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Numerical analysis of history-dependent variational–hemivariational inequalities with applications to contact problems†

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A new class of history-dependent variational–hemivariational inequalities was recently studied in Migórski *et al.* (2015 *Nonlinear Anal. Ser. B: Real World Appl.* **22**, 604–618). There, an existence and uniqueness result was proved and used in the study of a mathematical model which describes the contact between a viscoelastic body and an obstacle. The aim of this paper is to continue the analysis of the inequalities introduced in Migórski *et al.* (2015 *Nonlinear Anal. Ser. B: Real World Appl.* **22**, 604–618) and to provide their numerical analysis. We start with a continuous dependence result. Then we introduce numerical schemes to solve the inequalities and derive error estimates. We apply the results to a quasistatic frictional contact problem in which the material is modelled with a viscoelastic constitutive law, the contact is given in the form of normal compliance, and friction is described with a total slip-dependent version of Coulomb's law.

Key words: Variational–hemivariational inequality; Clarke subdifferential; history-dependent operator; Continuous dependence; Finite element method; Convergence; Error estimates; Viscoelastic material; Frictional contact; Weak solution

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1 Introduction

Variational and hemivariational inequalities play an important role in the study of both qualitative and quantitative analysis of nonlinear boundary value problems. The study of variational inequalities started in the early sixties and is based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. References in the field are [2–4, 11, 12, 21, 22, 28], among others. The theory of hemivariational inequalities was introduced in the early eighties and is based on the properties of generalized gradient introduced and studied in [6–8]. References in the field include [9, 17, 24, 26, 29]. Applications of the variational and hemivariational inequalities in mechanics and engineering, especially in contact mechanics, can be found in [1, 10, 14–16, 18, 20, 23, 24, 28, 30–32], among others. Variational–hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are involved. Interest in their study is motivated by various problems in mechanics, as illustrated in [26, 27, 29].

A class of stationary variational–hemivariational inequalities was studied in [13]. An inequality in the class involves two nonlinear operators and two nondifferentiable functionals, among which at least one is convex. An existence and uniqueness result was proved for a solution of the inequality, through arguments of surjectivity for pseudomonotone operators and the Banach fixed point theorem. Continuous dependence of the solution on the data was shown. Numerical methods for solving the inequality were introduced, and their convergence was established rigorously. Moreover, an error estimate was derived which is of an optimal order for the linear finite element method under appropriate solution regularity assumptions. Finally, results on the well-posedness and error estimation of numerical solutions were applied to a variational–hemivariational inequality arising in the study of a new model of elastic contact.

An evolutionary version of the variational–hemivariational inequalities studied in [13] was considered in [25]. There, the structure of the inequalities involves two history-dependent operators and two nondifferentiable functionals, one convex and the other nonconvex. An existence and uniqueness result was proved by using arguments of surjectivity for pseudomonotone operators and fixed point. Then, these abstract results were used in the study of a new model of viscoelastic frictionless contact, in which both the instantaneous and the memory effects of the foundation were taken into account. We note that, unlike a large number of references, including [24], the inequality problems considered in [25] are formulated on the unbounded interval of time $\mathbb{R}_+ = [0, \infty)$. This requires the use of the framework of Fréchet spaces of continuous functions, instead of that of the classical Banach spaces of continuous functions defined on a bounded interval of time, used in many papers.

The present paper represents a continuation of [25] and parallels part of [13]. Its aim is threefold. The first one is to show the continuous dependence of the solutions of the variational–hemivariational inequalities on the problem data. The second one is to provide the numerical analysis for solving such history-dependent inequalities. The third aim is to apply our abstract result in the study of a new model of quasistatic contact, which describes the equilibrium of a nonlinear viscoelastic body in contact with a foundation. In contrast to the model considered in [25], here the contact is assumed frictional.

The class of variational–hemivariational inequalities studied in this paper represents a general framework in which a large number of quasistatic contact problems, associated with various constitutive laws and frictional or frictionless contact conditions, can be cast. Therefore, our work provides arguments and tools useful for the unique solvability of a large number of quasistatic contact problems together with their numerical solution.

The rest of the paper is structured as follows. In Section 2, we review some material from [25]. In Section 3, we state and prove the continuous dependence of the solution with respect to the data. In Section 4, we study a fully discrete scheme. We derive error estimates and prove convergence results. In Section 5, we introduce a quasistatic frictional contact problem in which the material is modelled with a viscoelastic constitutive law, the contact is given in the form of normal compliance and friction is described with a total slip-dependent friction law. We show that this problem leads to a history-dependent quasivariational inequality for the velocity field. Then, in Section 6, we use our theoretical results in the variational and numerical analysis of this contact problem. Finally, in Section 7, we present some concluding remarks.

2 History-dependent variational inequalities

In this section, we introduce the class of history-dependent variational–hemivariational inequalities and recall an existence and uniqueness result proved in [25]. We start by presenting some notation, definitions and preliminary results used later in this paper.

We use $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ for the set of non-negative real numbers, i.e., $\mathbb{R}_+ = [0, +\infty)$. Let X be a normed space. We denote its norm by $\|\cdot\|_X$, its topological dual by X^* and the duality pairing of X and X^* by $\langle\cdot,\cdot\rangle_{X^*\times X}$. Let $h : X \to \mathbb{R}$ be a locally Lipschitz function. Then, the generalized (Clarke) directional derivative of h at $x \in X$ in the direction $v \in X$, denoted by $h^0(x; v)$, is defined by

$$h^0(x;v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{h(y+\lambda v) - h(y)}{\lambda}.$$

The generalized gradient (subdifferential) of h at x, denoted by $\partial h(x)$, is a subset of the dual space X^* given by

$$\partial h(x) = \{\, \zeta \in X^* \mid h^0(x;v) \geqslant \langle \zeta, v\rangle_{X^*\times X} \text{ for all } v \in X \,\}.$$

An operator $A : X \to X^*$ is called pseudomonotone, if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \leqslant 0$ imply

$$\langle Au, u - v\rangle_{X^*\times X} \leqslant \liminf \langle Au_n, u_n - v\rangle_{X^*\times X},$$

for all $v \in X$.

We also use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. It is well known that, if X is a Banach space, then $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e., as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Moreover, the convergence of a sequence $\{x_k\}_k$ to the element x, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} x_k \to x \quad \text{in } C(\mathbb{R}_+;X) \ \text{ as } \ k\to\infty \ \text{ if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r)-x(r)\|_X \to 0 \ \text{ as } k\to\infty, \ \text{ for all } \ n\in\mathbb{N}. \end{cases} \tag{2.1}$$

In other words, the sequence $\{x_k\}_k$ converges to the element x in the space $C(\mathbb{R}_+;X)$ if and only if it converges to x in the space $C([0,n];X)$ for all $n \in \mathbb{N}$, $C([0,n];X)$ being the space of continuous functions defined on the compact interval $[0,n]$ with values in X, endowed with its canonical norm.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$ and let Γ be a measurable subset of $\partial\Omega$. Below, we use the symbol $m(\Gamma)$ to denote the $d-1$ dimensional Lebesgue measure of Γ. Let V be a closed subspace of $H^1(\Omega;\mathbb{R}^s)$, $s \geqslant 1$, and let $H = L^2(\Omega;\mathbb{R}^s)$. We denote the trace operator by $\gamma: V \to L^2(\Gamma;\mathbb{R}^s)$, its norm in $\mathscr{L}(V, L^2(\Gamma;\mathbb{R}^s))$ by $\|\gamma\|$ and the adjoint operator to γ by $\gamma^*: L^2(\Gamma;\mathbb{R}^s) \to V^*$. From the theory of Sobolev spaces, we know that (V,H,V^*) forms an evolution triple of spaces and the embedding $V \subset H$ is compact.

Consider the operators $A: V \to V^*$, $\mathscr{S}: C(\mathbb{R}_+;V) \to C(\mathbb{R}_+;V^*)$, $\mathscr{R}: C(\mathbb{R}_+;V) \to C(\mathbb{R}_+;L^2(\Gamma))$, the functions φ, $j: \Gamma\times\mathbb{R}^s \to \mathbb{R}$ and $f: \mathbb{R}_+ \to V^*$. The problem under consideration is to find a function $u: \mathbb{R}_+ \to V$ such that

$$\begin{aligned} &\langle Au(t), v-u(t)\rangle_{V^*\times V} + \langle(\mathscr{S}u)(t), v-u(t)\rangle_{V^*\times V} \\ &\quad + \int_\Gamma (\mathscr{R}u)(t)\big(\varphi(\gamma v)-\varphi(\gamma u(t))\big)\, d\Gamma \\ &\quad + \int_\Gamma j^0(\gamma u(t); \gamma v-\gamma u(t))\, d\Gamma \geqslant \langle f(t), v-u(t)\rangle_{V^*\times V}, \end{aligned} \tag{2.2}$$

for all $v \in V$ and all $t \in \mathbb{R}_+$.

