

PAPER

Rogers semilattices of punctual numberings

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Abstract

The paper works within the framework of punctual computability, which is focused on eliminating unbounded search from constructions in algebra and infinite combinatorics. We study *punctual numberings*, that is, uniform computations for families S of primitive recursive functions. The *punctual reducibility* between numberings is induced by primitive recursive functions. This approach gives rise to upper semilattices of degrees, which are called *Rogers pr-semilattices*. We show that any infinite, uniformly primitive recursive family S induces an infinite Rogers pr-semilattice R . We prove that the semilattice R does not have minimal elements, and every nontrivial interval inside R contains an infinite antichain. In addition, every non-greatest element from R is a part of an infinite antichain. We show that the Σ_1 -fragment of the theory $Th(R)$ is decidable.

Keywords: Theory of numberings; upper semilattice; Rogers semilattice; primitive recursion; Friedberg numbering; decidability; online computation; punctual structure

1. Introduction

For a countable set \mathcal{S} , a *numbering* of \mathcal{S} is a surjective map ν from the set of natural numbers ω onto \mathcal{S} . Since the 1950s, *computable numberings* for families of computably enumerable (c.e.) sets have been extensively studied by computability theorists.

Consider a family \mathcal{S} of c.e. sets. A numbering ν of the family \mathcal{S} is *computable* if the set

$$\{\langle n, x \rangle : n \in \omega, x \in \nu(n)\}$$

is c.e. The family \mathcal{S} is *computable* if it has a computable numbering. Informally speaking, the computability of \mathcal{S} means that there is a procedure which provides a *uniform enumeration* of the family \mathcal{S} .

A measure of relative complexity is provided by the notion of *reducibility* between numberings: A numbering ν is *reducible* to another numbering μ , denoted by $\nu \leq \mu$, if there is total computable function $f(x)$ such that $\nu(n) = \mu(f(n))$ for all $n \in \omega$. In other words, there is an algorithmic procedure which, given a ν -index of an object from \mathcal{S} , computes a μ -index for the same object.

The notion of reducibility induces the *Rogers upper semilattice* (or *Rogers semilattice* for short) of a computable family \mathcal{S} : This semilattice contains the degrees of all computable numberings of \mathcal{S} . As usual, here we assume that two numberings have the same degree if they are reducible to each other. Rogers semilattices are widely used as a tool to classify properties of computable

numberings for different families. For known results on computable numberings, the reader is referred to the seminal monograph (Ershov 1977) and the papers (Ershov 1999; Eršov 1973, 1975, 1977).

Goncharov and Sorbi (1997) started developing the theory of generalized computable numberings. Their approach initiated a fruitful line of research, which is focused on numberings for families of sets which belong to various levels of recursion-theoretic hierarchies: the arithmetical hierarchy (Badaev et al. 2006; Bazhenov et al. 2019c; Podzorov 2008), the Ershov hierarchy (Badaev and Lempp 2009; Goncharov et al. 2002; Herbert et al. 2019; Ospichev 2015), the analytical hierarchy (Bazhenov et al. 2019d, 2020b; Dorzhieva 2019), etc. We refer the reader to Badaev and Goncharov (2008, 2000) for further background on numberings in these hierarchies.

In general, one can say that the theory of numberings is one of the branches of *computable* or *effective mathematics*. This research area aims to understand and calibrate the algorithmic content of mathematical objects. The roots of this direction go back to the introduction of non-recursive mathematical methods at the beginning of the 20th century, as discussed in Metakides and Nerode (1982). Working within the framework of effective mathematics, the theory of numberings generally employs the Turing computability model.

This paper is inspired by the recent developments in computable structure theory: Kalimullin et al. (2017) proposed a general framework for online computations in algebraic and combinatorial structures. The key object of this framework is the notion of a *punctual* (or *fully primitive recursive*) *structure*. An infinite structure \mathcal{S} in a finite signature is *punctual* if the domain of \mathcal{S} is equal to ω , and the basic functions and relations of \mathcal{S} are primitive recursive.

The notion of punctuality essentially *eliminates* all instances of unbounded search in Turing computable algorithms. This feature allows one to mimic a large class of “on-line” algorithms, that is, algorithms which have to make decisions on the fly. A typical example of such an algorithm is online coloring, say, of a tree: given the n -th vertex of an input tree, you have to decide its color right at the moment (you cannot wait for the $(n + 1)$ -th vertex to appear).

Our paper employs the punctuality paradigm of Kalimullin et al. (2017) in the studies of numberings.

Definition 1. Let \mathcal{S} be a family of total functions acting from ω to ω . We say that a numbering ν of the family \mathcal{S} is *punctual* if the function:

$$g_\nu(n, x) := (\nu(n))(x)$$

is primitive recursive. A family \mathcal{S} is punctual if it has a punctual numbering.

Informally, the punctuality of \mathcal{S} means that one can promptly (and uniformly) compute every function from \mathcal{S} . The punctuality approach requires that we have to *modify* the notion of the reduction between numberings accordingly:

Definition 2. Let ν and μ be numberings. We say that ν is *punctually reducible* to μ , denoted by $\nu \leq_{pr} \mu$, if there is a primitive recursive function $f : \omega \rightarrow \omega$ such that

$$\nu(n) = \mu(f(n)), \text{ for all } n \in \omega.$$

In this case, we say that f *punctually reduces* ν to μ .

In a natural way, Definitions 1 and 2 induce the notion of *Rogers pr-semilattice* for a punctual family \mathcal{S} , denoted by $\mathcal{R}_{pr}(\mathcal{S})$, see Section 2.1 for a formal definition.

The paper is arranged as follows. Section 2 provides the necessary preliminaries. In particular, we show that every finite punctual family induces a one-element Rogers pr-semilattice (Proposition 3). So, in the rest of the paper, we work only with infinite punctual families \mathcal{S} .

In Section 3, we introduce the notion of a strongly punctually decidable numbering (Definition 5), which plays a key role in most of our constructions.

Section 4 gives our first example of a punctual priority construction: we show that below an arbitrary degree from $\mathcal{R}_{pr}(\mathcal{S})$, one can find an infinite descending chain (Theorem 16). This fact already provides a contrast with the classical case of computable numberings: for example, the Rogers semilattice of a computable family $\mathcal{T}_0 = \{\emptyset\} \cup \{\{k\} : k \in \omega\}$ has a least element.

Section 5 proves that every non-greatest degree from $\mathcal{R}_{pr}(\mathcal{S})$ is a part of an infinite antichain (Theorem 20). In Section 6, we consider intervals inside $\mathcal{R}_{pr}(\mathcal{S})$. We prove that every interval $[a <_{pr} b]$ inside $\mathcal{R}_{pr}(\mathcal{S})$ contains an infinite antichain (Theorem 24). Note that the standard recursion-theoretic methods show that the classical Rogers semilattice of the family \mathcal{T}_0 has an initial segment containing precisely two degrees. We also establish that the Σ_1 -fragment of the theory $Th(\mathcal{R}_{pr}(\mathcal{S}))$ is decidable (Corollary 25).

Section 7 shows that there are infinite Rogers pr-semilattices which are lattices (Theorem 27). This contrasts with the result of Selivanov (1976): he established that a classical infinite Rogers semilattice cannot be a lattice. On the other hand, we build a Rogers pr-semilattice which contains a minimal pair (Proposition 29).

Section 8 discusses the existence of greatest elements. We provide examples of Rogers pr-semilattices with greatest element (Lemma 33) and without greatest element (Theorem 31).

In Section 9, Theorem 35 builds an infinite punctual family \mathcal{S} such that:

- (a) every punctual numbering ν of \mathcal{S} is \equiv_{pr} -equivalent to a strongly punctually decidable numbering, and
- (b) \mathcal{S} contains only characteristic functions of one-element sets.

We conclude our paper with Section 10, which discusses some results on punctual Friedberg numberings.

The paper is an extended version of the conference paper (Bazhenov et al. 2020a). New material in this paper includes the following. Subsection 3.1 introduces the notion of a persistently decidable family and discusses its properties. The material of Section 5 is new: it provides a generalization of Theorem 4.1 of Bazhenov et al. (2020a). Almost everything in Section 6 is new: the only old material is the formulation of Theorem 23 – in Bazhenov et al. (2020a) it was given as Theorem 6.1 (without a proof). In Section 7, Theorem 27 is a generalization of Proposition 5.1 of Bazhenov et al. (2020a). Note that Proposition 29 was given without a proof in Bazhenov et al. (2020a) (see Proposition 5.2 there). Theorem 31 generalizes Proposition 6.1 of Bazhenov et al. (2020a). Sections 9 and 10 are new.

2. Preliminaries

Given a total function $f : \omega \rightarrow \omega$ and a nonzero natural number m , by $f|m$ we denote the following tuple:

$$f|m := (f(0), f(1), \dots, f(m - 1)).$$

For a set $A \subseteq \omega$, by χ_A we denote the characteristic function of A . For a pair of natural numbers (k, ℓ) , the value $\langle k, \ell \rangle$ is its standard Cantor index, that is,

$$\langle k, \ell \rangle = \frac{(k + \ell)(k + \ell + 1)}{2} + k.$$

2.1 Theory of numberings

Let ν be a numbering of a family \mathcal{S}_0 and μ be a numbering of a family \mathcal{S}_1 . It is easy to see that the condition $\nu \leq \mu$ implies that \mathcal{S}_0 is a subfamily of \mathcal{S}_1 . Clearly, if $\nu \leq_{pr} \mu$, then $\nu \leq \mu$.

Numberings ν and μ are *equivalent*, denoted by $\nu \equiv \mu$, if $\nu \leq \mu$ and $\mu \leq \nu$. The *punctual equivalence* \equiv_{pr} is defined in a similar way. The numbering $\nu \oplus \mu$ of the family $\mathcal{S}_0 \cup \mathcal{S}_1$ is defined as follows:

$$(\nu \oplus \mu)(2x) = \nu(x), \quad (\nu \oplus \mu)(2x + 1) = \mu(x).$$

It is not difficult to obtain the following fact (see, e.g., Proposition 3 in Ershov 1977, p. 36). If $\sqsubseteq \in \{\leq, \leq_{pr}\}$ and ξ is an arbitrary numbering, then

$$(\nu \sqsubseteq \xi \ \& \ \mu \sqsubseteq \xi) \Leftrightarrow (\nu \oplus \mu \sqsubseteq \xi).$$

Let \mathcal{S} be a punctual family of functions. By $Com_{pr}(\mathcal{S})$, we denote the set of all punctual numberings of \mathcal{S} . Since the relation \equiv_{pr} is a congruence on the structure $(Com_{pr}(\mathcal{S}); \leq_{pr}, \oplus)$, we use the same symbols \leq_{pr} and \oplus on numberings of \mathcal{S} and on \equiv_{pr} -equivalence classes of the numberings.

