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Efficient and Stable Numerical Methods for Multi-Term Time Fractional Sub-Diffusion Equations

Jincheng Ren[1,2] and Zhi-zhong Sun[2,*]

[1] *College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450000, China.*
[2] *Department of Mathematics, Southeast University, Nanjing 210096, China.*

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Abstract. Some efficient numerical schemes are proposed for solving one-dimensional (1D) and two-dimensional (2D) multi-term time fractional sub-diffusion equations, combining the compact difference approach for the spatial discretisation and $L1$ approximation for the multi-term time Caputo fractional derivatives. The stability and convergence of these difference schemes are theoretically established. Several numerical examples are implemented, testifying to their efficiency and confirming their convergence order.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Multi-term time fractional sub-diffusion equations, compact /compact ADI difference scheme, discrete energy method, convergence.

1. Introduction

Fractional differential equations are now frequently invoked in various scientific and engineering applications. Physical phenomena in fields such as viscoelasticity, diffusion processes, relaxation vibrations and electrochemistry are successfully described by differential equations involving derivatives of fractional order [1–8]. Moreover, some underlying processes that cannot be described by single term time fractional partial differential equations can be described by multi-term equations — e.g. the multi-term time fractional diffusion-wave equation modelling various types of viscoelastic damping [9].

In this article, we provide some numerical difference schemes to solve multi-term time fractional sub-diffusion equations of the following form [9–11]:

$$P({}^C D_t)u(x,t) = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\,, \qquad 0 < x < L\,,\ 0 < t \le T\,, \tag{1.1}$$

*Corresponding author. *Email addresses:* `renjincheng2001@126.com` (J. Ren), `zzsun@seu.edu.cn` (Z. Sun)

where κ is a positive diffusion constant and the multi-term fractional operator $P({}^C D_t)$ here is defined by

$$P({}^C D_t)v(x,t) = \left({}_0^C D_t^\alpha + \sum_{i=1}^{s} a_i \, {}_0^C D_t^{\alpha_i} \right) v(x,t)$$

with $0 < \alpha_s < \cdots < \alpha_1 < \alpha < 1$ and $a_i \in R$, $i = 1, 2, \cdots, s$, and ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order α:

$${}_0^C D_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v'(s)}{(t-s)^\alpha} ds \, .$$

We note that Luchko [10] obtained *a priori* solution estimates for the generalised multi-term time fractional diffusion equation using an appropriate maximum principle, and established solution uniqueness by the Fourier method. Daftardar-Gejji & Bhalekar [11] used separation of variables to solve a multi-term fractional diffusion-wave equation with homogeneous/nonhomogeneous boundary conditions. From Luchko's Theorem and the equivalent relationship between the Laplacian operator and the Riesz fractional derivative, Jiang *et al.* [12, 13] derived analytical solutions for both multi-term time-space fractional advection-diffusion and multi-term modified power law wave equations, respectively.

Other authors have discussed the numerical solution of fractional partial differential equations, including the fractional anomalous diffusion equation [14–24]. Chen *et al.* [14] proved the stability and convergence of an implicit difference approximation scheme of the fractional sub-diffusion equation using Fourier analysis. Lynch *et al.* [15] studied the numerical properties of the partial differential equations of fractional order $1 < \alpha < 2$. Yuste [16, 17] presented an explicit scheme and weighted average finite difference methods for the time fractional diffusion equation, and analysed their stability by the von-Neumann method. Zhuang *et al.* [18] obtained an implicit numerical method to solve the sub-diffusion equation by integrating the original equation on both sides, and proved the stability and convergence of their scheme by the energy method. Zhang *et al.* [19] constructed a Crank-Nicolson-type difference scheme and a compact difference scheme, to solve the time fractional sub-diffusion equation with a Riemann-Liouville fractional derivative. They proved that these two difference schemes are unconditionally stable, and the numerical solution converges in the maximum norm. Zhao & Sun [20] provided a box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. Later, Ren *et al.* [21] proposed a compact difference scheme for the time fractional sub-diffusion equation with Neumann boundary conditions, and proved its unconditional stability and global convergence to be $O(\tau^{2-\alpha} + h^4)$ in the discrete L_2 norm.

There has also been some previous work on the numerical solution of problems with multiple fractional derivatives. Ford *et al.* [25] introduced a numerical method for solving the space-time fractional telegraph equation. Based on a quadrature formula approximation of the Caputo fractional derivative in spatial and temporal direction respectively, they proved the scheme was conditionally stable using the Fourier method. Liu *et al.* [26] proposed an implicit difference scheme for modified anomalous sub-diffusion equations with a nonlinear source term, and showed its convergence is $O(\tau + h^2)$ by the energy method.

Recently, Liu *et al.* [28] discussed some computationally effective numerical methods for the multi-term time fractional wave-diffusion equations, and Yang *et al.* [29] applied the variational iteration method to obtain approximate solutions for multi-term fractional differential equations.

Since fractional derivatives are non-local operators, any low-order finite difference or finite element scheme requires a large number of operations and a large memory storage capacity, so it is very desirable to use high-order methods for the efficient numerical solution of fractional derivative problems. However, there appear to be very few previous studies on efficient numerical methods and their relevant stability and convergence analysis, for problems involving multi-term fractional partial differential equations. The main purpose of this article is to construct effective and fast numerical methods for the 1D and 2D forms of the multi-term time fractional sub-diffusion equation (1.1), and to establish corresponding error estimates. To reduce the computational burden, we adopt fourth-order compact differences for the spatial approximation [19,21,33], such that fewer grid points are required to produce accurate solutions. The $L1$ approximation proposed by Xu [30] and Sun [31] is adopted to deal with the multi-term temporal Caputo fractional derivatives. Using the discrete energy method, we prove that our resulting compact difference scheme is unconditionally stable and globally convergent, with the convergence $O(\tau^{2-\alpha}+h^4)$ in the maximum norm. Furthermore, another new scheme with second-order spatial accuracy is also presented, and its corresponding stability and convergence discussed.

In Section 2, we first give some auxiliary lemmas, and then derive our compact difference scheme. In Section 3, by introducing a new inner product and using the energy method, we prove the stability and convergence of the compact difference scheme. In Section 4, the second-order scheme with unconditional stability and maximum norm convergence is discussed. Some results on the two-dimensional multi-term time fractional diffusion equation are given in Sections 5 and 6, respectively. In Section 7, some numerical experiments are presented to support the theoretical analysis, and to show the efficiency of the difference scheme. Some final comments are made in the concluding section.

2. Construction of the Compact Finite Difference Scheme

2.1. Notation and auxiliary lemmas

We first give some notation and auxiliary lemmas, to be used in the construction of the compact finite difference scheme. Without loss of generality, we may take $a_1 = 1$ and $\kappa = 1$ in Eq. (1.1) and consider the following problem involving the 1D two-term time fractional sub-diffusion equation:

$$
{}_0^C D_t^{\alpha_1} u(x,t) + {}_0^C D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\,, \quad 0<x<L\,,\ 0<t\le T\,, \tag{2.1}
$$

$$
u(x,0) = \varphi(x)\,, \quad 0 \le x \le L\,, \tag{2.2}
$$

$$
u(0,t) = \psi_1(t)\,, u(L,t) = \psi_2(t)\,, \quad 0<t\le T\,, \tag{2.3}
$$

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where $0 < \alpha_1 < \alpha < 1$ and $\varphi(x), \psi_1(t), \psi_2(t)$ and $f(x,t)$ are known smooth functions.

For the finite difference approximation, we set $h = L/M$ and $\tau = T/N$, where M and N are two positive integers. The domain $[0,L] \times [0,T]$ is covered by $\Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid x_i = ih,\ 0 \le i \le M\}$ and $\Omega_\tau = \{t_n \mid t_n = n\tau,\ 0 \le n \le N\}$. For any grid function $u = \{u_j^k \mid 0 \le j \le M,\ 0 \le k \le N\}$ defined on $\Omega_h \times \Omega_\tau$, we introduce the notation

$$\delta_x u_{j-\frac{1}{2}}^k = \frac{1}{h}\left(u_j^k - u_{j-1}^k\right), \quad \delta_x^2 u_j^k = \frac{1}{h}\left(\delta_x u_{j+\frac{1}{2}}^k - \delta_x u_{j-\frac{1}{2}}^k\right).$$

For convenience, we denote a discrete fractional derivative operator D_τ^α and an average operator A respectively as follows:

$$D_\tau^\alpha u_i^k = \frac{1}{\mu}\left[u_i^k - \sum_{j=1}^{k-1}(a_{k-j-1}^\alpha - a_{k-j}^\alpha)u_i^j - a_{k-1}^\alpha \varphi(x_i)\right], \quad 0 \le i \le M, \quad 1 \le k \le N,$$

where $\mu = \tau^\alpha \Gamma(2-\alpha)$ and $a_k^\alpha = (k+1)^{1-\alpha} - k^{1-\alpha}$; and

$$Au_i^k = \begin{cases} \dfrac{1}{12}\left(u_{i-1}^k + 10u_i^k + u_{i+1}^k\right) = \left(I + \dfrac{h^2}{12}\delta_x^2\right)u_i^k, & 1 \le i \le M-1, \quad 0 \le k \le N, \\ u_i^k, \quad i = 0, M, \quad 0 \le k \le N, & \end{cases}$$

where I denotes the identity operator.

The following lemmas will be used in deriving the difference schemes.

