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CLASSIFICATION OF FINITE GROUPS VIA THEIR BREADTH

HERMANN HEINEKEN
Department of Mathematics, University of Würzburg, Campus Hubland Nord, Email-Fischer-Strasse 30 97074, Würzburg, Germany
e-mail: heineken@mathematik.uni-wuerzburg.de

FRANCESCO G. RUSSO
Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1, Rondebosch 7701, Cape Town, South Africa
e-mail: francescog.russo@yahoo.com

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Abstract. Let k be a divisor of a finite group G and $L_k(G) = \{x \in G \mid x^k = 1\}$. Frobenius proved that the number $|L_k(G)|$ is always divisible by k. The following inverse problem is considered: for a given integer n, find all groups G such that $\max\{k^{-1}|L_k(G)| \mid k \in \mathrm{Div}(G)\} = n$, where $\mathrm{Div}(G)$ denotes the set of all divisors of $|G|$. A procedure beginning with (in a sense) minimal members and deducing the remaining ones is outlined and executed for $n = 8$.

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1. Introduction and statement of the main results. The present paper deals with finite groups and their layers of elements

$$L_k(G) = \{x \in G \mid x^k = 1\},$$

where $k \geq 1$. In 1895, Frobenius [3, Satz 1] proved that, for all divisors k of $|G|$, the number $|L_k(G)|$ is divisible by k. We consider the integers

$$\mathbf{b}_k(G) = k^{-1}\ |L_k(G)|,$$

called *local breadth* of G, and in particular groups satisfying

$$\mathbf{B}(G) = \max\{\mathbf{b}_k(G) \mid k \in \mathrm{Div}(G)\} = n,$$

for some $n \geq 1$, referring to $\mathbf{B}(G)$ as the *global breadth* of G. Denoting by $\exp(G)$ the exponent of G and by $\mathrm{Div}(\exp(G))$ the set of divisors of $\exp(G)$, we note that $\mathbf{b}_h(G) \leq \mathbf{b}_k(G)$ if $k = \gcd(h, \exp(G))$, so for finding $\mathbf{B}(G)$ only integers $k \in \mathrm{Div}(\exp(G))$ are relevant.

Certainly, $\mathbf{B}(G)$ is defined for every group G, and this gives rise for a classification. Meng and Shi [9] characterized groups of global breadth at most two (see Theorem 2.1), while Meng, Shi, and Chen [10] described all groups G with $\mathbf{B}(G) = 3$. Successively Meng [11] considered the case $\mathbf{B}(G) = 4$, and a more recent work [12] shows that the groups G with $\mathbf{B}(G) \leq 7$ must be solvable. These contributions belong to the line of research indicated by Frobenius in [3, 4]. For a given integer n, the procedure outlined and executed here considers first members which are minimal in the following sense:

DEFINITION 1.1. Let $\mathbf{B}(G) = n$. A group G is *refined*, if $\mathbf{B}(G) \neq \mathbf{B}(G/N)$ for all proper normal subgroups N of G.

The second step is to deduce the remaining groups of same global breadth.

DEFINITION 1.2. If there is a proper normal subgroup N of G such that $\mathbf{B}(G) = \mathbf{B}(G/N)$, then G is *deduced* from G/N.

A table of refined groups for small global breadth was laid down in [6, pp. 220, 221] with the exception of $n = 8$. In case of $\mathbf{B}(G) = 8$, we have nonsoluble groups for the first time and we execute the described method in this case as an example.

The notation will be mostly standard; see [1, 5, 6, 7]. In particular, C_k, D_k, and Q_k denote the cyclic, dihedral, and generalized quaternion groups, respectively, of order $k \geq 1$. With the symbols $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$, we denote the alternating and the symmetric group on n elements, respectively. The *holomorph* $\mathrm{Hol}(G)$ of a group G (see [1, 7]) is the extension of G by its automorpshim group, and

$$L(\mathbb{F}_q) = \{\phi : x \in \mathbb{F}_q \mapsto ax + b \in \mathbb{F}_q \mid a \in \mathbb{F}_q^{\times}, b \in \mathbb{F}_q\}$$

is the group of affine mappings from $\mathbb{F}_q$ onto $\mathbb{F}_q$, where q is a prime power and $\mathbb{F}_q$ the finite field of order q (see [7, Kapitel V, Section 8]). Note that we indicate the special linear group $\mathrm{SL}(2, q)$ of 2-by-2 matrices with coefficients in $\mathbb{F}_q$ by $\mathrm{SL}(2, \mathbb{F}_q)$ and similarly $\mathrm{PSL}(2, q)$ by $\mathrm{PSL}(2, \mathbb{F}_q)$ for the projective special linear group with coefficients in $\mathbb{F}_q$. The symbol $\pi(G) = \mathrm{Div}(G) \cap \mathbb{P}$ denotes the set of all prime divisors of $|G|$.

The main results, continuing previous investigations in [5, 6, 11], are listed in the following.

THEOREM 1.3 (Theorem of Deduction). *Let G be a group with normal subgroup N. Then*

(i) $\mathbf{B}(G/N) \leq \mathbf{B}(G)$,
(ii) *N is cyclic, if* $\mathbf{B}(G/N) = \mathbf{B}(G)$.

Notice that there is no restriction on G/N in Theorem 1.3.
We denote in Theorems 1.4 and 1.6

$$\mathcal{D} = \{T \text{ nilpotent group of } \mathbf{B}(T) = 2 \text{ without subgroups isomorphic to } D_8 \text{ or } Q_{16}\},$$

which is contained in the class of all nilpotent groups. In fact $\mathcal{D}$ will be used for one of the cases, where 2-groups G satisfying $\mathbf{B}(G) = 4$ are involved. In [11, Main Theorem (ii)] Meng has shown, among other facts, that $|G| = 4 \exp(G)$ for these groups. We will need a closer look and show the following theorem.

THEOREM 1.4 (Classification of Refined 2-Groups of Global Breadth Four). *Assume that G is a 2-group satisfying $\mathbf{B}(G) = 4$. If G is refined, then either*

(a) $G \simeq C_2 \times C_2 \times C_2$*; or*
(b) $G \simeq C_4 \times C_4$*; or*
(c) $G \simeq \langle a, b, c \mid a^2 = b^2 = c^4 = b[c, a] = [b, c] = [a, b] = 1\rangle$*; or*
(d) $G \simeq \langle a, b \mid a^4 = b^4 = b^2[a, b] = 1\rangle$.

If H is deduced from a refined group G listed above, then either

(e) $H \simeq C_2 \times T$*, where* $T \in \mathcal{D}$*; or*
(f) $H \simeq \langle a, b \mid a^4 = b^{4m} = b^{tm}[a, b] = 1\rangle$ *with* $t \in \{1, 2, 4\}$*; or*
(g) $H \simeq \langle a, b, c \mid a^2 = b^2 = c^{4m} = b[c, a] = 1\rangle$*; or*

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(h) $H \simeq \langle a, b, c \mid a^4 = b^4 = a^2b^2 = a^2[a, b] = c^{4m} = c^{2m}[a, b] = 1\rangle$; *or*
(i) $H \simeq \langle a, b \mid a^{4m} = b^4 = b^2[a, b] = 1\rangle$.

The importance of classifying groups, restricting to refined groups, is explained by the condition (ii) of Theorem 1.3. This motivates our third main theorem, which deals with refined groups only.

THEOREM 1.5 (Main Theorem for Refined Groups). *Assume that G is a refined group of $\mathbf{B}(G) = 8$.*

(i) *If G is nilpotent, then G is a 2-group.*
(ii) *If G possesses a noncyclic Sylow subgroup S of odd order, then $|S| = 9$ and either $G \simeq L(\mathbb{F}_9)$ or $G \simeq D_6 \times D_6$.*
(iii) *If $5 \in \pi(G)$, then either $G \simeq D_{30}$, or $G \simeq \mathrm{Hol}(C_5) \times C_2$, or $G \simeq \mathrm{Alt}(5)$.*
(iv) *If $7 \in \pi(G)$, then either $G \simeq D_{14} \times C_2$ or $G \simeq L(\mathbb{F}_8) \times C_2$.*
(v) *If $\pi(G) = \{2, 3\}$ and G is nonabelian with cyclic Sylow 3-subgroup S and with Sylow 2-subgroup T, then $|S| = 3$ and one of the following conditions is satisfied:*
    (1) *S is normal and either $T \simeq C_2 \times C_2 \times C_2$, or $T \simeq C_4 \times C_4$;*
    (2) *T is normal and either $T \simeq C_2 \times C_2 \times C_2 \times C_2$, or $T \simeq Q_8 \times C_4$;*
    (3) *neither S nor T are normal and $G = HK$ with $H \simeq \mathrm{Alt}(4)$, $K \simeq C_4$ and $G/Z(G) \simeq \mathrm{Sym}(4)$;*
(vi) *there is no G such that $p \in \pi(G)$ and $p \geq 11$.*

The items (iii), (iv), and (v) above may be formulated in terms of homomorphic images. This motivates the following result.

THEOREM 1.6 (Main Theorem for Deduced Groups). *Assume that G is a nonabelian group of $\mathbf{B}(G) = 8$.*

(j) *If $p \in \pi(G)$ is odd and G is deduced from one of the groups in Theorem 1.5 (ii), then one of the following conditions is true:*
    (a) *G is an extension of $C_3 \times C_3$ by C_{8m} with $\gcd(m, 3) = 1$;*
    (b) *G is an extension of $C_3 \times C_3$ by a $3'$-group $T \in \mathcal{D}$.*
(jj) *If $5 \in \pi(G)$ and G is deduced from one of the groups in Theorem 1.5 (iii), then one of the following conditions is true:*
    (a) *G is the extension of C_5 by a $5'$-group $T \in \mathcal{D}$;*
    (b) *G is the extension of C_5 by C_{2m} with $\gcd(15, m) = 1$;*
    (c) *$G \simeq \mathrm{Alt}(5) \times C_m$ with $\gcd(30, m) = 1$;*
    (d) *$G \simeq \mathrm{SL}(2, \mathbb{F}_5) \times C_m$ with $\gcd(30, m) = 1$.*
(jjj) *If $7 \in \pi(G)$ and G is deduced from one of the groups in Theorem 1.5 (iv), then one of the following conditions is true:*
    (a) *G is an extension of $C_2 \times C_2 \times C_2$ by C_{14m};*
    (b) *G is an extension of C_7 by a $7'$-group $T \in \mathcal{D}$.*
(jv) *If $G = K \times D$, where D is cyclic of $\gcd(|K|, |D|) = 1$ and K is a $\{2, 3\}$-group in Theorem 1.5 (v), then one of the following conditions is satisfied:*
    (a) *either $G \simeq \mathrm{Alt}(4) \times V$, where $\mathbf{B}(V) = 2$ and V is a 2-group, or $G \simeq (\mathrm{SL}(2, \mathbb{F}_3) \times W)/Z$, where Z is a suitable normal subgroup and either $W \simeq D_{16}$, or $\mathbf{B}(W) = 2$;*
    (b) *$G \simeq \mathrm{SL}(2, \mathbb{F}_3) \times C$ with C a cyclic 2-group of $|C| \geq 4$;*
    (c) *G is a split extension of $U \simeq \mathrm{PSL}(2, \mathbb{F}_3)$ by a cyclic 2-group T with $|T| \geq 4$ and G is as in (2);*