The quotient structure $\mathcal{R}_{pr}(\mathcal{S}) := (Com_{pr}(\mathcal{S})/\equiv_{pr}; \leq_{pr}, \oplus)$ is an upper semilattice. We call the structure $\mathcal{R}_{pr}(\mathcal{S})$ the *Rogers pr-semilattice* of the punctual family \mathcal{S} .

Let \mathcal{T} be a family of total (Turing) computable functions acting from ω to ω . A numbering ν of the family \mathcal{T} is *computable* if the function g_ν from Definition 1 is computable. Note that this definition is consistent with the notion of computable numbering discussed in the introduction: if we identify functions from \mathcal{T} with their graphs, then we will get precisely the same notions.

We say that a family \mathcal{T} is *Turing computable* if it has a computable numbering. The definition of *Rogers semilattice* $\mathcal{R}_c(\mathcal{T})$ is obtained in a similar way to the semilattice $\mathcal{R}_{pr}(\mathcal{S})$, modulo the following modification: one needs to consider all computable numberings of \mathcal{T} and the standard reducibility \leq between them.

Another way to look at things is the following. Suppose that \mathcal{T} is still a *Turing computable* family of total functions, *but* our reducibility is *punctual*. This gives rise to another Rogers semilattice $\mathcal{R}^{(c,pr)}(\mathcal{T})$. In this setting, the semilattices $\mathcal{R}_{pr}(\mathcal{S})$ constitute a special case of $\mathcal{R}^{(c,pr)}(\mathcal{T})$. Throughout the paper, we give several comments on how our results on $\mathcal{R}_{pr}(\mathcal{S})$ could be extended to a more general case of $\mathcal{R}^{(c,pr)}(\mathcal{T})$.

2.2 The punctuality paradigm

Kalimullin et al. (2017) noted that many known proofs from polynomial time structure theory (see, e.g., the survey Cenzer and Remmel 1998) are focused on making the operations and relations on a given algebraic structure \mathcal{S} primitive recursive, and then observing that the obtained presentation of \mathcal{S} is in fact computable in polynomial time.

Kalimullin et al. (2017) proposed that a reasonable “online” presentation of a structure must minimally satisfy the following: a countably infinite structure is *punctual* if its domain equals ω and the operations and predicates of the structure are (uniformly) primitive recursive. Roughly speaking, the intuition behind this notion is the following: punctual presentations do not allow any delays – or in other words, there are no instances of truly unbounded search.

Nowadays, the theory of punctual structures is a flourishing research direction. Its intricate constructions have provided a deep insight into the algorithmic complexity of isomorphisms (Downey et al. 2020a,b; Melnikov and Ng 2019). The developed techniques found applications in the theory of automatic structures (Bazhenov et al. 2019b). For further results on punctual structures, the reader is referred to the surveys (Bazhenov et al. 2019a; Downey et al. 2021; Melnikov 2017).

The *restricted Church–Turing thesis* for primitive recursive functions says the following: A function is primitive recursive if and only if it can be described by an algorithm that uses only bounded loops. Informally speaking, one needs to eliminate all instances of `while . . . do`, `repeat . . . until`, and `goto` in a Pascal-like programming language.

Our proofs will exploit the restricted Church–Turing thesis without an explicit reference.

Let k be a nonzero natural number. We fix a computable list $(p_e^{(k)})_{e \in \omega}$ of all k -ary primitive recursive functions. We emphasize that the list is computable, but it *cannot* be primitive recursive. Nevertheless, the following function *can* be treated as a punctual object:

$$p_e^{(k)}[t](\bar{x}) := \begin{cases} p_e^{(k)}(\bar{x}), & \text{if the value } p_e^{(k)}(\bar{x}) \text{ is computed in} \\ & \text{at most } t \text{ computational steps,} \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Without loss of generality, one may also assume the following: if $p_e^{(k)}(\bar{x})$ is equal to N , then for any $t \leq \max(e, \bar{x}, N)$, the value $p_e^{(k)}[t](\bar{x})$ is undefined. The formal details can be recovered from Bazhenov et al. (2019a, Section 10).

If $k = 1$, then we omit the superscript – we write p_e in place of $p_e^{(1)}$. For $e \in \omega$, we define a numbering ρ_e :

$$(\rho_e(k))(x) := p_e^{(2)}(k, x).$$

It is clear that $(\rho_e)_{e \in \omega}$ is a computable list of all punctual numberings.

2.3 Semilattices for finite families

We note the following – when a punctual family \mathcal{S} is finite, the situation is very simple:

Proposition 3. *Let \mathcal{S} be a finite punctual family. Then the semilattice $\mathcal{R}_{pr}(\mathcal{S})$ contains precisely one element.*

Proof. Suppose that $\mathcal{S} = \{f_0, f_1, \dots, f_m\}$. We fix a natural number N such that the strings $f_i|N$ and $i \leq m$ are pairwise different.

It is sufficient to establish the following fact: for arbitrary punctual numberings ν and μ of \mathcal{S} , one can build a function g , which punctually reduces ν to μ .

The desired g is constructed as follows. We fix indices $a_i, i \leq m$, such that $\mu(a_i) = f_i$. For an arbitrary index $k \in \omega$, we promptly find the number $j \leq m$ such that $\nu(k)|N = f_j|N$. Then we define $g(k) := a_j$. □

3. Strongly Punctually Decidable Numberings

In this section, we introduce the notion of a strongly punctually decidable numbering and discuss its properties. This notion will play a key role in most of our further constructions.

The background behind the notion comes from some classical results of the theory of numberings. An arbitrary numbering ν is traditionally associated with an equivalence relation η_ν on ω – the relation η_ν is defined as follows:

$$m \eta_\nu n \Leftrightarrow \nu(m) = \nu(n).$$

Sometimes, the relation η_ν is called the *numeration equivalence* of ν .

A numbering ν is called *negative* if the relation η_ν is co-c.e., that is, η_ν is the complement of a c.e. set. A numbering ν is *decidable* if the relation η_ν is (Turing) computable. A numbering ν is *Friedberg* if η_ν is the identity relation.

Let \mathcal{S} be an infinite, Turing computable family of functions. Mal'cev (1965) showed that every computable numbering of \mathcal{S} is negative. Ershov (1967) proved that for any computable numbering ν of \mathcal{S} , there is a computable Friedberg numbering μ of \mathcal{S} such that $\mu \leq \nu$.

The notion of a strongly punctually decidable numbering comes from a “punctual version” of the results of Mal'cev and Ershov. First, we note the following:

Observation 4. If ν is a punctual numbering, then ν is negative.

Indeed, this fact easily follows from the result of Mal'cev (1965). After that, we introduce our new notions:

Definition 5. We say that a numbering ν is punctually decidable if the relation η_ν is primitive recursive.

A punctually decidable numbering ν is strongly punctually decidable (or spd for short) if every η_ν -equivalence class C satisfies the following: either C contains only one element, or $C = [0]_{\eta_\nu}$.

Roughly speaking, the intuition behind the notion of an spd numbering is as follows. When one builds a punctual algebraic structure \mathcal{P} , one of the basic strategies is “time-filling”: while we wait for some Turing computable (not necessarily primitive recursive) process to finish its computations, we still *have to* promptly construct larger and large finite pieces of \mathcal{P} . Typically, this is achieved via copying some (relatively) simple parts of a given punctual structure \mathcal{A} . For example, if $\mathcal{A} = (\omega, s, 0)$, where s is the successor function, then one could promptly take fresh numbers and declare that inside our \mathcal{P} , they are equal to $0_{\mathcal{P}}, s_{\mathcal{P}}(0_{\mathcal{P}}), s_{\mathcal{P}}(s_{\mathcal{P}}(0_{\mathcal{P}})), \dots, s_{\mathcal{P}}^{(n)}(0_{\mathcal{P}}), \dots$.

The notion of an spd numbering mirrors the time-filling strategy. In general, the construction of a punctual Friedberg numbering for a given punctual family \mathcal{S} could be impossible (see Proposition 38 in the last section). Nevertheless, “stalling” the construction process by defining

$$\nu(1) := \nu(0), \nu(2) := \nu(0), \dots, \nu(n) := \nu(0), \dots,$$

allows to build an spd numbering for \mathcal{S} .

We prove the following general result about numberings:

Proposition 6.

- (i) If ν is a punctually decidable numbering, then any $\mu \leq_{pr} \nu$ is also punctually decidable.
- (ii) If ν is a punctually decidable numbering, then there is a spd numbering μ such that $\mu \equiv_{pr} \nu$.
- (iii) If ν is a negative numbering of a family \mathcal{S} , then there is a spd numbering μ of the same family such that $\mu \leq_{pr} \nu$.

Proof. (i) Suppose that ν is a punctually decidable numbering, and a function f punctually reduces μ to ν . Then clearly, we have

$$(k \eta_\mu \ell) \Leftrightarrow (f(k) \eta_\nu f(\ell)).$$

Since η_ν is primitive recursive, the relation η_μ is also primitive recursive.

(ii) Let ν be a punctually decidable numbering. We define a new numbering μ as follows:

$$\mu(0) := \nu(0), \quad \mu(k+1) := \begin{cases} \nu(k+1), & \text{if } (\forall i \leq k) \neg [(i, k+1) \in \eta_\nu], \\ \nu(0), & \text{if } (\exists i \leq k) [(i, k+1) \in \eta_\nu]. \end{cases}$$

Since the relation η_ν is primitive recursive, a unary relation

$$R(k) := (\exists i \leq k) [(i, k+1) \in \eta_\nu]$$

is also primitive recursive. Using this fact, it is easy to show that the numbering μ is spd, and $\mu = \nu \circ f$ for a primitive recursive function f . Moreover,

$$\nu(k+1) = \begin{cases} \mu(k+1), & \text{if } (\forall i \leq k) \neg [(i, k+1) \in \eta_\nu], \\ \mu(0), & \text{otherwise;} \end{cases}$$

hence, $\nu \leq_{pr} \mu$. Therefore, we deduce that $\mu \equiv_{pr} \nu$.

(iii) Let ν be a negative numbering of a family \mathcal{S} . By item (ii), it is sufficient to find a punctually decidable numbering μ of \mathcal{S} such that $\mu \leq_{pr} \nu$.

