbf{p}) &:= \{0, \ldots, p_1\} \times \{p_2 - l + 1, \ldots, p_2\}, \\
V_l(\mathbf{p}) &:= \{1, \ldots, p_1\} \times \{-l\}, \\
W_l(\mathbf{p}) &:= A(\mathbf{p}) \cap (\mathbb{Z} \times \{-l+1, \ldots, p_2 - l\}).
\end{aligned}$$

Formula (5.9) generalizes (5.2). In the present section we have used the fact that m-dependent fields satisfy (5.10) with $\mathbf{p}(n) := (m, m)$.

6. Example: moving maxima

Below, we use the results from Sections 3 and 4 to describe the asymptotics of partial maxima for the moving maximum field. We note that approaches to the problem using different methods can be found in Basrak and Tafro [3] or Jakubowski and Soja-Kukieła [12]. In the first paper compound Poisson point process approximation is applied, while in the second paper the authors combine a Bonferroni-like inequality and max-m-approximability.

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In the following, $\{Z_{\mathbf{n}}\}$ is an array of independent, identically distributed random variables satisfying

$$\mathbb{P}(|Z_{\mathbf{0}}|>x)=x^{-\alpha}L(x)$$

for some index $\alpha>0$ and slowly varying function L, and

$$\frac{\mathbb{P}(Z_{\mathbf{0}}>x)}{\mathbb{P}(|Z_{\mathbf{0}}|>x)}=p \quad \text{as } x\to\infty \quad \text{for some } p\in[0,1].$$

We define

$$a_n:=\inf\{y>0\colon \mathbb{P}(|Z_{\mathbf{0}}|>y)\le n^{-d}\}$$

and $v_n:=a_n v$ with fixed $v>0$. Then

$$n^d\mathbb{P}(|Z_{\mathbf{0}}|>v_n)\to v^{-\alpha} \quad \text{as } n\to\infty.$$

Let us consider the *moving maximum field* $\{X_{\mathbf{n}}\}$ defined as

$$X_{\mathbf{n}}=\sup_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}}Z_{\mathbf{n}+\mathbf{j}},$$

where $c_{\mathbf{j}}\in\mathbb{R}$, not all equal to zero, satisfy

$$\sum_{\mathbf{j}\in\mathbb{Z}^d}|c_{\mathbf{j}}|^{\beta}<\infty \quad \text{for some } 0<\beta<\alpha. \tag{6.1}$$

From Cline [7, Lemma 2.2] it follows that the field $\{X_{\mathbf{n}}\}$ is well-defined and

$$\begin{aligned}\lim_{x\to\infty}\frac{\mathbb{P}(X_{\mathbf{0}}>x)}{\mathbb{P}(|Z_{\mathbf{0}}|>x)}&=\lim_{x\to\infty}\frac{\mathbb{P}(\sup_{\mathbf{j}\in\mathbb{Z}^d}c_{\mathbf{j}}Z_{\mathbf{j}}>x)}{\mathbb{P}(|Z_{\mathbf{0}}|>x)}\\&=\lim_{x\to\infty}\frac{\sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{P}(c_{\mathbf{j}}Z_{\mathbf{j}}>x)}{\mathbb{P}(|Z_{\mathbf{0}}|>x)}\\&=p\sum_{c_{\mathbf{j}}>0}c_{\mathbf{j}}^{\alpha}+q\sum_{c_{\mathbf{j}}<0}|c_{\mathbf{j}}|^{\alpha},\end{aligned} \tag{6.2}$$

with $q:=1-p$.

Since the moving maximum field is max-*m-approximable*, there exists a *phantom distribution function* for $\{X_{\mathbf{n}}\}$ (see Jakubowski and Soja-Kukieła [12]) and hence the field is weakly dependent in the sense of (2.2). We will apply Theorem 3.1, with $\preccurlyeq$ being the lexicographic order on $\mathbb{Z}^d$, to describe the asymptotics of partial maxima. Let us observe that the exponent in (3.2) satisfies

$$\begin{aligned}&n^d\mathbb{P}(X_{\mathbf{0}}>v_n,\ M_{A(\mathbf{p}(n))}\le v_n)\\&\quad=n^d\mathbb{P}\bigg(\bigcup_{\mathbf{j}\in\mathbb{Z}^d}\{c_{\mathbf{j}}Z_{\mathbf{j}}>v_n\},\ \bigcap_{\mathbf{k}\in\mathbb{Z}^d}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}})\le v_n\Big\}\bigg)\\&\quad=n^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{P}\bigg(c_{\mathbf{j}}Z_{\mathbf{j}}>v_n,\ \bigcap_{\mathbf{k}\in\mathbb{Z}^d}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}})\le v_n\Big\}\bigg)+\mathrm{o}(1)\\&\quad=n^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{P}\bigg(c_{\mathbf{j}}Z_{\mathbf{j}}>v_n\ge\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}}),\ \bigcap_{\mathbf{k}\ne\mathbf{j}}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}})\le v_n\Big\}\bigg)+\mathrm{o}(1)\\&\quad=n^d\sum_{\mathbf{j}\in\mathbb{Z}^d}\mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}}>v_n\ge\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}})\Big)\mathbb{P}\bigg(\bigcap_{\mathbf{k}\ne\mathbf{j}}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}})\le v_n\Big\}\bigg)+\mathrm{o}(1),\end{aligned}$$