In the study of equation (2.2), we assume the following hypotheses:

$$\begin{cases} A: V\to V^* \text{ is such that} \\ \quad \text{(a) } A \text{ is pseudomonotone and there exists } \alpha>0 \text{ such that} \\ \qquad \langle Av,v\rangle_{V^*\times V} \geqslant \alpha\,\|v\|_V^2 \quad \text{for all } v\in V; \\ \quad \text{(b) } A \text{ is strongly monotone, i.e., there exists } m_A>0 \text{ such that} \\ \qquad \langle Av_1-Av_2, v_1-v_2\rangle_{V^*\times V} \geqslant m_A\|v_1-v_2\|_V^2 \\ \qquad \text{for all } v_1,v_2\in V. \end{cases} \tag{2.3}$$

$$\begin{cases} \mathscr{S}: C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;V^*) \text{ is such that} \\ \quad \text{for every } n\in\mathbb{N} \text{ there exists } s_n>0 \text{ such that} \\ \qquad \|(\mathscr{S}u_1)(t)-(\mathscr{S}u_2)(t)\|_{V^*} \leqslant s_n\displaystyle\int_0^t \|u_1(s)-u_2(s)\|_V\, ds \\ \quad \text{for all } u_1,\ u_2\in C(\mathbb{R}_+;V), \text{ for all } t\in[0,n]. \end{cases} \tag{2.4}$$

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$$\begin{cases} \mathscr{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma)) \text{ is such that} \\ \quad \text{(a) for every } n \in \mathbb{N} \text{ there exists } r_n > 0 \text{ such that} \\ \qquad \|(\mathscr{R}u_1)(t) - (\mathscr{R}u_2)(t)\|_{L^2(\Gamma)} \leqslant r_n \int_0^t \|u_1(s) - u_2(s)\|_V \, ds \\ \qquad \text{for all } u_1, u_2 \in C(\mathbb{R}_+; V), \text{ for all } t \in [0, n]; \\ \quad \text{(b) } (\mathscr{R}u)(t) \geqslant 0 \text{ a.e. on } \Gamma, \text{ for all } u \in C(\mathbb{R}_+; V), \text{ all } t \in \mathbb{R}_+. \end{cases} \tag{2.5}$$

$$\begin{cases} \varphi : \Gamma \times \mathbb{R}^s \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } \varphi(\cdot, \xi) \text{ is measurable on } \Gamma \text{ for all } \xi \in \mathbb{R}^s \text{ and there exists} \\ \qquad \tilde{e} \in L^2(\Gamma; \mathbb{R}^s) \text{ such that } \varphi(\cdot, \tilde{e}(\cdot)) \in L^2(\Gamma); \\ \quad \text{(b) } \varphi(\boldsymbol{x}, \cdot) \text{ is convex for a.e. } \boldsymbol{x} \in \Gamma; \\ \quad \text{(c) there exists } L_\varphi > 0 \text{ such that} \\ \qquad |\varphi(\boldsymbol{x}, \xi_1) - \varphi(\boldsymbol{x}, \xi_2)| \leqslant L_\varphi \|\xi_1 - \xi_2\|_{\mathbb{R}^s} \\ \qquad \text{for all } \xi_1, \xi_2 \in \mathbb{R}^s, \text{ a.e. } \boldsymbol{x} \in \Gamma. \end{cases} \tag{2.6}$$

$$\begin{cases} j : \Gamma \times \mathbb{R}^s \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j(\cdot, \xi) \text{ is measurable on } \Gamma \text{ for all } \xi \in \mathbb{R}^s \text{ and there exists} \\ \qquad e \in L^2(\Gamma; \mathbb{R}^s) \text{ such that } j(\cdot, e(\cdot)) \in L^1(\Gamma); \\ \quad \text{(b) } j(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^s \text{ for a.e. } \boldsymbol{x} \in \Gamma; \\ \quad \text{(c) } \|\partial j(\boldsymbol{x}, \xi)\|_{\mathbb{R}^s} \leqslant c_0 + c_1 \|\xi\|_{\mathbb{R}^s} \text{ for a.e. } \boldsymbol{x} \in \Gamma, \\ \qquad \text{all } \xi \in \mathbb{R}^s \text{ with } c_0, c_1 \geqslant 0; \\ \quad \text{(d) } j^0(\boldsymbol{x}, \xi_1; \xi_2 - \xi_1) + j^0(\boldsymbol{x}, \xi_2; \xi_1 - \xi_2) \leqslant \beta \|\xi_1 - \xi_2\|^2_{\mathbb{R}^s} \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma, \text{ all } \xi_1, \xi_2 \in \mathbb{R}^s \text{ with } \beta \geqslant 0. \end{cases} \tag{2.7}$$

$$f \in C(\mathbb{R}_+; V^*). \tag{2.8}$$

Concerning the above assumptions we have the following comments. First, following the terminology in [33], conditions (2.4) and (2.5)(a) show that the operators $\mathscr{S}$ and $\mathscr{R}$ are *history-dependent operators*. On the other hand, we note that the function φ is assumed to be convex with respect to its second argument, while the function j is locally Lipschitz in the second argument and could be nonconvex. For this reason, inequality (2.2) represents, in fact, a *variational–hemivariational inequality*. To combine these two ingredients, we refer to inequality (2.2) as a *history-dependent variational–hemivariational inequality*.

We have the following existence and uniqueness result.

Theorem 1 *Assume the hypotheses* (2.3)–(2.8) *and the smallness condition*

$$\beta \|\gamma\|^2 < m_A. \tag{2.9}$$

If one of the following hypotheses

$$\alpha > c_1 \sqrt{2} \|\gamma\|^2, \tag{2.10}$$

$$j^0(\boldsymbol{x}, \xi; -\xi) \leqslant d\,(1 + \|\xi\|_{\mathbb{R}^s}) \quad \textit{for all} \quad \xi \in \mathbb{R}^s, \ \textit{a.e. } \boldsymbol{x} \in \Gamma \ \textit{with } d \geqslant 0, \tag{2.11}$$

is satisfied, then inequality (2.2) *has a unique solution* $u \in C(\mathbb{R}_+; V)$.

The proof of Theorem 1 was given in [25]. It is based on arguments of surjectivity for pseudomonotone operators and a fixed point result for a special class of operators defined on the Fréchet space $C(\mathbb{R}_+;V)$.

3 Continuous dependence on data

We now study the continuous dependence of the solution of the history-dependent variational–hemivariational inequality (2.2) with respect to the problem data. Assume in what follows that equations (2.3)–(2.8) hold and denote by $u \in C(\mathbb{R}_+;V)$ the solution of equation (2.2) stated in Theorem 1. For each $\rho > 0$ let $\mathscr{S}_\rho$, $\mathscr{R}_\rho$, j_ρ and f_ρ represent perturbed data corresponding to $\mathscr{S}$, $\mathscr{R}$, j and f, which satisfy conditions (2.4), (2.5), (2.7) and (2.8), respectively. We denote by $s_{\rho n}$, $r_{\rho n}$ the constants involved in assumptions (2.4) and (2.5), respectively, and β_ρ will represent the constant involved in assumptions (2.7). In addition, assume that there exists m_0 such that

$$\beta_\rho \|\gamma\|^2 \leqslant m_0 < m_A \quad \text{for all } \rho > 0. \tag{3.1}$$

Then it follows from Theorem 1 that there exists a unique function $u_\rho \in C(\mathbb{R}_+;V)$ such that

$$\begin{aligned}\langle Au_\rho(t), v - u_\rho(t)\rangle_{V^*\times V} &+ \langle(\mathscr{S}_\rho u_\rho)(t), v - u_\rho(t)\rangle_{V^*\times V} \\ &+ \int_\Gamma (\mathscr{R}_\rho u_\rho)(t)\big(\varphi(\gamma v) - \varphi(\gamma u_\rho(t))\big)\, d\Gamma \\ &+ \int_\Gamma j_\rho^0(\gamma u_\rho(t); \gamma v - \gamma u_\rho(t))\, d\Gamma \geqslant \langle f_\rho(t), v - u_\rho(t)\rangle_{V^*\times V},\end{aligned} \tag{3.2}$$

for all $v \in V$ and all $t \in \mathbb{R}_+$.

Our interest lies in the behaviour of the solution u_ρ as ρ tends to zero. To this end we consider the following additional assumptions.

$$\left\{\begin{array}{l}\text{For each } n \in \mathbb{N} \text{ there exist } G_n : C([0,n];V) \to \mathbb{R}_+ \\ \ \text{and } g_n : (0,\infty) \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } \|(\mathscr{S}_\rho u)(t) - (\mathscr{S}u)(t)\|_{V^*} \leqslant g_n(\rho) G_n(u) \\ \qquad \text{for all } u \in C(\mathbb{R}_+;V), \ \text{for all } t \in [0,n], \ \text{for all } \rho > 0. \\ \text{(b) } g_n(\rho) \to 0 \quad \text{as} \quad \rho \to 0.\end{array}\right. \tag{3.3}$$

$$\left\{\begin{array}{l}\text{For each } n \in \mathbb{N} \text{ there exist } H_n : C([0,n];V) \to \mathbb{R}_+ \\ \ \text{and } h_n : (0,\infty) \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } \|(\mathscr{R}_\rho u)(t) - (\mathscr{R}u)(t)\|_{L^2(\Gamma)} \leqslant h_n(\rho) H_n(u) \\ \qquad \text{for all } u \in C(\mathbb{R}_+;V), \ \text{for all } t \in [0,n], \ \text{for all } \rho > 0. \\ \text{(b) } h_n(\rho) \to 0 \quad \text{as} \quad \rho \to 0.\end{array}\right. \tag{3.4}$$

$$\left\{\begin{array}{l}\text{There exist } K \in \mathbb{R}_+ \text{ and } k : (0,\infty) \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } j^0(\boldsymbol{x}, \xi; \eta) - j_\rho^0(\boldsymbol{x}, \xi; \eta) \leqslant k(\rho)(\|\xi\|_{\mathbb{R}^s} + K)\|\eta\|_{\mathbb{R}^s} \\ \qquad \text{for all } \xi, \eta \in \mathbb{R}^s, \ \text{a.e. } \boldsymbol{x} \in \Gamma, \ \text{for all } \rho > 0. \\ \text{(b) } k(\rho) \to 0 \quad \text{as } \rho \to 0.\end{array}\right. \tag{3.5}$$

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For each $n \in \mathbb{N}$ there exists $\widetilde{s}_n > 0$ such that $s_{\rho n} \leqslant \widetilde{s}_n \quad \forall\, \rho > 0.$ (3.6)

For each $n \in \mathbb{N}$ there exists $\widetilde{r}_n > 0$ such that $r_{\rho n} \leqslant \widetilde{r}_n \quad \forall\, \rho > 0.$ (3.7)

$$f_\rho \to f \quad \text{in} \quad C(\mathbb{R}_+; V^*) \quad \text{as} \quad \rho \to 0. \tag{3.8}$$

Examples of operators $\mathscr{S}$ and $\mathscr{S}_\rho$, $\mathscr{R}$ and $\mathscr{R}_\rho$ which satisfy conditions (3.3), (3.4), (3.6) and (3.7) will be provided in Section 6. Examples of functions j and j_ρ which satisfy the condition (3.5) have been provided in [13].