(d) *G is a split extension of* $V \simeq \mathrm{SL}(2, \mathbb{F}_3)$ *by a cyclic* 2*-group* T *of* $|T| > 4$*,* $T \cap V = Z(V)$ *and G is as in (2);*
(e) *G is a split extension of S by* $C_2 \times W$ *with* $\mathbf{B}(W) = 2$ *and* $W \not\simeq D_8$ *but W is a* 2*-group;*
(f) *G is a split extension of S either by* $C_4 \times C_{2^{2+n}}$ *with* $n \geq 0$*, or by* $\langle a, b | a^4 = b^8 = [a, b]b^4 = 1\rangle$*, or by any of* $\langle a, b | a^4 = b^{4t} = [a, b]b^{-st} = 1\rangle$ *with* $t = 2^k \geq 2$*,* $s \in \{1, 2, 4\}$*;*
(g) *G is a split extension of S by* $\langle a, b | a^4 = b^{2^k} = [a, b]a^2 = 1\rangle$ *with* $2 \leq k \leq m + 1$.

Preparations for the proof of the main results are executed in Sections 2 and 3. The proof of the Theorem of Deduction is placed in Section 4, while the classification of 2-groups of global breadth four is placed in Section 5. The proof of Main Theorem, and of its weak form, is done via a careful analysis on the prime divisors of the order of the group. The role of odd primes smaller than 7 is discussed in Section 6, while Section 7 deals with the case of {2, 3}-groups. The proofs of Theorems 1.5 and 1.6 are placed in Section 8, along with open problems, which we encountered in the course of our investigations.

2. Some previous results and description of refined groups. The origin of our investigations is due to a recent characterization of Meng and Shi [9] of groups with $|L_e(G)| \leq 2e$ for all $e \in \mathrm{Div}(\exp(G))$.

THEOREM 2.1 (See [9], Main Theorem and [6], list in Section 4). *Let G be a group and* $m \geq 1$*. Then* $|L_e(G)| \leq 2e$ *for all* $e \in \mathrm{Div}(\exp(G))$ *if and only if one of the following statements holds:*

(i) *G is cyclic;*
(ii) $G \simeq C_m \times C_{2^{k-1}} \times C_2$ *with m odd and* $k \geq 2$*;*
(iii) $G \simeq C_m \times Q_8 = C_m \times \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$ *with m odd;*
(iv) $G \simeq C_m \times \langle a, b \mid a^{2^{t-1}} = b^2 = 1, b^{-1}ab = a^{1+2^{t-2}}\rangle$ *with* $t \geq 4$ *and m odd;*
(v) $G \simeq C_m \times \langle a, b \mid a^3 = b^{2^s} = 1, b^{-1}ab = a^{-1}\rangle$ *with* $s \geq 1$ *and* $\gcd(m, 6) = 1$.

An easy consequence is the following:

COROLLARY 2.2. *If G is a nilpotent group of* $\mathbf{B}(G) = 2$*, then* D_8 *and* Q_{16} *cannot be subgroups of G. Moreover* $|G| = 2 \cdot \exp(G)$.

Meng, Shi, and Chen [10, Theorems 1.1, 1.2] characterize groups satisfying a bound of the form $|L_e(G)| = 3e$ with $e \in \mathrm{Div}(\exp(G))$ and so we have examples of groups of local breadth at most three. More generally, we introduced the notion of *Q-group* in [5] (i.e., a group G is a $\mathcal{Q}$-group if $|L_n(G)| \leq n^2$, whenever $n \in \mathrm{Div}(\exp(G))$), in order to classify groups of local breadth at most n with $n \geq 4$. Theorems 3.2, 3.5, 3.6, 3.8, 3.12, and 3.14 in [5] show a complete classification of these groups. Successively, we did the same in [6] working on the global breadth, instead of the local breadth. Here we involved the aforementioned notion of refined group. The following statements give examples how refinements may arise.

PROPOSITION 2.3. *Let G be a group. If N is a central subgroup of G of prime order p such that all elements of order p are contained in N, then* $\mathbf{B}(G) = \mathbf{B}(G/N)$.

Proof. We have $|N| = p$ and $N \subseteq Z(G)$. Let $s \in \mathrm{Div}(\exp(G))$. If $\gcd(s, p) \neq 1$ and $H \simeq G/N$, then $p \cdot |L_s(H)| = |L_{ps}(G)|$ for all $ps \in \mathrm{Div}(\exp(G))$, so $\mathbf{b}_{ps}(G) = \mathbf{b}_s(H)$, hence $\mathbf{B}(G) = \mathbf{B}(H)$. If $\gcd(s, p) = 1$, then $\mathbf{b}_s(G) = \mathbf{b}_{ps}(G) = \mathbf{b}_s(H)$. Since we have the same

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collection of local breadths both for G and for H, the maxima $\mathbf{B}(G)$ and $\mathbf{B}(H)$ are the same. The result follows. □

Proposition 2.3 gives a criterion for detecting refined groups and there is an interesting consequence in terms of special linear groups.

COROLLARY 2.4. *Let $n \geq 2$ and q a prime power. Then* $\mathbf{B}(\mathrm{SL}(2, \mathbb{F}_q)) = \mathbf{B}(\mathrm{PSL}(2, \mathbb{F}_q))$ *and* $\mathbf{B}(Q_{2^{n+1}}) = \mathbf{B}(D_{2^n})$.

Proof. Apply Proposition 2.3 for $p = 2$. □

The next result provides a criterion for detecting nonrefined groups.

THEOREM 2.5 (See [6], Theorem 2.8). *If a group G possesses some $p \in \pi(Z(G))$ such that the Sylow p-subgroups of G are cyclic or quaternion, then $\mathbf{B}(G) = \mathbf{B}(G/N)$, where N is the Sylow p-subgroup of $Z(G)$. Moreover, if N is nontrivial, then G is not refined.*

More detailed information can be observed in presence of direct products.

PROPOSITION 2.6. *Let A be a group and p prime. If $G \simeq A \times C_{p^n}$ and* $\gcd(p^n, \exp(A)) < p^n$, *then* $\mathbf{B}(G) = \mathbf{B}(A \times C_{p^{n-1}})$.

Proof. For all $m \in \mathrm{Div}(\exp(A))$ with $p \notin \mathrm{Div}(m)$ we have

$$\mathbf{b}_{p^n m}(G) = \mathbf{b}_{p^{n-1} m}(G) = \mathbf{b}_{p^{n-1} m}(A \times C_{p^{n-1}}),$$

so the collection of numbers $\{\mathbf{b}_t(G)\}$ and $\{\mathbf{b}_t(A \times C_{p^{n-1}})\}$ is the same and so is the global breadth. □

With this concept of refinement, we aim at making the set of all groups with the same global breadth more transparent. For instance, Theorem 2.1 can now be formulated in the following form.

COROLLARY 2.7. *If G is a refined group of $\mathbf{B}(G) = 2$, then either $G \simeq C_2 \times C_2$ or $G \simeq D_6$.*

It may be useful to recall here another relation between refined groups and global breadth. This can be found in [6].

THEOREM 2.8 (See [6], Theorem 3.1). *Let G be a refined group of $\mathbf{B}(G) = m \geq 2$. Then the following is true:*

(i) *$|G|$ is not divisible by primes $p > 2m - 1$.*
(ii) *If a prime p with $m < p \leq 2m - 1$ divides $|G|$, then G is a subgroup of* $\mathrm{Hol}(C_p)$ *and* $p - m \in \mathrm{Div}(m - 1)$.
(iii) *If $m \in \pi(G)$, then one of the following conditions may happen:*
  (1) *$G \simeq C_m \times C_m$;*
  (2) *$G \simeq C_m \times U$, where U is a nonabelian subgroup of* $\mathrm{Hol}(C_m)$*;*
  (3) *$m + 1$ is a prime power and $G \simeq L(\mathbb{F}_{m+1})$.*
(iv) *If G is abelian, then* $\exp(G) \in \mathrm{Div}(m)$. *Furthermore, there exists an abelian subgroup of G such that $G/R \simeq C_t \times C_t$, where $t = \exp(G)$.*

3. Preliminaries. In order to formulate the result of the present section, we denote by S_p the Sylow p-subgroup of G, where $p \in \pi(G)$, and will omit the subscript "p" from S_p when the meaning seems clear. Concerning classical results about Sylow's Theory and Hall's Theory, we refer directly to [7, Kapitel I, Kapitel VI].

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The first lemma collects information about the local and global breadth of direct products and subgroups. It turns out to be very useful for computational scopes.

LEMMA 3.1. *Let* X_1 *and* X_2 *be two groups,* $e \in \mathrm{Div}(\exp(X_1 \times X_2))$ *and* $e_i = \gcd(e, \exp(X_i))$ *for* $i = 1, 2$.

(i) $\mathbf{b}_e(X_1 \times X_2) = \gcd(e_1, e_2) \cdot \mathbf{b}_{e_1}(X_1) \cdot \mathbf{b}_{e_2}(X_2)$*;*
(ii) $\mathbf{B}(X_1 \times X_2) \leq \gcd(e_1, e_2) \cdot \mathbf{B}(X_1) \cdot \mathbf{B}(X_2)$*;*
(iii) *if* $\gcd(|X_1|, |X_2|) = 1$*, then* $\mathbf{B}(X_1 \times X_2) = \mathbf{B}(X_1) \cdot \mathbf{B}(X_2)$*, and* $\mathbf{B}(X_1 \times X_2) = \mathbf{B}(X_1)$ *if and only if* X_2 *is cyclic;*
(iv) *if* $\gcd(|X_1|, |X_2|) \neq 1$*, then* $\mathbf{B}(X_1 \times X_2) > \max\{\mathbf{B}(X_1), \mathbf{B}(X_2)\}$.

Proof. See Lemmas 2.1 and 2.2 of [6] for (i) and (ii).

