Ergod. Th. & Dynam. Sys., (2021), **41**, 3722–3739 © The Author(s), 2020. Published by Cambridge University Press.
doi:10.1017/etds.2020.116

Non-singular $\mathbb{Z}^d$-actions: an ergodic theorem over rectangles with application to the critical dimensions

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(*Received* 6 *June* 2016 *and accepted in revised form* 15 *December* 2019)

Abstract. We adapt techniques developed by Hochman to prove a non-singular ergodic theorem for $\mathbb{Z}^d$-actions where the sums are over rectangles with side lengths increasing at arbitrary rates, and in particular are not necessarily balls of a norm. This result is applied to show that the critical dimensions with respect to sequences of such rectangles are invariants of metric isomorphism. These invariants are calculated for the natural action of $\mathbb{Z}^d$ on a product of d measure spaces.

Key words: classical ergodic theory, group actions, critical dimensions
2020 Mathematics Subject Classification: 37A40 (Primary); 37A35, 37A15 (Secondary)

1. *Introduction*

Let G be a countable group with a non-singular left action on a standard probability space $(X, \mathcal{B}, \mu)$, which is assumed to be ergodic throughout. Each $g \in G$ induces a non-singular map on X which we also denote by g. The measures μ and $\mu \circ g$ are equivalent and so the Radon–Nikodm derivative

$$\omega_g = \frac{d\mu \circ g}{d\mu}$$

is well defined and strictly positive almost everywhere. In turn each $g \in G$ induces a linear isometry on L^1 given by $\hat{g}\phi(x) = \phi(gx)\omega_g(x)$. Note that this is not the usual transfer operator, which in the context of group actions is given by $\phi(g^{-1}x)d(\mu \circ g^{-1})/d\mu$, but fulfils essentially the same role and simplifies notation significantly.

1.1. *Critical dimensions.* For conservative integer actions the Hurewicz ergodic theorem ensures that, for $\phi \in L^1$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \phi(ix)\omega_i(x)}{\sum_{i=1}^{n} \omega_i(x)} = \int \phi \, d\mu$$

almost everywhere. Since the action is conservative, if $\phi > 0$almost surely then both the numerator and the denominator in the theorem diverge to infinity. Therefore the ergodic theorem says that both are diverging at the same rate. This suggests that the growth rate of $\sum_{i=1}^{n} \omega_i$ may encode some intrinsic behaviour of the system. This motivated work by Dooley and Mortiss [3–5, 9] in which they conducted a rigorous study of the growth rate of $\sum_{i=1}^{n} \omega_i$ and created invariants called the upper and lower critical dimensions. We aim to extend this study from the context of $\mathbb{Z}$-actions to those of other countable groups, with $\mathbb{Z}^d$-actions being the focus of this paper. The critical dimensions are defined for a countable group G as follows.

Fix a sequence $e \in B_1 \subseteq B_2 \subseteq \cdots$ of finite subsets of G; we will refer to such a sequence as a *summing sequence*. For $t \in \mathbb{R}$ write

$$L_t = \left\{ x \in X : \liminf_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g \in B_n} \omega_g(x) > 0 \right\}$$

and

$$U_t = \left\{ x \in X : \limsup_{n\to\infty} \frac{1}{|B_n|^t} \sum_{g \in B_n} \omega_g(x) = 0 \right\}.$$

Observe that L_t and U_t are decreasing and increasing respectively with t, and are disjoint.

Definition 1.1. The *lower critical dimension* of (X, μ, G) with respect to summing sequence $\mathcal{S} = \{B_n\}_{n=1}^{\infty}$ is defined by

$$\alpha = \alpha(\mathcal{S}) = \sup\{t : \mu(L_t) = 1\}.$$

The *upper critical dimension* of (X, μ, G) with respect to $\{B_n\}_{n=1}^{\infty}$ is defined by

$$\beta = \beta(\mathcal{S}) = \inf\{t : \mu(U_t) = 1\}.$$

When α and β coincide we let $\gamma = \alpha = \beta$ and call it the *critical dimension*.