Since the numbering ν is negative, the set

$$I_\nu := \{n \in \omega : (\forall i < n)(\nu(i) \neq \nu(n))\}$$

is computably enumerable. Choose a primitive recursive function $h(x)$ such that $\text{range}(h) = I_\nu$. Then it is not hard to show that the numbering $\mu := \nu \circ h$ is punctually decidable. Hence, μ has the desired properties. □

The following consequence of Proposition 6 will play a crucial part in our constructions:

Corollary 7. *If ν is a punctual numbering of a family \mathcal{S} , then there is a spd numbering $\mu \in \text{Com}_{pr}(\mathcal{S})$ such that $\mu \leq_{pr} \nu$.*

3.1 Punctual families with spd numberings

After Proposition 6 and Corollary 7 are obtained, it is natural to consider the following problem:

Problem 8. We say that a punctual family \mathcal{S} is *persistently decidable* if every punctual numbering of \mathcal{S} is punctually decidable. Characterize persistently decidable families.

First, we observe the following easy sufficient condition for being persistently decidable.

Definition 9. *We say that a punctual family \mathcal{S} is strongly finitely discrete (or sf-discrete for short) if there exists a natural number N such that for any functions $f, g \in \mathcal{S}$, the following conditions are equivalent:*

$$f = g \Leftrightarrow f|N = g|N.$$

Lemma 10. *Every sf-discrete punctual family \mathcal{S} is persistently decidable.*

Proof. Let ν be an arbitrary punctual numbering of \mathcal{S} . It is clear that we have

$$(k \eta_\nu \ell) \Leftrightarrow \nu(k)|N = \nu(\ell)|N,$$

and this condition is primitive recursive. □

Note that every finite punctual family is sf-discrete.

Example 11. A simple example of an infinite sf-discrete family is the family containing the following functions: for $i \in \omega$,

$$g_i(x) = i, \text{ for all } x.$$

Second, we obtain a sufficient condition for being *not* persistently decidable.

Definition 12. *Let ν be a punctual numbering of a family \mathcal{S} . We say that a primitive recursive function f is a quickly spoiling limit point for ν if it satisfies the following: there is a primitive recursive function $sp(n, t)$ such that for any n ,*

$$\nu(sp(n, 0))|n = \nu(sp(n, 1))|n = f|n \text{ and } \nu(sp(n, 0)) \neq \nu(sp(n, 1)).$$

The intuition behind the definition is as follows. Consider a construction, which promptly defines a function $g(x)$ bit by bit. A default action of this construction is copying the function

f – we put $g(0) := f(0), g(1) := f(1), g(2) := f(2), \dots$. Then the function $sp(n, t)$ always provides two different opportunities for a quick “diagonalization”: at any stage s , one can abandon copying f and set either $g := v(sp(s, 0))$ or $g := v(sp(s, 1))$.

Observation 13. Let ν be a punctual numbering of a family \mathcal{S} , and let μ be a punctual numbering of a family \mathcal{T} . If f is a quickly spoiling limit point for ν , then f is also quickly spoiling for $\nu \oplus \mu$.

Example 14. A simple example for Definition 12 can be recovered as follows. Consider a punctual Friedberg numbering:

$$\nu(k) := \chi_{\{k\}}.$$

Then the function $f := \chi_{\emptyset}$ is a quickly spoiling limit point for ν . This is witnessed by a function sp satisfying $sp(n, 0) := n + 1$ and $sp(n, 1) := n + 2$. Note that in this example, f is the *unique* limit point of \mathcal{S} in the Cantor space 2^ω .

Proposition 15. Suppose that ν is a punctual numbering of a family \mathcal{S} , and f is a quickly spoiling limit point for ν . Then the family \mathcal{S} is not persistently decidable.

Proof. Choose a primitive recursive function $sp(n, t)$, which witnesses the spoiledness of f . Let A be a computable set which is not primitive recursive. Fix two primitive recursive functions $p(x)$ and $q(x)$ such that $A = \text{range}(p)$ and $\bar{A} = \text{range}(q)$.

We define a punctual numbering μ of some subfamily $\mathcal{S}_0 \subseteq \mathcal{S}$. An informal idea of the construction is as follows. Consider a number $k \in \omega$. We look at two quick listings $A = \{p(0), p(1), p(2), \dots\}$ and $\bar{A} = \{q(0), q(1), q(2), \dots\}$, and we wait for the first stage s_0 , at which we see that k appeared on one of the two lists. At every stage $s < s_0$, the functions $\mu(2k)$ and $\mu(2k + 1)$ just copy (the initial segments of) the limit point f . At the stage s_0 , we consider two cases:

- (1) If $k \in A$, then we put $\mu(2k) = \mu(2k + 1) := v(sp(s_0, 0))$.
- (2) If $k \notin A$, then we set $\mu(2k) := v(sp(s_0, 0))$ and $\mu(2k + 1) := v(sp(s_0, 1))$.

The properties of the function $sp(n, t)$ ensure that the described construction builds a punctual numbering, and every function $\mu(\ell)$ belongs to \mathcal{S} . Moreover, we deduce that

$$k \in A \Leftrightarrow \mu(2k) = \mu(2k + 1).$$

Since the set A is not primitive recursive, the numbering μ cannot be punctually decidable.

For the sake of completeness, we provide a more formal description of the construction. We build the numbering μ and auxiliary prompt functions $u(k, s)$ and $w(k, s)$. We put $\mu(k)(0) = f(0)$ and $u(k, 0) = w(k, 0) = -1$ for all $k \in \omega$.

At a stage $s + 1$, for each k we proceed as follows:

- (1) If $u(k, s) \geq 0$, then we put $u(k, s + 1) := u(k, s)$ and $w(k, s + 1) := w(k, s)$. Otherwise, we check the following conditions:
 - If $k \in \{p(0), p(1), \dots, p(s + 1)\}$, then set $u(k, s + 1) = w(k, s + 1) := sp(s + 1, 0)$.
 - If $k \in \{q(0), q(1), \dots, q(s + 1)\}$, then define $u(k, s + 1) := sp(s + 1, 0)$ and $w(k, s + 1) := sp(s + 1, 1)$.
 - Otherwise, do not change the values of u and v .

(2) Define

$$\mu(2k)(s+1) := \begin{cases} v(u(k, s+1))(s+1), & \text{if } u(k, s+1) \geq 0, \\ f(s+1), & \text{if } u(k, s+1) = -1; \end{cases}$$

$$\mu(2k+1)(s+1) := \begin{cases} v(w(k, s+1))(s+1), & \text{if } w(k, s+1) \geq 0, \\ f(s+1), & \text{if } w(k, s+1) = -1. \end{cases}$$

This concludes the description of the construction. The argument given before the description shows that μ promptly indexes some subfamily \mathcal{S}_0 of \mathcal{S} , and the numbering μ is not punctually decidable. Therefore, the numbering $\mu_1 := v \oplus \mu$ belongs to $Com_{pr}(\mathcal{S})$, and μ_1 is not punctually decidable. Hence, the family \mathcal{S} is not persistently decidable. \square

In general, it seems that Problem 8 is pretty hard to solve. After the reader familiarizes themselves with further punctual constructions, in Section 9, we will build an infinite, persistently decidable family \mathcal{S} such that:

- For every function $f \in \mathcal{S}$, $range(f) \subseteq \{0, 1\}$. This implies that \mathcal{S} is not sf-discrete.
- The family \mathcal{S} has a unique limit point f in the Cantor space, and this f is primitive recursive. By Proposition 15, f cannot be quickly spoiling for any punctual numbering v of \mathcal{S} .

4. Warming Up: Absence of Minimal Elements

The section contains an introduction to punctual priority constructions: we give a detailed proof of the result below, which serves as a good starting point.

Theorem 16. *Let \mathcal{S} be an infinite punctual family. Then the semilattice $\mathcal{R}_{pr}(\mathcal{S})$ does not contain minimal elements. Consequently, the structure $\mathcal{R}_{pr}(\mathcal{S})$ is infinite.*

Proof. Let α be a punctual numbering of \mathcal{S} . By Corollary 7, there is a spd numbering $v \in Com_{pr}(\mathcal{S})$ such that $v \leq_{pr} \alpha$. In order to prove the theorem, we build a numbering $\mu \in Com_{pr}(\mathcal{S})$ such that $\mu \leq_{pr} v$ and $v \not\leq_{pr} \mu$.

Our construction will satisfy the following series of requirements:

P_e : The function p_e does not punctually reduce v to μ .

The key difference between our construction and a typical priority argument (of recursion theory) is the following: our requirements *do not injure* each other, and we will satisfy only one requirement P_e at a time.

Strategy for P_e . Suppose that the P_e -strategy starts working at a stage s^e of the construction, and N^e is the least index such that the object $\mu(N^e)$ is still undefined at the beginning of the stage s^e .

We wait until the first stage $t > s^e$ with the following properties:

- (a) There is a (least) number $w^e \leq t$ such that $w^e \geq e$, $v(w^e) \neq v(0)$, and we have not used the object $v(w^e)$ in our definition of μ before.
- (b) For this particular w^e , the value $p_e[t](w^e)$ is already defined.

Note that checking whether $v(w^e)$ is equal to $v(0)$ is a punctual procedure, since the numbering v is spd.

While waiting for this t to appear, we should not delay the definition of the numbering μ , so, one by one, we put

$$\mu(N^e) := v(0), \mu(N^e + 1) := v(0), \mu(N^e + 2) := v(0), \dots$$

When the desired stage t is achieved, we proceed as follows:

- (1) For each $k \leq p_e(w^e)$, if the object $\mu(k)$ is still undefined, then put $\mu(k) := v(0)$.
- (2) Let m be the least index such that at this moment, $\mu(m)$ is still undefined. Set $\mu(m + \ell) := v(\ell)$, for every $\ell \leq w^e$.

It is clear that the described actions ensure that the requirement P_e is forever satisfied: our choice of the witness w^e guarantees that we have $v(w^e) \neq \mu(p_e(w^e))$.

The construction is arranged as follows: we start the P_0 -strategy and wait until it is satisfied. When P_0 is satisfied, we immediately start the P_1 -strategy. After P_1 is satisfied, we start P_2 , etc.

Verification. Since the family \mathcal{S} is infinite, each strategy P_e will eventually find its witness w^e , and after that, P_e will eventually become satisfied. Therefore, we deduce $v \not\leq_{pr} \mu$.

The constructed numbering μ is punctual: indeed, for an index $k \in \omega$, one can just look at the stage $k + 1$ of the described construction. At this stage $k + 1$, we can promptly find an index $r(k)$ such that $\mu(k)$ is equal to $v(r(k))$. This shows the punctuality of μ , and furthermore, the function r punctually reduces μ to v .