Lemma 2.1 (cf. Ref. [31]). *Suppose that* $0 < \alpha < 1$, $g \in C^2[0, t_n]$, *and*

$$\begin{aligned} R^\alpha(g(t_n)) \equiv & \frac{1}{\Gamma(1-\alpha)}\int_0^{t_n} \frac{g'(s)}{(t_n - s)^\alpha} ds \\ & - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^\alpha g(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1}^\alpha - a_{n-k}^\alpha)g(t_k) - a_{n-1}^\alpha g(0)\right]. \end{aligned}$$

Then

$$\left|R^\alpha(g(t_n))\right| \le \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})\right] \max_{0 \le t \le t_n} |g''(t)| \tau^{2-\alpha}.$$

Lemma 2.2 (cf. Ref [32]). *Denote* $\zeta(s) = (1-s)^3[5-3(1-s)^2]$. *Then if* $f(x) \in C^6[x_{i-1}, x_{i+1}]$ *where* $1 \le i \le M-1$,

$$\begin{aligned} & \frac{1}{12}\left[f''(x_{i-1}) + 10f''(x_i) + f''(x_{i+1})\right] - \frac{1}{h^2}\left[f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\right] \\ = & \frac{h^4}{360}\int_0^1 \left[f^{(6)}(x_i - sh) + f^{(6)}(x_i + sh)\right]\zeta(s) ds. \end{aligned}$$

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2.2. Derivation of the compact difference scheme

Let us now construct a compact difference scheme to solve the problem (2.1)–(2.3). On $\Omega_h \times \Omega_\tau$, we define the grid functions

$$U_i^k = u(x_i, t_k)\,, \quad f_i^k = f(x_i, t_k)\,, \quad 0 \le i \le M\,, \quad 0 \le k \le N\,.$$

Suppose $u(x,t) \in C_{x,t}^{6,2}([0,L]\times[0,T])$. Considering Eq. (2.1) at the point (x_i, t_k), we have

$${}_0^C D_t^{\alpha_1} u(x_i,t_k) + {}_0^C D_t^{\alpha} u(x_i,t_k) = \frac{\partial^2 u(x_i,t_k)}{\partial x^2} + f(x_i,t_k)\,, \quad 0 \le i \le M\,,\ 1 \le k \le N\,,$$

and on applying the average operator A to both sides:

$$A\,{}_0^C D_t^{\alpha_1} u(x_i,t_k) + A\,{}_0^C D_t^{\alpha} u(x_i,t_k) = A\frac{\partial^2 u(x_i,t_k)}{\partial x^2} + Af(x_i,t_k),\ 1 \le i \le M-1,\ 1 \le k \le N\,.$$

From Lemmas 2.1 and 2.2 and recalling $D_\tau^{\alpha_1}$ and D_τ^{α},

$$AD_\tau^{\alpha_1} U_i^k + AD_\tau^{\alpha} U_i^k = \delta_x^2 U_i^k + Af_i^k + R_i^k\,, \quad 1 \le i \le M-1\,,\ 1 \le k \le N\,, \tag{2.4}$$

where

$$\begin{aligned} R_i^k =& \frac{h^4}{360}\int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_i - sh, t_k) + \frac{\partial^6 u}{\partial x^6}(x_i + sh, t_k)\right]\zeta(s)ds \\ &+ AR^{\alpha_1}\left(u(x_i,t_k)\right) + AR^{\alpha}(u(x_i,t_k))\,. \end{aligned}$$

Noting that $0 < \alpha_1 < \alpha < 1$, there exists a positive constant c_1 independent of h and τ such that

$$|R_i^k| \le c_1(\tau^{2-\alpha} + h^4)\,, \quad 1 \le i \le M-1\,,\ 1 \le k \le N\,. \tag{2.5}$$

In addition, it follows from the initial and boundary conditions (2.2) and (2.3) that

$$U_i^0 = \varphi(x_i)\,, \quad 0 \le i \le M, \tag{2.6}$$

$$U_0^k = \psi_1(t_k)\,, \quad U_M^k = \psi_2(t_k)\,, \quad 1 \le k \le N\,. \tag{2.7}$$

Omitting the small term R_i^k in Eq. (2.4) and replacing the function U_i^k with its numerical approximation u_i^k, on noting (2.6) and (2.7) we construct the following compact difference scheme (L1-CD):

$$AD_\tau^{\alpha_1} u_i^k + AD_\tau^{\alpha} u_i^k = \delta_x^2 u_i^k + Af_i^k\,, \quad 1 \le i \le M-1\,,\ 1 \le k \le N\,, \tag{2.8}$$

$$u_i^0 = \varphi(x_i)\,, \quad 0 \le i \le M, \tag{2.9}$$

$$u_0^k = \psi_1(t_k)\,, \quad u_M^k = \psi_2(t_k)\,, \quad 1 \le k \le N\,. \tag{2.10}$$

It is easy to see that at each time level, this L1-CD scheme (2.8)–(2.10) is a linear tridiagonal system with a strictly diagonally dominant coefficient matrix, so we have the following theorem:

Theorem 2.1. *The L1-CD scheme* (2.8)–(2.10) *is uniquely solvable.*

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3. Stability and Convergence of the L1-CD Scheme

We now proceed to investigate of the stability and convergence of the L1-CD scheme (2.8)–(2.10). Let $V_h = \{v \mid v = (v_0, v_1, \cdots, v_{M-1}, v_M), v_0 = v_M = 0\}$. For any $u, v \in V_h$, we define the discrete inner product, L_2 norm, H^1 semi-norm and maximum norm as follows:

$$(u,v) = h\sum_{i=1}^{M-1} u_i v_i\,, \quad \|u\| = \sqrt{(u,u)}\,,$$

$$\langle \delta_x u, \delta_x v\rangle = h\sum_{i=0}^{M-1}\left(\delta_x u_{i+\frac{1}{2}}\right)\left(\delta_x v_{i+\frac{1}{2}}\right) - \frac{h^3}{12}\sum_{i=1}^{M-1}(\delta_x^2 u_i)(\delta_x^2 v_i)\,, \quad \|\delta_x u\|_h = \sqrt{\langle \delta_x u, \delta_x u\rangle}\,,$$

$$|u|_1 = \sqrt{h\sum_{i=0}^{M-1}\left(\delta_x u_{i+\frac{1}{2}}\right)^2}\,, \quad \|u\|_\infty = \max_{1\le i\le M-1}|u_i|\,.$$

Lemma 3.1 (cf. Refs. [14,19]). *If* $0<\gamma<1$ *and* $a_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}$, $k = 0,1,2,\cdots$. *Then*

$$1 = a_0^\gamma > a_1^\gamma > a_2^\gamma > \cdots > a_k^\gamma > \cdots \to 0\,,$$
$$(1-\gamma)(k+1)^{-\gamma} < a_k^\gamma < (1-\gamma)k^{-\gamma}\,.$$

Lemma 3.2. *Let* $0<\gamma<1$. *For any grid function* $u = \{u_i^k \mid 0\le i\le M,\ 0\le k\le N\}$ *defined on* $\Omega_h \times \Omega_\tau$, *it holds that*

$$-(AD_\tau^\gamma u^k, \delta_x^2 u^k) = \frac{1}{\mu}\left[\|\delta_x u^k\|_h^2 - \sum_{j=1}^{k-1}(a_{k-j-1}^\gamma - a_{k-j}^\gamma)\langle \delta_x u^j, \delta_x u^k\rangle - a_{k-1}^\gamma\langle \delta_x u^0, \delta_x u^k\rangle\right],$$

where $\mu = \tau^\gamma\Gamma(2-\gamma)$.

Proof. From the definition of D_τ^γ and the discrete Green formula,

$$\begin{aligned}
&-\mu(AD_\tau^\gamma u^k, \delta_x^2 u^k)\\
=&-\left(\left(I+\frac{h^2}{12}\delta_x^2\right)u^k, \delta_x^2 u^k\right) + \sum_{j=1}^{k-1}\left(a_{k-j-1}^\gamma - a_{k-j}^\gamma\right)\left(\left(I+\frac{h^2}{12}\delta_x^2\right)u^j, \delta_x^2 u^k\right)\\
&+a_{k-1}^\gamma\left(\left(I+\frac{h^2}{12}\delta_x^2\right)u^0, \delta_x^2 u^k\right)\\
=&\left(\|\delta_x u^k\|^2 - \frac{h^2}{12}\|\delta_x^2 u^k\|^2\right) - \sum_{j=1}^{k-1}\left(a_{k-j-1}^\gamma - a_{k-j}^\gamma\right)\left[\left(\delta_x u^j, \delta_x u^k\right) - \frac{h^2}{12}\left(\delta_x^2 u^j, \delta_x^2 u^k\right)\right]\\
&-a_{k-1}^\gamma\left[\left(\delta_x u^0, \delta_x u^k\right) - \frac{h^2}{12}\left(\delta_x^2 u^0, \delta_x^2 u^k\right)\right]\\
=&\|\delta_x u^k\|_h^2 - \sum_{j=1}^{k-1}\left(a_{k-j-1}^\gamma - a_{k-j}^\gamma\right)\langle \delta_x u^j, \delta_x u^k\rangle - a_{k-1}^\gamma\langle \delta_x u^0, \delta_x u^k\rangle\,.
\end{aligned}$$

Dividing the above equality by μ then yields the desired result. $\square$

Lemma 3.3 (cf. Ref. [33]). *For any mesh function $u \in V_h$,*

$$\frac{2}{3}|u|_1^2 \le \|\delta_x u\|_h^2 \le |u|_1^2 .$$

The semi-norm $\|\cdot\|_h$ is equivalent to the standard H^1 semi-norm, but the semi-norm $\|\cdot\|_h$ is more convenient than the standard H^1 semi-norm for the stability and convergence analysis. We now prove the following Theorem on an *a priori* estimate.

Theorem 3.1. *Let $\{v_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ be the solution of the difference system*

$$AD_\tau^{\alpha_1} v_i^k + AD_\tau^{\alpha} v_i^k = \delta_x^2 v_i^k + g_i^k, \qquad 1 \le i \le M-1\,,\ 1 \le k \le N\,, \tag{3.1}$$

$$v_i^0 = v^0(x_i)\,, \qquad 0 \le i \le M\,, \tag{3.2}$$

$$v_0^k = 0\,, \quad v_M^k = 0\,, \qquad 1 \le k \le N\,, \tag{3.3}$$

where $v^0(x_0) = v^0(x_M) = 0$. Then

$$\|\delta_x v^k\|_h^2 \le \|\delta_x v^0\|_h^2 + \frac{c_{max}}{4} \max_{1 \le l \le N} \|g^l\|^2 ,$$

where $c_{max} = \max\{T^{\alpha_1}\Gamma(1-\alpha_1), T^{\alpha}\Gamma(1-\alpha)\}$.