We have the following convergence result.

Theorem 2 *Assume the hypotheses of Theorem 1. Moreover, let $\mathscr{S}_\rho$, $\mathscr{R}_\rho$, j_ρ and f_ρ satisfy assumptions of Theorem 1 and, in addition, assume that equations* (3.1) *and* (3.3)–(3.8) *hold. Then the solution u_ρ of the inequality* (3.2) *converges to the solution u of the inequality* (2.2), *i.e.,*

$$u_\rho \to u \quad \text{in} \quad C(\mathbb{R}_+; V) \quad \text{as } \rho \to 0. \tag{3.9}$$

Proof. Let $\rho > 0$, $n \in \mathbb{N}$ and $t \in [0, n]$. We take $v = u(t)$ in equation (3.2) and $v = u_\rho(t)$ in equation (2.2) and add the resulting inequalities to obtain

$$\begin{aligned}
&\langle Au_\rho(t) - Au(t), u_\rho(t) - u(t)\rangle_{V^*\times V} \\
&\quad \leqslant \langle (\mathscr{S}_\rho u_\rho)(t) - (\mathscr{S} u)(t), u(t) - u_\rho(t)\rangle_{V^*\times V} \\
&\qquad + \int_\Gamma \big((\mathscr{R}_\rho u_\rho)(t) - (\mathscr{R} u)(t)\big)\big(\varphi(\gamma u) - \varphi(\gamma u_\rho)\big)\, d\Gamma \\
&\qquad + \int_\Gamma \Big(j_\rho^0(\gamma u_\rho(t); \gamma u(t) - \gamma u_\rho(t)) + j^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma \\
&\qquad + \langle f_\rho(t) - f(t), u_\rho(t) - u(t)\rangle_{V^*\times V}.
\end{aligned} \tag{3.10}$$

Let us bound each term in equation (3.10). First, it follows from assumption (2.3)(b) that

$$\langle Au_\rho(t) - Au(t), u_\rho(t) - u(t)\rangle_{V^*\times V} \geqslant m_A \|u_\rho(t) - u(t)\|_V^2. \tag{3.11}$$

Next, we write

$$\begin{aligned}
&\langle (\mathscr{S}_\rho u_\rho)(t) - (\mathscr{S} u)(t), u(t) - u_\rho(t)\rangle_{V^*\times V} \\
&\quad = \langle (\mathscr{S}_\rho u_\rho)(t) - (\mathscr{S}_\rho u)(t), u(t) - u_\rho(t)\rangle_{V^*\times V} \\
&\qquad + \langle (\mathscr{S}_\rho u)(t) - (\mathscr{S} u)(t), u(t) - u_\rho(t)\rangle_{V^*\times V} \\
&\quad \leqslant \Big(\|(\mathscr{S}_\rho u_\rho)(t) - (\mathscr{S}_\rho u)(t)\|_V + \|(\mathscr{S}_\rho u)(t) - (\mathscr{S} u)(t)\|\Big)\|u_\rho(t) - u(t)\|_V,
\end{aligned}$$

then we use the assumptions (2.4), (3.3) and (3.6). As a result we find that

$$\begin{aligned}
&\langle (\mathscr{S}_\rho u_\rho)(t) - (\mathscr{S} u)(t), u(t) - u_\rho(t)\rangle_{V^*\times V} \\
&\quad \leqslant \Big(\widetilde{s}_n \int_0^t \|u_\rho(s) - u(s)\|_V\, ds + g_n(\rho) G_n(u)\Big)\|u_\rho(t) - u(t)\|_V.
\end{aligned} \tag{3.12}$$

Similar arguments, based on assumptions (2.5), (2.6), (3.4) and (3.7), lead to the inequality

$$\int_\Gamma \big((\mathscr{R}_\rho u_\rho)(t) - (\mathscr{R}u)(t)\big)\big(\varphi(\gamma u) - \varphi(\gamma u_\rho)\big)\, d\Gamma$$
$$\leqslant L_\varphi \|\gamma\| \Big(\widetilde{r}_n \int_0^t \|u_\rho(s) - u(s)\|_V\, ds + h_n(\rho) H_n(u)\Big) \|u_\rho(t) - u(t)\|_V. \tag{3.13}$$

On the other hand, the property (2.7)(d) of the function j_ρ combined with assumption (3.5) yields

$$\begin{aligned}
&\int_\Gamma \Big(j_\rho^0(\gamma u_\rho(t); \gamma u(t) - \gamma u_\rho(t)) + j^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma \\
&\quad = \int_\Gamma \Big(j_\rho^0(\gamma u_\rho(t); \gamma u(t) - \gamma u_\rho(t)) + j_\rho^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma \\
&\qquad + \int_\Gamma \Big(j^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t)) - j_\rho^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma \\
&\quad \leqslant \beta_\rho \int_\Gamma \|\gamma u_\rho(t) - \gamma u(t)\|_{\mathbb{R}^s}^2\, d\Gamma + k(\rho) \int_\Gamma (\|\gamma u(t)\|_{\mathbb{R}^s} + K)\|\gamma u_\rho(t) - \gamma u(t)\|_{\mathbb{R}^s} d\Gamma.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Gamma \Big(j_\rho^0(\gamma u_\rho(t); \gamma u(t) - \gamma u_\rho(t)) + j^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma \\
&\quad \leqslant \beta_\rho \|\gamma\|^2 \|u_\rho(t) - u(t)\|_V^2 \\
&\qquad + k(\rho)\, \|\gamma\| \big(\|\gamma\|\, \|u(t)\|_V + K\sqrt{m(\Gamma)}\big)\, \|u_\rho(t) - u(t)\|_V,
\end{aligned}$$

which implies that

$$\int_\Gamma \Big(j_\rho^0(\gamma u_\rho(t); \gamma u(t) - \gamma u_\rho(t)) + j^0(\gamma u(t); \gamma u_\rho(t) - \gamma u(t))\Big)\, d\Gamma$$
$$\leqslant \beta_\rho \|\gamma\|^2 \|u_\rho(t) - u(t)\|_V^2 + k(\rho) F_n(u) \|u_\rho(t) - u(t)\|_V, \tag{3.14}$$

where

$$F_n(u) = \max_{t \in [0,n]} \|\gamma\| \big(\|\gamma\|\, \|u(t)\|_V + K\sqrt{m(\Gamma)}\big).$$

Finally, note that

$$\langle f_\rho(t) - f(t), u_\rho(t) - u(t)\rangle_{V^* \times V} \leqslant \delta_n(\rho) \|u_\rho(t) - u(t)\|_V, \tag{3.15}$$

where

$$\delta_n(\rho) = \max_{t \in [0,n]} \|f_\rho(t) - f(t)\|_{V^*}. \tag{3.16}$$

We now combine inequalities (3.10)–(3.15) to deduce that

$$\begin{aligned}\big(m_A - \beta_\rho \|\gamma\|^2\big)\|u_\rho(t) - u(t)\|_V \leqslant \widetilde{s}_n \int_0^t \|u_\rho(s) - u(s)\|_V\, ds + g_n(\rho) G_n(u) \\ + L_\varphi \|\gamma\| \widetilde{r}_n \int_0^t \|u_\rho(s) - u(s)\|_V ds \\ + h_n(\rho) L_\varphi \|\gamma\| H_n(u) + k(\rho) F_n(u) + \delta_n(\rho).\end{aligned}$$

Denote

$$\zeta_n = \frac{\widetilde{s}_n + L_\varphi \|\gamma\| \widetilde{r}_n}{m_A - m_0},$$

$$m_n(\rho) = \frac{1}{m_A - m_0}\Big(g_n(\rho) G_n(u) + h_n(\rho) L_\varphi \|\gamma\| H_n(u) + k(\rho) F_n(u) + \delta_n(\rho)\Big).$$

Then, using assumption (3.1) we see that

$$\|u_\rho(t) - u(t)\|_V \leqslant \zeta_n \int_0^t \|u_\rho(s) - u(s)\|_V ds + m_n(\rho).$$

Therefore, the Gronwall argument yields

$$\|u_\rho(t) - u(t)\|_V \leqslant m_n(\rho)\, e^{\zeta_n t}.$$

We conclude from this inequality that

$$\max_{t \in [0,n]} \|u_\rho(t) - u(t)\|_V \leqslant m_n(\rho)\, e^{\zeta_n n}. \tag{3.17}$$

Note that assumption (3.8), definitions (2.1) and (3.16) imply that

$$\delta_n(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \tag{3.18}$$

Therefore, using equations (3.3)(b), (3.4)(b), (3.5)(b) and (3.18), it follows from equation (3.17) that

$$\max_{t \in [0,n]} \|u_\rho(t) - u(t)\|_V \to 0 \quad \text{as} \quad \rho \to 0. \tag{3.19}$$

The convergence (3.9) is now a consequence of equations (3.19) and (2.1). This completes the proof of the theorem. □