Informally speaking, the punctuality of μ is ensured by elimination of unbounded searches: Surely, the P_e -strategy wants to “catch” a particularly good stage t , but this quest for t does not delay the construction at all – while doing the t -search, our definition of μ just executes a straightforward filler action (copying $v(0)$ for appropriate μ -indices).

Now it is enough to show that the numbering μ has an index for every element of \mathcal{S} . This is ensured by the assignment $\mu(m + \ell) := v(\ell)$ given above – after satisfying P_e , μ copies the long initial segment $v(0), v(1), \dots, v(w^e)$. □

The classical result of Khutoretskii (1971) shows that for any computable family \mathcal{T} , its semilattice $\mathcal{R}_c(\mathcal{T})$ is either one element or infinite. Proposition 3 and Theorem 16 together imply that the punctual setting exhibits a similar behavior:

Corollary 17. *For an arbitrary punctual family \mathcal{S} , its Rogers pr-semilattice is either one element or infinite.*

For a Turing computable family of total functions \mathcal{T} , we consider the semilattice $\mathcal{R}^{(c,pr)}(\mathcal{T})$ defined in Section 2.1. A simple analysis of the proof of Theorem 16 shows the following:

Corollary 18. *If \mathcal{T} is a Turing computable infinite family of total functions, then the semilattice $\mathcal{R}^{(c,pr)}(\mathcal{T})$ does not contain minimal elements.*

Proof Sketch. In the construction of Theorem 16, one builds a new numbering μ only by looking at the v -indices, that is, we always declare that $\mu(k) := v(\ell_k)$ for some $\ell_k \in \omega$, and we never really work with the “internal content” of our functions $v(\ell_k)$. By working with the internal content, here we mean the following: say, one can choose to define

$$\mu(0)(0) = v(\ell_0)(m_0), \mu(0)(1) = v(\ell_1)(m_1), \mu(0)(2) = v(\ell_2)(m_2), \dots,$$

that is, to copy different finite pieces from different functions $v(\ell_t)$, $t \in \omega$.

This observation implies that the constructed new numbering μ is a Turing computable numbering of our family \mathcal{T} such that $\mu \leq_{pr} v$ and $v \not\leq_{pr} \mu$. □

5. Antichains

In this section, we establish that any non-greatest element of a Rogers pr-semilattice is a part of an infinite antichain. First, we prove the following.

Theorem 19. *Let \mathcal{S} be an infinite punctual family. Let α and μ be punctual numberings of \mathcal{S} such that $\mu <_{pr} \alpha$, that is, $\mu \leq_{pr} \alpha$ and $\alpha \not\leq_{pr} \mu$. Then there is a numbering $\nu \in Com_{pr}(\mathcal{S})$ such that $\nu \leq_{pr} \alpha$ and ν is \leq_{pr} -incomparable with μ .*

Proof. We apply Corollary 7 and fix a spd numbering $\tilde{\mu}$ of the family \mathcal{S} such that $\tilde{\mu} \leq_{pr} \mu$. Our numbering ν will copy different pieces of α and $\tilde{\mu}$. Choose a function $f_{\tilde{\mu}}$, which punctually reduces $\tilde{\mu}$ to α .

We satisfy the following series of requirements:

P_e : $p_e : \tilde{\mu} \not\leq_{pr} \nu$, that is, p_e does not punctually reduce $\tilde{\mu}$ to ν .

Q_i : $p_i : \nu \not\leq_{pr} \mu$.

The P_e -requirements ensure that $\tilde{\mu} \not\leq_{pr} \nu$. Since $\tilde{\mu} \leq_{pr} \mu$, this implies $\mu \not\leq_{pr} \nu$.

We also have a global requirement

R : ν is punctually reducible to α .

Informally, this will be satisfied via the following action: at a stage s , we promptly define the value $f_\nu(s)$ such that $\nu(s) = \alpha(f_\nu(s))$.

We fix a (punctual) ordering of requirements: $P_0 < Q_0 < P_1 < Q_1 < \dots$. This means that we will satisfy P_0 , then Q_0 , then P_1 , etc.

The P_e -strategy. By the *background action* of the P_e -strategy, we mean the following: whenever we are waiting for some object to be found, we do not delay our construction, and we just put

$$\nu(N) := \tilde{\mu}(0), \nu(N + 1) := \tilde{\mu}(0), \nu(N + 2) := \tilde{\mu}(0) \dots,$$

starting with an appropriate index N . This N is typically clear from the context (recall the N^e from Theorem 16).

The strategy P_e waits until the first (large enough) stage t with the following properties:

- (a) There are $N + 1$ indices a_0, a_1, \dots, a_N such that $a_i \leq t$ and the objects $\tilde{\mu}(a_i)$ are pairwise different. Recall that the relation $\eta_{\tilde{\mu}}$ is primitive recursive, so this condition can be verified in a prompt way.
- (b) For each $i \leq N$, the value $p_e(a_i)[t]$ is defined.

When this t is found, we compute $M := \max(p_e(a_0), p_e(a_1), \dots, p_e(a_N))$. For every $k \leq M$, if $\nu(k)$ is still undefined, then we put $\nu(k) := \tilde{\mu}(0)$.

Consider the finite set $F := \{j : j \leq M\}$. On one hand, it is clear that the family $\{\nu(j) : j \in F\}$ contains at most N different objects. On the other hand, there are $(N + 1)$ different functions $\tilde{\mu}(a_i), i \leq N$, and $p_e(a_i) \in F$ for all i .

Therefore, we deduce that p_e cannot punctually reduce $\tilde{\mu}$ to our ν , and the P_e -requirement is satisfied.

The Q_i -strategy. Suppose that the strategy starts working at a stage s_0 of the construction. Let M_0 be the least index such that $\nu(M_0)$ is still undefined. We put

$$\nu(M_0 + k) := \alpha(k) \text{ for every } k \leq s_0.$$

This action ensures that every object from the family \mathcal{S} eventually obtains a ν -index.

Put $N := M_0 + s_0 + 1$. The *background action* of the Q_i -strategy is as follows – it defines

$$v(N) := \alpha(0), v(N + 1) := \alpha(1), v(N + 2) := \alpha(2), \dots$$

We wait until the first stage t such that there are an index $w_0 \leq t$ and a number $L \leq t$ with the following properties:

- The value $p_i(w_0)[t]$ is defined.
- The object $v(w_0)$ is defined.
- We have $v(w_0)|L \neq \mu(p_i(w_0))|L$.

If these w_0 and L are found, then $v(w_0) \neq \mu(p_i(w_0))$. This shows that p_i does not reduce v to μ , and the Q_i -strategy is satisfied.

Now we only need to show that the desired stage t will be eventually reached. Toward a contradiction, assume that there is no such stage t . Then for every w and L , we have $v(w)|L = \mu(p_i(w))|L$. In other words, the function p_i punctually reduces v to μ . Then our background actions imply that

$$\alpha(x) = v(N + x) = \mu(p_i(N + x)) \text{ for all } x \in \omega;$$

hence $\alpha \leq_{pr} \mu$, which contradicts the conditions of the theorem.

The *construction* is arranged similarly to Theorem 16: our requirements are satisfied one by one, according to their priority ordering.

Verification. We have already shown that every requirement will be eventually satisfied; hence, the numberings v and μ are \leq_{pr} -incomparable. Moreover, our description of a Q_i -strategy guarantees that v provides an index for every element from \mathcal{S} .

Note that there is a primitive recursive function $f_v(x)$ such that $v = \alpha \circ f_v$. This is ensured by the background actions: at each stage, we promptly set either $v(x) := \tilde{\mu}(0) = \alpha(f_{\tilde{\mu}}(0))$ or $v(x) := \alpha(\ell_x)$ for appropriately chosen index ℓ_x . Therefore, the numbering v has desired properties. \square

A not too difficult modification of Theorem 19 provides the following result:

Theorem 20. *Let \mathcal{S} be an infinite punctual family. Let α and μ be punctual numberings of \mathcal{S} such that $\mu <_{pr} \alpha$. Then there is a computable list $(v_k)_{k \in \omega}$ of punctual numberings of \mathcal{S} such that:*

- (a) *for every k , $v_k \leq_{pr} \alpha$ and v_k is \leq_{pr} -incomparable with μ ;*
- (b) *the numberings v_k are pairwise \leq_{pr} -incomparable.*

Corollary 21. *If μ is a non-greatest element of a Rogers pr -semilattice \mathcal{R} , then μ is a part of an infinite antichain inside \mathcal{R} .*

Proof of Theorem 20. Here we discuss a construction, which builds only two \leq_{pr} -incomparable punctual numberings v_0 and v_1 . This construction admits a straightforward generalization to the case of countably many incomparable elements. We use the same notation as in Theorem 19.

For $\varepsilon \in \{0, 1\}$, we need to satisfy the following requirements:

R^ε : v_ε is punctually reducible to α .

P^ε_e : $p_e: \tilde{\mu} \not\leq_{pr} v_\varepsilon$.

Q^ε_i : $p_i: v_\varepsilon \not\leq_{pr} \mu$.

N_j^0 : $p_j: v_0 \not\leq_{pr} v_1$.

N_j^1 : $p_j: v_1 \not\leq_{pr} v_0$.

The strategies for the requirements R^ε , P_e^ε , and Q_j^ε are essentially the same as those in the proof of Theorem 19. Since N^1 -requirements are similar to N^0 -requirements, it is sufficient to discuss only a N_j^0 -strategy.

The N_j^0 -strategy. The background action of the strategy is the following: we put

$$\begin{aligned} v_0(N) &:= \tilde{\mu}(0), & v_0(N + 1) &:= \tilde{\mu}(1), & v_0(N + 2) &:= \tilde{\mu}(2), & \dots \\ v_1(N) &:= \tilde{\mu}(0), & v_1(N + 1) &:= \tilde{\mu}(0), & v_1(N + 2) &:= \tilde{\mu}(0), & \dots \end{aligned}$$

Our strategy waits until the first stage t with the following properties:

- (a) There are $N + 1$ indices b_0, b_1, \dots, b_N such that $N \leq b_i \leq t$ and the objects $v_0(b_i)$ are pairwise different. Since the numbering $\tilde{\mu}$ is spd and (in the background) $v_0(N + x) = \tilde{\mu}(x)$, these indices will eventually appear.
- (b) For each $i \leq N$, the value $p_j(b_i)[t]$ is defined.

When the number t is found, we set $M := \max\{p_j(b_i) : i \leq N\}$. For each $k \leq M$, if the value $v_0(k)$ is still undefined, then put $v_0(k) := \tilde{\mu}(0)$. We work with $v_1(k)$, $k \leq M$, in a similar way.

An argument similar to the one given in the description of the P_e -strategy (Theorem 19) shows that p_j does not punctually reduce v_0 to v_1 .