Intuitively, the lower critical dimension gives the slowest growth rate of all the subsequences of $\sum_{g\in B_n} \omega_g(x)$, and the upper critical dimension the fastest. It follows from the definitions that $0 \leq \alpha \leq \beta$ and from Fatou's lemma that $\alpha \leq 1$.

For real numbers $a \leq b$ let $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$. When $G = \mathbb{Z}$ the sets B_n are normally taken to be the discrete intervals $[\![1, n]\!] = \{1, \ldots, n\}$ in analogy with the range of the sums in the ergodic theorem. However, in the case of a general countable group there is no such standard choice for B_n. This raises the question: how does the choice of the summing sequence affect the critical dimensions?

We start to address this question in §3. As the summing sequences we consider are almost exclusively symmetric about the origin, for reasons which will become apparent below, we first look at how the critical dimensions of T and T^{-1} with respect to the

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intervals $[\![1, n]\!]$ affect the critical dimension of T with respect to $[\![-n, n]\!]$. We then move on to examine product $\mathbb{Z}^d$-actions on spaces $X = X_1 \times \cdots \times X_d$, where each standard generator e_i acts by applying a transformation $T_i : X_i \to X_i$ to the ith coordinate. We consider the critical dimensions with respect to sequences of *rectangles* $B_n = B_n^1 \times \cdots \times B_n^d$ where each $B_n^i = [\![-s_i(n), s_i(n)]\!]$ for some increasing $s_i : \mathbb{N}_0 \to \mathbb{N}_0$. Note the requirement that these rectangles are symmetric about the origin. For each $1 \leq i \leq d$ we write γ_i for the single critical dimension (if it exists) of T_i with respect to $[\![-n, n]\!]$. Our main result in this section, Theorem 3.4, shows that for these actions the critical dimension $\gamma(\mathcal{S})$ of the product action is a weighted average of the γ_i, with weightings determined by relative growth rates of the functions s_i.

THEOREM 1.2. *Suppose that for an increasing function $s : \mathbb{N} \to \mathbb{N}$ the limits $c_i = \lim_{n\to\infty}(\log s_i(n)/\log s(n))$ exist, and that one of these is non-zero. Then*

$$\gamma(\mathcal{S}) = \frac{\sum_{i=1}^{d} c_i \gamma_i}{\sum_{i=1}^{d} c_i}.$$

A pair of illustrative applications of this result in the case $d = 2$ are that

$$\gamma(\mathcal{S}_{\text{sq}}) = \frac{\gamma_1 + 2\gamma_2}{3} \quad \text{and} \quad \gamma(\mathcal{S}_{\text{exp}}) = \gamma_2$$

where $\mathcal{S}_{\text{sq}} = [\![-n, n]\!] \times [\![-n^2, n^2]\!]$ and $\mathcal{S}_{\text{exp}} = [\![-n, n]\!] \times [\![-\lfloor e^n - 1\rfloor, \lfloor e^n - 1\rfloor]\!]$.

For integer actions, the first and simplest demonstration of the intrinsic nature of the critical dimensions is due to Mortiss who proved that when $B_n = [\![1, n]\!]$ they are invariants of metric isomorphism [9].

Definition 1.3. Two non-singular actions of a group G on probability spaces (X, μ) and (X', μ') are *metrically isomorphic* if there exist null sets $X_0 \subset X$, $X_0' \subset X'$ and a bimeasurable map $\phi : X\backslash X_0 \to X'\backslash X_0'$ such that $\mu' \circ \phi \sim \mu$ and for all $g \in G$ we have $\phi(gx) = g\phi(x)$ almost everywhere.