4 Analysis of a numerical method

In this section, we introduce and study a numerical scheme for the inequality (2.2) where the time variable is discretized by finite difference and the spatial variable by finite elements. We solve the inequality (2.2) on a time interval $[0, T]$. For simplicity in exposition, we use uniform partition of the time interval $[0, T]$ and comment that analysis of the numerical scheme can be extended in a straightforward way to the case of non-uniform partitions. For a positive integer N, let $k = T/N$ be the time step size and define

$$t_n = nk, \quad 0 \leqslant n \leqslant N.$$

For a continuous function $v(t)$ with values in a function space, we write $v_j = v(t_j)$, $0 \leqslant j \leqslant N$. Note that the condition (2.4) is satisfied for operators $\mathscr{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V^*)$ of the form

$$(\mathscr{S}v)(t) = S\left(\int_0^t q(t,s)\, v(s)\, ds + a_S\right) \quad \text{for all } v \in C(\mathbb{R}_+; V),\ t \in \mathbb{R}_+, \tag{4.1}$$

where $S \in \mathscr{L}(V, V^*)$, $q \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathscr{L}(V))$, $a_S \in V$. For an operator $\mathscr{S}$ of the form (4.1), we approximate the integral $\int_0^t q(t,s)\, v(s)\, ds$ by numerical integration. We take the trapezoidal rule as an example for the discretization of the time integration. All the discussion and results below can be extended to schemes based on other numerical quadratures. Recall the trapezoidal rule

$$\int_0^{t_n} z(s)\, ds \approx k \sum_{j=0}^{n}{}' z(t_j), \tag{4.2}$$

where a prime indicates that the first and last terms in the summation are to be halved. Then the operator $\mathscr{S}_n := \mathscr{S}(t_n)$ is approximated by $\mathscr{S}_n^k$ defined as follows:

$$\mathscr{S}_n^k v^k := S\left(k \sum_{j=0}^{n}{}' q(t_n, t_j)\, v_j^k + a_S\right) \quad \text{for } 0 \leqslant n \leqslant N,\ v^k = \{v_j^k\}_{j=0}^N. \tag{4.3}$$

While it is possible to consider numerical methods for the problem (2.2) for a general operator $\mathscr{R}$ satisfying the condition (2.5), in the following we focus on the special case where

$$\mathscr{R}v = R_0 \in L^2(\Gamma), \quad R_0 \geqslant 0 \text{ a.e. on } \Gamma, \tag{4.4}$$

for which an optimal order error estimate will be derived for the linear element solution.

For the spatial discretization, we let $\{V^h\}_{h>0}$ be a family of finite dimensional subspaces of V, indexed by a discretization parameter $h > 0$. Then the fully discrete scheme for the problem (2.2) is to find the discrete solution $u^{kh} := \{u_n^{kh}\}_{n=0}^N \subset V^h$ such that

$$\begin{aligned} &\langle Au_n^{hk}, v^h - u_n^{hk}\rangle + \langle \mathscr{S}_n^k u^{hk}, v^h - u_n^{hk}\rangle + \int_\Gamma R_0 \left(\varphi(\gamma v^h) - \varphi(\gamma u_n^{hk})\right) d\Gamma \\ &\quad + \int_\Gamma j^0(\gamma u_n^{hk}; \gamma v^h - \gamma u_n^{hk})\, d\Gamma \geqslant \langle f_n, v^h - u_n^{hk}\rangle \quad \text{for all } v^h \in V^h. \end{aligned} \tag{4.5}$$

Here and below, we use $\langle\cdot,\cdot\rangle$ to stand for $\langle\cdot,\cdot\rangle_{V^*\times V}$. It can be shown that under the assumptions of Theorem 1, the discrete problem has a unique solution u^{hk}. The rest of the section is devoted to an error estimation.

Consider the inequality (2.2) at $t = t_n$, which takes the form

$$\begin{aligned} &\langle Au_n, v - u_n\rangle + \langle \mathscr{S}_n u, v - u_n\rangle + \int_\Gamma R_0 \left(\varphi(\gamma v) - \varphi(\gamma u_n)\right) d\Gamma \\ &\quad + \int_\Gamma j^0(\gamma u_n; \gamma v - \gamma u_n)\, d\Gamma \geqslant \langle f_n, v - u_n\rangle \quad \text{for all } \ v \in V. \end{aligned} \tag{4.6}$$

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For any $v^h \in V^h$, write

$$\langle Au_n - Au_n^{hk}, u_n - u_n^{hk}\rangle = \langle Au_n, u_n - u_n^{hk}\rangle - \langle Au_n^{hk}, v^h - u_n^{hk}\rangle - \langle Au_n^{hk}, u_n - v^h\rangle.$$

Apply equation (4.6) with $v = u_n^{hk}$ to the first term on the right side and apply equation (4.5) with an arbitrary $v^h \in V^h$ to the second term on the right side to obtain

$$\begin{aligned}
&\left\langle Au_n - Au_n^{hk}, u_n - u_n^{hk}\right\rangle \\
&\quad \leqslant \left\langle Au_n - Au_n^{hk}, u_n - v^h\right\rangle \\
&\qquad + \left\langle Au_n, v^h - u_n\right\rangle + \left\langle \mathscr{S}_n u, v^h - u_n\right\rangle - \left\langle f_n, v^h - u_n\right\rangle \\
&\qquad + \left\langle \mathscr{S}_n u, u_n^{hk} - v^h\right\rangle + \left\langle \mathscr{S}_n^k u^{hk}, v^h - u_n^{hk}\right\rangle \\
&\qquad + \int_\Gamma R_0\left[\varphi(\gamma v^h) - \varphi(\gamma u_n)\right] d\Gamma \\
&\qquad + \int_\Gamma \left[j^0\left(\gamma u_n; \gamma u_n^{hk} - \gamma u_n\right) + j^0\left(\gamma u_n^{hk}; \gamma v^h - \gamma u_n^{hk}\right)\right] d\Gamma .
\end{aligned} \tag{4.7}$$

To proceed further, we replace v by $2u_n - v$ in equation (4.6) and get

$$\begin{aligned}
&\langle Au_n, v - u_n\rangle + \langle \mathscr{S}_n u, v - u_n\rangle - \langle f_n, v - u_n\rangle \\
&\quad \leqslant \int_\Gamma \left[R_0\left(\varphi(2\gamma u_n - \gamma v) - \varphi(\gamma u_n)\right) + j^0(\gamma u_n; \gamma u_n - \gamma v)\right] d\Gamma .
\end{aligned}$$

We use this inequality with $v = v^h$ in equation (4.7),

$$\left\langle Au_n - Au_n^{hk}, u_n - u_n^{hk}\right\rangle \leqslant \left\langle Au_n - Au_n^{hk}, u_n - v^h\right\rangle + I_{\mathscr{S},n} + I_{\mathscr{R},n} + I_{j,n}, \tag{4.8}$$

where

$$I_{\mathscr{S},n} = \left\langle \mathscr{S}_n u - \mathscr{S}_n^k u^{hk}, u_n^{hk} - v^h\right\rangle , \tag{4.9}$$

$$I_{\mathscr{R},n} = \int_\Gamma R_0\left(\varphi(\gamma v^h) + \varphi(2\gamma u_n - \gamma v^h) - 2\,\varphi(\gamma u_n)\right) d\Gamma , \tag{4.10}$$

$$I_{j,n} = \int_\Gamma \left[j^0(\gamma u_n; \gamma u_n - \gamma v^h) + j^0(\gamma u_n; \gamma u_n^{hk} - \gamma u_n) + j^0(\gamma u_n^{hk}; \gamma v^h - \gamma u_n^{hk})\right] d\Gamma . \tag{4.11}$$

We now bound each of the terms defined in equations (4.9)–(4.11). First, write

$$\|\mathscr{S}_n u - \mathscr{S}_n^k u^{hk}\|_V \leqslant \|\mathscr{S}_n u - \mathscr{S}_n^k u\|_V + \|\mathscr{S}_n^k u - \mathscr{S}_n^{hk} u^{hk}\|_V .$$

We will assume the regularities

$$q \in C^2(\mathbb{R}_+ \times \mathbb{R}_+; \mathscr{L}(V)), \quad u \in W^{2,\infty}(\mathbb{R}_+; V). \tag{4.12}$$

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Then following [19], we have

$$\|\mathscr{S}_n u - \mathscr{S}_n^k u\|_V \leqslant c\,k^2 \|u\|_{W^{2,\infty}(0,T;V)},$$
$$\|\mathscr{S}_n^k u - \mathscr{S}_n^k u^{hk}\|_V \leqslant c\,k \sum_{j=0}^{n} \|u_j - u_j^{hk}\|.$$

So

$$|I_{\mathscr{S},n}| \leqslant c \left(k^2 \|u\|_{W^{2,\infty}(0,T;V)} + k \sum_{j=0}^{n} \left\|u_j - u_j^{hk}\right\|_V \right) \left\|u_n^{hk} - v^h\right\|_V. \tag{4.13}$$

The term $I_{\mathscr{R},n}$ of equation (4.10) is bounded as follows:

$$|I_{\mathscr{R},n}| \leqslant c\, L_\varphi \|v^h - u_n\|_{L^2(\Gamma)}. \tag{4.14}$$

To bound $I_{j,n}$ of equation (4.11), we first write

$$j^0\left(\gamma u_n^{hk}; \gamma v^h - \gamma u_n^{hk}\right) \leqslant j^0\left(\gamma u_n^{hk}; \gamma v^h - \gamma u_n\right) + j^0\left(\gamma u_n^{hk}; \gamma u_n - \gamma u_n^{hk}\right).$$

Note that, by equation (2.7) (d), we have

$$j^0\left(\gamma u_n; \gamma u_n^{hk} - \gamma u_n\right) + j^0\left(\gamma u_n^{hk}; \gamma u_n - \gamma u_n^{hk}\right) \leqslant \beta\, \|\gamma\|^2 \left\|u_n - u_n^{hk}\right\|_V^2.$$

We assume additionally that $j(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^s$ for a.e. $\boldsymbol{x} \in \Gamma$, with a Lipschitz constant $L_j > 0$ independent of $\boldsymbol{x}$. Then,

$$j^0\left(\gamma u_n^{hk}; \gamma v^h - \gamma u_n\right) \leqslant L_j \|v^h - u_n\|_{L^2(\Gamma)},$$
$$j^0(\gamma u_n; \gamma u_n - \gamma v^h) \leqslant L_j \|v^h - u_n\|_{L^2(\Gamma)}.$$