(iii). Assume $\gcd(|X_1|, |X_2|) = 1$. Of course, $\mathbf{B}(X_1 \times X_2) = \mathbf{B}(X_1) \cdot \mathbf{B}(X_2)$ follows from (i). In particular, if $\mathbf{B}(X_1 \times X_2) = \mathbf{B}(X_1)$, then $\mathbf{B}(X_2) = 1$ implies that X_2 is cyclic. Vice versa, if X_2 is cyclic, then even $\mathbf{B}(X_2) = 1$ is true, and so $\mathbf{B}(X_1 \times X_2) = \mathbf{B}(X_1)$.

(iv). We claim that $\mathbf{B}(X_1 \times X_2) > \mathbf{B}(X_1)$. Let $p \in \pi(X_1) \cap \pi(X_2)$ and $\mathbf{B}(X_1) = \mathbf{b}_k(X_1)$ for some multiple k of p. Then

$$\mathbf{B}(X_1 \times C_p) \geq \mathbf{b}_k(X_1 \times C_p) = p\, \mathbf{b}_k(X_1) = p\, \mathbf{B}(X_1).$$

Since $X_1 \times C_p$ is isomorphic to a subgroup of $X_1 \times X_2$, $\mathbf{B}(X_1 \times X_2) \geq \mathbf{B}(X_1 \times C_p) = p\, \mathbf{B}(X_1) > \mathbf{B}(X_1)$ and the claim is true. On the other hand, if k is not a multiple of p, then

$$\mathbf{b}_{pk}(X_1 \times C_p) \geq \mathbf{b}_k(X_1) + p\, \mathbf{b}_p(X_1) > \mathbf{b}_k(X_1) = \mathbf{B}(X_1),$$

and again $\mathbf{B}(X_1 \times X_2) \geq \mathbf{B}(X_1 \times C_p) > \mathbf{B}(X_1)$, as claimed. Symmetrically, we find that $\mathbf{B}(X_1 \times X_2) > \mathbf{B}(X_2)$, hence (iv) follows. □

The importance of refined groups is emphasized by the previous lemma, because the following logic applies, when we want to describe all groups of given global breadth

REMARK 3.2. In this article, we have mainly two types of refined groups:

(A) G such that G/G' is cyclic, $\exp(G) = k|G/G'|$ and $\gcd(k, |G/G'|) = 1$,
(B) G such that G/G' is a 2-group, $\mathbf{B}(G/G') = 2$, and $\exp(G/G') = \exp(S)$ for some Sylow 2-subgroup S of G.

In Case (A), for $|G/G'| = d$, we have firstly the extensions of G' by $C = C_{dm}$ with m prime to k. If r divides the exponent of the Schur multiplier of G and r divides m, we consider an element $y \in C$ of order r and form the central extension H^+ of H via an element z of order r. Then also $H^+/\langle yz \rangle$ is deduced from G. This describes all deduced groups in case (A).

In Case (B), we know that there are nilpotent 2-groups T with $\mathbf{B}(T) = 2$ and quotient group T/V isomorphic to G/G'. Then we form the extension H of G' by T such that $H/V \simeq G$. If the Schur multiplier of G is of even order, then the procedure in Case (A) may also lead to examples of deduced groups. There are no more deduced groups.

We will refer to these descriptions in the following statements whenever applicable. On the other hand, we will find very often extensions of abelian groups when we will compute the breadth of deduced groups, so it is useful to recall [6, Corollary 4.1] in this perspective. This result helps with the computation of the global breadth of $L(\mathbb{F}_q)$, which may be expressed in terms of extensions, but it is useful also for the computation of the global breadth of semidirect products like $G = A \ltimes B$ of two abelian groups A and B such that $C_G(B) = B$ and A isomorphic to a subgroup of Aut(B) (i.e., for instance, $D_{2n} = C_2 \ltimes C_n$ with n odd).

LEMMA 3.3 (See [6], Corollary 4.1). *If $s = p_1^{n_1} \dots p_r^{n_r}$ is a factorization in the product of prime powers of the integer $s \geq 1$ such that $p_i^{n_i} \equiv 1 \mod k$ for all $i = 1, 2, \dots, r$, there is an abelian group N of order s and a cyclic subgroup $W \subseteq \mathrm{Aut}(N)$ such that $|W| = k$ and nontrivial elements of W operate without fixed points on N. The extension G of N by W satisfies the relation $k \cdot \mathbf{B}(G) = (k-1)s + 1$; in particular $2 \cdot \mathbf{B}(G) > s$.*

A starting point for most of the proofs in Section 4 is the following.

LEMMA 3.4. *Let G be a group with $\mathbf{B}(G) = n$ and p an odd prime such that $p^k \geq n$ and S a Sylow p-subgroup of G. If $A \subseteq G$ is an elementary abelian p-subgroup of G of order p^k and $S \supseteq A$ a Sylow p-subgroup of G, then*

(i) *$p^{k-1} < n$;*
(ii) *A and S are normal in G;*
(iii) *S/A is cyclic and $S = \langle A, t\rangle$ with $|A \cap \langle t\rangle| = p$.*

Proof. (i) follows from $p^{k-1} = \mathbf{b}_p(A) \leq \mathbf{b}_p(G) = n$.
For (ii) we claim the following:

If $A \subseteq U \subseteq G$ and A is subnormal in U, then A is normal in U.

Assume this is false and $N_U(A) \neq A$. We choose $x \in N_U(N_U(A)) \setminus N_U(A)$ and derive that $A \neq A^x$ and both normalizes each other. So A^xA is nilpotent of class 2 and of exponent p. Also $|A^xA| \geq p^{k+1}$ and $\mathbf{B}(A^xA) > \mathbf{B}(U)$, a contradiction. So A is a normal subgroup of U and in particular of S. Let $\exp(S) = p^m$. Then

$$p^m\, \mathbf{b}_{p^m}(S) = |S| \leq p^m\, \mathbf{B}(G),$$

and $\mathbf{b}_{p^m}(S) = p^{k-1}$. Put $|S| = p^{m+k-1} = p^w$, where $w = m + k - 1$. Choose a Sylow p-subgroup $T \neq S$ of G such that $|T \cap S| = p^d$ is maximal. There are p^{w-d} conjugates of T with respect to S with pairwise intersection of order $|T \cap S|$ leading to

$$|S| + p^{w-d}(p^w - p^d) = p^{2w-d},$$

p-elements at least in G, and

$$\mathbf{b}_{p^m}(G) \geq p^{2w-d-m} = p^{m+2(k-1)-d} > p^{2(k-1)} > n.$$

This shows that S is a normal subgroup of G, and A is subnormal and normal in G.
For (iii), take $t \in S$ of maximal order p^l. Then

$$p^l\, \mathbf{B}(G) = p^l\, n \geq |S|,$$

and $|S : \langle t\rangle| = p^{k-1}$, showing that $\langle t, A\rangle = S$. Moreover $|S : \langle t\rangle| = p^{k-1}$ implies $A \cap \langle t\rangle \neq 1$ and so $|A \cap \langle t\rangle| = p$. □

When G has odd order, can we say that $\mathbf{B}(G)$ is odd as well? The answer is positive and turns out to be a consequence of the next lemma.

LEMMA 3.5. *Assume that G is a group, $n \in \mathrm{Div}(\exp(G))$, $k \geq 1$, $s \geq 1$, $p, p_1, \dots, p_s \in \mathbb{P}$.*

(i) *If $n = p^k$, then we have $p - 1 \in \mathrm{Div}(\mathbf{b}_n(G) - 1)$.*

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(ii) *If* $n = p_1^{k_1} \dots p_s^{k_s}$ *and* $t \in \mathrm{Div}(p_i - 1)$ *for all* $i = 1, \dots, s$, *then we have* $t \in \mathrm{Div}(\mathbf{b}_n(G) - 1)$.

Proof. (i) If $y \in G$ and $|\langle y \rangle| = p^k$, then $\langle y \rangle$ contains $(p-1)p^{k-1}$ many elements of (exact) order p^k. Now it follows by induction on k, that the number a of all elements satisfying $y^n = 1 \neq y$ is divisible by $p-1$, and the addition of 1 leads to

$$a + 1 = n \cdot \mathbf{b}_n(G) \Rightarrow 1 \equiv \mathbf{b}_n(G) \mod (p-1).$$

From this, the result follows.

(ii) Let $n = pm$ and $ps \in \mathrm{Div}(n) - \mathrm{Div}(m)$ for some $m \geq 1$. If $y \in G$ and $|\langle y \rangle| = ps$, then $\langle y \rangle$ contains $(p-1)s$ many elements of order not dividing s and this number is divisible by t. In analogy to (i) above, we obtain

$$\mathbf{b}_n(G) \equiv 1 \mod t.$$

□

Corollary 3.6. *Let G be a group. If* $|G|$ *is odd, so is* $\mathbf{B}(G)$.

Proof. Application of Lemma 3.5 (ii) for $t = 2$. □

We note that the converse of Corollary 3.6 is false, as shown in the following.

Remark 3.7. $\mathbf{B}(G)$ is odd for $G \simeq D_{2^n}$ for all $n \geq 3$, but here $|G|$ is even.

The use of Corollary 3.6 allows us to say more on the nilpotent case.

Proposition 3.8. *If G is a refined nilpotent group of* $\mathbf{B}(G) = 8$, *then G is a* 2-*group.*

Proof. G is the direct product of its Sylow 2-subgroup S and a group W of odd order, so $\mathbf{B}(W)$ is odd by Corollary 3.6 and $\mathbf{B}(G) = \mathbf{B}(S) \cdot \mathbf{B}(W)$ by Lemma 3.1 (iii). Now $\mathbf{B}(W) = 1$; since G is refined, we have $W = 1$. Therefore, $\mathbf{B}(G) = \mathbf{B}(S)$ and G is a 2-group. □

Proposition 3.8 gives a strong arithmetic condition for refined nilpotent groups of global breadth eight. In the next section, we will prove similar arithmetic conditions, but removing the assumption of being nilpotent.

Remark 3.9. Notice the necessity of being refined by Lemma 3.1(iii). In fact Proposition 3.8 is not true if the group G is nonrefined. An example is $\mathbf{B}(G \times C_q) = \mathbf{B}(G)$ by Lemma 3.1, when q is a prime not dividing $|G|$.

We end with a lemma, which will be very useful.

Lemma 3.10. *If S is a cyclic Sylow p-subgroup of a group G and* $|G : N_G(S)| = p + 1$, *then*

(i) $C_G(S)^p = Z(G)$,
(ii) *for any conjugate* $T \neq S$ *of S*, $N_G(S) \cap N_G(T) = Z(G)$ *or*

$$|G : N_G(N_G(S) \cap N_G(T))| = \frac{p(p+1)}{2}.$$

Proof. By conjugation, G operates as a group of permutations on the $p+1$ Sylow p-subgroups of G; therefore, we have a homomorphism $\sigma : G \to R \subseteq \mathrm{Sym}(p+1)$ and $R \simeq G/\ker(\sigma)$. Now $\ker(\sigma)$consists of all elements fixing all Sylow p-subgroups. In $\mathrm{Sym}(p+1)$ and in R, all Sylow p-subgroups are self-centralizing, so $C_G(S) \cap C_G(T) \subseteq \ker(\sigma) = Z(G)$ and also $Z(G) = C_G(S)^p$. This shows (i).