As usual, our construction fixes some punctual ordering of all requirements (except R^0 and R^1) and satisfies these requirements one by one. The requirements R^0 and R^1 are satisfied via “global” actions. The verification proceeds similarly to Theorem 19. This concludes the proof of Theorem 20. □

Similarly to Corollary 18, the proof of Theorem 20 implies the following:

Corollary 22. *Let \mathcal{T} be a Turing computable infinite family of total functions, and let μ be a non-greatest element in the semilattice $\mathcal{R}^{(c,pr)}(\mathcal{T})$. Then μ is a part of an infinite antichain inside $\mathcal{R}^{(c,pr)}(\mathcal{T})$.*

6. Density

In this section, we consider intervals inside a Rogers pr-semilattice $\mathcal{R}_{pr}(\mathcal{S})$. First, we prove that the poset $\mathcal{R}_{pr}(\mathcal{S})$ is dense:

Theorem 23. *Let \mathcal{S} be an infinite punctual family. Suppose that v and μ are punctual numberings of \mathcal{S} such that $v <_{pr} \mu$. Then there is a numbering $\xi \in \text{Com}_{pr}(\mathcal{S})$ such that $v <_{pr} \xi <_{pr} \mu$.*

Proof. Fix a function f , which provides a punctual reduction $v \leq_{pr} \mu$. We build a punctual numbering α of some subfamily $\mathcal{S}_0 \subseteq \mathcal{S}$, while satisfying the following requirements:

- R : α is punctually reducible to μ .
- P_e : $p_e : \alpha \not\leq_{pr} v$.
- Q_i : $p_i : \mu \not\leq_{pr} \alpha \oplus v$.

These requirements are enough for our goals: indeed, assume that all requirements are satisfied and consider the punctual numbering $\xi := \alpha \oplus v$ of the whole family \mathcal{S} . Then clearly, we have $v \leq_{pr} \xi \leq_{pr} \mu$. The P_e -requirements guarantee that $\xi \not\leq_{pr} v$. The Q_i -requirements ensure $\mu \not\leq_{pr} \xi$.

As in Theorem 19, the R -strategy acts globally: at a stage s , we promptly define the value $g_\alpha(s)$ with $\alpha(s) = \mu(g_\alpha(s))$.

The P_e -strategy. Suppose that the strategy starts working at a stage s_0 , and N is the least index with $\alpha(N)$ still being undefined.

The background action of the P_e -strategy is as follows. We propagate

$$\alpha(N) := \mu(0), \alpha(N + 1) := \mu(1), \alpha(N + 2) := \mu(2), \dots$$

We wait for the least stage $s_1 > s_0$ such that there is a number $w \leq s_1$ with the following properties:

- The object $\alpha(w)$ is already defined, and $\alpha(w) = \mu(g_\alpha(w))$. The verification will demonstrate that given w , the desired μ -index $g_\alpha(w)$ can be calculated in a prompt way.
- The value $p_e[s_1](w)$ is defined.
- By the stage s_1 , we have already witnessed that $\mu(g_\alpha(w)) \neq \mu f(p_e(w))$.

The last item requires a little bit of clarification: How does one check such a condition?

Recall that by Observation 4, the numbering μ is negative. In other words, the set $\{(k, \ell) : \mu(k) \neq \mu(\ell)\}$ is c.e., and hence, it can be enumerated by a punctual function. Thus, if our $g_\alpha(w)$ and $f(p_e(w))$ are μ -indices of different objects, then we will eventually witness this fact.

We emphasize the following: If the desired stage s_1 is found, then the requirement P_e will be forever satisfied. Indeed, we know that $\alpha(w) = \mu(g_\alpha(w)) \neq \mu f(p_e(w)) = v(p_e(w))$, and hence, p_e cannot punctually reduce α to v . Thus, after the stage s_1 , one can proceed to working with other requirements.

The Q_i -strategy. Again, we assume that the strategy starts working at a stage s_0 , and N is the least index with $\alpha(N)$ still undefined.

The background action of Q_i is propagating the following:

$$\alpha(N) := v(0) = \mu(f(0)), \alpha(N + 1) := v(1) = \mu(f(1)), \dots$$

Wait for the least stage $s_1 > s_0$ such that there is a number $w \leq s_1$ satisfying:

- The value $p_i[s_1](w)$ is defined.
- If the value $p_i(w)$ is even, then the object $\alpha(p_i(w)/2)$ is already defined. Set

$$\ell(w) := \begin{cases} f(\lfloor p_i(w)/2 \rfloor), & \text{if } p_i(w) \text{ is odd,} \\ g_\alpha(p_i(w)/2), & \text{if } p_i(w) \text{ is even.} \end{cases}$$

Here, again, we assume that $\alpha(k) = \mu(g_\alpha(k))$.

- By the stage s_1 , we have already witnessed that $\mu(w) \neq \mu(\ell(w))$.

If such a stage s_1 is found, then the requirement Q_i will be satisfied. Indeed, we witness that

$$\begin{aligned} \mu(w) \neq \mu(\ell(w)) &= \begin{cases} \mu f(\lfloor p_i(w)/2 \rfloor), & \text{if } p_i(w) \text{ is odd,} \\ \mu g_\alpha(p_i(w)/2), & \text{if } p_i(w) \text{ is even} \end{cases} \\ &= \begin{cases} v(\lfloor p_i(w)/2 \rfloor), & \text{if } p_i(w) \text{ is odd,} \\ \alpha(p_i(w)/2), & \text{if } p_i(w) \text{ is even} \end{cases} = (\alpha \oplus v)(p_i(w)). \end{aligned}$$

The construction is arranged as follows. We fix the ordering of our requirements $P_0 < Q_0 < P_1 < Q_1 < \dots$, and satisfy them one by one.

Verification. It is clear that $\alpha(x) = \mu(g_\alpha(x))$, where g_α is a primitive recursive function. This is ensured by our background actions: we always promptly set either $\alpha(N) := \mu(\ell_N)$ or

$\alpha(N) := v(\ell_N) = \mu(f(\ell_N))$ for appropriately chosen index ℓ_N . Hence, $\alpha \leq_{pr} \mu$, and the strategy descriptions above are well defined.

In order to finish the proof, we need to establish the following fact: *every* requirement is eventually satisfied.

Toward a contradiction, assume that there is the least requirement T , which is never satisfied. There are two possible cases.

Case 1. Assume that $T = P_e$ for some $e \in \omega$. This implies that the desired stage s_1 of the P_e -strategy is never found. Therefore, for *each* index w , we have $\alpha(w) = \mu(g_w(w)) = \mu f(p_e(w)) = v(p_e(w))$.

Since the P_e -strategy works forever, for some fixed index N , we have $\alpha(N + \ell) = \mu(\ell)$, for all $\ell \in \omega$. Therefore,

$$\mu(\ell) = \alpha(N + \ell) = v(p_e(N + \ell)).$$

This means that $\mu \leq_{pr} v$, which gives a contradiction.

Case 2. Suppose that $T = Q_i$. The desired s_1 of the Q_i -strategy cannot be found, and this implies that for every index w , we have $\mu(w) = \mu(\ell(w)) = (\alpha \oplus v)(p_i(w))$.

The background actions of the Q_i -strategy ensure that there is N such that $\alpha(N + m) = v(m)$, for all $m \in \omega$. This shows that $\alpha \leq_{pr} v$, since for each $k < N$, one can nonuniformly find a v -index of the object $\alpha(k)$. Hence, we have

$$\mu \leq_{pr} \alpha \oplus v \leq_{pr} v,$$

which contradicts the conditions of the theorem.

Therefore, we deduce that every requirement is satisfied. This concludes the proof of Theorem 23. □

With a little bit more work, we obtain the following result.

Theorem 24. *Let \mathcal{S} be an infinite punctual family. Suppose that v and μ are punctual numberings of \mathcal{S} such that $v <_{pr} \mu$. Then there is a computable list $(\xi_i)_{i \in \omega}$ of numberings such that:*

- (a) $\xi_i \in \text{Com}_{pr}(\mathcal{S})$ and $v <_{pr} \xi_i <_{pr} \mu$.
- (b) For any non-empty finite sets F and G , we have

$$\bigoplus_{i \in F} \xi_i \leq_{pr} \bigoplus_{j \in G} \xi_j \Leftrightarrow F \subseteq G.$$

Proof. First, we give a detailed construction for only two incomparable numberings ξ_0 and ξ_1 . After that, we discuss the modifications that are needed for the general case.

We use the same notation as in Theorem 23. We build punctual numberings $\alpha_0 \in \text{Com}_{pr}(\mathcal{S}_0)$ and $\alpha_1 \in \text{Com}_{pr}(\mathcal{S}_1)$ for some subfamilies \mathcal{S}_0 and \mathcal{S}_1 of the family \mathcal{S} . We satisfy the following requirements:

- R^0 : $\alpha_0 \leq_{pr} \mu$ via a function g_{α_0} .
- R^1 : $\alpha_1 \leq_{pr} \mu$ via a function g_{α_1} .
- N_e^0 : $p_e : \alpha_0 \oplus v \not\leq_{pr} \alpha_1 \oplus v$.
- N_i^1 : $p_i : \alpha_1 \oplus v \not\leq_{pr} \alpha_0 \oplus v$.

Then the numberings $\xi_0 := \alpha_0 \oplus \nu$ and $\xi_1 := \alpha_1 \oplus \nu$ have all the desired properties. Indeed, R^0 ensures that $\xi_0 \leq_{pr} \mu$. It is obvious that $\nu \leq_{pr} \xi_0$. The N_e^0 - and N_i^1 -requirements guarantee that ξ_0 and ξ_1 are \leq_{pr} -incomparable. Hence, it is clear that $\nu <_{pr} \xi_j <_{pr} \mu$ for $j \in \{0, 1\}$.

It is sufficient to discuss only an N_e^0 -strategy: The R^j -satisfaction is similar to Theorem 23, and an N_i^1 -strategy is essentially the same as that for N_e^0 .

The N_e^0 -strategy. Suppose that the strategy starts working at a stage s_0 , and M is the least index with both $\alpha_0(M)$ and $\alpha_1(M)$ undefined.