Therefore,

$$|I_{j,n}| \leqslant \beta\, \|\gamma\|^2 \|u_n - u_n^{hk}\|_V^2 + c\, \|u_n - v^h\|_{L^2(\Gamma)}. \tag{4.15}$$

In addition to equation (2.3), we further assume that $A : V \to V^*$ is Lipschitz continuous. Then from equations (2.3) (b), (4.8), (4.13)–(4.15), we have the inequality

$$m_A \|u_n - u_n^{hk}\|_V^2 \leqslant c\, \|u_n - u_n^{hk}\|_V \|u_n - v^h\|_V + \beta\, \|\gamma\|^2 \|u_n - u_n^{hk}\|_V^2 + c\, \|u_n - v^h\|_{L^2(\Gamma)}$$
$$+ c \left(k^2 \|u\|_{W^{2,\infty}(0,T;V)} + k \sum_{j=0}^{n} \|u_j - u_j^{hk}\|_V \right) \|u_n^{hk} - v^h\|_V.$$

Here and below, c represents a constant independent of h, k whose value may change from place to place. Using the smallness assumption (2.9), the above inequality yields

$$\|u_n - u_n^{hk}\|_V \leqslant c \Big(\|u_n - v^h\|_V + \|u_n - v^h\|_{L^2(\Gamma)}^{1/2}$$
$$+ k^2 \|u\|_{W^{2,\infty}(0,T;V)} + k \sum_{j=0}^{n} \left\|u_j - u_j^{hk}\right\|_V \Big), \quad 0 \leqslant n \leqslant N. \tag{4.16}$$

We recall a discrete Gronwall's inequality ([15]). Let $\{e_n\}_{n=0}^N$ and $\{g_n\}_{n=0}^N$ be two sequences of non-negative numbers with

$$e_n \leqslant c_1 g_n + c_2 k \sum_{j=0}^{n} e_j, \quad 0 \leqslant n \leqslant N,$$

where c_1 and c_2 are constants. Then for some constant c_3, we have

$$\max_{0\leqslant n\leqslant N} e_n \leqslant c_3 \max_{0\leqslant n\leqslant N} g_n.$$

Applying the discrete Gronwall's inequality, we conclude from equation (4.16) that

$$\max_{0\leqslant n\leqslant N} \left\|u_n - u_n^{hk}\right\|_V \leqslant c \max_{0\leqslant n\leqslant N} \inf_{v^h\in V^h} \left(\|u_n - v^h\|_V + \|u_n - v^h\|_{L^2(\Gamma)}^{1/2}\right) + c\,k^2 \|u\|_{W^{2,\infty}(0,T;V)}. \tag{4.17}$$

Summarizing the above arguments, we have the following result.

Theorem 3 *Assume the conditions stated in Theorem* 1. *Moreover, assume* $A: V \to V^*$ *is Lipschitz continuous,* $j(\boldsymbol{x},\cdot)$ *is locally Lipschitz on* $\mathbb{R}^s$ *for a.e.* $\boldsymbol{x} \in \Gamma$ *with a Lipschitz constant independent of* $\boldsymbol{x}$, *and consider the particular cases* (4.1) *and* (4.4). *Then under the regularity assumptions* (4.12), *we have the error bound* (4.17).

The error bound (4.17) leads to convergence order error estimates, as will be seen in Section 6.

5 A frictional contact problem

A large number of quasistatic contact problems with elastic, viscoelastic or viscoplastic materials lead to a variational–hemivariational inequality of the form (2.2) in which the unknown is either the displacement or the velocity field. For a variety of such inequalities, the results in Sections 3 and 4 can be applied. We illustrate this point here on a viscoelastic contact problem. To this end, we need some specific notation that we present in what follows.

Given $d \in \mathbb{N}$, we use the symbol $\mathbb{S}^d$ for the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i), \quad \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Everywhere below Ω will represents a regular domain of $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that the measure of Γ_1, denoted $m(\Gamma_1)$, is positive. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \partial\Omega$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at $\partial\Omega$. Here and below, the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. We denote by $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes we do not indicate explicitly the dependence of the variables on the spatial variable $\boldsymbol{x}$. Recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where $u_{i,j} = \partial u_i / \partial x_j$. For a vector field, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on $\partial\Omega$ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

The physical setting is the following. A viscoelastic body occupies, in its reference configuration, a regular domain Ω. The body is clamped on Γ_1 and so the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and time-dependent volume forces of density $\boldsymbol{f}_0$ act in Ω. The body is in permanent contact on Γ_3 with a device, say a piston. The contact is modelled with a nonmonotone normal compliance condition associated with a total slip-dependent version of Coulomb's law of dry friction. We are interested in the equilibrium process of the mechanical state of the body, in the time interval of interest $\mathbb{R}_+$. Then, the mathematical model of the contact problem is stated as follows.

Problem 4 *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{5.4}$$

$$-\sigma_\nu(t) \in \partial j_\nu(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{5.5}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \leqslant F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big), \\ &-\boldsymbol{\sigma}_\tau = F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\frac{\boldsymbol{u}_\tau(t)}{\|\boldsymbol{u}_\tau(t)\|} \quad \text{if} \quad \boldsymbol{u}_\tau(t) \neq \boldsymbol{0} \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{5.6}$$

for all $t \in \mathbb{R}_+$.

We now present a short description of the equations and conditions in Problem 4 and we refer the reader to [15, 24, 32] for more details and mechanical interpretation. Equation (5.1) is the constitutive law for viscoelastic materials in which $\mathscr{A}$ represents the elasticity operator and $\mathscr{B}$ is the relaxation tensor. Equation (5.2) is the equilibrium equation for the quasistatic contact process. On Γ_1, we have the clamped boundary condition (5.3) and, on Γ_2, the surface traction boundary condition (5.4). Relation (5.5) is the contact condition in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν, and equation (5.6) represents a version of Coulomb's law of dry friction in which F_b is a given positive function, the friction bound.

Note that the friction bound is assumed to depend on the quantity

$$S(\boldsymbol{x},t) = \int_0^t \|\boldsymbol{u}_\tau(\boldsymbol{x},s)\|\,ds,$$

which represents the total slip (or, alternatively, the accumulated slip) at the point $\boldsymbol{x} \in \Gamma_3$ over the time period $[0,t]$. Considering such a dependence is reasonable from the physical point of view, since it incorporates the changes on the contact surface resulting from sliding. Indeed, when slip arises the asperities of the contact surface are flattening and, therefore, the friction bound evolves and, usually, it decreases. Moreover, considering such a dependence makes the frictional contact model more interesting from a mathematical point of view.

In the study of Problem 4, we use standard notation for Lebesgue and Sobolev spaces. For all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$ we still denote by $\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on $\partial\Omega$ and, recall, we use the notation v_ν and $\boldsymbol{v}_\tau$ for its normal and tangential traces. In addition, we introduce spaces V and $\mathcal{H}$ defined by

$$V = \{\ \boldsymbol{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1\ \},$$

$$\mathcal{H} = \{\ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega;\mathbb{S}^d) \mid \tau_{ij} = \tau_{ji},\ 1 \leqslant i,j \leqslant d\ \}.$$

The space $\mathcal{H}$ is a real Hilbert space with the canonical inner product given by

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\,\tau_{ij}(\boldsymbol{x})\,dx,$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. Since $m(\Gamma_1) > 0$, it is well known that V is a real Hilbert space with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \boldsymbol{u},\boldsymbol{v} \in V, \tag{5.7}$$

and the associated norm $\|\cdot\|_V$. Moreover, by the Sobolev trace theorem we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leqslant \|\gamma\|\,\|\boldsymbol{v}\|_V \quad \text{for all } \ \boldsymbol{v} \in V. \tag{5.8}$$

Here and below, $\|\gamma\|$ represents the norm of the trace operator $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$.

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth-order tensor fields given by

$$\mathbf{Q}_\infty = \{\ \mathscr{E} = (\mathscr{E}_{ijkl}) \mid \mathscr{E}_{ijkl} = \mathscr{E}_{jikl} = \mathscr{E}_{klij} \in L^\infty(\Omega),\ \ 1 \leqslant i,j,k,l \leqslant d\ \}.$$

We note that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathscr{E}\|_{\mathbf{Q}_\infty} = \sum_{0\leqslant i,j,k,l\leqslant d} \|\mathscr{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

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Moreover, a simple calculation shows that

$$\|\mathscr{E}\boldsymbol{\tau}\|_{\mathscr{H}} \leqslant \|\mathscr{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_{\mathscr{H}} \quad \text{for all } \mathscr{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in \mathscr{H}. \tag{5.9}$$

To derive a variational formulation of Problem 4, we list assumptions on the problem data. First, we assume that the elasticity operator $\mathscr{A}$ and the relaxation tensor $\mathscr{B}$ satisfy the following conditions.