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On the other hand, if $N_G(S) \cap N_G(T) \neq Z(G)$ for some conjugate T, in particular $N_G(S)/C_G(S)$ is cyclic and, if $\langle t, C_G(S)\rangle = N_G(S)$, then $(x^{-1}\langle t\rangle x) \cap \langle t\rangle \subseteq Z(G)$ for all $x \in C_G(S) \setminus (C_G(S))^p$. The intersections of any two different pairs of conjugates of normalizers of Sylow p-subgroups are conjugate since for every third conjugate U we have $y^{-1}Ty = U$ for some $y \in S$ and $N_G(S) \cap N_G(T) = t^{-1}(N_G(S) \cap N_G(U))t$, and the same can be done for every other Sylow p-subgroup. Since we have $\frac{p(p+1)}{2}$ such pairs, (ii) is true. □

4. Proof of the Theorem of Deduction. The present section is devoted to show Theorem 1.3 and an instructive example, motivated by the argument that is illustrated in the following.

Proof of Theorem 1.3. We begin with a special case.

(a) Let N be an elementary abelian normal subgroup of G of order p^n.

(i) If $n \geq 2$, then $\mathbf{B}(G/N) < \mathbf{B}(G)$.

(ii) If $n = 1$, then $\mathbf{B}(G/N) \leq \mathbf{B}(G)$.

(iii) If $\mathbf{b}_k(G/N) = \mathbf{B}(G/N) = \mathbf{B}(G)$, then $n = 1$ and $\mathbf{B}(G) = \mathbf{b}_{pk}(G)$.

In order to show (i) and (ii), put $\mathbf{B}(G/N) = \mathbf{b}_k(G/N)$ for some divisor k of $\exp(G/N)$, where we recall that $k \cdot \mathbf{b}_k(G/N) = |L_k(G/N)|$ by definition. Then $L_k(G/N)\ N \subseteq L_{pk}(G)$ and so

$$|L_{pk}(G)| = |\{g \in G \mid g^{pk} = 1\}| \leq |\{gN \mid (gN)^k = N\}\ N| = |L_k(G/N)\ N|.$$

Now, either $\mathbf{b}_{pk}(G)$ does not exist and $\mathbf{b}_p(G) = p^n \cdot \mathbf{b}_k(G/N)$, or $\mathbf{b}_{kp}(G) \geq p \cdot \mathbf{b}_k(G/N)$. In both cases, we have $\mathbf{B}(G) > \mathbf{B}(G/N)$ and (i) follows. Note that if $n = 1$, then the same argument shows that $\mathbf{B}(G) \geq \mathbf{B}(G/N)$ is possible. Therefore, (ii) follows. For the proof of (iii), we have

$$\mathbf{B}(G/N) = \mathbf{b}_k(G/N) = \frac{|L_k(G/N)|}{k} = \frac{|L_{pk}(G)|}{k\ |N|} = \mathbf{B}(G)\ \Rightarrow\ n = 1.$$

On the other hand, we may consider $M = \{x \in G \mid x^k \in N\}$. Clearly $L_k(G) \subseteq M \subseteq L_{pk}(G)$ and $|M| = p \cdot |L_k(G)|$, so $\mathbf{b}_k(G) > \mathbf{b}_k(G/N) = \mathbf{B}(G/N)$ if $M = L_k(G)$. On the other hand, if $M \neq L_{pk}(G)$, then $\mathbf{b}_{pk}(G) > \mathbf{b}_{pk}(G/N) = \mathbf{B}(G/N)$. Now $\mathbf{B}(G) = \mathbf{B}(G/N)$ yields $M = L_{pk}(G)$ and (iii) follows completely.

Up to this point, we may conclude that the result is proved when G contains a subgroup like N in (a). This case happens when G is a soluble group, looking at N as a minimal normal subgroup of G. In order to attack the general case, another step is the proof of the following claim:

(b) Let N be a normal subgroup of G and $\mathbf{B}(G) = \mathbf{B}(G/N)$. Then all Sylow subgroups of N are cyclic.

Assume that $|N|$ is even and that D is a Sylow 2-subgroup of N. Of course, if D is cyclic, there is nothing to prove. Consider $N_G(D) = T$ with D noncyclic. By the Frattini Argument [7, Satz 7.8, p.35], we have $TN = G$ and $G/N = TN/N \simeq T/(T \cap N)$. Since $(T \cap N)/D$ is of odd order, $(T \cap N)/D$ is soluble by a well-known theorem of Thompson and Feit [7, See p. 128]. On the other hand, $T \cap N$ is soluble, because the derived series of D and the derived series of $(T \cap N)/D$ allows us to have a normal series of $T \cap N$ with abelian factors. Note that D is a normal subgroup of T and $\Phi(D)$ is a characteristic subgroup of D,

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so we may form the quotient $T/\Phi(D)$. Since D is noncyclic, $D/\Phi(D)$ must be elementary abelian of rank ≥ 2 and (i) of (a) may be applied, getting

$$\mathbf{B}(T/D) = \mathbf{B}((T/\Phi(D))/(D/\Phi(D))) < \mathbf{B}(T/\Phi(D)).$$

Therefore, there is a normal series of T through $T \cap N$ with a noncyclic elementary abelian 2-quotient such that

$$\mathbf{B}(T) > \mathbf{B}(T/(T \cap N)) = \mathbf{B}(G/N).$$

On the other hand, T is a subgroup of G; therefore, $\mathbf{B}(T) \leq \mathbf{B}(G)$ and

$$\mathbf{B}(T) \leq \mathbf{B}(G) = \mathbf{B}(G/N) = \mathbf{B}(T/(T \cap N)),$$

which gives contradiction.

We have deduced that the Sylow 2-subgroup D of N must be cyclic. This means also that N is 2-nilpotent and so soluble. Note that groups of odd order are soluble, as mentioned before. Then the argument works perfectly beginning from N soluble and p arbitrary prime dividing $|N|$ and D arbitrary Sylow p-subgroup of N. Therefore, all Sylow subgroups of N are cyclic, regardless N is of even order or of odd order, and (b) follows.

Now we attack the general case.

(c) If N is an arbitrary normal subgroup of G, then (i) and (ii) are true.

If N is of odd order, then it is soluble by a well-known result of Thompson and Feit (see [7]). A minimal normal subgroup M of G is a chief factor of G, so elementary abelian. Then we may apply (a) to M and we get $\mathbf{B}(G/M) \leq \mathbf{B}(G)$ by (i) of (a) and M cyclic by (iii) of (a). The result follows in this case.

If N is of even order, the idea is to adapt the argument which we have seen in (b) above. First of all, we may choose a Sylow 2-subgroup D of N and consider $N_G(D)$, noting that D is normal in G. Using the Frattini Argument, we have $N_G(D)N = G$ and therefore $G/N \simeq N_G(D)/(N_G(D) \cap N)$. We have

$$\mathbf{B}(G/N) = \mathbf{B}(N_G(D)/(N_G(D) \cap N)) \leq \mathbf{B}(N_G(D)) \leq \mathbf{B}(G),$$

and so $\mathbf{B}(G/N) \leq \mathbf{B}(G)$ follows.

Assume $\mathbf{B}(G/N) = \mathbf{B}(G)$. By (b), all Sylow subgroups of N are cyclic and, in particular, N is metacyclic. If N is abelian, then the result follows. We are going to see that it is not possible that $N' \neq 1$.

Assume that $N' \neq 1$ and consider a Hall subgroup H of N which is a complement of N' in N. Hall subgroups of the same order of a soluble group are conjugate (see [7, Kapitel VI, Sections 1, 2, and 3] for the properties of Hall subgroups in soluble groups). By Frattini's Argument $N_G(H)N = N_G(H)N' = G$ and by construction, H is self-normalizing in N, so $N_G(H)$ is a complement of N' in G. Let $\mathbf{B}(G/N) = \mathbf{b}_k(G/N)$ for some $k \geq 1$ and $w = |N/N'|$. Here $N_G(H)$ is not a normal subgroup of G and $H \neq x^{-1}Hx$ for all $x \in N' \setminus \{1\}$. This shows that N has more than $|H|$ elements of order dividing w than $|H|$ and accordingly G has more elements of order dividing kw than $N_G(H)$. Therefore, we may argue as in (iii) of Case (a) above, in fact

$$\mathbf{B}(G) \geq \mathbf{b}_{kw}(G) > \mathbf{b}_{kw}(N_G(H)) = \mathbf{B}(N_G(H)) = \mathbf{B}(G/N') = \mathbf{B}(G),$$

so that $L_{kw}(N_G(H)) = L_{kw}(G)$ and $L_w(H) = L_w(N)$. This shows that H is a normal subgroup of N, $N' = 1$, and $N = H$, contrary to our assumption. The result follows completely. □

As previously noted, we end with an instructive example.

EXAMPLE 4.1. Looking at the first part of the proof above, if G is a group possessing a cyclic normal subgroup N of order p and k is a divisor of $\exp(G/N)$, then we had the following implication:

$$\mathbf{B}(G/N) = \mathbf{B}(G) \text{ and } \mathbf{b}_k(G/N) = \mathbf{B}(G/N) \Rightarrow \mathbf{B}(G) = \mathbf{b}_{pk}(G).$$

One may wonder whether the same implication is true removing the condition $\mathbf{B}(G/N) = \mathbf{B}(G)$. This is not possible.

In fact, if $G = \mathrm{Alt}(4) \times C_2$ and $N \simeq C_2$ with $p = 2$ and $k = 2$, then one can use [**6**, List at p. 220], in order to see that $\mathbf{B}(G) = 4$ and $\mathbf{B}(G/N) = \mathbf{B}(\mathrm{Alt}(4)) = 3$. Then $\mathbf{B}(G) \neq \mathbf{B}(G/N)$ and $\mathbf{b}_4(G/N) = \mathbf{B}(G/N) = 3$, but $\mathbf{B}(G) = \mathbf{b}_2(G) = 4$ is different from $\mathbf{b}_4(G) = 2$.