One of the purposes of this paper is to show that the same holds for $\mathbb{Z}^d$-actions when the B_n are given by certain rectangles. More precisely, we take the B_n to be integer balls of a metric of the form

$$\rho(u, v) = \max_{1\leq i\leq d} F_i(|u_i - v_i|) \tag{1.1}$$

where each $F_i : [0, \infty) \to [0, \infty)$ satisfies $F_i(0) = 0$, is subadditive and strictly increasing. The first two properties ensure ρ is a metric and the latter guarantees that F_i has an inverse, which we denote by f_i, and which is superadditive on $[0, \infty)$. We call such a metric ρ on $\mathbb{Z}^d$ *rectangular*. For a subset $S \subseteq \mathbb{Z}^d$ with metric ρ we say ρ is a *rectangular metric on S* if it is the restriction of a rectangular metric on $\mathbb{Z}^d$ to S. In this case we say (S, ρ) is a *rectangular metric space*.

We will refer to the balls $B_r(z)$ of rectangular metrics as *rectangular balls*. We assume that rectangular balls carry the information of their radius with them, and observe that the centre can then be determined from the ball. Rectangular metrics are translation invariant

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and so $B_r(z) = z + B_r$ where

$$B_r = B_r(0) = \prod_{i=1}^{d} [\![-\lfloor f_i(r)\rfloor, \lfloor f_i(r)\rfloor]\!].$$

We will mainly be focused on rectangular balls with $r \in \mathbb{N}_0$. We call a summing sequence $B_1 \subseteq B_2 \subseteq \cdots$ *rectangular* if it is constructed in this way for some rectangular metric.

Restricting temporarily to $d = 2$ by taking $F_1(s) = s$ and $F_2(s) = \sqrt{s}$ or $F_2(s) = \log(1 + s)$, we define rectangular metrics with integer balls $[\![-n, n]\!] \times [\![-n^2, n^2]\!]$ and $[\![-n, n]\!] \times [\![-\lfloor e^n - 1\rfloor, \lfloor e^n - 1\rfloor]\!]$, respectively. In particular, the critical dimensions calculated earlier are associated to rectangular summing sequences, and will be seen to be invariants.

Mortiss's proof that the critical dimensions are invariants of metric isomorphism made use of the ergodic theorem.

1.2. *Ergodic theorem.* Given an ergodic action of a group G on a probability space (X, μ) and a summing sequence $B_1 \subseteq B_2 \subseteq \cdots$ of finite subsets of G, the ergodic theorem is satisfied if for every integrable function ϕ,

$$\lim_{n\to\infty} \frac{\sum_{g\in B_n} \hat{g}\phi}{\sum_{g\in B_n} \hat{g}1} = \int \phi \, d\mu$$

almost everywhere.

For non-singular actions of countable groups the question of when the ergodic theorem holds is an area of current research. The foremost positive result is due to Hochman [7], who proved it holds for free, non-singular and ergodic $\mathbb{Z}^d$-actions and $B_n = \{u \in \mathbb{Z}^d : \|u\| \le n\}$ where $\|\cdot\|$ is a norm on $\mathbb{R}^d$. Crucially, this does not include the case where (B_n) is rectangular because the $f_i(n)$ may have completely different orders of growth. For example, it excludes both the sequences $\mathcal{S}_{\text{sq}}$ or $\mathcal{S}_{\text{exp}}$ for $\mathbb{Z}^2$. To apply the arguments of Mortiss verbatim it is therefore necessary to show the ergodic theorem extends to rectangular summing sequences.

This requires care because there are natural choices of B_n for which the ergodic theorem is known to fail. One such, due to Brunel and Krengel [8], shows the ratio ergodic theorem (a consequence of the ergodic theorem in this context) fails for $B_n = [\![0, n]\!]^d$ and $d > 1$. The generally cited reason for this failure is that the sets $[\![0, n]\!]^d$ fail to satisfy the Besicovitch property, as defined in [7]. However, as noted in [2], sequences of symmetric rectangles with increasing side lengths do have the Besicovitch property.