$$\left\{\begin{array}{l}
\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that}\\
\quad \text{(a) there exists } L_{\mathscr{A}} > 0 \text{ such that}\\
\qquad \|\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathscr{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\\
\qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
\quad \text{(b) there exists } m_{\mathscr{A}} > 0 \text{ such that}\\
\qquad (\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathscr{A}}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2\\
\qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
\quad \text{(c) the mapping } \ \boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d.\\
\quad \text{(d) } \mathscr{A}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0} \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{5.10}$$

$$\mathscr{B} \in C(\mathbb{R}_+;\mathbf{Q}_\infty). \tag{5.11}$$

The potential function j_ν and the friction bound F_b satisfy

$$\left\{\begin{array}{l}
j_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that}\\
\quad \text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there}\\
\qquad \text{exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\overline{e}(\cdot)) \in L^1(\Gamma_3).\\
\quad \text{(b) } j_\nu(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3.\\
\quad \text{(c) } |\partial j_\nu(\boldsymbol{x},r)| \leqslant \overline{c}_0 + \overline{c}_1\,|r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3,\\
\qquad \text{for all } r \in \mathbb{R} \text{ with } \overline{c}_0,\ \overline{c}_1 \geqslant 0.\\
\quad \text{(d) } j_\nu^0(\boldsymbol{x},r_1;r_2 - r_1) + j_\nu^0(\boldsymbol{x},r_2;r_1 - r_2) \leqslant \overline{\beta}\,|r_1 - r_2|^2\\
\qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3, \text{ all } r_1,\ r_2 \in \mathbb{R} \text{ with } \overline{\beta} \geqslant 0.
\end{array}\right. \tag{5.12}$$

$$\left\{\begin{array}{l}
F_b : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ is such that}\\
\quad \text{(a) there exists } L_{F_b} > 0 \text{ such that}\\
\qquad |F_b(\boldsymbol{x},r_1) - F_b(\boldsymbol{x},r_2)| \leqslant L_{F_b}|r_1 - r_2|\\
\qquad\quad \text{for all } \ r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.\\
\quad \text{(b) the mapping } \boldsymbol{x} \mapsto F_b(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\
\qquad \text{for any } r \in \mathbb{R}.\\
\quad \text{(c) } F_b(\boldsymbol{x},0) = 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{5.13}$$

Finally, we assume that the densities of body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+;L^2(\Omega;\mathbb{R}^d)), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+;L^2(\Gamma_2;\mathbb{R}^d)). \tag{5.14}$$

We now turn to the variational formulation of Problem 4. Let $\boldsymbol{v} \in V$ and $t \in \mathbb{R}_+$. We perform integrations by parts, decompose the resulting surface integral on three integrals

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on Γ_1, Γ_2 and Γ_3 and then we use the boundary conditions (5.3), (5.4) and the equation (5.2) to obtain

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx = \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,d\Gamma. \tag{5.15}$$

Next, we use equations (5.5), (5.6) and the definition of the subdifferential to find that

$$F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\big(\|\boldsymbol{v}_\tau(t)\|-\|\boldsymbol{u}_\tau(t)\|\big) + j_\nu^0(u_\nu(t);v_\nu-u_\nu(t)) + \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t)) \geqslant 0 \quad \text{a.e. on } \Gamma_3,$$

which implies that

$$\int_{\Gamma_3} F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\big(\|\boldsymbol{v}_\tau(t)\|-\|\boldsymbol{u}_\tau(t)\|\big)\,d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))\,d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,d\Gamma \geqslant 0. \tag{5.16}$$

Consider the function $\boldsymbol{f}:\mathbb{R}_+\to V^*$ given by

$$\langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V^*\times V} = (\boldsymbol{f}_0(t),\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2(t),\gamma\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \tag{5.17}$$

for all $\boldsymbol{v}\in V$ and all $t\in\mathbb{R}_+$. Then, combining equations (5.15)–(5.17), we find that

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathscr{H}} + \int_{\Gamma_3} F\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\big(\|\boldsymbol{v}_\tau(t)\|-\|\boldsymbol{u}_\tau(t)\|\big)\,d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))\,d\Gamma \geqslant \langle \boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t)\rangle_{V^*\times V}. \tag{5.18}$$

We now use the constitutive law (5.1) and inequality (5.18) to obtain the following variational formulation of Problem 4, in terms of displacement.

Problem 5 *Find a displacement field $\boldsymbol{u}:\mathbb{R}_+\to V$ such that*

$$(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathscr{H}} + \Big(\int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_{\mathscr{H}} + \int_{\Gamma_3} F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\big(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_\tau(t)\|\big)\,d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu(t);v_\nu-u_\nu(t))\,d\Gamma \geqslant \langle \boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t)\rangle_{V^*\times V}, \tag{5.19}$$

for all $\boldsymbol{v}\in V$ and all $t\in\mathbb{R}_+$.

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Analysis of Problem 5, including its unique solvability and its numerical approximation will be presented in the next section.

6 Analysis of the contact problem

We start with the following existence and uniqueness result in which, recall, $\|\gamma\|$ represents the norm of the trace operator $\gamma : V \to L^2(\Gamma_3;\mathbb{R}^d)$.

Theorem 6 *Assume the hypotheses* (5.10)–(5.14) *and the smallness condition*

$$\overline{\beta}\,\|\gamma\|^2 < m_{\mathscr{A}}. \tag{6.1}$$

If one of the following hypotheses

$$m_{\mathscr{A}} > \overline{c}_1\sqrt{2}\,\|\gamma\|^2, \tag{6.2}$$

$$j_\nu^0(\boldsymbol{x},r;-r) \leqslant \overline{d}\,(1+|r|) \ \ \textit{for all } r\in\mathbb{R},\ \textit{a.e. } \boldsymbol{x}\in\Gamma_3 \textit{ with } \overline{d}\geqslant 0, \tag{6.3}$$

is satisfied, then Problem 5 *has a unique solution* $\boldsymbol{u}\in C(\mathbb{R}_+;V)$.

Proof We apply Theorem 1 with $\Gamma=\Gamma_3\subset\partial\Omega$ and $s=d$. To this end, we consider the operators $A : V\to V^*$, $\mathscr{S} : C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;V^*)$ and $\mathscr{R} : C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;L^2(\Gamma_3))$ defined by

$$\langle A\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}} \quad \text{for all } \ \boldsymbol{u},\,\boldsymbol{v}\in V, \tag{6.4}$$

$$\langle(\mathscr{S}\boldsymbol{u})(t),\boldsymbol{v}\rangle_{V^*\times V} = \Big(\int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_{\mathscr{H}} \tag{6.5}$$
$$\text{for all } \ \boldsymbol{u}\in C(\mathbb{R}_+;V),\ \boldsymbol{v}\in V,\ t\in\mathbb{R}_+,$$

$$(\mathscr{R}\boldsymbol{u})(t) = F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big) \quad \text{for all } \ \boldsymbol{u}\in C(\mathbb{R}_+;V),\ t\in\mathbb{R}_+, \tag{6.6}$$

respectively. Also, we consider the functions $\varphi : \Gamma_3\times\mathbb{R}^d\to\mathbb{R}$ and $j : \Gamma_3\times\mathbb{R}^d\to\mathbb{R}$ given by

$$\varphi(\boldsymbol{x},\boldsymbol{\xi}) = \|\boldsymbol{\xi}_\tau\| \qquad \text{for all } \ \boldsymbol{\xi}\in\mathbb{R}^d,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3, \tag{6.7}$$

$$j(\boldsymbol{x},\boldsymbol{\xi}) = j_\nu(\boldsymbol{x},\xi_\nu) \qquad \text{for all } \ \boldsymbol{\xi}\in\mathbb{R}^d,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3. \tag{6.8}$$

We will check that the hypotheses (2.3)–(2.8) are satisfied. First, from equation (5.10)(b), we get

$$\langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle_{V^*\times V} \geqslant m_{\mathscr{A}}\,\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V^2 \tag{6.9}$$

for all $\boldsymbol{v}_1$, $\boldsymbol{v}_2\in V$. Hence A is strongly monotone, i.e., it satisfies condition (2.3)(b) with $m_A = m_{\mathscr{A}}$. Moreover, using equation (5.10)(a), we easily establish the Lipschitz continuity of A, i.e.,

$$\|A\boldsymbol{u}-A\boldsymbol{v}\|_{V^*} \leqslant L_{\mathscr{F}}\|\boldsymbol{u}-\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in V.$$

On the other hand, from the assumption (5.10)(d), it is clear that $A\boldsymbol{0}_V = \boldsymbol{0}_{V^*}$ which means that $\|A\boldsymbol{u}\|_{V^*}\leqslant L_{\mathscr{A}}\|\boldsymbol{u}\|_V$ for all $\boldsymbol{u}\in V$. We conclude from this that A is bounded, monotone and hemicontinuous and, therefore, A is pseudomonotone. Finally, equation (6.9) combined with equality $A\mathbf{0}_V = \mathbf{0}_{V^*}$ shows that

$$\langle A\boldsymbol{v}, \boldsymbol{v}\rangle_{V^*\times V} \geqslant m_{\mathscr{F}} \|\boldsymbol{v}\|_V^2 \quad \text{for all } \boldsymbol{v} \in V.$$

We conclude from above that A satisfies condition (2.3)(a) with $\alpha = m_{\mathscr{F}}$.