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as $n \to \infty$, where the second equality follows from (6.2) combined with the choice of $\{v_n\}$ and the last equality is a consequence of the independence of $Z_{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{Z}^d$. Note that we have

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{\mathbf{k}\neq\mathbf{j}}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}}) \le v_n\Big\}\Big) &\ge \mathbb{P}\Big(\bigcap_{\mathbf{k}\in\mathbb{Z}^d}\Big\{\max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{k}-\mathbf{i}}Z_{\mathbf{k}}) \le v_n\Big\}\Big)\\ &\ge \mathbb{P}(M_{A(\mathbf{p}(n))} \le v_n)\\ &\ge 1 - \mathrm{o}(n^d)\mathbb{P}(X_{\mathbf{0}} > v_n)\\ &= 1 + \mathrm{o}(1).\end{aligned}$$

Moreover, for $p_{\min}(n) := \min\{p_l(n) \colon 1 \le l \le d\}$ and for $q(n) \in \mathbb{N}$ chosen so that

$$q(n) \to \infty, \quad q(n) \le p_{\min}(n)/2 \quad \text{and} \quad q(n)^d n^d \mathbb{P}(\max\{c_{\mathbf{i}}Z_{\mathbf{0}} \colon \|\mathbf{i}\| > p_{\min}(n)/2\} > v_n) \to 0,$$

it follows that

$$\begin{aligned}&\Big| n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n \ge \max_{\mathbf{i}\in A(\mathbf{p}(n))}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}})\Big) - n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n \ge \sup_{\mathbf{0}\prec\mathbf{i}}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}})\Big)\Big|\\ &\quad\le n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n, \sup_{\mathbf{0}\prec\mathbf{i}\notin A(\mathbf{p}(n))}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}}) > v_n\Big)\\ &\quad\le n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n, \sup_{\|\mathbf{i}\|>p_{\min}(n)}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}}) > v_n\Big)\\ &\quad\le n^d \sum_{\|\mathbf{j}\|\le q(n)} \mathbb{P}\Big(\sup_{\|\mathbf{i}\|>p_{\min}(n)}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}}) > v_n\Big) + n^d \sum_{\|\mathbf{j}\|>q(n)} \mathbb{P}(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n)\\ &\quad\le n^d(2q(n)+1)^d \mathbb{P}\Big(\sup_{\|\mathbf{i}\|>p_{\min}(n)/2}(c_{\mathbf{i}}Z_{\mathbf{0}}) > v_n\Big) + n^d \sum_{\|\mathbf{j}\|>q(n)} \mathbb{P}(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n).\end{aligned}$$

The first summand on the right-hand side tends to zero due to the choice of $q(n)$ and the second summand tends to zero by properties (6.1), (6.2), and the definition of v_n. We conclude that

$$\begin{aligned}&n^d \mathbb{P}(X_{\mathbf{0}} > v_n, M_{A(\mathbf{p}(n))} \le v_n)\\ &\quad= \Big(n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n \ge \sup_{\mathbf{0}\prec\mathbf{i}}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}})\Big) + \mathrm{o}(1)\Big)(1+\mathrm{o}(1)) + \mathrm{o}(1).\end{aligned}$$

To complete the above calculation, it is sufficient to observe that

$$\begin{aligned}n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{j}} > v_n \ge \sup_{\mathbf{0}\prec\mathbf{i}}(c_{\mathbf{j}-\mathbf{i}}Z_{\mathbf{j}})\Big) &= n^d \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbb{P}\Big(c_{\mathbf{j}}Z_{\mathbf{0}} > v_n \ge \sup_{\mathbf{i}\prec\mathbf{j}}(c_{\mathbf{i}}Z_{\mathbf{0}})\Big)\\ &= n^d \mathbb{P}\Big(\sup_{\mathbf{j}\in\mathbb{Z}^d}(c_{\mathbf{j}}Z_{\mathbf{0}}) > v_n\Big)\\ &\to (p(c^+)^\alpha + q(c^-)^\alpha)v^{-\alpha},\end{aligned}$$

with $c^+ := \max_{\mathbf{i}\in\mathbb{Z}^d} \max\{c_{\mathbf{i}}, 0\}$ and $c^- := \max_{\mathbf{i}\in\mathbb{Z}^d} \max\{-c_{\mathbf{i}}, 0\}$. By (3.2) we obtain

$$\mathbb{P}(M_{\mathbf{N}(n)} \le v_n) \to \exp\big(-(p(c^+)^\alpha + q(c^-)^\alpha)v^{-\alpha}\big) \quad \text{as } n \to \infty.$$

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Applying formula (4.2) and property (6.2), we calculate the extremal index of $\{X_{\mathbf{n}}\}$ as follows:

$$\theta = \frac{p(c^+)^\alpha + q(c^-)^\alpha}{p\sum_{c_{\mathbf{j}}>0} c_{\mathbf{j}}^\alpha + q\sum_{c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^\alpha},$$

whenever the denominator is positive, which is the only interesting case.

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