Proof. Taking the inner product of Eq. (3.1) with $-\delta_x^2 v^k$, we have

$$-(AD_\tau^{\alpha_1} v^k, \delta_x^2 v^k) - (AD_\tau^{\alpha} v^k, \delta_x^2 v^k) = -(\delta_x^2 v^k, \delta_x^2 v^k) - (g^k, \delta_x^2 v^k)\,, \quad 1 \le k \le N\,; \tag{3.4}$$

and hence from Lemma 3.2

$$\begin{aligned} -(AD_\tau^{\alpha_1} v^k, \delta_x^2 v^k) =& \frac{1}{\mu_1}\|\delta_x v^k\|_h^2 - \frac{1}{\mu_1}\sum_{j=1}^{k-1}(a_{k-j-1}^{\alpha_1} - a_{k-j}^{\alpha_1})\langle \delta_x v^j\,, \delta_x v^k\rangle \\ & - \frac{a_{k-1}^{\alpha_1}}{\mu_1}\langle \delta_x v^0, \delta_x v^k\rangle\,, \end{aligned} \tag{3.5}$$

$$\begin{aligned} -(AD_\tau^{\alpha} v^k, \delta_x^2 v^k) =& \frac{1}{\mu_0}\|\delta_x v^k\|_h^2 - \frac{1}{\mu_0}\sum_{j=1}^{k-1}(a_{k-j-1}^{\alpha} - a_{k-j}^{\alpha})\langle \delta_x v^j, \delta_x v^k\rangle \\ & - \frac{a_{k-1}^{\alpha}}{\mu_0}\langle \delta_x v^0, \delta_x v^k\rangle\,. \end{aligned} \tag{3.6}$$

Substituting Eqs. (3.5) and (3.6) into Eq. (3.4), from Lemma 3.1 and the Cauchy-Schwarz inequality we consequently deduce that

$$\begin{aligned} &\Big(\frac{1}{\mu_1} + \frac{1}{\mu_0}\Big)\|\delta_x v^k\|_h^2 \\ \le& \sum_{j=1}^{k-1}\Big[\frac{1}{\mu_1}(a_{k-j-1}^{\alpha_1} - a_{k-j}^{\alpha_1}) + \frac{1}{\mu_0}(a_{k-j-1}^{\alpha} - a_{k-j}^{\alpha})\Big]\|\delta_x v^j\|_h^2 \\ & + \frac{a_{k-1}^{\alpha_1}}{\mu_1}\Big(\|\delta_x v^0\|_h^2 + \frac{\mu_1}{4a_{k-1}^{\alpha_1}}\|g^k\|^2\Big) + \frac{a_{k-1}^{\alpha}}{\mu_0}\Big(\|\delta_x v^0\|_h^2 + \frac{\mu_0}{4a_{k-1}^{\alpha}}\|g^k\|^2\Big)\,. \end{aligned} \tag{3.7}$$

Applying Lemma 3.1 and noting that $0 < \alpha_1 < \alpha < 1$, we also obtain

$$\begin{aligned} \frac{\mu_1}{4a_{k-1}^{\alpha_1}} &< \frac{k^{\alpha_1}\tau^{\alpha_1}\Gamma(2-\alpha_1)}{4(1-\alpha_1)} \le \frac{T^{\alpha_1}\Gamma(1-\alpha_1)}{4}\,, \\ \frac{\mu_0}{4a_{k-1}^{\alpha}} &< \frac{k^{\alpha}\tau^{\alpha}\Gamma(2-\alpha)}{4(1-\alpha)} \le \frac{T^{\alpha}\Gamma(1-\alpha)}{4}\,. \end{aligned}$$

Letting

$$B = \|\delta_x v^0\|_h^2 + \frac{c_{max}}{4}\|g^l\|^2$$

and substituting the above estimates into Eq. (3.7), we have

$$\begin{aligned} \left(\frac{1}{\mu_1}+\frac{1}{\mu_0}\right)\|\delta_x v^k\|_h^2 \le& \sum_{j=1}^{k-1}\left[\frac{1}{\mu_1}\left(a_{k-j-1}^{\alpha_1}-a_{k-j}^{\alpha_1}\right)+\frac{1}{\mu_0}\left(a_{k-j-1}^{\alpha}-a_{k-j}^{\alpha}\right)\right]\|\delta_x v^j\|_h^2 \\ &+\left(\frac{a_{k-1}^{\alpha_1}}{\mu_1}+\frac{a_{k-1}^{\alpha}}{\mu_0}\right)B\,, \quad 1 \le k \le N\,. \end{aligned}$$

Using the same arguments as in [20, 21], we obtain the desired result. □

We obtain the following stability statement from Theorem 3.1.

Theorem 3.2. *The L1-CD scheme* (2.8)–(2.10) *is unconditionally stable to the initial value φ and the inhomogeneous term f.*

Let us now consider the convergence of the L1-CD scheme (2.8)–(2.10). Writing

$$e_i^k = U_i^k - u_i^k\,, \quad 0 \le i \le M\,, \quad 0 \le k \le N\,,$$

and subtracting Eqs. (2.8)–(2.10) from Eq. (2.4) and Eqs. (2.6)–(2.7) respectively, we get the error equations

$$\begin{aligned} &AD_\tau^{\alpha_1}e_i^k + AD_\tau^{\alpha}e_i^k = \delta_x^2 e_i^k + R_i^k\,, \quad 1 \le i \le M-1\,,\ 1 \le k \le N\,, \\ &e_i^0 = 0\,, \quad 0 \le i \le M\,, \\ &e_0^k = 0\,, \quad e_M^k = 0\,, \quad 1 \le k \le N\,. \end{aligned}$$

From Eq. (2.5) and Theorem 3.1 we have

$$\|\delta_x e^k\|_h^2 \le \frac{c_{max}}{4}\max_{1\le l\le N}\|R^l\|^2 \le \frac{c_{max}Lc_1^2}{4}(\tau^{2-\alpha}+h^4)^2\,,$$

and then applying the embedding inequality $\|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1$ (cf. [34]) and Lemma 3.3 we obtain the following convergence result.

Theorem 3.3. *Assume that $u(x,t) \in C_{x,t}^{6,2}([0,L]\times[0,T])$ is the solution of Eqs.* (2.1)–(2.3) *and $\{u_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ is the solution of the L1-CD scheme Eqs.* (2.8)–(2.10). *Then*

$$\|e^k\|_\infty \le \frac{c_1 L}{8}\sqrt{6c_{max}}(\tau^{2-\alpha}+h^4)\,, \quad 0 \le k \le N\,.$$

4. A Second-Order Finite Difference Scheme

We now present a finite difference scheme second-order in the spatial direction that is computationally efficient when the storage is inexpensive. Thus for the problem (2.1)–(2.3) we construct the L1-SOD scheme

$$D_\tau^{\alpha_1} u_i^k + D_\tau^{\alpha} u_i^k = \delta_x^2 u_i^k + f_i^k\,, \quad 1 \le i \le M-1\,,\ 1 \le k \le N\,, \tag{4.1}$$
$$u_i^0 = \varphi(x_i)\,, \quad 0 \le i \le M\,, \tag{4.2}$$
$$u_0^k = \psi_1(t_k)\,, \quad u_M^k = \psi_2(t_k)\,, \quad 1 \le k \le N\,. \tag{4.3}$$

Lemma 4.1. *Suppose $u(x,t) \in C_{x,t}^{4,2}([0,L]\times[0,T])$ is the solution of the problem* (2.1)–(2.3)*. Then the truncation error of the L1-SOD scheme* (4.1)–(4.3) *satisfies*

$$|\hat{R}_i^k| \le \hat{c}_R(\tau^{2-\alpha}+h^2)\,, \quad 0 \le i \le M\,, \quad 1 \le k \le N\,,$$

where $\hat{c}_R$ is a positive constant independent of τ and h.

Using the analytical method of the compact difference scheme in Section 3, we can also prove that the L1-SOD scheme (4.1)–(4.3) is stable to the initial value φ and the forcing term f, and convergent with the convergence order of $O(\tau^{2-\alpha}+h^2)$ in the maximum norm.

Theorem 4.1. *Suppose $\{u_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ is the solution of the L1-SOD scheme* (4.1)–(4.3)*. Then*

$$\|\delta_x u^k\|^2 \le \|\delta_x u^0\|^2 + \frac{c_{max}}{4}\max_{1\le l\le N}\|f^l\|^2\,.$$

Theorem 4.2. *If the problem* (2.1)–(2.3) *has smooth solution $u(x,t) \in C_{x,t}^{4,2}([0,L]\times[0,T])$ and $\{u_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ is the solution of the L1-SOD scheme* (4.1)–(4.3)*, then*

$$\|U^k - u^k\|_\infty \le \frac{\hat{c}_R L}{4}\sqrt{c_{max}}(\tau^{2-\alpha}+h^2)\,, \quad 0 \le k \le N\,.$$

5. Extension to 2D Multi-Term Time-Fractional Sub-Diffusion Equations

Let is now consider the numerical solution of the following problem involving the 2D multi-term time fractional sub-diffusion equations:

$${}_0^C D_t^{\alpha_1} u(x,y,t) + {}_0^C D_t^{\alpha} u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), \quad (x,y)\in\Omega,\ 0<t\le T\,, \tag{5.1}$$
$$u(x,y,0) = \varphi(x,y)\,, \quad (x,y)\in\bar{\Omega} = \Omega\cup\partial\Omega\,, \tag{5.2}$$
$$u(x,y,t) = \psi(x,y,t)\,, \quad (x,y)\in\partial\Omega\,,\ 0<t\le T\,, \tag{5.3}$$

where $\Omega = (0,L_1)\times(0,L_2)$, $\partial\Omega$ is the boundary, $0<\alpha_1<\alpha<1$, and $\varphi(x,y), \psi(x,y,t)$ and $f(x,y,t)$ are known smooth functions.