Prior to Hochman's work, Feldman [6] used a simpler method to prove a weaker result; it was assumed that each of the standard generators $e_1, \ldots, e_d$ of $\mathbb{Z}^d$ acted conservatively on X and $\|\cdot\|$ was taken to be the supremum norm on $\mathbb{R}^d$. Both methods follow the standard approach: one produces a dense subset of L^1 for which the theorem holds and then applies a maximal inequality to extend this to all of L^1.

Upon a quick examination of Feldman's proof it becomes apparent that the sets $B_n = [\![-n, n]\!]^d$ can be replaced by the rectangles $\prod_{i=1}^{d} [\![-s_i(n), s_i(n)]\!]$ in the production of an appropriate dense set of functions. The maximal inequality is then proved using

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two properties. The first is that balls of norms in $\mathbb{R}^d$ satisfy the Besicovitch property; see [2] for a proof. The second is that they satisfy the doubling condition, $|B_{2n}| \leq C|B_n|$, for some fixed constant C. We have already noted that our rectangles satisfy the Besicovitch property. Moreover, rectangles B_n satisfy an additive version of the doubling condition,

$$|2B_n| = |B_n + B_n| \leq 2^d |B_n|, \tag{1.2}$$

where for rectangles B_n and $\lambda \in \mathbb{N}$ we let $\lambda B_n = \prod_{i=1}^{d} [\![-\lambda s_i(n), \lambda s_i(n)]\!]$. This coincides with the sum of λ copies of B_n. Note that when (B_n) is rectangular the translation invariance of the metric ensures that $\lambda B_n \subseteq B_{\lambda n}$. These sets are very different; even in the simple case above with $F_1(s) = s$ and $F_2(s) = \log(1+s)$ we have $2B_n = [\![-2n, 2n]\!] \times [\![-2\lfloor e^n - 1 \rfloor, 2\lfloor e^n - 1 \rfloor]\!]$ and $B_{2n} = [\![-2n, 2n]\!] \times [\![-\lfloor e^{2n} - 1 \rfloor, \lfloor e^{2n} - 1 \rfloor]\!]$. This example exhibits that we cannot just use the metric doubling condition.

Despite using (1.2), one can still deduce that the maximal inequality holds for rectangles. We explain this modification in §2. This means that Feldman's result can be extended so that the sums can be taken over rectangles.

This extension is, in fact, sufficient to show that the critical dimensions taken with respect to rectangles are invariants of metric isomorphism *between product* $\mathbb{Z}^d$*-actions* of the form considered in Theorem 1.2. This is because for actions of this type ergodicity of the whole action implies ergodicity of the T_i, with respect to the marginals, which in turn gives the conservativity of the generators as required by Feldman. However, we would like to show the dimensions are invariants independent of the form of the measure space and the particular group action.

It is then natural to ask whether similar changes can be made to Hochman's method for producing a dense set of functions. His approach consistently views $\mathbb{Z}^d$ as a translation-invariant metric space, which is our reason for considering rectangular metrics. It also makes use of both the doubling and Besicovitch properties to produce the appropriate dense set of functions, in addition to a type of finite-dimensionality property of $\mathbb{Z}^d$ with respect to balls of norms. In §2 we will set out how one can use (1.2) and a corresponding finite-dimensionality property with respect to rectangular metrics to adapt Hochman's method to prove the following ergodic theorem.

THEOREM 1.4. *Let* $\mathbb{Z}^d$ *have a non-singular and ergodic action on a probability space* (X, μ) *and* $B_n = \{u \in \mathbb{Z}^d : \rho(u, 0) \leq n\}$ *for some rectangular metric* ρ *on* $\mathbb{Z}^d$*. Then for every* $\phi \in L^1$*, as* $n \to \infty$*,*

$$\frac{\sum_{u \in B_n} \hat{u}\phi}{\sum_{u \in B_n} \hat{u}1} \to \int \phi \, d\mu.$$

With this result in hand, the arguments of Mortiss can be applied to see that the critical dimensions of summing sequences of rectangles are invariants of metric isomorphism.