Our background action defines

$$\begin{aligned} \alpha_0(M) &:= \mu(0), \alpha_0(M + 1) := \mu(1), \alpha_0(M + 2) := \mu(2), \dots; \\ \alpha_1(M) &:= \nu(0) = \mu f(0), \alpha_1(M + 1) := \nu(1) = \mu f(1), \alpha_1(M + 2) := \nu(2) = \mu f(2), \dots \end{aligned}$$

We wait for a stage $s_1 > s_0$ such that there is $w \leq s_1$ with the following properties:

- The object $\alpha_0(w) = \mu(g_{\alpha_0}(w))$ is already defined.
- The value $p_e[s_1](2w)$ is defined.
- By the stage s_1 , we have already witnessed that

$$\mu g_{\alpha_0}(w) \neq \begin{cases} \mu g_{\alpha_1}(p_e(2w)/2), & \text{if } p_e(2w) \text{ is even,} \\ \mu f(\lfloor p_e(2w)/2 \rfloor), & \text{if } p_e(2w) \text{ is odd} \end{cases} = (\alpha_1 \oplus \nu)(p_e(2w)).$$

If this stage s_1 is found, then the N_e^0 -requirement is satisfied: indeed, we have

$$(\alpha_0 \oplus \nu)(2w) = \alpha_0(w) = \mu g_{\alpha_0}(w) \neq (\alpha_1 \oplus \nu)(p_e(2w)).$$

We show that s_1 will be eventually reached. Toward a contradiction, assume that there is no such s_1 . Then we deduce the following:

- (1) $\mu(x) = \alpha_0(M + x)$ for all $x \in \omega$.
- (2) The function $q(k) := p_e(2k)$ provides a punctual reduction from α_0 to $(\alpha_1 \oplus \nu)$.
- (3) $\nu(x) = \alpha_1(M + x)$ for all x . By nonuniformly choosing a finite set of ν -indices, one can construct a punctual reduction from α_1 to ν .

These facts together imply that $\mu \leq_{pr} \alpha_0 \leq_{pr} \alpha_1 \oplus \nu \leq_{pr} \nu$. Thus, μ is punctually reducible to ν , which gives a contradiction.

The construction is arranged in a straightforward way. It is not hard to prove that the requirements R^0 and R^1 are satisfied. We showed above that every N_e^ε , where $\varepsilon \in \{0, 1\}$, will be eventually satisfied. This concludes the discussion for the case of two incomparable numberings ξ_0 and ξ_1 .

The general case is obtained as follows. Here, for every $k \in \omega$ and every non-empty finite set $F \subset \omega$, we need to satisfy the requirements:

$$R^k: \alpha_k \leq_{pr} \mu \text{ via a function } g_{\alpha_k}.$$

$$N_e^{k,F}: \text{If } k \notin F, \text{ then } p_e: \alpha_k \oplus \nu \not\leq_{pr} (\bigoplus_{j \in F} \alpha_j) \oplus \nu$$

Then one can show that the numberings $\xi_k := \alpha_k \oplus \nu$ satisfy the theorem.

The $N_e^{k,F}$ -strategy is similar to the N_e^0 -strategy described above, modulo its background actions. For all $j \in F$ and $\ell \notin F \cup \{k\}$, here we put

$$\begin{aligned} \alpha_k(M) &:= \mu(0), \alpha_k(M + 1) := \mu(1), \alpha_k(M + 2) := \mu(2), \dots; \\ \alpha_j(M) &:= \nu(0), \alpha_j(M + 1) := \nu(1), \alpha_j(M + 2) := \nu(2), \dots; \\ \alpha_\ell(M) &:= \mu(0), \alpha_\ell(M + 1) := \mu(0), \alpha_\ell(M + 2) := \mu(0), \dots \end{aligned}$$

All other modifications are pretty straightforward. This concludes the proof of Theorem 24. \square

As a consequence, we obtain the following:

Corollary 25. *Let \mathcal{S} be an infinite punctual family. Suppose that ν and μ are punctual numberings of \mathcal{S} such that $\nu <_{pr} \mu$. Then every finite upper semilattice is isomorphically embeddable into the interval $[\text{deg}_{pr}(\nu); \text{deg}_{pr}(\mu)]$ inside $\mathcal{R}_{pr}(\mathcal{S})$ (this embedding preserves suprema). In particular, this implies that the Σ_1 -fragment of the theory $\text{Th}(\mathcal{R}_{pr}(\mathcal{S}))$ (in the signature of upper semilattices) is decidable.*

Proof Sketch. It is sufficient to show that every finite Boolean algebra (treated as an upper semilattice) can be isomorphically embedded into the interval $[\text{deg}_{pr}(\nu); \text{deg}_{pr}(\mu)]$.

Let B be a finite Boolean algebra with precisely $n + 1$ atoms. Consider the sequence $(\xi_i)_{i \in \omega}$ constructed in Theorem 24. Then it is not hard to prove that the subsemilattice of $\mathcal{R}_{pr}(\mathcal{S})$ with domain

$$\left\{ \text{deg}_{pr}(\nu) \right\} \cup \left\{ \text{deg}_{pr} \left(\bigoplus_{j \in F} \xi_j \right) : F \neq \emptyset, F \subseteq \{0, 1, \dots, n\} \right\}$$

is isomorphic to B . □

Corollary 26. *Let \mathcal{T} be a Turing computable infinite family of total functions, and let $\nu <_{pr} \mu$ be Turing computable numberings of \mathcal{T} . Then the interval $[\text{deg}_{pr}(\nu); \text{deg}_{pr}(\mu)]$ inside $\mathcal{R}^{(c,pr)}(\mathcal{T})$ contains an infinite antichain.*

Proof Sketch. We follow the proof of Theorem 24. The only nontrivial detail is the following: recall that in Theorem 23, we use the fact that the numbering μ is negative. In this corollary, our μ is a Turing computable numbering, and it is also negative. □

7. Lattices

Note that Corollary 25 implies that for arbitrary infinite punctual families \mathcal{S}_0 and \mathcal{S}_1 , the Σ_1 -fragments of the theories $\text{Th}(\mathcal{R}_{pr}(\mathcal{S}_0))$ and $\text{Th}(\mathcal{R}_{pr}(\mathcal{S}_1))$ are the same. In this section, we witness that this is not the case already for the Σ_2 -fragments. On the one hand, we show that some Rogers pr-semilattices are lattices (Theorem 27). On the other hand, some semilattices contain minimal pairs (Proposition 29).

Selivanov (1976) obtained the following classical result: for any computable family \mathcal{T} , if the semilattice $\mathcal{R}_c(\mathcal{T})$ is infinite, then it *cannot* be a lattice. This section illustrates that the result of Selivanov *cannot be transferred* to the punctual setting.

Our first theorem shows that persistently decidable families \mathcal{S} (see Problem 8) exhibit a pretty tame behavior.¹

Theorem 27. *Let \mathcal{S} be an infinite persistently decidable family. Then the structure $\mathcal{R}_{pr}(\mathcal{S})$ is an infinite lattice.*

Proof. Given two punctual numberings ν and μ of \mathcal{S} , we build their infimum α .

First, notice the following: since the family \mathcal{S} is persistently decidable, the equivalence relation $\eta_{\nu \oplus \mu}$ is primitive recursive. Therefore, given $k, \ell \in \omega$, one can promptly compute whether the equality $\nu(k) = \mu(\ell)$ is true: indeed, $\nu(k) = \mu(\ell)$ if and only if $(2k, 2\ell + 1) \in \eta_{\nu \oplus \mu}$.

For each $s \in \omega$, at the stage s , we will promptly define an index w_s such that $\alpha(s) = \nu(w_s)$.

At a stage s of the construction, we find a finite subset of ω :

$$I[s] := \left\{ k \leq s : (\exists \ell \leq s)[\nu(k) = \mu(\ell)] \ \& \ (\forall k' < k)[\nu(k') \neq \nu(k)] \right\}.$$

It is not hard to show that the procedure of finding $I[s]$ is punctual.

We (punctually) search for the least number $z \in I[s]$ such that $z \notin \{w_t : t < s\}$. If such a number z exists, then put $w_s := z$ and $\alpha(s) := v(z)$. Otherwise, set $w_s := 0$ and $\alpha(s) := v(0)$.

The construction of α is described. It is not difficult to prove that the numbering α is punctual. Moreover, since we have

$$\bigcup_s I[s] = \{k : (\forall k' < k)[v(k') \neq v(k)]\},$$

α indexes the whole family \mathcal{S} . In addition, the function $s \mapsto w_s$ provides a punctual reduction from α to v .

We prove that $\alpha \leq_{pr} \mu$. Fix a number a_0 with $\mu(a_0) = v(0)$. A punctual reduction g from α to μ is defined as follows:

- If $w_x = 0$, then put $g(x) := a_0$.
- If $w_x \neq 0$, then notice that $w_x \in I[x]$, and hence, one can promptly find an index $\ell \leq x$ with $\mu(\ell) = v(w_x)$. We set $h(x) := \ell$.

In order to finish the proof, we need to establish the following fact: suppose that $\beta \leq_{pr} v$ via a function g_0 , and $\beta \leq_{pr} \mu$ via a function g_1 . Then β is punctually reducible to α .

Since the numbering $\beta \oplus v$ is punctually decidable, we deduce that for $k, \ell \in \omega$, one can quickly compute whether $\beta(k) = v(\ell)$ holds.

Let $x \in \omega$. If $\beta(x) = v(0)$, then one can nonuniformly choose an α -index for $v(0)$, and this is enough for us. Assume that $\beta(x) \neq v(0)$. It is clear that there exists k such that $k \in \widehat{I}_x := I[\max(g_0(x), g_1(x))]$ and $\beta(x) = v(k)$. Let M be the greatest element of \widehat{I}_x . One can show that there is (the least) index ℓ such that $\alpha(\ell) = v(k)$, and

$$\ell \leq \max(g_0(x), g_1(x)) + M + 1.$$

Thus, this ℓ can be found in a prompt way, and one can construct a punctual reduction from β to α . □

Remark 28. It is unclear whether one can obtain an analog of Theorem 27 for some semilattice $\mathcal{R}^{(c,pr)}(\mathcal{T})$, where \mathcal{T} is an infinite Turing computable family of total functions. For such a family \mathcal{T} , one can easily construct a Turing computable numbering v such that the equivalence η_v is not primitive recursive.

Indeed, let μ be some fixed computable numbering of the family \mathcal{T} such that $\mu(0) \neq \mu(1)$. Then for an arbitrary computable, not primitive recursive set $X \subseteq \omega$, we define

$$v_X(2k + 1) = \mu(k), \quad v_X(2k) = \begin{cases} \mu(0), & \text{if } k \notin X, \\ \mu(1), & \text{if } k \in X. \end{cases}$$

The numbering v_X is computable, and η_{v_X} is not primitive recursive.