Set $x_i = ih_1$ and $y_j = jh_2$ with $h_1 = L_1/M_1$ and $h_2 = L_2/M_2$, where M_1 and M_2 are positive integers. Define $\Omega_{h_1} = \{x_i|\ 0 \le i \le M_1\}$ and $\Omega_{h_2} = \{y_j|\ 0 \le j \le M_2\}$, $\bar{\Omega}_h = \Omega_{h_1} \times \Omega_{h_2}$, $\Omega_h = \bar{\Omega}_h \cup \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cup \partial\Omega$. The definitions of τ, t_k are the same in Section 2. For any grid function $v = \{v_{ij} \mid 0 \le i \le M_1, 0 \le j \le M_2\}$, we denote

$$\delta_x v_{i-\frac{1}{2},j} = \frac{1}{h_1}\left(v_{ij} - v_{i-1,j}\right), \qquad \delta_x^2 v_{ij} = \frac{1}{h_1}\left(\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}\right),$$

$$\delta_y\delta_x v_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{h_2}\left(\delta_x v_{i-\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j-1}\right), \quad \delta_y\delta_x^2 v_{i,j-\frac{1}{2}} = \frac{1}{h_2}\left(\delta_x^2 v_{ij} - \delta_x^2 v_{i,j-1}\right).$$

Similarly, the notations $\delta_y v_{i,j-\frac{1}{2}}$, $\delta_y^2 v_{ij}$, $\delta_x\delta_y^2 v_{i-\frac{1}{2},j}$, $\delta_x^2\delta_y^2 v_{ij}$ can be defined. The discrete Laplace operator is denoted $\Delta_h v_{ij} = \delta_x^2 v_{ij} + \delta_y^2 v_{ij}$, and the spatial average difference operators are defined as

$$H_x v_{ij} = \begin{cases} \dfrac{1}{12}(v_{i-1,j} + 10v_{ij} + v_{i+1,j}) = \left(I + \dfrac{h_1^2}{12}\delta_x^2\right)v_{ij}, & 1 \le i \le M_1 - 1,\ 0 \le j \le M_2, \\ v_{ij}, & i = 0 \quad \text{or} \quad M_1,\ 0 \le j \le M_2, \end{cases}$$

$$H_y v_{ij} = \begin{cases} \dfrac{1}{12}(v_{i,j-1} + 10v_{ij} + v_{i,j+1}) = \left(I + \dfrac{h_2^2}{12}\delta_y^2\right)v_{ij}, & 1 \le j \le M_2 - 1,\ 0 \le i \le M_1, \\ v_{ij}, & j = 0 \quad \text{or} \quad M_2,\ 0 \le i \le M_1. \end{cases}$$

We introduce the space of grid functions on $\bar{\Omega}_h$:

$$V_h^* = \left\{ v \mid v = \{v_{ij} \mid (x_i, y_j) \in \bar{\Omega}_h\} \text{ and } v_{ij} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h \right\}.$$

For any grid functions $u, v \in V_h^*$, we define the discrete inner product

$$(u,v) = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} u_{ij} v_{ij}$$

and denote $\|v\| = \sqrt{(v,v)}$. Similarly, we define $\|\delta_x^2 v\|$, $\|\delta_y^2 v\|$ and $\|\delta_x^2\delta_y^2 v\|$; and denote

$$\|\delta_x v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1} \left|\delta_x v_{i-\frac{1}{2},j}\right|^2},$$

$$\|\delta_x\delta_y v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} \left|\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}\right|^2},$$

$$\|\delta_y\delta_x^2 v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2} \left|\delta_y\delta_x^2 v_{ij-\frac{1}{2}}\right|^2},$$

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and $\|\delta_y v\|$, $\|\delta_x\delta_y^2 v\|$ similarly. The discrete H^1 semi-norm and H^1 norm are

$$\|\nabla_h v\| = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}\,, \qquad \|v\|_{H^1} = \sqrt{\|v\|^2 + \|\nabla_h v\|^2}\,.$$

Finally, for any grid functions $u, v \in V_h^*$ we denote

$$\begin{aligned}
&\langle \nabla_h u, \nabla_h v\rangle_{h^*} \\
=&(\delta_x u, \delta_x v) + (\delta_y u, \delta_y v) - \frac{h_1^2}{12}\left[(\delta_x^2 u, \delta_x^2 v) + (\delta_x\delta_y u, \delta_x\delta_y v)\right] \\
&-\frac{h_2^2}{12}\left[(\delta_y^2 u, \delta_y^2 v) + (\delta_x\delta_y u, \delta_x\delta_y v)\right] + \frac{h_1^2 h_2^2}{144}\left[(\delta_x^2\delta_y u, \delta_x^2\delta_y v) + (\delta_x\delta_y^2 u, \delta_x\delta_y^2 v)\right]
\end{aligned}$$

and

$$\|\nabla_h u\|_{h^*} = \sqrt{\langle \nabla_h u, \nabla_h u\rangle_{h^*}}\,.$$

5.1. Derivation of the compact ADI difference scheme

We suppose that $u(x,y,t) \in C_{x,y,t}^{6,6,2}(\Omega \times [0,T])$, and define the grid functions

$$U_{ij}^n = u(x_i, y_j, t_n)\,,\ f_{ij}^n = f(x_i, y_j, t_n)\,, \quad (x_i, y_j) \in \Omega_h\,, \quad 1 \le n \le N\,.$$

Considering Eq. (5.1) at the point (x_i, y_j, t_n), we have

$$\begin{aligned}
{}_0^C D_t^{\alpha_1} u(x_i, y_j, t_n) + {}_0^C D_t^{\alpha} u(x_i, y_j, t_n) = \Delta u(x_i, y_j, t_n) + f(x_i, y_j, t_n), \\
(x_i, y_j) \in \Omega_h, \quad 1 \le n \le N.
\end{aligned}$$

Applying the average operator $H_x H_y$ on both sides of this equation,

$$\begin{aligned}
&H_x H_y {}_0^C D_t^{\alpha_1} u(x_i, y_j, t_n) + H_x H_y {}_0^C D_t^{\alpha} u(x_i, y_j, t_n) \\
=&H_x H_y \left(u_{xx}(x_i, y_j, t_n) + u_{yy}(x_i, y_j, t_n)\right) + H_x H_y f(x_i, y_j, t_n)\wr\,, \quad (x_i, y_j) \in \Omega_h\,,\ 1 \le n \le N.
\end{aligned}$$

From Lemmas 2.1 and 2.2,

$$\begin{aligned}
H_x H_y D_\tau^{\alpha_1} U_{ij}^n + H_x H_y D_\tau^{\alpha} U_{ij}^n = H_y \delta_x^2 U_{ij}^n + H_x \delta_y^2 U_{ij}^n + H_x H_y f_{ij}^n + \bar{R}_{ij}^n\,, \\
(x_i, y_j) \in \Omega_h\,,\ 1 \le n \le N\,, \qquad (5.4)
\end{aligned}$$

where

$$\begin{aligned}
\bar{R}_{ij}^n =& \frac{h_1^4}{360}\int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_i - \lambda h_1, y_j, t_n) + \frac{\partial^6 u}{\partial x^6}(x_i + \lambda h_1, y_j, t_n)\right]\zeta(\lambda) d\lambda \\
&+ \frac{h_2^4}{360}\int_0^1 \left[\frac{\partial^6 u}{\partial y^6}(x_i, y_j - \lambda h_2, t_n) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j + \lambda h_2, t_n)\right]\zeta(\lambda) d\lambda \\
&+ H_x H_y R^{\alpha_1}\left(u(x_i, t_k)\right) + H_x H_y R^{\alpha}(u(x_i, t_k))\,.
\end{aligned}$$

Adding a small term $\frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}$ into Eq. (5.4), we then have

$$
\begin{aligned}
&H_xH_yD_\tau^{\alpha_1}U_{ij}^n+H_xH_yD_\tau^{\alpha}U_{ij}^n+\frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\delta_x^2\delta_y^2\delta_tU_{ij}^{n-\frac{1}{2}}\\
=&H_y\delta_x^2U_{ij}^n+H_x\delta_y^2U_{ij}^n+H_xH_yf_{ij}^n+R_{ij}^n,\quad (x_i,y_j)\in\Omega_h\,,\ 1\le n\le N\,,
\end{aligned}
\tag{5.5}
$$

where

$$
R_{ij}^n=\bar{R}_{ij}^n+\frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\delta_x^2\delta_y^2\delta_tU_{ij}^{n-\frac{1}{2}}\,,
$$

so there exists a positive constant c_2 independent of h_1,h_2 and τ such that

$$
|R_{ij}^{n-\frac{1}{2}}|\le c_2(\tau^{\min\{1+\alpha,2-\alpha\}}+h_1^4+h_2^4)\,,\quad (x_i,y_j)\in\Omega_h\,,\ 1\le n\le N\,. \tag{5.6}
$$

In addition, on noting the initial and boundary value conditions (5.2)–(5.3) we find that

$$
U_{ij}^0=\varphi(x_i,y_j)\,,\quad (x_i,y_j)\in\Omega_h\,, \tag{5.7}
$$

$$
U_{ij}^n=\phi(x_i,y_j,t_n)\,,\quad (x_i,y_j)\in\partial\Omega_h\,,\quad 1\le n\le N\,. \tag{5.8}
$$

Omitting small terms R_{ij}^n in Eq. (5.5), and replacing the function U_{ij}^n with its numerical approximation u_{ij}^n in Eqs. (5.5), (5.7) and (5.8), we obtain the following compact difference scheme (L1-CADI):

$$
\begin{aligned}
&H_xH_yD_\tau^{\alpha_1}u_{ij}^n+H_xH_yD_\tau^{\alpha}u_{ij}^n+\frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\delta_x^2\delta_y^2\delta_tu_{ij}^{n-\frac{1}{2}}\\
&\quad=H_y\delta_x^2u_{ij}^n+H_x\delta_y^2u_{ij}^n+H_xH_yf_{ij}^n\,,\quad (x_i,y_j)\in\Omega_h\,,\ 1\le n\le N\,,
\end{aligned}
\tag{5.9}
$$

$$
u_{ij}^0=\varphi(x_i,y_j)\,,\quad (x_i,y_j)\in\Omega_h\,, \tag{5.10}
$$

$$
u_{ij}^n=\phi(x_i,y_j,t_n)\,,\quad (x_i,y_j)\in\partial\Omega_h\,,\quad 1\le n\le N\,. \tag{5.11}
$$