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C(\mathbb{R}_+, V)$, $n \in \mathbb{N}$ and $t \in [0, n]$. Then, using equations (6.5), (5.9) and (5.11), we have

$$\begin{aligned}
&\langle (\mathscr{S}\boldsymbol{u}_1)(t) - (\mathscr{S}\boldsymbol{u}_2)(t), \boldsymbol{v}\rangle_{V^*\times V}\\
&\quad = \Big(\int_0^t \mathscr{B}(t-s)\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))\big)\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_{\mathscr{H}}\\
&\quad \leqslant \Big\| \int_0^t \mathscr{B}(t-s)(\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s)))\, ds \Big\|_{\mathscr{H}} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathscr{H}}\\
&\quad \leqslant \int_0^t \|\mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|_{\mathscr{H}}\, ds\, \|\boldsymbol{v}\|_V\\
&\quad \leqslant \max_{s\in[0,n]} \|\mathscr{B}(s)\|_{\mathbf{Q}_\infty} \Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \Big) \|\boldsymbol{v}\|_V
\end{aligned}$$

for all $\boldsymbol{v} \in V$, which implies that

$$\|(\mathscr{S}\boldsymbol{u}_1)(t) - (\mathscr{S}\boldsymbol{u}_2)(t)\|_{V^*} \leqslant \max_{s\in[0,n]} \|\mathscr{B}(s)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(r) - \boldsymbol{u}_2(r)\|_V\, dr. \tag{6.10}$$

We conclude from this that the operator $\mathscr{S}: C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V^*)$ given by (6.5) satisfies the hypothesis (2.4). Next, using equation (5.13) and the properties of the integral we obtain

$$\begin{aligned}
&\|(\mathscr{R}\boldsymbol{u}_1)(t) - (\mathscr{R}\boldsymbol{u}_2)(t)\|_{L^2(\Gamma_3)}\\
&\quad = \Big\| F_b\Big(\int_0^t \|\boldsymbol{u}_{1\tau}(s)\|\, ds\Big) - F_b\Big(\int_0^t \|\boldsymbol{u}_{2\tau}(s)\|\, ds\Big)\Big\|_{L^2(\Gamma_3)}\\
&\quad = \Big(\int_{\Gamma_3} \Big| F_b\Big(\int_0^t \|\boldsymbol{u}_{1\tau}(s)\|\, ds\Big) - F_b\Big(\int_0^t \|\boldsymbol{u}_{2\tau}(s)\|\, ds\Big)\Big|^2 d\Gamma \Big)^{1/2}\\
&\quad \leqslant L_{F_b} \Big(\int_{\Gamma_3} \Big| \int_0^t (\|\boldsymbol{u}_{1\tau}(s)\| - \|\boldsymbol{u}_{2\tau}(s)\|)\, ds \Big|^2 d\Gamma \Big)^{1/2}\\
&\quad = L_{F_b} \Big\| \int_0^t (\|\boldsymbol{u}_{1\tau}(s)\| - \|\boldsymbol{u}_{2\tau}(s)\|)\, ds \Big\|_{L^2(\Gamma_3)}\\
&\quad \leqslant L_{F_b} \int_0^t \|\boldsymbol{u}_{1\tau}(s) - \boldsymbol{u}_{2\tau}(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\, ds\\
&\quad \leqslant L_{F_b} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\, ds.
\end{aligned}$$

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Therefore, using the trace inequality (5.8) we obtain

$$\|(\mathscr{R}\boldsymbol{u}_1)(t) - (\mathscr{R}\boldsymbol{u}_2)(t)\|_{L^2(\Gamma_3)} \leqslant L_{F_b}\,\|\gamma\| \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds. \tag{6.11}$$

Inequality (6.11) shows that the operator $\mathscr{R}\colon C(\mathbb{R}_+;V) \to C(\mathbb{R}_+;L^2(\Gamma_3))$ satisfies the hypothesis (2.5)(a). Moreover, using the definition (6.6) and assumption (5.13), it follows that $\mathscr{R}$ satisfies equation (2.5)(b), too.

Subsequently, we observe that the function $\varphi\colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ defined by equation (6.7) satisfies condition (2.6). In addition, it can be immediately seen that for the function j given by equation (6.8), the conditions (2.7)(a) and (b) follow from equations (5.12)(a) and (b), respectively. The properties (2.7)(c) and (d) are now the consequences of the relations

$$\partial j(\boldsymbol{x},\boldsymbol{\xi}) \subset \partial j_\nu(\boldsymbol{x},\xi_\nu)\cdot \boldsymbol{\nu},\quad j^0(\boldsymbol{x},\boldsymbol{\xi};\boldsymbol{\eta}) \leqslant j_\nu^0(\boldsymbol{x},\xi_\nu;\eta_\nu)\ \text{ for all } \boldsymbol{\xi},\,\boldsymbol{\eta} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,$$

combined with the hypothesis (5.12)(c) and (d). Hence, we infer that equation (2.7) holds with $c_0 = \overline{c}_0$, $c_1 = \overline{c}_1$ and $\beta = \overline{\beta}$. Note also that the condition (2.8) is satisfied, due to the definition (5.17) and the regularity hypotheses (5.14). Finally, the conditions (2.9) follow from conditions (6.1) and (2.10), equation (2.11) represent consequences of equations (6.2) and (6.3), respectively.

We conclude from above that we are in a position to use Theorem 1. Therefore, we deduce the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+;V)$ such that

$$\begin{aligned}
&\langle A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V} + \langle(\mathscr{S}\boldsymbol{u})(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V}\\
&\quad + \int_{\Gamma_3} (\mathscr{R}\boldsymbol{u})(t)(\varphi(\gamma\boldsymbol{v}) - \varphi(\gamma\boldsymbol{u}(t)))\, d\Gamma + \int_{\Gamma_3} j^0(\gamma\boldsymbol{u}(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{u}(t))\, d\Gamma\\
&\quad \geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^*\times V},
\end{aligned} \tag{6.12}$$

for all $\boldsymbol{v} \in V$ and all $t \in \mathbb{R}_+$. Theorem 6 is now a consequence of inequality (6.12) and notation (6.4)–(6.8). □

We note that Theorem 6 provides the unique solvability of Problem 5 under the condition (6.1). Even if this condition is restrictive from physical point of view, we observe that it is automatically satisfied if j_ν is a convex function. Indeed, in this case assumption (5.12)(d) holds with $\overline{\beta} = 0$. Moreover, we recall that smallness assumptions of the from equation (6.1) are widely used in the analysis of static and quasistatic frictional contact problems, as explained in [32].

A couple of functions $(\boldsymbol{u},\boldsymbol{\sigma})$ which satisfies equations (5.1) and (5.19) is called a *weak solution* to Problem 4. We conclude that, under the assumptions of Theorem 6, Problem 4 has a unique weak solution. Moreover, the regularity of the weak solution is $\boldsymbol{u} \in C(\mathbb{R}_+;V)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+;\mathscr{H})$.

We now use Theorem 2 to study the dependence of the solution with respect to perturbations of the data. Various cases could be considered but, for simplicity, we restrict ourselves to proving the continuous dependence of the solution with respect to the relaxation tensor $\mathscr{B}$ and the friction bound F_b. Therefore, we assume in what follows that equations (5.10)–(5.14), (6.1) and (6.2) hold and we denote by $\boldsymbol{u}$ the solution of Problem 5

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obtained in Theorem 6. For each $\rho > 0$, let $\mathscr{B}_\rho$ and F_{b_ρ} represent perturbations of $\mathscr{B}$ and F_b respectively, which satisfy conditions (5.11) and (5.13), respectively. And, we denote by $L_{F_{b_\rho}}$ the Lipschitz constant of the function F_{b_ρ}, see equation (5.13). With these data, we consider the following perturbation of Problem 5.

Problem 7 *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$ *such that*

$$(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_{\mathscr{H}} + \Big(\int_0^t \mathscr{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) \Big)_{\mathscr{H}}$$
$$+ \int_{\Gamma_3} F_{b_\rho}\Big(\int_0^t \|\boldsymbol{u}_{\rho\tau}(s)\|\, ds \Big) \big(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{\rho\tau}(t)\| \big)\, d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_{\rho\nu}(t); v_\nu - u_{\rho\nu}(t))\, d\Gamma \geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_{V^* \times V} \tag{6.13}$$

for all $\boldsymbol{v} \in V$ *and all* $t \in \mathbb{R}_+$.

Here and below, $u_{\rho\nu}$ and $\boldsymbol{u}_{\rho\tau}$ represent the normal and the tangential components of the function $\boldsymbol{u}_\rho$, respectively. It follows from Theorem 6 that, for each $\rho > 0$, Problem 7 has a unique solution $\boldsymbol{u}_\rho$ with the regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; V)$. Consider now the following assumptions:

$$\mathscr{B}_\rho \to \mathscr{B} \quad \text{in } C(\mathbb{R}_+; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0. \tag{6.14}$$

$$\begin{cases} \text{There exists } F : (0,\infty) \to \mathbb{R}_+,\ \delta \geqslant 0 \text{ and } L_0 \geqslant 0 \text{ such that} \\ \text{(a) } |F_{b_\rho}(\boldsymbol{x}, r) - F_b(\boldsymbol{x}, r)| \leqslant F(\rho)(|r| + \delta) \\ \qquad \text{for all } \ r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{for all } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \\ \text{(c) } L_{b_\rho} \leqslant L_0 \quad \text{for all} \quad \rho > 0. \end{cases} \tag{6.15}$$

We have the following convergence result.

Theorem 8 *Assume hypotheses of Theorem 6, and, in addition, that equations* (6.1)*, and* (6.2) *or* (6.3) *hold. Let* $\mathscr{B}_\rho$ *and* F_{b_ρ} *satisfy equations* (5.11), (5.13), (6.14) *and* (6.15). *Then the solution* $\boldsymbol{u}_\rho$ *of Problem 7 converges to the solution* $\boldsymbol{u}$ *of Problem 5, i.e.,*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V) \quad \text{as } \rho \to 0. \tag{6.16}$$

Proof We apply Theorem 2. To this end, besides the operators (6.5) and (6.6), for each $\rho > 0$ we consider the operators $\mathscr{S}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V^*)$ and $\mathscr{R}_\rho : C(\mathbb{R}_+; V) \to$

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$C(\mathbb{R}_+; L^2(\Gamma_3))$ defined by

$$\langle (\mathscr{S}_\rho \boldsymbol{u})(t), \boldsymbol{v}\rangle_{V^*\times V} = \Big(\int_0^t \mathscr{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_{\mathscr{H}} \quad \text{for all } \boldsymbol{u}\in C(\mathbb{R}_+;V),\ \boldsymbol{v}\in V,\ t\in\mathbb{R}_+, \tag{6.17}$$

$$(\mathscr{R}_\rho \boldsymbol{u})(t) = F_{b_\rho}\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big) \quad \text{for all } \boldsymbol{u}\in C(\mathbb{R}_+;V),\ t\in\mathbb{R}_+. \tag{6.18}$$

Let $n\in\mathbb{N}$ and $\rho>0$. Then, using arguments similar to those used to prove the inequality (6.10) we deduce that

$$\|(\mathscr{S}_\rho \boldsymbol{u})(t) - (\mathscr{S}\boldsymbol{u})(t)\|_{V^*} \leqslant \max_{s\in[0,n]} \|\mathscr{B}_\rho(s)-\mathscr{B}(s)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}(r)\|_V\,dr, \tag{6.19}$$

for all $\boldsymbol{u}\in C(\mathbb{R}_+;V)$, $t\in[0,n]$. Hence, we deduce that condition (3.3)(a) holds with

$$G_n(\boldsymbol{u}) = \int_0^n \|\boldsymbol{u}(r)\|_V\,dr \quad\text{and}\quad g_n(\rho) = \max_{s\in[0,n]} \|\mathscr{B}_\rho(s)-\mathscr{B}(s)\|_{\mathbf{Q}_\infty}.$$

In addition, assumption (6.14) combined with equation (2.1) shows that $g_n(\rho)\to 0$ as $\rho\to 0$ which implies that equation (3.3)(b) holds, too.