Our second result gives an example of a Rogers pr-semilattice, which is not a lattice:

Proposition 29. *There exists an infinite punctual family \mathcal{S} such that the semilattice $\mathcal{R}_{pr}(\mathcal{S})$ contains a minimal pair. Consequently, $\mathcal{R}_{pr}(\mathcal{S})$ is not a lattice.*

Proof. Our family \mathcal{S} is defined via its punctual Friedberg numbering: for $k, \ell \in \omega$, we set $v\langle k, 0 \rangle := \chi_{\{2k\}}$ and $v\langle k, \ell + 1 \rangle := \chi_{\{2k, 2\ell + 1\}}$.

The degree of the numbering v will be one of the components of the minimal pair. Now we build the other component: it will be induced by a Friedberg numbering μ .

Fix a non-computable c.e. set W , and a primitive recursive function q with $\text{range}(q) = W$.

For each number $k \in \omega$, the construction of the “ k -th layer” $\mu\langle k, \cdot \rangle$ proceeds as follows. Beforehand, set the values:

$$\mu\langle k, 0 \rangle(2k) := 1 \text{ and } \mu\langle k, 0 \rangle(2m) := 0$$

for all $m \neq k$.

Consider a stage s of the construction. If $k \notin \{q(0), q(1), \dots, q(s)\}$, then put

$$\mu\langle k, 0 \rangle(2s + 1) := 0 \text{ and } \mu\langle k, s + 1 \rangle := \chi_{\{2k, 2s+1\}}.$$

Suppose that s_0 is the least stage with $k \in \{q(0), q(1), \dots, q(s_0)\}$. Then we set

$$\mu\langle k, 0 \rangle := \chi_{\{2k, 2s_0+1\}}, \mu\langle k, s_0 + 1 \rangle := \chi_{\{2k\}}, \mu\langle k, t + 2 \rangle := \chi_{\{2k, 2t+3\}},$$

for all $t \geq s_0$. After the stage s_0 , we stop the construction of this particular k -th layer.

It is not hard to show that the constructed μ belongs to $Com_{pr}(\mathcal{S})$, and μ is Friedberg. We need to prove that the degrees of ν and μ form a minimal pair inside $\mathcal{R}_{pr}(\mathcal{S})$.

Toward a contradiction, assume that there is a punctual numbering α of \mathcal{S} with punctual reductions $f: \alpha \leq_{pr} \nu$ and $h: \alpha \leq_{pr} \mu$. Consider a c.e. set

$$V := \{k \in \omega : \exists x(f(x) = h(x) = \langle k, 0 \rangle)\}.$$

Clearly, for a number $k \in V$ and the corresponding x , we have $\nu(f(x)) = \mu(h(x)) = \chi_{\{2k\}}$. Since μ is a Friedberg numbering, the function $\chi_{\{2k\}}$ has a unique μ -index e_k .

Note that the construction ensures the following:

- If $k \in V$, then $e_k = \langle k, 0 \rangle$ and $k \notin \text{range}(q) = W$.
- If $k \notin V$, then for every $x \in \omega$, the condition $\alpha(x) = \chi_{\{2k\}}$ implies $f(x) = \langle k, 0 \rangle$ and $h(x) \neq \langle k, 0 \rangle$. Hence, $e_k = \langle k, s_0 + 1 \rangle$ for some $s_0 \in \omega$, and $k \in W$.

Thus, we deduce that $V = \overline{W}$, which contradicts the choice of W . □

Remark 30. Note that the family \mathcal{S} and its numbering ν (from the proposition above) have countably many quickly spoiling limit points (in the sense of Definition 12).

- (a) The function $\chi_{\emptyset} \notin \mathcal{S}$ is quickly spoiling – this is witnessed by the function $sp(n, t)$ with $sp(n, 0) = \langle n + 1, 0 \rangle$ and $sp(n, 1) = \langle n + 2, 0 \rangle$.
- (b) For $k \in \omega$, the function $\chi_{\{2k\}}$ is quickly spoiling. The witness can be defined as follows: $sp(n, 0) = \langle k, n + 2 \rangle$ and $sp(n, 1) = \langle k, n + 3 \rangle$.

Therefore, the family \mathcal{S} is not persistently decidable.

8. Existence of the Greatest Element

In this section, we provide examples of Rogers pr-semilattices with greatest element and without greatest element. This gives another difference in the Σ_2 -fragments of the first-order theories.

Theorem 31. *Let \mathcal{S} be an infinite punctual family, and let ν be its punctual numbering. Suppose that there exists a quickly spoiling limit point f for ν such that $f \notin \mathcal{S}$. Then the semilattice $\mathcal{R}_{pr}(\mathcal{S})$ does not have a greatest element.*

Proof. Fix a primitive recursive function $sp(n, t)$ witnessing the spoiledness of the limit point f .

Let μ be an arbitrary punctual numbering of the family \mathcal{S} . In order to prove the theorem, we build a punctual numbering α of some subfamily $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $\alpha \not\leq_{pr} \mu$. Indeed, the existence of such α is sufficient for us, since this implies that $\mu \leq_{pr} \mu \oplus \alpha$ and $\mu \oplus \alpha \not\leq_{pr} \mu$.

We satisfy the series of requirements:

$$P_e: p_e: \alpha \not\leq_{pr} \mu.$$

The P_e -strategy. Choose the least w such that we have not talked about the object $\alpha(w)$ before. Wait for the least stage s such that:

- the value $p_e[s](w)$ is defined, and
- our computations have already provided a number $N \leq s$ with $\mu(p_e(w))|N \neq f|N$.

Since f does not belong to the family \mathcal{S} , such a stage s will be eventually reached.

While waiting for this s , just propagate

$$\alpha(w)(0) := f(0), \alpha(w)(1) := f(1), \alpha(w)(2) := f(2), \dots$$

When the stage s and the corresponding N are found, define M as the least number such that $M \geq N$ and the value $\alpha(w)(M)$ is still undefined. Using the spoiledness of f , we promptly set $\alpha(w) := v(sp(M, 0))$.

This procedure is well defined, and it satisfies the P_e -requirement: indeed, we have

$$\mu(p_e(w))|M \neq f|M = v(sp(M, 0))|M = \alpha(w)|M.$$

The construction is arranged in a straightforward manner. We note that in order to ensure the punctuality of α , the P_e -strategy also has to implement some simple background actions – for example, one by one, we set $\alpha(w + 1) := v(0)$, $\alpha(w + 2) := v(0)$.

Clearly, α indexes a subfamily of \mathcal{S} . Since every P_e is eventually satisfied, we have $\alpha \not\leq_{pr} \mu$. \square

Remark 32. Note that the family from Example 14 and the family from Proposition 29 both satisfy the conditions of Theorem 31.

We give an example of an infinite punctual family such that its Rogers pr-semilattice has the greatest element.

Lemma 33. Consider the family \mathcal{S} from Example 11 – it can be defined via its punctual Friedberg numbering: for $i \in \omega$,

$$v(i)(x) = i \text{ for all } x \in \omega.$$

Any punctual numbering $\mu \in \text{Com}_{pr}(\mathcal{S})$ is punctually reducible to v .

Proof. Let μ be a punctual numbering of \mathcal{S} . Then a punctual reduction from μ to v is provided by the function $h(k) := \mu(k)(0)$. \square

We leave the following question open:

Problem 34. Is it true that for any sf-discrete family \mathcal{S} , the semilattice $\mathcal{R}_{pr}(\mathcal{S})$ contains the greatest element?

9. Persistently Decidable Families Revisited

Here we prove the result, which was announced at the end of Subsection 3.1:

Theorem 35. There exists an infinite, persistently decidable family \mathcal{S} such that for every function $f \in \mathcal{S}$, there is a number $k \in \omega$ with $f = \chi_{\{k\}}$. Consequently, the family \mathcal{S} has a unique limit point $g = \chi_{\emptyset}$ in the Cantor space.

Proof. Recall that $(\rho_e)_{e \in \omega}$ is the computable list of all punctual numberings (see Subsection 2.2). Without loss of generality, we assume that for every e and x , $\text{range}(\rho_e(x)) \subseteq \{0, 1\}$.

The desired family \mathcal{S} is constructed via its punctual numbering ν . Beforehand, we put $\nu(0) := \chi_{\{0\}}$.

We satisfy the following series of requirements:

R_e : If ρ_e is a numbering of \mathcal{S} , then η_{ρ_e} is primitive recursive.

We fix an ordering of the requirements: $R_0 < R_1 < R_2 < \dots$.

At a stage s , the R_e -readiness level is the maximal number $t \leq s$ such that for every $x \leq t$ and every $z \leq t + 2$, the value $\rho_e(x)(z)[s]$ is defined. Note that the R_e -readiness level tends to infinity.

At a stage s , we define the *construction readiness level* $\text{cr}[s] \in \omega$. We put $\text{cr}[0] := 1$. The intuition behind this parameter is as follows. We want to add a fresh element g to the family \mathcal{S} . This g will be always chosen as either $\chi_{\{\text{cr}[s]\}}$ or $\chi_{\{\text{cr}[s]+1\}}$. The choice of the g is motivated by the following: we want to diagonalize against a requirement R_e with the highest priority – this is done via ensuring that ρ_e does not index the family \mathcal{S} (to be elaborated below).

The construction. Consider stage $s + 1$. We work with the requirements $R_0, R_1, \dots, R_{\text{cr}[s]}$. For $e \leq \text{cr}[s]$, let $\text{rl}(e)$ be the R_e -readiness level at the stage $s + 1$.

If there is $x \leq \text{rl}(e)$ such that the string $\rho_e(x)|_{\text{rl}(e)}$ contains at least two ones, then the R_e -requirement is declared *inactive*. If R_e is inactive, then it is forever satisfied: indeed, we have already witnessed that the function $\rho_e(x)$ cannot be the characteristic function of a one-element set.

We compute the parameter:

$$r := \min\{\text{rl}(e) : e \leq \text{cr}[s], \text{ and } R_e \text{ is still active}\}.$$

If $r < \text{cr}[s]$, then put $\nu(s + 1) := \nu(0)$ and proceed to the next stage.

Suppose that $r \geq \text{cr}[s]$. Then, we define $\text{cr}[s + 1] := \text{cr}[s] + 2$. We search for the least $e \leq \text{cr}[s]$ such that there is $x \leq \text{cr}[s]$ with

$$\rho_e(x)|_{(\text{cr}[s] + 2)} = \chi_{\{\text{cr}[s]\}}|_{(\text{cr}[s] + 2)} \text{ or } \rho_e(x)|_{(\text{cr}[s] + 2)} = \chi_{\{\text{cr}[s]+1\}}|_{(\text{cr}[s] + 2)}.$$

If these e and x exist, we put

$$\nu(s + 1) := \begin{cases} \chi_{\{\text{cr}[s]+1\}}, & \text{if } \rho_e(x)|_{(\text{cr}[s] + 2)} = \chi_{\{\text{cr}[s]\}}|_{(\text{cr}[s] + 2)}, \\ \chi_{\{\text{cr}[s]\}}, & \text{otherwise.} \end{cases}$$

In this case, we say that R_e is *successfully defeated*. The requirement R_e is declared forever *inactive*, and R_e is satisfied – our construction ensures that $\rho_e(x)$ does not belong to \mathcal{S} .