To efficiently solve this formulation, the following techniques can be used. Introducing the intermediate variable u_{ij}^*, we obtain the D'Yakonov-type ADI scheme

$$
\begin{cases}
\Big[H_x-\dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\delta_x^2\Big]u_{ij}^*=\Big(\dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\Big)^2\delta_x^2\delta_y^2u_{ij}^{n-1}+\dfrac{\mu_0}{\mu_1+\mu_0}\displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_1}-a_{n-k}^{\alpha_1})H_xH_yu_{ij}^k\\
\qquad+\dfrac{\mu_0a_{n-1}^{\alpha_1}}{\mu_1+\mu_0}H_xH_y\psi_{ij}+\dfrac{\mu_1}{\mu_1+\mu_0}\displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha})H_xH_yu_{ij}^k\\
\qquad+\dfrac{\mu_0a_{n-1}^{\alpha}}{\mu_1+\mu_0}H_xH_y\psi_{ij}+\dfrac{\mu_1\mu_0}{\mu_1+\mu_0}H_xH_yf_{ij}^n\,,\\
\Big[H_y-\dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\delta_y^2\Big]u_{ij}^n=u_{ij}^*\,.
\end{cases}
$$

We determine $\{u_{ij}^n\}$ by solving two sets of independent one-dimensional problems. Since the coefficient matrices are strictly diagonally dominant, at each time level one can run the x-sweep of the procedure to compute $\{u_{ij}^*|(x_i,y_j)\in\Omega_h\}$ with the known data $\{u_{ij}^{n-1}|(x_i,y_j)\in\Omega_h\}$ using the fast tridiagonal (Thomas) algorithm — cf. Ref. [35] for example. Then once the solution $\{u_{ij}^*|(x_i,y_j)\in\Omega_h\}$ is available, the y-sweep can readily be performed to calculate the desired unique solution $\{u_{ij}^n|(x_i,y_j)\in\Omega_h\}$ to the compact ADI difference scheme (5.9)–(5.11).

5.2. Stability and convergence of the L1-CADI scheme

To analyse the stability and convergence of the L1-CADI scheme (5.9)–(5.11), we introduce some important lemmas.

Lemma 5.1. *If $0<\gamma<1$, for any grid function $\{u_{ij}^n \mid (x_i,y_j)\in\bar{\Omega}_h,\ 0\le n\le N\}$ defined on $\bar{\Omega}_h\times\Omega_\tau$ we have*

$$-(H_xH_yD_\tau^\gamma u^n,\Delta_h u^n)=\frac{1}{\mu}\Big[\|\nabla_h u^n\|_{h^*}^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\gamma-a_{n-k}^\gamma)\langle\nabla_h u^k,\nabla_h u^n\rangle_{h^*}$$
$$-a_{k-1}^\gamma\langle\nabla_h u^0,\nabla_h u^n\rangle_{h^*}\Big].$$

Proof. The result follows from the definition of $\|\cdot\|_{h^*}$ and the discrete Green formula. □

Lemma 5.2. *For any grid function $u\in V_h^*$,*

$$\Big((H_y\delta_x^2+H_x\delta_y^2)u,\Delta_h u\Big)\ge\frac{2}{3}\|\Delta_h u\|^2\,.$$

Proof. From the discrete Green formulation,

$$\begin{aligned}
&\left(\left(H_y\delta_x^2+H_x\delta_y^2\right)u,\Delta_h u\right)\\
=&\left(\left(I+\frac{h_2^2}{12}\delta_y^2\right)\delta_x^2u+\left(I+\frac{h_1^2}{12}\delta_x^2\right)\delta_y^2u,\Delta_h u\right)\\
=&\|\Delta_h u\|^2-\frac{h_1^2}{12}\left(\|\delta_x\delta_y^2u\|^2+\|\delta_x^2\delta_y u\|^2\right)-\frac{h_2^2}{12}\left(\|\delta_x\delta_y^2u\|^2+\|\delta_x^2\delta_y u\|^2\right)\\
\ge&\|\Delta_h u\|^2-\frac{1}{3}\left(\|\delta_y^2u\|^2+\|\delta_x\delta_y u\|^2\right)-\frac{1}{3}\left(\|\delta_x^2u\|^2+\|\delta_x\delta_y u\|^2\right)\\
\ge&\frac{2}{3}\|\Delta_h u\|^2\,,
\end{aligned}$$

where we have used

$$\begin{aligned}
&\|\delta_x\delta_y u\|\le 2h_1^{-1}\|\delta_y u\|\,, &&\|\delta_x^2u\|\le 2h_1^{-1}\|\delta_x u\|\,,\\
&\|\delta_x\delta_y u\|\le 2h_2^{-1}\|\delta_x u\|\,, &&\|\delta_y^2u\|\le 2h_2^{-1}\|\delta_y u\|\,.
\end{aligned}$$
□

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Lemma 5.3. *For any grid function $u \in V_h^*$, we have*

$$\frac{1}{3}\|\nabla_h u\|^2 \le \|\nabla_h u\|_{h^*}^2 \le \|\nabla_h u\|^2.$$

Proof. The result readily follows from the definition of $\|\cdot\|_{h^*}$ and inverse estimates. □

We may now obtain an *a priori* estimate in the following Theorem.

Theorem 5.1. *Let $\{v_{ij}^n \mid (x_i,y_j)\in\Omega_h,\ 0\le n\le N\}$ be the solution of the difference system*

$$\begin{aligned}
&H_xH_yD_\tau^{\alpha_1}v_{ij}^n + H_xH_yD_\tau^{\alpha}v_{ij}^n + \frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\delta_x^2\delta_y^2\delta_t v_{ij}^{n-\frac{1}{2}} \\
&\quad = H_y\delta_x^2 v_{ij}^n + H_x\delta_y^2 v_{ij}^n + g_{ij}^n, \quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N, && (5.12)\\
&v_{ij}^0 = v^0(x_i,y_j), \quad (x_i,y_j)\in\Omega_h, && (5.13)\\
&v_{ij}^n = 0, \quad (x_i,y_j)\in\partial\Omega_h, \quad 1\le n\le N, && (5.14)
\end{aligned}$$

where $v^0(x_i,y_j)\equiv 0$ when $(x_i,y_j)\in\partial\Omega_h$. Then it follows that

$$\begin{aligned}
\tau\sum_{k=1}^n \|\nabla_h v^k\|_{h^*}^2 \le c_{max}\Bigg\{&\frac{\mu_1\mu_0 T}{\mu_1+\mu_0}\left(\|\delta_x\delta_y^2 v^0\|^2 + \|\delta_x^2\delta_y v^0\|^2\right)\\
&+\left(\frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\nabla_h v^0\|_{h^*}^2 + \frac{3}{4}\tau\sum_{k=1}^n\|g^k\|_0^2\Bigg\}.
\end{aligned}$$

Proof. Taking the inner product of Eq. (5.12) with $-\Delta_h v^n$, we have

$$\begin{aligned}
&-\left(H_xH_yD_\tau^{\alpha_1}v^n,\Delta_h v^n\right) - \left(H_xH_yD_\tau^{\alpha}v^n,\Delta_h v^n\right) - \frac{\mu_1\mu_0\tau}{\mu_1+\mu_0}\left(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}},\Delta_h v^n\right)\\
&=-\left((H_y\delta_x^2 + H_x\delta_y^2)v^n,\Delta_h v^n\right) - \left(g^k,\Delta_h v^n\right), \qquad 1\le k\le N. && (5.15)
\end{aligned}$$

On using the discrete Green formula and Cauchy-Schwarz inequality, we rewrite the third term on the left-hand side of Eq. (5.15) as

$$\begin{aligned}
&-\tau\left(\delta_x^2\delta_y^2\delta_t v^{n-\frac{1}{2}},\Delta_h v^n\right)\\
&=-\left(\delta_x^2\delta_y^2 v^n,\Delta_h v^n\right) + \left(\delta_x^2\delta_y^2 v^{n-1},\Delta_h v^n\right)\\
&=\left(\|\delta_x^2\delta_y v^n\|^2 + \|\delta_x\delta_y^2 v^n\|^2\right) - \left[\left(\delta_x\delta_y^2 v^{n-1},\delta_x\delta_y^2 v^n\right) + \left(\delta_x^2\delta_y v^{n-1},\delta_x^2\delta_y v^n\right)\right]\\
&\ge\left(\|\delta_x^2\delta_y v^n\|^2 + \|\delta_x\delta_y^2 v^n\|^2\right) - \frac{1}{2}\Big[\left(\|\delta_x\delta_y^2 v^{n-1}\|^2 + \|\delta_x\delta_y^2 v^n\|^2\right)\\
&\quad+\left(\|\delta_x^2\delta_y v^{n-1}\|^2 + \|\delta_x^2\delta_y v^n\|^2\right)\Big]\\
&\ge\frac{1}{2}\left[\left(\|\delta_x^2\delta_y v^n\|^2 + \|\delta_x\delta_y^2 v^n\|^2\right) - \left(\|\delta_x^2\delta_y v^{n-1}\|^2 + \|\delta_x\delta_y^2 v^{n-1}\|^2\right)\right]. && (5.16)
\end{aligned}$$