5. The classification of 2-groups of global breadth four.

Proof of Theorem 1.4. Assume first that G is a direct product. Then $4 = \mathbf{B}(G) = \mathbf{B}(X \times Y) \geq 2\ \mathbf{B}(X)\ \mathbf{B}(Y)$. Without loss of generality, we may assume $\mathbf{B}(X) = 1$. For $X \simeq C_2$, we obtain $\mathbf{B}(Y) = 2$, while for $X \simeq C_4$ we obtain $Y \simeq C_{4k}$ or $Y \simeq C_2 \times C_2$. This shows (a) and (b).

Assume now that G is not a direct product. Let $U = \langle c \rangle$ be a cyclic subgroup of maximal order, that is, $|U| = \exp(G)$. Now $\mathbf{b}_{|U|}(G) = |G : U| \leq 4$ and G is nonabelian. If $|U| > 8$ and $|G : U| = 2$, then G is dihedral or quaternion and $\mathbf{B}(G)$ is odd, a contradiction. So $|G : U| = 4$. If U is not invariant in G, then $U \subset V \subset G$, where $V = \langle b, U \rangle$ and without loss of generality $b^2 = 1$ and V is abelian or $|[V, b]| = 2$. Further $G = \langle a, V \rangle$ and $[G, a] \not\subseteq U$. So without loss of generality $[a, c] = b$ and we have case (c). If $U = Z(G)$ and G is nonabelian, then G/U is noncyclic and

$$G \simeq \langle a, b, c \mid a^2 = b^2 = c^{4k} = c^{2k}[a, b] = [a, c] = [b, c] = 1 \rangle,$$

which is (d). Notice that in case (d) we have $\langle a, bc^{2k} \rangle \simeq D_8$ and $\langle ac^{2k}, bc^{2k} \rangle \simeq Q_8$, so all cases with three generators are considered. The remaining cases are easily seen to be belong to this class of groups, also it is clear that the list is exhaustive.

By Meng [**11**, Main Theorem (ii)], we have $|G| = 4\ \exp(G)$ for 2-groups of global breadth four. The only refined abelian groups H with $\mathbf{B}(H) = 4$ are those mentioned in (a) and (b). The nonabelian groups of exponent 4 are isomorphic to those mentioned in (c) and (d). The remaining cases follow from the claim.

Claim. There is no refined 2-group H such that $\exp(H) \geq 4$ and $\mathbf{B}(H) = 4$.

Let $\exp(H) = 8$ and choose $x \in H$ of order 8, further let $\langle x \rangle = L \subset K \subset H$. Then $K = \langle x, y \rangle$ for some y and $y^2 \in L$. We obtain $[y, x] = x^m$ for even m. For $m = 2$, we obtain D_{16} or Q_{16} and $\mathbf{B}(D_{16}) = 5$ makes the first one impossible. In the second case, $\mathbf{B}(Q_{16}) = \mathbf{B}(D_8) = 3$ and L is characteristic in K. If $C_H(K)L = H$, then $H \simeq C_2 \times D_8$ and $\mathbf{B}(H) = 6$, which gives contradiction. If $C_H(K) = K$, then we have $z \in H$ such that $z^2 = 1$ and $[z, x] = x^4$. We compute the breadth: elements of exact order 2 are x^4, z, zx^2, zx^4, zx^6, so $\mathbf{b}_2(H) = 3$, elements of exact order 4 are $x^2, x^6, zx, zx^3, zx^5, zx^7$ and all elements of yL, so $\mathbf{b}_4(H) = 5$, so these two cases are excluded for H. It remains to consider the case that K is abelian or $K' = L^4 = \langle l^4 \mid l \in L \rangle$. If H is abelian, it is not refined. If K (and not H) is abelian, we have z with $[x, z] = y$ or $[x, y] = x^4$. Here $\mathbf{B}(H) = \mathbf{B}(H/L^4)$ and H is not refined. The same is true in the two remaining cases. We found that all groups H with

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$\exp(H) = 8$ are not refined. For H with $\exp(H) \geq 16$, the only new two-generator group H appearing is

$$\langle x, y \mid x^4 = y^{16m} = y^{4m}[x, y] = 1 \rangle,$$

and here $\mathbf{B}(H/\langle y^4 \rangle) = \mathbf{B}(H)$. □

Note that if N is a normal subgroup of a group G of $\mathbf{B}(G) = 4$ such that $G/N \simeq C_2 \times C_2$, then the condition $\exp(N) = \exp(G)$ is possible in cases (a),(c),(d) of Theorem 1.4.

6. The role of odd primes less than 7. Before entering into the details of the proofs, we recall a well-known fact.

REMARK 6.1. If H is a subgroup of the group G, then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H (see [7, Kapitel I, Satz 4.5]). Note also that $\mathrm{Aut}(C_p^n) = \mathrm{GL}(n, \mathbb{F}_p)$, where C_p^n is the elementary abelian p-group of rank n (see [7, Kapitel II, Section 6]). Specializing to $p = 3$ and $n = 2$, we find $|\mathrm{GL}(2, \mathbb{F}_3)| = 48$ and its noncyclic subgroups are isomorphic to one of the following: D_6, D_8, $C_2 \times C_2$, D_{12}, Q_8, SD_{16}, $\mathrm{SL}(2, \mathbb{F}_3)$, where SD_{16} denotes the semidihedral group of order 16 (see [7, Satz 13.10, Kapitel III, Section 13] and [7, Satz 14.9, Kapitel I, Section 14]).

We have the following restriction on the primes dividing $|G|$, if $\mathbf{B}(G) = 8$.

PROPOSITION 6.2. *A group G cannot be refined if $\mathbf{B}(G) = 8$ and $p \in \mathrm{Div}(\exp(G))$ where $p > 7$.*

Proof. Assume that G is refined and p is a divisor of $|G|$. Then $p > 15$ is impossible by Theorem 2.8 (i), and $p \in \{11, 13\}$ is impossible by Theorem 2.8 (ii) since $p - 8 \notin \mathrm{Div}(7)$ in both cases. Then the result follows. □

Now we turn to the groups themselves, characterizing first the refined groups and then, for every case, the deduced groups, so that the first half of each statement is connected with Theorem 1.5, while the second half with Theorem 1.6. Whenever the word *extension* is used here, it means an extension having the refined group as epimorphic image. In some cases, we will note different non-isomorphic possibilities. Then all of these possibilities are included.

THEOREM 6.3. *Let G be a nonabelian group of $\mathbf{B}(G) = 8$, and p odd prime. If the noncyclic Sylow p-subgroups of G are noncyclic, then $p = 3$. Moreover,*

(i) *if G is refined, then either $G \simeq L(\mathbb{F}_9)$ or $G \simeq D_6 \times D_6$;*
(ii) *if G is deduced from $L(\mathbb{F}_9)$ or from $D_6 \times D_6$, then either G is an extension of $C_3 \times C_3$ by C_{8m} with $\gcd(m, 3) = 1$, or G is an extension of $C_3 \times C_3$ by a $3'$-group $T \in \mathcal{D}$.*

Proof. (i). Applying Lemma 3.4 (with $k = 2$), we have $|N| = p^2$ and there is a maximal subgroup M of S which is cyclic, further $S' = [S, S] \subseteq M \cap N$ and S is nilpotent of class at most 2 and $M^p \subseteq Z(S)$.

Let T be a complement of S in G. There are p maximal subgroups of S that are cyclic, and at least one of them is normalized by T; let this be M. Note that $T \subseteq C_G(S)$ is impossible, since then $G = S \times T$ and $\mathbf{B}(G) = p \cdot \mathbf{B}(T) \neq 8$ by Lemma 3.1. If $T \not\subseteq C_G(M)$, then there is an element $x \in T$ such that $\langle x, M \rangle$ is nonabelian and

$$\mathbf{B}(\langle x, M \rangle) \geq \frac{|M| + 1}{2},$$

so this happens only if $|M| \leq 15$, since $\mathbf{B}(G) = 8$.

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We have derived for $|M|>p$: $|M|=9$ or $M\subseteq C_G(T)$. If $M\subseteq C_G(T)$, we consider the p subgroups of order p not contained in M. Not all of them are fixed by conjugation by elements in T, otherwise $S\subseteq C_G(T)$. Let V be one such subgroup. Then $V\subseteq G'$ and $[V,T]=1$ since $G/C_G(M)$ is abelian as subgroup of $\mathrm{Aut}(M)$. We deduce that $M\subseteq Z(G)$ and $G=VT\times M$. This is not refined if $|M|>p$. We summarize $|M|=9$ or $N=S$. Now we split into the cases $p\in\{5,7\}$ and $p=3$.

Case 1: The prime p equals either 5 *or* 7.

Here we have reduced to $S=N\simeq C_p\times C_p$, elementary abelian of order p^2, and, as before, $S\not\subseteq Z(G)$. Since

$$8\geq \mathbf{B}(C_G(S))=\mathbf{B}(S)\cdot\mathbf{B}(C_T(S)),$$

we have $C_T(S)=1$, so $C_T(S)$ is cyclic. We consider first the case $S\cap Z(G)=1$. If we have $x\in T\setminus C_T(S)$ with $x^q\in C_T(S)$ with q a prime, and $[x,S]=S$, we obtain $\mathbf{B}(\langle x,S\rangle)>8$, a contradiction. If $x^4\in C_T(S)$ and $[x,S]=S$, we have $2p^2$ elements of order 4 modulo $C_T(S)$ and

$$\mathbf{B}(\langle x,S\rangle)\geq\frac{p^2}{2}>8$$

again a contradiction. For $p=7$, we have to consider also $x^6\in C_T(S)$ and $[x^3,S][x^2,S]=S$, $[x^3,S]\cap[x^2,S]=1$ and this yields to the presence of two subgroups $X\simeq D_{14}$ and Y of $\langle x,S\rangle$ such that

$$\mathbf{B}(\langle x,S\rangle)=\mathbf{B}(D_{14})\cdot\mathbf{B}(Y)=4\cdot 5=20>8.$$

Note that $D_{14}=C_2\ltimes C_7$ has $\mathbf{B}(D_{14})=4$ by Lemma 3.3. For every other case of $G/C_G(S)$ with $[x,S]=S$, we have a subgroup of G which is of one of the previous forms. The contradictions above lead to the following restriction: $S\cap Z(G)\neq 1$. This means $G=\langle x,C_G(S)\rangle$, $x^{p-1}\in C_G(S)$, and $|[x,S]|=p$. As mentioned before $C_T(S)$ is cyclic, where T is a complement of S in G. The quotient group G/S cannot be cyclic since otherwise $\mathbf{B}(G)=p$. If G/S is abelian and noncyclic, then $C_T(G)\subseteq Z(G)$ and there is a noncyclic q-subgroup in G/S, where q divides $|G/C_G(S)|$; therefore

$$\mathbf{B}(G)\geq qp>8.$$

So G/S is nonabelian and the number of conjugates of x increases by a factor,

$$\mathbf{B}(G)\geq 2\cdot\mathbf{B}(\langle x,S\rangle)=2p>8.$$

Again this is impossible. Then S is cyclic of order p if $p=5$ or 7. So we have just proved that there is no refined group G of $\mathbf{B}(G)=8$ and noncyclic Sylow p-subgroup (when $p\in\{5,7\}$).