Next, let $\boldsymbol{u}\in C(\mathbb{R}_+;V)$, $n\in\mathbb{N}$ and $t\in[0,n]$. Using equation (6.15) and the properties of the integral we obtain

$$\begin{aligned}
&\|(\mathscr{R}_\rho \boldsymbol{u})(t) - (\mathscr{R}\boldsymbol{u})(t)\|_{L^2(\Gamma_3)}\\
&\quad= \Big\|F_{b_\rho}\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big) - F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\Big\|_{L^2(\Gamma_3)}\\
&\quad= \Big(\int_{\Gamma_3}\Big|F_{b_\rho}\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big) - F_b\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds\Big)\Big|^2 d\Gamma\Big)^{1/2}\\
&\quad\leqslant F(\rho)\Big\|\int_0^t \|\boldsymbol{u}_\tau(s)\|\,ds + \delta\Big\|_{L^2(\Gamma_3)}\\
&\quad\leqslant F(\rho)\Big(\int_0^t \|\boldsymbol{u}_\tau(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)} + \delta\sqrt{m(\Gamma_3)}\Big)\\
&\quad\leqslant F(\rho)\Big(\|\gamma\|\int_0^t \|\boldsymbol{u}(s)\|_V + \delta\sqrt{m(\Gamma_3)}\Big).
\end{aligned}$$

Hence, it follows that equation (3.4)(a) holds with

$$H_n(\boldsymbol{u}) = \|\gamma\|\int_0^n \|\boldsymbol{u}(s)\|_V\,ds + \delta\sqrt{m(\Gamma_3)} \quad\text{and}\quad h_n(\rho)=F(\rho).$$

Therefore assumption (6.15)(b) shows that $h_n(\rho)\to 0$ as $\rho\to 0$ which implies that equation (3.4)(b) holds, too.

Finally, we note that the proof of inequalities (6.10) and (6.11) show that the operators $\mathscr{S}_\rho$ and $\mathscr{R}_\rho$ satisfy conditions (2.4) and (2.5) with

$$s_{\rho n} = \max_{s\in[0,n]} \|B_\rho(s)\|_{\mathbf{Q}_\infty} \quad \text{and} \quad r_{\rho n} = L_{F_{b_\rho}} \|\gamma\|,$$

respectively. We now use assumptions (6.14) and (6.15)(c) to see the sequences $(s_{\rho n})_\rho$ and $(r_{\rho n})_\rho$ are bounded and, therefore, conditions (3.6) and (3.7) hold.

Theorem 8 is now a consequence of Theorem 2, which concludes the proof. □

Turning now to the numerical approximation of the contact Problem 5, as in Section 4, we again focus on the case where $F_b(\boldsymbol{x}, r) = F_0(\boldsymbol{x})$ is a non-negative $L^2(\Gamma_3)$ valued function. We solve the problem on $[0, T] \times \Omega$ for some $T > 0$ and use the uniform partition for the time interval $[0, T]$ as in Section 4. For the spatial discretization, we assume that Ω is a polygonal domain and introduce a regular family of triangular finite element partitions $\{\mathscr{T}^h\}_{h>0}$ of $\overline{\Omega}$ that are compatible with the boundary decomposition $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$ in the sense that a side of an element lies on the boundary $\partial\Omega$, then the side lies entirely on $\overline{\Gamma_1}$, or $\overline{\Gamma_2}$, or $\overline{\Gamma_3}$. The parameter h is the mesh size of $\mathscr{T}^h$. For each partition $\mathscr{T}^h$, we then use the linear element space V^h to approximate V: a generic function $\boldsymbol{v}^h$ in V^h is a continuous, piecewise linear function that vanishes at the nodes on $\overline{\Gamma_1}$.

The numerical scheme is to find a discrete displacement field $\boldsymbol{u}^{hk} := \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ such that

$$\begin{aligned}
&\left(\mathscr{A}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)_{\mathscr{H}} + \left(k\sum_{j=0}^{n}{}' \mathscr{B}(t_n - t_j)\boldsymbol{\varepsilon}\left(\boldsymbol{u}_j^{hk}\right), \boldsymbol{\varepsilon}(\boldsymbol{v}^h) - \boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)_{\mathscr{H}} \\
&\quad + \int_{\Gamma_3} F_0\left(\|\boldsymbol{v}_\tau^h\| - \|\boldsymbol{u}_{n,\tau}^{hk}\|\right) d\Gamma + \int_{\Gamma_3} j_\nu^0\left(u_{n,\nu}^{hk}; v_\nu^h - u_{n,\nu}^{hk}\right) d\Gamma \\
&\geqslant \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}\rangle_{V^*\times V} \quad \text{for all} \quad \boldsymbol{v}^h \in V^h,\ 0 \leqslant n \leqslant N. \qquad (6.20)
\end{aligned}$$

Then under the assumptions stated in Theorem 6, where equation (5.14) is replaced by the condition that $F_b(\boldsymbol{x}, r) = F_0(\boldsymbol{x})$ is a non-negative $L^2(\Gamma_3)$ valued function, there is a unique discrete displacement field satisfying equation (6.20). For error estimation, in equation (5.12)(b), we assume the local Lipschitz constant for $j_\nu(\boldsymbol{x}, \cdot)$ is independent of $\boldsymbol{x}$. Then with the solution regularity assumption

$$\boldsymbol{u} \in W^{2,\infty}(\mathbb{R}_+; V), \qquad (6.21)$$

we have the following slight variant of the error bound (4.17):

$$\begin{aligned}
\max_{0\leqslant n\leqslant N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V &\leqslant c \max_{0\leqslant n\leqslant N} \inf_{\boldsymbol{v}^h\in V^h} \left(\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{u}_{n,\tau} - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma;\mathbb{R}^d)}^{1/2}\right) \\
&\quad + c\,k^2 \|\boldsymbol{u}\|_{W^{2,\infty}(0,T;V)}. \qquad (6.22)
\end{aligned}$$

In addition to equation (6.21), we further assume

$$\boldsymbol{u} \in C(\mathbb{R}_+; H^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{u}_\tau \in C(\mathbb{R}_+; \tilde{H}^2(\Gamma_3; \mathbb{R}^d)). \qquad (6.23)$$

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Here the space $\tilde{H}^2(\Gamma_3;\mathbb{R}^d)$ is defined as follows. Let $\overline{\Gamma_3}$ be represented as $\overline{\Gamma_3} = \cup_{1\leqslant i\leqslant I}\Gamma_{3,i}$ with each $\Gamma_{3,i}$ a closed subset of an affine hyperplane in $\mathbb{R}^d$. Then $v \in \tilde{H}^2(\Gamma_3;\mathbb{R}^d)$ means $v \in H^2(\Gamma_{3,i})$, $1 \leqslant i \leqslant I$. Under the solution regularity assumption (6.23), we apply the standard finite element interpolation theory ([1, 5]) to get

$$\max_{0\leqslant n\leqslant N} \inf_{\boldsymbol{v}^h\in V^h} \left(\|\boldsymbol{u}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{u}_{n,\tau} - \boldsymbol{v}^h_\tau\|^{1/2}_{L^2(\Gamma;\mathbb{R}^d)} \right) \leqslant c\, h,$$

where the constant c depends only on the seminorms of $\boldsymbol{u}$ in $C([0,T];H^2(\Omega;\mathbb{R}^d))$ and of $\boldsymbol{u}_\tau$ in $C([0,T];H^2(\Gamma_{3,i};\mathbb{R}^d))$, $1 \leqslant i \leqslant I$. Then by equation (6.22), we have the optimal order error estimate

$$\max_{0\leqslant n\leqslant N} \|\boldsymbol{u}_n - \boldsymbol{u}^{hk}_n\|_V \leqslant c\left(h + k^2\right), \tag{6.24}$$

i.e., the method is of first-order in spatial mesh size and of second-order in the time step.

7 Conclusion

This paper provides results in the study of a new class of variational–hemivariational inequalities with history-dependent operators, motivated by the development of the mathematical theory of contact mechanics. This concerns the continuous dependence of the solution with respect to the data and analysis of a discrete numerical scheme, in which the time variable is discretized by finite difference and the spatial variable by finite elements. Our results can be applied in the study of a large class of nonlinear boundary value problems. For a concrete example, we present a new model of quasistatic frictional contact with viscoelastic materials, which leads to a history-dependent variational–hemivariational inequality for the displacement field. We apply the abstract results in the study of this contact problem. Several open problems related to the contents of this manuscript remain to be investigated and resolved in the future. The first one would be to consider evolutionary versions of the inequality (2.2) in which the derivatives of the unknown u are involved. This would open a way to the study of dynamic contact problems. Note also that the solution of the inequality (2.2) is defined on the positive real line. Therefore, it would be of interest to study its asymptotic behaviour as $t \to \infty$.

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