On substituting Eq. (5.16) into Eq. (5.15), and using Lemmas 5.1 and 5.2 together with the Cauchy-Schwarz inequality, we deduce that

$$
\begin{aligned}
&\left(\frac{1}{\mu_1}+\frac{1}{\mu_0}\right)\|\nabla_h v^n\|_{h^*}^2+\frac{2}{3}\|\Delta_h v^n\|^2+\frac{\mu_1\mu_0}{2(\mu_1+\mu_0)}\Big[\left(\|\delta_x^2\delta_y v^n\|^2+\|\delta_x\delta_y^2 v^n\|^2\right)\\
&-\left(\|\delta_x^2\delta_y v^{n-1}\|^2+\|\delta_x\delta_y^2 v^{n-1}\|^2\right)\Big]\\
\le&\frac{1}{\mu_1}\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_1}-a_{n-k}^{\alpha_1})\langle\nabla_h v^k,\nabla_h v^n\rangle_{h^*}+\frac{a_{n-1}^{\alpha_1}}{\mu_1}\langle\nabla_h v^0,\nabla_h v^n\rangle_{h^*}\\
&+\frac{1}{\mu_0}\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha})\langle\nabla_h v^k,\nabla_h v^n\rangle_{h^*}+\frac{a_{n-1}^{\alpha}}{\mu_0}\langle\nabla_h v^0,\nabla_h v^n\rangle_{h^*}-(g^n,\nabla_h v^n)\\
\le&\frac{1}{2\mu_1}\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_1}-a_{n-k}^{\alpha_1})\left(\|\nabla_h v^k\|_{h^*}^2+\|\nabla_h v^n\|_{h^*}^2\right)+\frac{a_{k-1}^{\alpha_1}}{2\mu_1}\|\nabla_h v^0\|_{h^*}^2+\frac{a_{k-1}^{\alpha_1}}{2\mu_1}\|\nabla_h v^n\|_{h^*}^2\\
&+\frac{1}{2\mu_0}\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha})\left(\|\nabla_h v^k\|_{h^*}^2+\|\nabla_h v^n\|_{h^*}^2\right)+\frac{a_{n-1}^{\alpha}}{2\mu_0}\|\nabla_h v^0\|_{h^*}^2+\frac{a_{n-1}^{\alpha}}{2\mu_0}\|\nabla_h v^n\|_{h^*}^2\\
&+\frac{3}{8}\|g^n\|^2+\frac{2}{3}\|\Delta_h v^n\|^2,
\end{aligned}
$$

implying that

$$
\begin{aligned}
&\left(\frac{1}{\mu_1}+\frac{1}{\mu_0}\right)\|\nabla_h v^n\|_{h^*}^2+\frac{\mu_1\mu_0}{\mu_1+\mu_0}\Big[\left(\|\delta_x^2\delta_y v^n\|^2+\|\delta_x\delta_y^2 v^n\|^2\right)\\
&-\left(\|\delta_x^2\delta_y v^{n-1}\|^2+\|\delta_x\delta_y^2 v^{n-1}\|^2\right)\Big]\\
\le&\sum_{k=1}^{n-1}\left[\frac{1}{\mu_1}\left(a_{n-k-1}^{\alpha_1}-a_{n-k}^{\alpha_1}\right)+\frac{1}{\mu_0}(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha})\right]\|\nabla_h v^k\|_{h^*}^2\\
&+\left(\frac{a_{n-1}^{\alpha_1}}{\mu_1}+\frac{a_{n-1}^{\alpha}}{\mu_0}\right)\|\nabla_h v^0\|_{h^*}^2+\frac{3}{4}\|g^n\|.
\end{aligned}
\tag{5.17}
$$

On denoting

$$
G^n=\sum_{k=1}^{n-1}\left(\frac{a_{n-k}^{\alpha_1}}{\mu_1}+\frac{a_{n-k}^{\alpha}}{\mu_0}\right)\|\nabla_h v^k\|_{h^*}^2+\frac{\mu_1\mu_0}{\mu_1+\mu_0}\left(\|\delta_x^2\delta_y v^n\|^2+\|\delta_x\delta_y^2 v^n\|^2\right),\quad 1\le n\le N
$$

and

$$
G^0=\frac{\mu_1\mu_0}{\mu_1+\mu_0}\left(\|\delta_x^2\delta_y v^0\|^2+\|\delta_x\delta_y^2 v^0\|^2\right),
$$

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Eq. (5.17) can be written as

$$\begin{aligned} G^n &\leq G^{n-1} + \left(\frac{a_{n-1}^{\alpha_1}}{\mu_1} + \frac{a_{n-1}^{\alpha}}{\mu_0} \right) \|\nabla_h v^0\|_{h^*}^2 + \frac{3}{4}\|g^n\|^2 \\ &\leq G^0 + \sum_{k=1}^{n} \left(\frac{a_{k-1}^{\alpha_1}}{\mu_1} + \frac{a_{k-1}^{\alpha}}{\mu_0} \right) \|\nabla_h v^0\|_{h^*}^2 + \frac{3}{4}\sum_{k=1}^{n}\|g^k\|^2. \end{aligned} \tag{5.18}$$

On the one hand, noting that $a_{n-k}^{\gamma} \geq (1-\gamma)(n-k+1)^{1-\gamma} \geq (1-\gamma)N^{-\gamma}$ where $\gamma = \alpha_1$, α we arrive at

$$\begin{aligned} \frac{a_{n-k}^{\alpha_1}}{\mu_1} &= \frac{a_{n-k}^{\alpha_1}}{\tau^{\alpha_1}\Gamma(2-\alpha_1)} \geq \frac{(1-\alpha_1)N^{-\alpha_1}}{\tau^{\alpha_1}\Gamma(2-\alpha_1)} = \frac{1}{T^{\alpha_1}\Gamma(1-\alpha_1)}, \\ \frac{a_{n-k}^{\alpha}}{\mu_0} &= \frac{a_{n-k}^{\alpha}}{\tau^{\alpha}\Gamma(2-\alpha)} \geq \frac{(1-\alpha)N^{-\alpha}}{\tau^{\alpha}\Gamma(2-\alpha)} = \frac{1}{T^{\alpha}\Gamma(1-\alpha)} . \end{aligned}$$

On the other hand, noting that $a_k^{\gamma} = (k+1)^{1-\gamma} - k^{1-\gamma}$ we have

$$\begin{aligned} \tau\sum_{k=1}^{n}\frac{a_{k-1}^{\alpha_1}}{\mu_1} &\leq \frac{\tau n^{1-\alpha_1}}{\tau^{\alpha_1}\Gamma(2-\alpha_1)} \leq \frac{t_n^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \leq \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)}, \\ \tau\sum_{k=1}^{n}\frac{a_{k-1}^{\alpha}}{\mu_0} &\leq \frac{\tau n^{1-\alpha}}{\tau^{\alpha}\Gamma(2-\alpha)} \leq \frac{t_n^{1-\alpha}}{\Gamma(2-\alpha)} \leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} . \end{aligned}$$

Multiplying Eq. (5.18) by τ and invoking the above inequality yields the desired result. □

From Theorem 5.1, we obtain the following stability statement.

Theorem 5.2. *The L1-CADI scheme* (5.9)–(5.11) *is unconditionally stable to the initial value* φ *and the inhomogeneous term* f.

Let us now consider the convergence of the L1-CADI scheme (5.9)–(5.11).

Theorem 5.3. *Assume that* $u(x,y,t) \in C_{x,y,t}^{6,6,2}(\Omega \times [0,T])$ *is the solution of Eqs.* (5.1)–(5.3), *and* $\{u_{ij}^n \mid (x_i,y_j) \in \Omega_h,\ 0 \leq n \leq N\}$ *is the solution of the L1-CADI scheme* (5.9)–(5.11). *Then*

$$\sqrt{\tau\sum_{k=1}^{n}\|\nabla_h e^k\|^2} \leq \frac{3c_2}{2}\sqrt{TL_1L_2}\left(\tau^{\min\{1+\alpha,2-\alpha\}} + h_1^4 + h_2^4\right), \quad 1 \leq n \leq N .$$

Proof. Let

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (x_i,y_j) \in \bar{\Omega}_h, \quad 0 \leq n \leq N .$$

Subtracting Eqs. (5.9)–(5.11) from Eqs. (5.5), (5.7) and (5.8) respectively, we obtain the error equations

$$\begin{aligned} &H_xH_yD_\tau^{\alpha_1}e_{ij}^n + H_xH_yD_\tau^{\alpha}e_{ij}^n + \frac{\mu_1\mu\tau}{\mu_1+\mu}\delta_x^2\delta_y^2\delta_t e_{ij}^{n-\frac{1}{2}} \\ &\quad = H_y\delta_x^2 e_{ij}^n + H_x\delta_y^2 e_{ij}^n + R_{ij}^n , \quad (x_i,y_j) \in \Omega_h,\ 1 \leq n \leq N , \\ &e_{ij}^0 = 0 , \quad (x_i,y_j) \in \Omega_h , \\ &e_{ij}^n = 0 , \quad (x_i,y_j) \in \partial\Omega_h , \quad 0 \leq n \leq N , \end{aligned}$$

where $D_\tau^\gamma e_{ij}^n = D_\tau^\gamma (U_{ij}^n - u_{ij}^n)$ and $\gamma = \alpha, \alpha_1$. Using inequality (5.6), it follows from Theorem 5.1 and Lemma 5.3 that

$$\begin{aligned}\tau \sum_{k=1}^{n} \|\nabla_h e^k\|^2 \leq & \frac{9\tau}{4} \sum_{k=1}^{n} \|R^k\|^2 \leq \frac{9\tau}{4} \sum_{k=1}^{n} L_1 L_2 c_2^2 (\tau^{\min\{1+\alpha, 2-\alpha\}} + h_1^4 + h_2^4)^2 \\ \leq & \frac{9T}{4} c_2^2 L_1 L_2 \left(\tau^{\min\{1+\alpha,2-\alpha\}} + h_1^4 + h_2^4\right)^2, \quad 1 \leq n \leq N .\end{aligned}$$

□

6. A Second-Order ADI Finite Difference Scheme for the 2D Problem

We consider a difference scheme L1-ADI for the problem defined by Eqs. (5.1)–(5.3) as follows:

$$D_\tau^{\alpha_1} u_{ij}^n + D_\tau^\alpha u_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_t u_{ij}^{n-\frac{1}{2}}$$
$$= \Delta_h u_{ij}^n + f_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 1 \leq n \leq N, \tag{6.1}$$
$$u_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{6.2}$$
$$u_{ij}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leq n \leq N. \tag{6.3}$$

The difference scheme for Eq. (6.1) can be written in the ADI form

$$\begin{cases} \left[I - \dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\delta_x^2\right] u_{ij}^* = \left(\dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\right)^2 \delta_x^2\delta_y^2 u_{ij}^{n-1} + \dfrac{\mu_0}{\mu_1+\mu_0}\displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha_1} - a_{n-k}^{\alpha_1}) u_{ij}^k \\ \qquad + \dfrac{\mu_0 a_{n-1}^{\alpha_1}}{\mu_1+\mu_0}\psi_{ij} + \dfrac{\mu_1}{\mu_1+\mu_0}\displaystyle\sum_{k=1}^{n-1}(a_{n-k-1}^{\alpha} - a_{n-k}^{\alpha}) u_{ij}^k + \dfrac{\mu_0 a_{n-1}^{\alpha}}{\mu_1+\mu_0}\psi_{ij} + \dfrac{\mu_1\mu_0}{\mu_1+\mu_0} f_{ij}^n, \\ \left[I - \dfrac{\mu_1\mu_0}{\mu_1+\mu_0}\delta_y^2\right] u_{ij}^n = u_{ij}^* . \end{cases}$$

Lemma 6.1. *Suppose that $u(x,y,t) \in C_{x,y,t}^{4,4,2}(\Omega \times [0,T])$ is the solution of the problem* (5.1)–(5.3). *Then the truncation error of the L1-ADI scheme* (6.1)–(6.3) *satisfies*

$$|\tilde{R}_{ij}^k| \leq \tilde{c}_R \left(\tau^{\min\{1+\alpha,2-\alpha\}} + h_1^2 + h_2^2\right), \quad (x_i, y_j) \in \bar{\Omega}_h, \ 1 \leq n \leq N,$$

where $\tilde{c}_R$ is a positive constant independent of τ and h_1, h_2.