Case 2: The prime p equals 3.

We claim the following:

Any noncyclic Sylow 3-subgroup of G with $\mathbf{B}(G)=8$ is of order 9.

Clearly this means in the notation above $S=N$. Assume that this is false and there is a subgroup M, maximal in S, and cyclic and normal in G, with $|M|>3$. If $G/C_G(M)$ is a 3-group, then G is not refined; in particular, S cannot be a direct factor of G. If $G/C_G(M)$ is not a 3-group, there is $x\in G\setminus C_G(M)$ of order a power of 2 and $\mathbf{B}(\langle x,M\rangle)=\frac{1}{2}(|M|+1)$. This shows $|M|=9$. The quotient group G/S must be a 2-group with $\mathbf{B}(G/S)\leq 2$;

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therefore, G is supersoluble, because of the series $1 \lhd S^3 \lhd M \lhd S \lhd G$ with $|S^3| = |S/S^3| = 3$ and $|G/S| = 2$.

Note from Theorem 2.1 that a 2-group D with $\mathbf{B}(D) = 2$ is always generated by two elements, since otherwise the formula $\mathbf{B}(\langle u, D\rangle) = \frac{1}{2}(|D| + 1)$ would give contradiction (with u such that $D \subset \langle u, D\rangle$).

Therefore, we may conclude that $G/SC_G(S)$ is of order 2 or elementary abelian of order 4. If $|G/SC_G(S)| = 4$, then G possesses an element y such that $C_G(y) \cap S = 1$ and $\mathbf{B}(\langle y, S\rangle) = 14$, which is impossible. If $|G/C_G(M)|$ is not even, then $\mathbf{B}(G) = \mathbf{B}(G/M^3)$ and G is not refined. It remains $G' = M$ and $\mathbf{B}(G/S) = \mathbf{B}(G/M)$ since $G/S \simeq G/S \times S/M$. Let us call again D a Sylow 2-subgroup of G; obviously $\mathbf{B}(D) = \mathbf{B}(G/S)$. If $\mathbf{B}(D) = 1$, then $|D| = 2$ since G is refined, and $\mathbf{b}_2(G) = 5$; $\mathbf{b}_3(G) = 3$; $\mathbf{b}_6(G) = 6$; $\mathbf{b}_{18}(G) = 3$. If $\mathbf{B}(D) = 2$, then D is not a quaternion group since G is refined (recall that Q_8 is not refined because $\mathbf{B}(Q_8) = \mathbf{B}(Q_8/Z(Q_8)) = 2$, see [6, Example 2.9 (iv)] for details), so $\mathbf{b}_2(D) = 2$ and $D \simeq C_2 \times C_2$, again since G is refined. We obtain $G \simeq D_{18} \times C_6$, and putting $e_1 = \gcd(12, 18) = 6$, $e_2 = \gcd(12, 6) = 6$, we find from Lemma 3.1 that

$$\mathbf{B}(G) \geq \mathbf{b}_{12}(D_{18} \times C_6) = \gcd(e_1, e_2) \cdot \mathbf{b}_6(D_{18}) \cdot \mathbf{b}_6(C_6) = 6 \cdot 2 \cdot 1 = 12 > 8.$$

This contradiction proves the claim.

Therefore, we have $S = N$ if $p = 3$, and $\mathbf{B}(G/S) \leq 2$. If G/S is cyclic, we deduce again that $C_G(S) = S$ since G is refined, and $\mathbf{B}(G) = 8$ if $|G/S| = 8$. If $\mathbf{B}(G/S) = 2$, we see at once $\mathbf{B}(D_6 \times D_6) = 8$ (for instance, Corollary 2.7 shows $\mathbf{B}(D_6) = \mathbf{b}_2(D_6) = 2$ and Lemma 3.1 shows that $\mathbf{B}(D_6 \times D_6) = \mathbf{b}_6(D_6 \times D_6) = \gcd(6, 2) \cdot \mathbf{b}_2(D_6) \cdot \mathbf{b}_2(D_6) = 2 \cdot 2 \cdot 2$) for $G/S \simeq C_2 \times C_2$. Clearly $G/C_G(S)$ must be isomorphic to a 2-subgroup of $\mathrm{GL}(2, \mathbb{F}_3)$ (note Remark 6.1) and it cannot be D_8 or the full Sylow 2-subgroup K of S since $\mathbf{b}_2(D_8) = 3$ and $\mathbf{b}_2(K) = 7$. The extension E of S by Q_8 (with $Z(E) = 1$) is not allowed, since $\mathbf{B}(E) = 16$. Collecting all these information, plus the fact that G is refined and $Z(G) = 1$, whenever $\mathbf{B}(G/S) = 2$, we deduce that the only possible solutions are $G \simeq L(\mathbb{F}_9)$, which has $\mathbf{B}(L(\mathbb{F}_9)) = 8$ by Lemma 3.3, or $G \simeq D_6 \times D_6$, as we have seen.

(ii) If the refined group G is isomorphic to $L(\mathbb{F}_9)$, we are in a situation like Case (A) of Remark 3.2. The Schur multiplier of G is trivial, so all deduced groups are split extensions of G' by a group isomorphic to some C_{8m} with $gcd(m, 3) = 1$. If the refined group is isomorphic to $D_6 \times D_6$, then we have Case (B) of Remark 3.2. The Schur multiplier has order 2, we obtain here the extension of $C_3 \times C_3$ by Q_8 with center of order 2. The result follows. □

When we do not have information on the presence of noncyclic Sylow subgroups, a different argument must be used. The following result fits this scope, illustrating what happens when the global breadth is 8 and the prime 7 appears among the divisors of the order of the group.

THEOREM 6.4. *Let G be a nonabelian group of* $\mathbf{B}(G) = 8$ *and* $7 \in \pi(G)$.

(i) *If G is refined and* $7 \in \pi(G)$*, then either* $G \simeq L(\mathbb{F}_8) \times C_2$ *or* $G \simeq D_{14} \times C_2$.
(ii) *If G is deduced from one of the groups in (i) above, then either G is an extension of* $C_2 \times C_2 \times C_2$ *by* C_{14m}*, or G is an extension of* C_7 *by a* $7'$*-group* $T \in \mathcal{D}$.

Proof. (i). The Sylow 7-subgroup S of G is cyclic of order 7 by Theorem 6.3. Now $|G : N_G(S)| = 7k + 1$ by the third Sylow Theorem, and

$$\mathbf{b}_7(G) = \frac{6(7k+1)+1}{7} = 6k + 1 \leq 8,$$

implies $k < 2$. We have to consider $k = 1$ and $k = 0$.

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If $k = 1$ and $|N_G(S) \cap N_G(T)| = d \neq 1$ for some conjugate T of S, then we apply Lemma 3.10 and obtain 28 conjugates of this intersection, and

$$\mathbf{b}_d(G) \geq \frac{28(d-1)+1}{d} > 14,$$

since $d \in \{2, 3, 6\}$. So $d = 1$. Again by Lemma 3.10, $|G/Z(G)| = 56$ and G is 7-nilpotent. Thus $G/Z(G) \cong L(\mathbb{F}_8)$ and $G \simeq L(\mathbb{F}_8) \times C_2$.

If $k = 0$, then $|G : N_G(S)| = 1$ and S is a normal subgroup of G. Here S cannot be contained in $Z(G)$, otherwise we apply Proposition 2.3 and G cannot be refined. Looking at Remark 6.1, $|G/C_G(S)| = |N_G(S)/C_G(S)|$ divides $6 = |\mathrm{Aut}(C_7)| = |\mathrm{Aut}(S)|$. If $|G/C_G(S)^7| = 3$, then $\mathbf{B}(G/C_S(G)^7) = \mathbf{b}_3(G/C_S(G)^7) = 5$, and the elements of order 3 generate the quotient group $G/C_S(G)^7$. Let $x \in G \setminus C_G(S)^7$ be an element of order 3^d for some $d \geq 1$. The index $|G : \langle x\rangle|$ must be a multiple of 7. Assume that the index is at least 14. Then the number of elements of order 3^d is at least

$$3^{d-1} + 14(3^d - 3^{d-1}) = 29(3^{d-1}) > 9 \cdot 3^d,$$

and this implies that

$$\mathbf{B}(G) \geq \mathbf{b}_{3^d}(G) = \frac{|\{y \in G \mid y^{3^d} = 1\}|}{3^d} \geq \frac{9 \cdot 3^d}{3^d} = 9,$$

which is in contradiction with $\mathbf{B}(G) = 8$. So $|G : \langle x\rangle| = |G : N_G(\langle x\rangle)| = 7$ and $\langle x, S\rangle$ is a normal subgroup of G, that is, $G = \langle x, S\rangle \times C_G(S)^7$, but now again $\mathbf{B}(G) = 8$ is impossible, because (as before) the presence of $\langle x, S\rangle$ implies that the number of elements of order 3^d is at least $9 \cdot 3^d$ and this gives a $\mathbf{b}_{3^d}(G) > 8$. Then $|G/C_G(S)^7| = 3$ cannot happen. By a similar argument, we can see that $|G/C_G(S)^7| = 6$ is impossible.

It remains the case $|G/C_G(S)^7| = 2$. Here we obtain $\mathbf{B}(D_{14}) = 4$ (for instance, apply Lemma 3.3 to $D_{14} = W \ltimes N = C_2 \ltimes C_7$ with $k = 2$ and $s = 7$, in order to find $\mathbf{B}(D_{14}) = 4$) and so $\mathbf{B}(D_{14} \times C_2)$ as the refined example. Then (i) is shown.

(ii) If $G \simeq L(\mathbb{F}_7) \times C_2$, we have case Remark 3.2 (A) and the deduced groups are the extensions of $C_2 \times C_2 \times C_2$ by C_{14m} (no restrictions to m). If $G \simeq D_{14} \times C_2$, we have case Remark 3.2 (B). The Schur multiplier has order 2, the corresponding extension of C_7 by Q_8 is included in the extension by nilpotent $7'$-groups T, where $T \in \mathcal{D}$. Notice that here two non-isomorphic extensions may appear for $16 \in \mathrm{Div}(T)$ according to the centralizer of C_7 being cyclic or not. The result follows. □

In the same spirit of Theorem 6.4, we describe what happens when the global breadth is 8 and the prime 5 appears between the divisors of the order of the group.