If there are no such e and x , then $\nu(s + 1) := \chi_{\{\text{cr}[s]\}}$. This concludes the description of the construction.

Verification. It is not hard to show that ν is a punctual numbering. The construction ensures that for any $x \in \omega$, there is a number k such that $\nu(x) = \chi_{\{k\}}$.

Since for each e , the R_e -readiness level tends to infinity, we have $\text{cr}[s] \xrightarrow{s \rightarrow \infty} \infty$. This implies that the numbering ν indexes an infinite family \mathcal{S} . Now it is sufficient to establish the following fact.

Claim 1. *If ρ_e is a numbering of the family \mathcal{S} , then the equivalence relation η_{ρ_e} is primitive recursive.*

Proof. Let $\rho_e \in \text{Com}_{pr}(\mathcal{S})$. It is clear that the requirement R_e is never declared inactive. Let s_0 be the first stage with the following properties:

- (a) $\text{cr}[s_0] \geq e$.
- (b) If a requirement R_i , where $i < e$, is declared inactive at some stage s^* , then this s^* is strictly less than s_0 .

Given numbers $x < y$, we describe a prompt procedure, which checks whether $\rho_e(x) = \rho_e(y)$. If $y \leq s_0$, then this checking is realized nonuniformly.

Suppose that $y > s_0$. Compute the strings $\sigma := \rho_e(x)|y$ and $\tau := \rho_e(y)|y$. There are two cases:

Case 1. If one of the strings σ or τ contains 1, then it is clear that

$$\rho_e(x) = \rho_e(y) \Leftrightarrow \sigma = \tau.$$

Case 2. Otherwise, we have $\sigma = \tau = \chi_\emptyset|y$. We argue that in this case, our construction ensures that $\rho_e(x) = \rho_e(y)$.

Toward a contradiction, assume that $\rho_e(x) \neq \rho_e(y)$. Since ρ_e indexes the family \mathcal{S} , there are numbers $k, \ell \geq y$ such that $k \neq \ell$, $\rho_e(x) = \chi_{\{k\}}$, and $\rho_e(y) = \chi_{\{\ell\}}$. Without loss of generality, one may assume that $k < \ell$.

It is not hard to show that there is a unique ν -index $s^* > 0$ such that $\nu(s^*) = \chi_{\{\ell\}}$. Using the description of the construction, we deduce the following properties (at the stage s^*):

- $\ell \in \{\text{cr}[s^* - 1], \text{cr}[s^* - 1] + 1\}$.
- $s^* \geq \text{rl}(e) \geq r \geq \text{cr}[s^* - 1] \in \{\ell, \ell - 1\}$; hence $s^* \geq \ell - 1 \geq k \geq y > s_0$.
- Since $s^* - 1 \geq s_0$, we have $\text{cr}[s^* - 1] \geq \text{cr}[s_0] \geq e$.
- $\text{cr}[s^* - 1] \geq y$.

Summarizing, at the stage $s^* > s_0$, we have $e \leq \text{cr}[s^* - 1]$ and $y \leq \text{cr}[s^* - 1]$ such that

$$\rho_e(y) = \chi_{\{\text{cr}[s^*-1]\}} \text{ or } \rho_e(y) = \chi_{\{\text{cr}[s^*-1]+1\}}.$$

Recall that after stage s_0 , no additional requirement R_i with $i < e$ can become inactive. Therefore, at the stage s^* , we successfully defeat R_e , and ρ_e does not index \mathcal{S} . This contradicts the choice of ρ_e . Therefore, we deduce that the equality $\sigma = \tau = \chi_\emptyset|y$ implies $\rho_e(x) = \rho_e(y)$.

So, we established that the two discussed cases provide a prompt procedure for checking if $(x, y) \in \eta_{\rho_e}$. Therefore, the numbering ρ_e is punctually decidable. □

Claim 1 shows that our family \mathcal{S} is persistently decidable. This concludes the proof of Theorem 35. □

After the first version of the paper was submitted, Sergey Goncharov introduced the following simple example of a persistently decidable family which is not sf-discrete.

Example 36. Goncharov. The family \mathcal{S} contains the following functions: for each $k \in \omega$,

$$\begin{aligned} g_k^0(0) &= k + 1, & g_k^0(x + 1) &= 0 \text{ for all } x; \\ g_k^1(0) &= k + 1, & g_k^1(k + 1) &= 1, & g_k^1(x) &= 0 \text{ for } x \notin \{0, k + 1\}. \end{aligned}$$

Since $g_k^0(k + 1) \neq g_k^1(k + 1)$ for all k , it is clear that the family \mathcal{S} is not sf-discrete.

On the other hand, let f and h be arbitrary functions from the family \mathcal{S} . Then $f = h$ if and only if $f(0) = h(0)$ and $f(f(0)) = h(h(0))$. This implies that every punctual numbering of \mathcal{S} is punctually decidable; hence, the family \mathcal{S} is persistently decidable.

10. Friedberg Numberings

Recall that a numbering ν is *Friedberg* if η_ν equals to the identity relation. Friedberg numberings constitute an important classical object of study. One of the reasons behind this is the following fact: one can show that for any computable family \mathcal{S} , an arbitrary computable Friedberg numbering of \mathcal{S} induces a minimal element inside the semilattice $\mathcal{R}_c(\mathcal{S})$.

We have already seen that this fact *fails* in the punctual setting: recall that an infinite $\mathcal{R}_{pr}(\mathcal{S})$ is downward dense (Theorem 16). Essentially, the reason behind this is the following fact: the inverse of a primitive recursive function is not necessarily primitive recursive. Furthermore, one can obtain the following:

Proposition 37. *Let \mathcal{S} be a punctual family, and let ν be a punctual Friedberg numbering of \mathcal{S} . Then below the degree of ν (inside $\mathcal{R}_{pr}(\mathcal{S})$), one can build*

- (a) *An infinite descending chain of degrees, such that each of them is induced by a Friedberg numbering.*
- (b) *An infinite antichain of degrees, such that each of them is induced by a Friedberg numbering.*

Proof Sketch. First, we observe the following simple fact: If ν and μ are Friedberg numberings of \mathcal{S} , and a function f provides a punctual reduction from ν to μ , then f is a primitive recursive permutation of ω .

Let α be a punctual Friedberg numbering of \mathcal{S} .

(a) It is known that there is a primitive recursive permutation f of ω such that its inverse f^{-1} is not primitive recursive (Kuznetsov 1950, a proof sketch in English can be found, for example, in Paolini et al. 2016). This implies that the sequence:

$$\mu_0 := \alpha \circ f, \mu_1 := \mu_0 \circ f, \dots, \mu_{k+1} := \mu_k \circ f, \dots$$

induces a strictly \leq_{pr} -descending chain below α .

(b) Informally speaking, here one needs to modify the proof of Theorem 20. In the notations of the theorem, we need to build two Friedberg numberings ν_0 and ν_1 , while satisfying the requirements R^ε , N_j^0 , and N_j^1 .

The numberings ν_i , which are being constructed, copy different pieces of the given Friedberg numbering α . We modify the N_j^0 -strategy appropriately.

The new N_j^0 -strategy. Let M be the least index such that the object $\alpha(M)$ has not been used in the construction yet. The background action is the following: we put

$$\begin{aligned} \nu_0(N) &:= \alpha(M), \nu_0(N + 1) := \alpha(M + 1), \nu_0(N + 2) := \alpha(M + 2), \dots \\ \nu_1(N) &:= \alpha(M + 1), \nu_1(N + 1) := \alpha(M + 2), \nu_1(N + 2) := \alpha(M + 3), \dots \end{aligned}$$

Our strategy waits until the first stage t with the following properties: The value $p_j(N)[t]$ is defined, and the object $\nu_1(p_j(N))$ is also defined.

When the number t is found, it is clear that the N_j^0 -requirement is satisfied: we have

$$\nu_0(N) = \alpha(M) \neq \alpha(\ell) = \nu_1(p_j(N)),$$

where $\ell \in \omega \setminus \{M\}$.

Note that we have a little technical problem here: since we require $\nu_1 \in \text{Com}_{pr}(\mathcal{S})$, we need to provide a unique ν_1 -index k such that $\nu_1(k) = \alpha(M)$. Surely, this should be done after the N_j^0 -requirement is satisfied. It is not difficult to resolve this problem via a careful arrangement of the construction. Proposition 37 is proved. □

Recall that below any (Turing) computable negative numbering of an infinite family, one can always find a computable Friedberg numbering (see Section 3). This fact fails in the punctual setting:

Proposition 38. *There is an infinite punctual family \mathcal{S} , which does not have punctual Friedberg numberings.*

Proof. Choose an infinite c.e. set W , which cannot be the range of a primitive recursive injective function. For the sake of completeness, we sketch the construction of such a set W .

The desired set W is built in stages. Note that these stages have to be arranged in a *non-punctual* way. At a stage e , we define finite sets $W[e]$ and $V[e]$.

Stage $e + 1$. Find the least index k such that:

- either the string $p_e|k$ witnesses that p_e is not injective,
- or there are two indices ℓ and m with $\ell, m \leq k$ and $\{p_e(\ell) \neq p_e(m)\} \subseteq \omega \setminus (W[e] \cup V[e])$.

If these indices are found, then put $p_e(\ell)$ into W and $p_e(m)$ into V .

It is not hard to show that both W and V are infinite, $W \cap V = \emptyset$, and for any injective p_e , we have $\text{range}(p_e) \neq W$. This concludes the construction of W .

Choose a primitive recursive (non-injective) function h with $\text{range}(h) = W$. The desired family \mathcal{S} is defined via its punctual numbering: for $i, x \in \omega$, set $(v(i))(x) := h(i)$.

Assume that μ is a punctual Friedberg numbering of \mathcal{S} . Then the primitive recursive injection $\psi(i) := (\mu(i))(0)$ satisfies $\text{range}(\psi) = W$, which contradicts the choice of the set W . \square

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Conflicts of Interest. The authors declare none.

Note

1 In the first version of the paper, Theorem 27 was proved only for sf-discrete families \mathcal{S} (see Definition 9). We are grateful to Sergey Goncharov for pointing out that the theorem can be generalized to the persistently decidable case.

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