The proofs for the stability and convergence for the L1-ADI scheme (6.1)–(6.3) are essentially the same as for Theorems 5.1 and 5.3, so we omit the details and simply state the relevant results.

Theorem 6.1. *The L1-ADI scheme* (6.1)–(6.3) *is unconditionally stable to the initial value* φ *and the inhomogeneous term* f *— i.e.*

$$\tau\sum_{k=1}^{n}\|\nabla_h u^k\|^2 \le c_{max}\left\{\frac{\mu_1\mu_0 T}{\mu_1+\mu_0}\left(\|\delta_x\delta_y^2 u^0\|^2+\|\delta_x^2\delta_y u^0\|^2\right)\right.$$
$$\left.+\left(\frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)}+\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)\|\nabla_h u^0\|^2+\frac{3}{4}\tau\sum_{k=1}^{n}\|f^k\|^2\right\}.$$

Theorem 6.2. *Suppose* $u(x,y,t)$ *is the smooth solution of* (5.1)–(5.3) *and* $\{u_{ij}^n \mid (x_i,y_j)\in \bar{\Omega}_h,\ 0\le n\le N\}$ *is the solution of the L1-ADI scheme* (6.1)–(6.3), *respectively. Then*

$$\sqrt{\tau\sum_{k=1}^{n}\|\nabla_h e^k\|^2}\le\frac{\tilde{c}_R}{2}\sqrt{3TL_1L_2}\left(\tau^{\min\{1+\alpha,2-\alpha\}}+h_1^2+h_2^2\right),\quad 1\le n\le N.$$

7. Numerical Experiments

We now present numerical solutions of the problem using the numerical methods previously discussed, where the errors involved are measured by comparing the numerical solutions with exact solutions.

Example 7.1. Let $L=\pi$, $T=1$. In order to test the convergence rate of the proposed methods, we refer to the exact solution of the problem (2.1)–(2.3) — viz.

$$u(x,t)=t^{1+\alpha_1+\alpha}\sin x.$$

It can readily be checked that the corresponding source term $f(x,t)$ and the initial and boundary conditions are respectively

$$f(x,t)=\left(\frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha)}t^{2+\alpha}+\frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha_1)}t^{2+\alpha_1}+t^{1+\alpha_1+\alpha}\right)\sin x,$$

and

$$\varphi(x)=0,\quad \psi_1(t)=0,\quad \psi_2(t)=0.$$

We compute the maximum norm errors of the numerical solution

$$e_\infty(h,\tau)=\max_{0\le k\le N}\|U^k-u^k\|_\infty,$$

and respectively characterise the temporal convergence order and the spatial convergence order as

$$\text{Order1}=\log_2\left(\frac{e_\infty(h,2\tau)}{e_\infty(h,\tau)}\right),\quad \text{Order2}=\log_2\left(\frac{e_\infty(2h,\tau)}{e_\infty(h,\tau)}\right).$$

The first computational investigation concerns the temporal errors and convergence orders. In order to find the temporal convergence order, the space step h should be chosen

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Table 1: Numerical convergence of the L1-CD and L1-SOD schemes in the temporal direction.

$\alpha_1, \quad \alpha$	τ	the L1-CD scheme $e_\infty(h,\tau)$	Order1	the L1-SOD scheme $e_\infty(h,\tau)$	Order1
$\alpha_1=0.15,\ \alpha=0.95$	1/10	3.776e-2	*	3.776e-2	*
	1/20	1.836e-2	1.040	1.836e-2	1.040
	1/40	8.890e-3	1.047	8.890e-3	1.047
	1/80	4.296e-3	1.049	4.296e-3	1.049
	1/160	2.075e-3	1.050	2.075e-3	1.050
$\alpha_1=0.35,\ \alpha=0.65$	1/10	1.242e-2	*	1.242e-2	*
	1/20	4.817e-3	1.366	4.817e-3	1.366
	1/40	1.858e-3	1.374	1.858e-3	1.374
	1/80	7.159e-4	1.376	7.160e-4	1.376
	1/160	2.759e-4	1.376	2.761e-4	1.375

Table 2: Numerical convergence of the L1-CD scheme in the spatial direction with $\tau = 1/200000$.

$\alpha_1, \quad \alpha$	h	$e_\infty(h,\tau)$	Order2
$\alpha_1=0.35,\ \alpha=0.65$	$\pi/2$	6.413e-3	*
	$\pi/4$	3.788e-4	4.082
	$\pi/8$	2.328e-5	4.024
	$\pi/16$	1.464e-6	3.991
$\alpha_1=0.45,\ \alpha=0.55$	$\pi/2$	6.450e-3	*
	$\pi/4$	3.809e-4	4.082
	$\pi/8$	2.340e-5	4.025
	$\pi/16$	1.461e-6	4.002

sufficiently small to prevent the effect of the spatial discretisation error entering into the calculation. The computational results of the L1-CD scheme (2.8)–(2.10) and the L1-SOD scheme (4.1)–(4.3) with $h = \pi/100$ and $h = \pi/1000$ are presented in Table 1, respectively. It is observed that both schemes generate $2-\alpha$ temporal convergence order, consistent with our theoretical analysis.

Secondly, we test the spatial errors and convergence orders of the two schemes by letting h vary, but fixing the time step τ sufficiently small to avoid contamination of the spatial error. Tables 2 and 3 show the maximum norm errors and spatial convergence orders of the L1-CD scheme and the L1-SOD scheme with different α_1, α. As predicted by our theoretical estimates, the L1-CD scheme attains fourth-order spatial accuracy while the L1-SOD scheme has second-order spatial accuracy.

Next, in order to quantify some features of the computational efficiencies of the L1-CD scheme more precisely, we investigate the CPU time of both schemes. As mentioned before, since fractional derivatives are non-local operators they require a large memory storage capacity if low-order finite difference methods are employed for the spatial approximation. For the L1-SOD scheme, the optimal step sizes satisfy $\tau^{2-\alpha} \approx h^2$, or $N \approx [M^{\frac{2}{2-\alpha}}]$. For the L1-CD scheme, the optimal step sizes satisfy $\tau^{2-\alpha} \approx h^4$, or $N \approx [M^{\frac{4}{2-\alpha}}]$.

From Table 4, it is clear that the two schemes provide almost the same accuracy for

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Table 3: Numerical convergence of the L1-SOD scheme in the spatial direction with $\tau = 1/20000$.

$\alpha_1,\quad \alpha$	h	$e_\infty(h,\tau)$	Order2
$\alpha_1 = 0.35,\ \alpha = 0.65$	$\pi/2$	4.612e-2	$*$
	$\pi/4$	1.188e-2	1.956
	$\pi/8$	2.993e-3	1.990
	$\pi/16$	7.497e-4	1.997
$\alpha_1 = 0.45,\ \alpha = 0.55$	$\pi/2$	4.640e-2	$*$
	$\pi/4$	1.195e-2	1.957
	$\pi/8$	3.009e-3	1.990
	$\pi/16$	7.537e-4	1.997

Table 4: The maximum norm error and CPU time of the L1-CD and L1-SOD schemes.

		the L1-CD scheme			the L1-SOD scheme		
$\alpha_1,\quad \alpha$	N	M	$e_\infty(h,\tau)$	CPU time(s)	M	$e_\infty(h,\tau)$	CPU time(s)
$\alpha_1 = 0.1,\ \alpha = 0.3$	225	10	1.678e-5	0.1669	100	2.940e-5	0.7005
	585	15	3.286e-6	0.7557	225	5.802e-6	4.8096
	1151	20	1.044e-6	2.3417	400	1.835e-6	20.8146
	1947	25	4.262e-7	5.9645	625	7.512e-7	70.2803
$\alpha_1 = 0.1,\ \alpha = 0.5$	464	10	2.159e-5	0.3780	100	3.331e-5	1.5963
	1368	15	4.229e-6	2.1757	225	6.572e-6	14.9198
	2947	20	1.343e-6	8.7419	400	2.078e-6	86.3564
	5344	25	5.485e-7	28.4608	625	8.505e-7	369.1998
$\alpha_1 = 0.1,\ \alpha = 0.7$	1194	10	3.070e-5	1.1944	100	4.131e-5	5.4915
	4157	15	6.024e-6	11.1884	225	8.156e-6	89.9025
	10073	20	1.916e-6	65.8156	400	2.580e-6	727.2762
	20015	25	7.829e-7	292.5059	625	1.057e-6	4146.7922

the same temporal grid size, but the L1-CD scheme needs fewer spatial grid points and less CPU time. Thus the L1-CD scheme reduces both the storage requirement and the necessary CPU time successfully.