THEOREM 6.5. *Let G be a nonabelian group of* $\mathbf{B}(G) = 8$ *and* $5 \in \pi(G)$.

(i) *If G is refined and* $|G|$ *is divisible by* 5, *then*

(a) $G \simeq \mathrm{Hol}(C_5) \times C_2$;
(b) $G \simeq D_{30}$;
(c) $G \simeq \mathrm{Alt}(5)$.

(ii) *If G is deduced from one of the group in (i) above, then*

(a) *G is the extension of* C_5 *by a* $5'$*-group* T *such that* $T \in \mathcal{D}$;
(b) *G is the extension of* C_{15} *by* C_{2m} *with* $\gcd(30, m) = 1$;
(c) $G \simeq \mathrm{Alt}(5) \times C_m$ *with* $\gcd(30, m) = 1$; *or*
(c') $G \simeq \mathrm{SL}(2, \mathbb{F}_5) \times C_m$ *with* $\gcd(30, m) = 1$.

Proof. (i) Repeating the argument of the proof of Theorem 6.4, we obtain that the Sylow 5-subgroup S of G has order 5, and so $|G:N_G(S)| \in \{1, 6\}$. Moreover, S is not contained in $Z(G)$ (otherwise Proposition 2.3 gives contradiction with the assumption that G is refined).

We consider first the situation that S is a normal subgroup of G, that is, $|G: N_G(S)| = 1$. Here $|N_G(S)/C_G(S)| = |G/C_G(S)|$ is a divisor of $4 = |\mathrm{Aut}(S)| = |\mathrm{Aut}(C_5)|$ (see Remark 6.1). If $|G/C_G(S)| = 4$, then $G \simeq \mathrm{Hol}(C_5) \times C_2$, and this is the only possibility. Note that $\mathbf{B}(\mathrm{Hol}(C_5)) = \mathbf{B}(C_4 \ltimes C_5) = 4$ by Lemma 3.3 with $W = C_4$, $N = C_5$, $k = 4$, and $s = 5$. Then Lemma 3.1 gives the following:

$$\mathbf{B}(G) = \gcd(|\mathrm{Hol}(C_5)|, |C_2|) \cdot \mathbf{B}(\mathrm{Hol}(C_5)) \cdot \mathbf{B}(C_2) = 2 \cdot 4 \cdot 1 = 8.$$

On the other hand, the structure of $\mathrm{Hol}(C_5) \times C_2 \simeq (C_4 \ltimes C_5) \times C_2$ shows that its nontrivial quotients are isomorphic either to $C_4 \ltimes C_5$, or to C_2, or to $C_5 \times C_2$, or to $C_4 \times C_2$. Applying Lemma 3.1, it is clear that we cannot get the value eight in these cases, so G is refined. If $|G/C_G(S)| = 2$, we obtain $G \simeq D_{30}$, again uniquely. One can check easily that $D_{30} \simeq C_2 \ltimes C_{15}$ has global breadth eight, again with Lemma 3.3. Here again the structure of semidirect product allows us to describe the possible quotients of D_{30}, so that we may easily conclude that D_{30} is refined.

Now assume that S is not a normal subgroup of G. This means that $|G:N_G(S)| = 6$, that is, there are six conjugates of S. If $N_G(S) = C_G(S)$, then we apply Burnside's Theorem [7, Hauptsatz 2.6, p. 419] and there is a normal subgroup of M of G with $G = SM$ and $S \cap M = 1$. If q is a prime dividing $|M|$, then there is a Sylow q-subgroup Q which is normalized by S. If $Q \not\subseteq C_G(S)$, then $|Q:C_Q(S)| = 6$, which is not a power of a prime, a contradiction. So for every prime dividing $|M|$, there is a corresponding Sylow subgroups of M which is centralized by S, contradicting the nonnormality of S. If $|N_G(S):C_G(S)| = 4$, then there is an element $x \in G$ of order a power of 2 such that $\langle x, S\rangle \subseteq N_G(S)$ and every conjugate of $\langle x\rangle$ normalizes two conjugates of S. So we get

$$\mathbf{b}_{o(x)}(G) \geq o(x) + 3\binom{14}{4} \cdot o(x) \geq 10 \cdot o(x),$$

contradicting $\mathbf{B}(G) = 8$. We are left with $|N_G(S):C_G(S)| = 2$. We may choose $y \in N_G(S) \setminus C_G(S)$ of order a power of 2, and we obtain in the same way

$$\mathbf{b}_{o(y)}(G) \geq o(y) + 3\binom{14}{4} \cdot o(y) \geq 8 \cdot o(y),$$

with equality if $\langle y^2\rangle$ is a normal subgroup of G. Let D be the intersection of all conjugates of $N_G(S)$ in G. Of course D is normal in G and we have that G/D is isomorphic to a subgroup of $\mathrm{Sym}(6)$ of order a multiple of 30. Note that there are no subgroups of order 30 in $\mathrm{Sym}(6)$, but, as multiple of 30, only of order 120 and 60. These subgroups are, respectively, isomorphic to $\mathrm{Sym}(5)$ or $\mathrm{Alt}(5)$. Now counting the involutions of $\mathrm{Sym}(5)$, one can see that $\mathbf{b}_2(\mathrm{Sym}(5)) > 8$, which is clearly in contrast with $\mathbf{B}(G) = 8$. Then the only case is now $G/D \simeq \mathrm{Alt}(5)$, and if G is refined, this implies $G \simeq \mathrm{Alt}(5)$. Here, we should note that $\exp(\mathrm{Alt}(5)) = 30$ and count $\mathbf{b}_e(\mathrm{Alt}(5))$ for all $e \in \mathrm{Div}(30)$, checking that $\mathbf{B}(\mathrm{Alt}(5)) = 8$.

(ii) In (a) we apply Remark 3.2 (B). Notice that here the centralizer of C_5 is always noncyclic. In (b) we have Case (A) of Remark 3.2, and the statement follows. In (c) here G is perfect, and for H with $H/N \simeq G$ with cyclic N, we have $N \subseteq Z(G)$. The Schur multiplier of G is of order 2. This leads to $\mathrm{SL}(2, \mathbb{F}_5)$. By Corollary 2.4 and Lemma 3.1, $\mathbf{B}(G \times C_m) = 8$ if and only if $\gcd(30, m) = 1$. This shows (c) and completes the proof.

□

7. The case of {2,3}-groups. Here we prove (v) in Theorem 1.5 and (jv) in Theorem 1.6.

Proof of (v) in Theorem 1.5. Let S be a Sylow 3-subgroup of G. Note that S is cyclic by assumption. Moreover $|S|$ must be at most 9 by Theorems 6.3, 6.4, and 6.5. We are going to prove that $|S| = 3$, that is, $S \simeq C_3$.

The number of conjugates $|G : N_G(S)|$ is of the form $1 + 3k$ by the third theorem of Sylow and a power of 2 at the same time. If $|G : N_G(S)| \geq 16$, then

$$|S| \cdot \mathbf{b}_{|S|}(G) = \frac{|S| + 16 \cdot |S|}{3} = 11 \cdot |S|,$$

and so $\mathbf{b}_{|S|}(G) \geq 11$, and this contradicts $\mathbf{B}(G) = 8$. So $|G : N_G(S)| \in \{1, 4\}$. It follows that S^3 is a normal subgroup of G.

Assume $|G : N_G(S)| = 4$. If $C_G(S) \neq N_G(S)$, then there is $x \in N_G(S) \setminus C_G(S)$ of order a power of 2 such that $\mathbf{B}(\langle x, S\rangle) = \frac{|S|+1}{2}$. Since $\mathbf{B}(G) = 8$, we obtain $|S| \leq 9$. Assume $|S| = 9$. Then the element x normalizes S and some other conjugate S^+ of S, and the number of conjugates of x is at least $|S \cup S^+| = 15$ and not divisible by 5, since $|G|$ is not divisible by 5. This implies $\mathbf{b}_{o(x)}(G) > 8$, a contradiction. So $|S| = 3$ when $N_G(S) \neq C_G(S)$. If this does not happen, that is, if $C_G(S) = N_G(S)$, then G is 3-nilpotent and nonrefined, by the normality of S^3, and this gives a further contradiction. Hence $|S| = 3$ for $|G : N_G(S)| = 4$.

Consider now $S^G = \langle S^g \mid g \in G\rangle$ and $|G : N_G(S)| = 4$. We obtain $S^G \simeq \mathrm{Alt}(4)$ or $S^G \simeq \mathrm{SL}(2, \mathbb{F}_3)$. For the 3-nilpotent case, we find as only refined groups

$$G \simeq \mathrm{Alt}(4) \times C_2 \times C_2 \quad \text{or} \quad G \simeq \mathrm{SL}(2, \mathbb{F}_3) \times C_4.$$

If $|G : N_G(S)| = 4$ and $N_G(S) \neq C_G(S)$, then the intersection K of all conjugates of $N_G(S)$ is a normal subgroup of G such that $G/K \simeq \mathrm{Sym}(4)$. As only refined possibility, we would have either $G \simeq \mathrm{Sym}(4) \times C_2$ or the subdirect product of $\mathrm{Sym}(4)$ by C_4. The first case cannot happen, because $\mathbf{b}_2(\mathrm{Sym}(4) \times C_2) > 8$, but the second case has breadth 8 and is exactly the circumstance, described in (3).

It remains to consider the case that S is a normal subgroup of G. Again we may assume that $|S| \leq 9$. Now if $|S| = 3$ and the Sylow 2-subgroup D is abelian, we obtain $D \simeq C_4 \times C_4$ and $C_2 \times C_2 \times C_2 \times C_2$ as only possibilities. If D is nonabelian, we have either

$$D \simeq \langle a, b | a^2 = [a, [a, b]] = [a, b^2] = b^4 = 1\rangle \text{ or } D \simeq \langle a, b | a^4 = b^4 = abab^{-1}\rangle.$$

Here we should consider two non-isomorphic extensions of S by D; both are possible.

For $|S| = 9$, we have $|G : C_G(S)| = 2$. Let T be a Sylow 2-subgroup of G further $U = C_S(T)$ and $V = T \setminus C_G(T)$. Then $\mathbf{b}_m(T) = \mathbf{b}_m(U) + \mathbf{b}_m(V)$ and all terms are divisible by m as long as m divides $\exp(U)$. The number of elements of order a power of 2 is $|U| + 9|U| = 10|U| = 5|S|$. Since $\mathbf{B}(G) = 8$, we obtain for every m dividing $\exp(U)$

$$\mathbf{b}_m(G) = \mathbf{b}_m(U) + 9\mathbf{b}_m(V) \leq 8m,$$

where m divides $\mathbf{b}_m(V)$. So $\mathbf{b}_m(V) = 0$ for all m dividing $\exp(U)$ and $\exp(T) = 2\exp(U)$. So for $u = \exp(U)$, we obtain

$$\mathbf{b}_{2u}(G) = \mathbf{b}_{2u}(U) + 9\mathbf{b}_{2u}(V) = 10\mathbf{b}_{2u}(U),$$

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since $|U| = |V|$. Now $\mathbf{b}_{2u}(U) = |U| = \mathbf{b}_u(U)$ by construction, and u divides $\mathbf{b}_u(U)$. So $\mathbf{b}_{2u}(G)$ is divisible by 5 and $\mathbf{B}(G) = 8$ is impossible.