COROLLARY 1.5. *The upper and lower critical dimensions with respect to any summing sequence of balls* $B_n = \{u \in \mathbb{Z}^d : \rho(u, 0) \leq n\}$*, for some rectangular metric* ρ*, are invariants of metric isomorphism.*

2. *The ergodic theorem for rectangles*

In the standard proof for ergodic theorems there are two key ingredients. The first is a maximal inequality. For $\phi \in L^1(X)$ let

$$R_n\phi(x) = \frac{\sum_{g\in B_n} \phi(gx)\omega_g(x)}{\sum_{g\in B_n} \omega_g(x)}.$$

The maximal inequality holds if there exists $C > 0$ such that for any $\phi \in L^1$ and $\epsilon > 0$,

$$\mu\Big(\sup_{n\geq 1} |R_n\phi| > \epsilon\Big) \leq \frac{C}{\epsilon}\|\phi\|_1.$$

The second key ingredient is a dense subset H of L^1 such that for all $h \in H$ and all $\sigma \in G$,

$$\frac{\sum_{g\in B_n\setminus\sigma B_n} \hat{g}h - \sum_{g\in\sigma B_n\setminus B_n} \hat{g}h}{\sum_{g\in B_n} \hat{g}h} \to 0 \tag{2.1}$$

almost surely.

The latter condition is used to show that the ergodic theorem holds for functions in the set

$$D = \operatorname{span}\{c + h - \hat{\sigma}h : c \in \mathbb{R}, \sigma \in G, h \in H\}.$$

The maximal inequality is used to extend the convergence of $R_n\phi$ to the closure of D in L^1. It then suffices to show that D is dense in L^1. The details of this argument can be found in [1].

In the case where $H = L^\infty$ condition (2.1) is implied by

$$\frac{\sum_{g\in B_n\triangle\sigma B_n} \omega_g}{\sum_{g\in B_n} \omega_g} \to 0 \quad \text{almost surely,} \tag{nsFC}$$

which we call the *non-singular Følner condition*. For measure-preserving actions this reduces to the standard Følner condition for the sequence B_n, implying that G is amenable. For integer actions, if $B_n = [\![1, n]\!]$ then (nsFC) follows from the Chacon–Ornstein lemma, as in [1], and the assumption that the action is conservative. Hochman's variant of the Chacon–Ornstein lemma in [7], summing over balls of norms, also implies (nsFC).

It should be noted that Feldman's argument shows (2.1) directly for a smaller dense set than L^∞, rather than via (nsFC).

To see that the maximal inequality holds for $B_n = \{u \in \mathbb{Z}^d : \rho(u, 0) \leq n\}$ with ρ a rectangular metric we refer the reader to a concise proof of the maximal inequality for balls of norms as given in [6, Inequality 5.3], attributed to Aaronson and Becker. Upon examining this proof the reader will observe that the same argument, with two changes, goes through for rectangles. The first is that to apply the Besicovitch property in the proof of Inequality 5.2 one needs to intersect with a finite subset; this can be taken arbitrarily large at the end of the proof. That rectangular metric spaces have the Besicovitch covering property follows from a comment in [2, pp. 7]. The second is that one replaces each occurrence of B_{2n} with $2B_n$, and then applies the modified doubling condition (1.2).

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2.1. *The non-singular Følner condition.* With the maximal inequality in hand it is sufficient to show that (nsFC) holds. We will directly adapt the approach in [7]. First we will briefly explain the changes which need to be made to Hochman's approach, and then we show that rectangular metrics have finite coarse dimension, the property lying at its foundation.

2.1.1. *Modifications.* In order to apply the arguments from [7] to rectangular metric spaces one needs to make two changes and then check that these do not affect the nature of the rest of the argument.