Finally, we compute the long time behaviour of the L1-CD and the L1-SOD schemes. The optimal step sizes again satisfy $\tau^{2-\alpha} \approx h^2$ for the L1-SOD scheme and $\tau^{2-\alpha} \approx h^4$ for the L1-CD scheme, on fixing $T = 10$ and $M = 4, 6, 8, \cdots, 24$. Fig. 1 shows the maximum error and CPU time of the L1-CD scheme and the L1-SOD scheme for $t = 0, 1, 2, \cdots, 10$ when $\alpha_1 = 0.1$, $\alpha = 0.2$, and the compact difference scheme efficiency.

Example 7.2. Let $T = 1$, $\Omega = (0,\pi) \times (0,\pi)$. As before, we refer to the exact solution of the problem (5.1)–(5.3) — i.e. in this case

$$u(x,y,t) = t^{3+\alpha_1+\alpha} \sin x \sin y\ .$$

It is again not difficult to obtain the corresponding forcing term $f(x,y,t)$, and the initial and boundary conditions $\varphi(x,y)$ and $\psi(x,y,t)$.

In order to test the convergence rate of the proposed methods, we use the same spacing h in each direction ($h = h_1 = h_2$), and compute the maximum norm errors of the numerical

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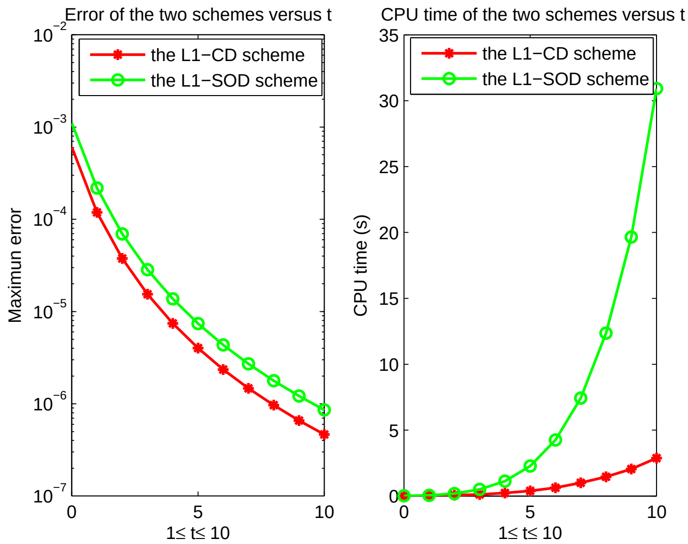

Figure 1: Error and CPU time of the L1-CD and L1-SOD schemes.

Table 5: Numerical convergence of the L1-CADI and L1-ADI schemes in the temporal direction.

α_1, α	τ	the L1-CADI scheme $E_\infty(h,\tau)$	Order1	the L1-ADI scheme $E_\infty(h,\tau)$	Order3
$\alpha_1=0.2,\ \alpha=0.9$	1/10	7.038e-2	$*$	7.038e-2	$*$
	1/20	3.452e-2	1.028	3.453e-2	1.027
	1/40	1.657e-2	1.059	1.658e-2	1.058
	1/80	7.857e-3	1.077	7.862e-3	1.076
	1/160	3.698e-3	1.087	3.703e-3	1.086
$\alpha_1=0.5,\ \alpha=0.7$	1/10	4.140e-2	$*$	4.141e-2	$*$
	1/20	1.757e-2	1.236	1.758e-2	1.236
	1/40	7.294e-3	1.269	7.299e-3	1.268
	1/80	2.985e-3	1.289	2.990e-3	1.287
	1/160	1.212e-3	1.301	1.217e-3	1.297

solution

$$E_\infty(h,\tau)=\max_{\substack{(x_i,y_j)\in\bar{\Omega}_h\\ 0\le n\le N}}\left|u(x_i,y_j,t_n)-u_{ij}^n\right|,$$

via

$$\text{Order3}=\log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right),\ \ \text{Order4}=\log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right).$$

Firstly, the numerical accuracy in time is verified. For fixed space step sizes $h=\pi/20$ and $h=\pi/200$ respectively, and varying the temporal step size τ, the computational results we obtain are displayed in Table 5. From this data, we conclude that there is $\min\{1+\alpha,2-\alpha\}$-order convergence in time.

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Table 6: Numerical convergence of the L1-CADI scheme in the spatial direction with $\tau = 1/200000$.

α_1, α	h	$E_\infty(h,\tau)$	Order4
$\alpha_1 = 0.1$, $\alpha = 0.2$	$\pi/2$	4.074e-3	$*$
	$\pi/4$	2.413e-4	4.078
	$\pi/8$	1.482e-5	4.025
	$\pi/16$	9.222e-7	4.006
$\alpha_1 = 0.2$, $\alpha = 0.3$	$\pi/2$	2.768e-3	$*$
	$\pi/4$	1.641e-4	4.076
	$\pi/8$	1.009e-5	4.024
	$\pi/16$	6.334e-7	3.993

Table 7: Numerical convergence of the L1-ADI scheme in the spatial direction with $\tau = 1/10000$.

α_1, α	h	$E_\infty(h,\tau)$	Order4
$\alpha_1 = 0.1$, $\alpha = 0.2$	$\pi/4$	2.314e-2	$*$
	$\pi/8$	5.769e-3	2.004
	$\pi/16$	1.436e-3	2.007
	$\pi/32$	3.528e-4	2.025
$\alpha_1 = 0.2$, $\alpha = 0.3$	$\pi/4$	2.124e-2	$*$
	$\pi/8$	5.306e-3	2.001
	$\pi/16$	1.324e-3	2.003
	$\pi/32$	3.287e-4	2.010

Secondly, we test the spatial errors and convergence orders of the two schemes, by letting h vary and fixing the time step τ sufficiently small in order to avoid significant contamination from the spatial error. Tables 6 and 7 give the maximum norm errors and spatial convergence orders for the two schemes. As predicted by the theoretical estimates, the L1-CADI scheme attains fourth-order spatial accuracy whereas the L1-ADI scheme has second-order spatial accuracy. In Table 8, we display some CPU time results for the L1-CADI and L1-ADI schemes. It is clear that the two schemes generate almost the same accuracy for the same temporal grid size, while the L1-CADI scheme needs fewer spatial grid points and less CPU time, so it requites less storage and CPU time.

Similar to Example 7.1, we compute the problem for a longer time by fixing $T = 10$ and $M = 4, 5, 6, \cdots, 14$, and still choosing the optimal step size $\tau^{\min\{1+\alpha, 2-\alpha\}} \approx h^2$ for the L1-ADI scheme and $\tau^{\min\{1+\alpha, 2-\alpha\}} \approx h^4$ for the L1-CADI scheme, respectively. Fig. 2 shows the maximum error and CPU time of both schemes for $t = 0, 1, 2, \cdots, 10$ when $\alpha_1 = 0.1$, $\alpha = 0.2$, and also the efficiency of the L1-CADI scheme.

8. Conclusions

In this article, we discuss some computationally effective numerical methods for solving 1D and 2D multi-term time fractional sub-diffusion equations. Based on the $L1$ approximation for the multi-term time Caputo fractional derivative in the temporal direction, and in order to reduce the storage requirement, a compact difference method for spatial

Table 8: The maximum norm error and CPU time of the L1-CADI and L1-ADI schemes.

$\alpha_1,\quad \alpha$	N	the L1-CADI scheme M	$E_\infty(h,\tau)$	CPU time(s)	the L1-ADI scheme M	$E_\infty(h,\tau)$	CPU time(s)
$\alpha_1=0.1,\ \alpha=0.3$	26	4	8.013e-6	0.0952	16	8.406e-6	0.1933
	68	6	6.330e-7	0.4022	36	6.355e-7	2.3234
	133	8	8.503e-8	1.7451	64	8.510e-8	15.2556
	225	10	1.659e-8	5.9884	100	1.660e-8	68.3041
$\alpha_1=0.1,\ \alpha=0.5$	40	4	9.825e-6	0.0979	16	9.861e-6	0.2762
	119	6	2.380e-7	0.7966	36	2.381e-7	4.3325
	256	8	1.658e-8	4.4757	64	1.658e-8	33.2143
	464	10	2.056e-9	18.5542	100	2.057e-9	179.8494
$\alpha_1=0.1,\ \alpha=0.7$	71	4	1.743e-6	0.1580	16	1.744e-6	0.4916
	248	6	1.668e-8	2.1985	36	1.669e-8	10.0185
	601	8	5.994e-10	17.3999	64	5.994e-10	100.2059
	1194	10	4.492e-11	95.8941	100	4.492e-11	917.2806

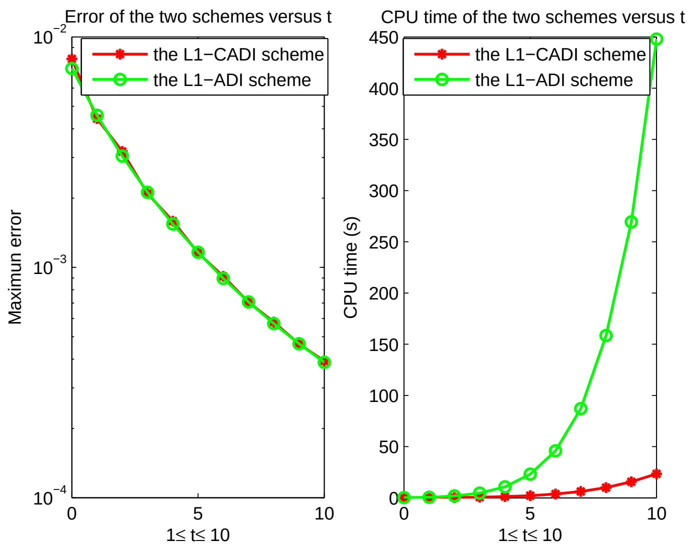

Figure 2: Error and CPU time of the L1-CADI and L1-ADI schemes.

approximation is derived that has fourth-order spatial accuracy [19, 21, 33]. Using some novel techniques, the unique solvability, unconditional stability and global convergence are proved rigorously for both the 1D and 2D cases. Numerical examples verify the effectiveness of the compact difference scheme. These methods and techniques could be extended to other kinds of multi-term time-space fractional equations and to equations with a nonlinear source term. Future work may concentrate on studying the effective numerical schemes for the multi-term fractional time-space equations.

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