We are left with the case that $G = N_G(S) \neq C_G(S)$ and $|S| = 3$. Using the same notation as above, we have that G has $|U| + 3|V| = 2|T|$ elements of order a power of 2. If $\mathbf{B}(T) = \mathbf{b}_{\exp(T)}(T)$, then, in analogy to the preceding case, $\mathbf{B}(G) = 2\mathbf{B}(T)$ and $\mathbf{B}(T) = 4$. If T is not refined, then $\mathbf{B}(T) = \mathbf{B}(T/N)$ for some normal subgroup N of order 2 contained in $T' = [T, T]$. Now N is also normal in G and $\mathbf{B}(G/T) = \mathbf{B}(G)$, against the fact that G is refined.

If $\mathbf{B}(X) = 4$ and X is refined and nilpotent, then $\exp(X) \leq 4$: let $Y \subseteq X$ be a cyclic subgroup of maximal order. Then $|\langle X, Y\rangle| \in \{2, 4\}$. If $|\langle X, Y\rangle| = 2$, then X is abelian or dihedral, or quaternion, or of nilpotency class 2. All of these are not refined except Q_8 and $\mathbf{B}(Q_8) \neq 4$. So $|\langle X, Y\rangle| = 4$. Let $|Y| > 4$ and $N \subseteq Y$ be a subgroup of order 2. If Y is a normal subgroup of G, then $\mathbf{B}(G) = \mathbf{B}(G/N)$ and G is not refined. If Y is not a normal subgroup of G, then there is an element $z \in G$ such that $Y^z \neq Y$ and $Y \subseteq YY^z \subseteq G$ with all indices 2, and YY^z of nilpotency class 2 and $N = (YY^z)^m$, where $2m = |Y|$. Again $\mathbf{B}(G) = \mathbf{B}(G/N)$ and G is not refined.

The refined groups X with $\mathbf{B}(X) = 4$ are either elementary abelian of order 8 or of order 16 and exponent 4; then we conclude that either

$$|X^2| = 8 \text{ and } X \simeq \langle u, v | u^4 = v^2 = [[v, u], u] = [[v, u], v] = 1\rangle.$$

or $|X^2| = 4$ with two isomorphism classes. The result follows. □

In order to modify the previous argument for strictly deduced groups, we note that these are always extensions of cyclic groups by the corresponding refined group. This is the main idea of the following argument.

Proof of (jv) in Theorem 1.6 . For (a) we have as first subcase that G is a direct product $\mathrm{PSL}(2, \mathbb{F}_3) \times V$ and $\mathbf{B}(V) = 2$ follows from Lemma 3.1 (ii).

If G is not a direct product, we consider first the case that the Sylow 2-subgroup D is an extension of a group of order 2 by an elementary abelian group of order 16 and $G' \cong Q_8$. Now $G'C_D(G') = D$ and $|G' \cap C_D(G')| = |D'| = 2$, and we have to decide which of the groups Q_8, D_8, $C_4 \times C_2$ are possible.

Clearly $\exp(D) = 4$ and $\mathbf{b}_4(D) = 4$ and we need $\mathbf{b}_2(D) \leq 4$. For $C_D(G') \simeq Q_8$, we would obtain $\mathbf{b}_2(D) = 10$, so this case is impossible. For the cases $C_D(G') \simeq D_8$, $C_4 \times C_2$, we have $\mathbf{b}_2(D) = 8$, so these lead to deduced groups. For Sylow 2-subgroups D of G with $C_D(G')$ of higher order, we obtain that they do not have a subgroup isomorphic to Q_8, but all other groups of breadth two are possible for $C_D(G')$. Finally $C_D(G') \cong D_{16}$ is possible. For the group in (b), the only possibility is obvious since the Schur multiplier of $\mathrm{SL}(2, \mathbb{F}_3)$ is trivial. In this situation, we have $G \simeq (\mathrm{SL}(2, \mathbb{F}_3) \times W)/Z$, where W^* is the unique minimal normal subgroup of W^2, $Z \subset (\mathrm{SL}(2, \mathbb{F}_3))''W^*$, $(\mathrm{SL}(2, \mathbb{F}_3)'' \neq Z \neq W^*$, and $W \simeq D_{16}$ or $\mathbf{B}(W) = 2$ but $W \not\simeq Q_8$.

For the groups in (c) and (d), there are only the possibilities given since split extensions analogous to (c) of $\mathrm{SL}(2, \mathbb{F}_3)$ lead to breadth 16. □

8. Proofs of main theorems and closing remarks. We begin by proving Theorems 1.5 and 1.6.

Proof of Theorem 1.5. The condition (i) is described by Proposition 3.8, while (ii), (iii), and (iv) follow from Theorems 6.3 (i), 6.4 (i), and 6.5 (i), respectively. Now (vi) follows from Theorem 2.8 (i) and (v) is shown in Section 7. □

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Proof of Theorem 1.6. The conditions (j), (jj), and (jjj) follow from Theorems 6.3 (ii), 6.4 (ii), and 6.5 (ii), respectively. Finally (jv) is shown in Section 7. □

There are a series of conjectures and open problems that we will list in the following, after we discussed the details of the proofs of the main theorems. On the other hand, the readers may have noted that in all cases

$$\exists k \in \mathbb{N} \text{ such that } \langle L_k(G)\rangle = G \text{ and } \mathbf{b}_k(G) = \mathbf{B}(G). \tag{8.1}$$

In addition to (8.1), if $\mathbf{B}(G) = \mathbf{B}(G/N)$ for some proper normal subgroup N of G, then N is contained in the hypercenter $Z_\infty(G)$ of G (i.e., the largest term of the upper central series of G, see [7]).

Both these facts are not true in general, as can be seen by the following example.

Example 8.1. The group

$$T = \langle x, y, z, w \mid x^2 = y^{15} = (xy)^2 = z^{17} = w^{17} = [z, w] = y^{-1}zyzw = y^{-1}wyz^{-1} = 1\rangle,$$

has

$$\mathbf{B}(T) = \mathbf{b}_3(T) = \mathbf{b}_{15}(T) = 193, \quad \mathbf{b}_2(T) = 128, \quad \mathbf{b}_6(T) = 139, \quad \mathbf{b}_5(T) = 1,$$

$$\mathbf{b}_{17}(T) = \mathbf{b}_{51}(T) = \mathbf{b}_{85}(T) = 17, \quad \mathbf{b}_{34}(T).$$

Here $N = \langle y^3\rangle$ has $\mathbf{B}(T/N) = \mathbf{B}(T) = \mathbf{b}_3(T) = 193$, but N is not contained in $Z_\infty(T)$. Moreover $L_3(T)$ does not generate T.

This property of generation of G via $L_k(G)$ is strictly related to the case of global breadth eight, but it may be related to different values of global breadth. If the number k is different from $\exp(G)$, this may lead to exclusion of primes as divisors of the order of the cyclic normal subgroups. In other words, it seems that the general theory of generation of groups (see [**8**] for recent results) may be related to that of the global breadth, under prescribed conditions. More precisely, it would be interesting to study the following class of groups:

$$\mathcal{E} = \{G \mid \exists k \in \mathbb{N} \text{ such that } \langle L_k(G)\rangle = G \text{ and } \mathbf{b}_k(G) = \mathbf{B}(G)\},$$

and see if the deduced groups, belonging to this class, have the property of inclusion in the hypercenter, as mentioned before.

Another observation, which may deserve further study, is the behavior of "minimal sets of global breadth." This idea is illustrated here.

Remark 8.2. A refined group G (for a given value of $\mathbf{B}(G)$) may contain proper subgroups S with $\mathbf{B}(G) = \mathbf{B}(S)$; for instance, this is the case for $C_2 \times C_2 \times C_2 \times C_2$ and $L(\mathbb{F}_8) \times C_2$. In this particular case, the class of deduced groups of one need not be the class of suitable subgroups of the other: $Q_8 \times C_2 \times C_2$ is not isomorphic to a subgroup of a group, deduced from $L(\mathbb{F}_8) \times C_2$ but is deduced from $C_2 \times C_2 \times C_2 \times C_2$.

The following remark illustrates again a general behavior which may be of independent interest.

Remark 8.3. As already mentioned, in addition to (8.1), the following conditions are satisfied by several deduced groups:

$$\text{If } \mathbf{B}(G) = \mathbf{B}(G/N), \quad \text{then } N \subseteq Z_\infty(G). \tag{8.2}$$

$$\text{If } \mathbf{B}(G) = \mathbf{B}(G/N) \text{ and } N \text{ is a Sylow subgroup of } G, \text{ then } \ N \subseteq Z(G). \tag{8.3}$$

$$\text{If } G \text{ is refined, then } |G| \leq \mathbf{B}(G)(\mathbf{B}(G)+1). \tag{8.4}$$

The study of the behavior of these implications is very interesting. Consequently, one could investigate the following classes of groups:

$$\mathcal{F} = \{G \mid G \text{ is deduced from } \ G/N \text{ and } N \subseteq Z_\infty(G)\};$$

$$\mathcal{G} = \{G \mid G \text{ is deduced from } \ G/N, \text{ and } N \text{ is a central Sylow subgroup of } G\};$$

$$\mathcal{H} = \{G \mid G \text{ satisfies the condition } \ |G| \leq \mathbf{B}(G)(\mathbf{B}(G)+1)\}.$$

What we have seen until now allows us to conclude that these classes are not empty and their intersection is not empty. Of course, it would be nice to see if there are relations with the theory of formations of Gaschütz and Lubeseder (see [**2**]).

One can see that (8.1), (8.2), (8.3), and (8.4) are not satisfied in general, producing examples which disprove these implications, nevertheless $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ and their properties remain interesting to study.

On the other hand, it seems that the orders of simple groups G with $\mathbf{B}(G) \leq M$ (for a given constant $M > 0$) are more restricted than refined groups in general, perhaps by $|G| \leq M^2$. Thompson's Conjecture points into the same direction: it says that every finite simple group G possesses a conjugacy class C such that $C^2 \cup \{1\} = G$. This would ease finding all nonsolvable refined groups for a given global breadth.

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