The first is that, as with the maximal inequality, wherever the metric doubling condition is applied, rather than considering balls $B_{\lambda r}(z)$ one instead looks at the rectangle $z + \lambda B_r$ and applies (1.2). This is required because rectangular metrics do not necessarily satisfy the metric doubling condition. Specifically, this change needs to be made to the proofs of [7], and it goes through essentially because the latter set is contained by the former.

The second is a change in the notion of a 'thickened sphere'. In Hochman's paper, where the metrics were norms, these thickened spheres are given by the sets $B_{r+t} \backslash B_{r-t}$ for $t \leq r$. The idea here is that t is very small compared to r, and so the thickened sphere looks almost like the sphere ∂B_r. In our situation this appears not to be the correct definition. For example, if one considers the case where one side of rectangle is growing exponentially and takes $t = \log 2$ then for large radii the thickened sphere, which is meant to be a slight thickening of the boundary, would consist of more than half of the points in the rectangle. Instead we take the following definition which emulates the behaviour in the case where the metric is given by a norm.

When $S = \mathbb{Z}^d$, for rectangular balls $B = z + \prod_{i=1}^{d} [\![-\lfloor f_i(r) \rfloor, \lfloor f_i(r) \rfloor]\!]$, let ∂B denote the set of points in $\mathbb{Z}^d$ which lie in the usual topological boundary of $z + \prod_{i=1}^{d} [-\lfloor f_i(r) \rfloor, \lfloor f_i(r) \rfloor]$ in $\mathbb{R}^d$, and call these sets *boxes*. Another perspective is that the box associated to a rectangle is the collection of points for which some coordinate takes the maximum or minimum value in that coordinate over the rectangle.

For $t \in \mathbb{N}$ we define the *t-boundary* $\partial_t B$ to be the collection of $z \in \mathbb{Z}^d$ which lie within distance t of ∂B with respect to the rectangular metric. Equivalently,

$$\partial_t B = \bigcup_{u \in \partial B} (u + B_t).$$

When $S \subseteq \mathbb{Z}^d$ we take ∂B and $\partial_t B$ to be the intersections of their $\mathbb{Z}^d$ counterparts with S. We refer to a collection of t-boundaries, possibly with different values of t, as *thick boxes*.

The impact of this change is seen in two ways. One is that these boundaries appear in the statements of and arguments for [7] and so one needs to check that these are unaffected. The key point here is that the only property of thickened boundaries used in these proofs is that it contains all the points within a given distance of the topological boundary, which is the definition we have taken above. The other change is to the definition of the course dimension. In the following section we show that rectangular metrics satisfy this slightly changed definition. Theorem 1.4 then follows by applying Hochman's arguments.

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2.1.2. *Coarse dimension.* What follows is a restatement of [7] but for rectangular metrics.

Definition 2.1. For a rectangular metric space S and $R > 1$ the relation $\mathrm{cdim}_R S = k$ (read: S has coarse dimension k at scales R or greater) is defined by recursion on k as follows:

(i) $\mathrm{cdim}_R S = -1$ for $S = \emptyset$ and any R;

(ii) $\mathrm{cdim}_R S$ is the minimum integer k for which $\mathrm{cdim}_{tR} \partial B_r(s) \leq k - 1$ for any $t \geq 1$, $r \geq tR$ and $s \in S$.

The only difference between this definition and the one in [7] is the change in the definition of the t-boundary.

The following proposition will be useful in the proof that $\mathbb{Z}^d$ has finite coarse dimension with respect to the redefined boundary and rectangular metrics.

For $e \in \{\pm e_i : 1 \leq i \leq d\}$ let $F_{r,u}(e)$ be the face of $B_r(u) = u + \prod_{i=1}^d [\![-\lfloor f_i(r) \rfloor, \lfloor f_i(r) \rfloor]\!]$ in direction e from u, that is, those points in $B_r(u)$ whose projection onto e is maximal. The *face* of the thickened boundary $\partial_t B_r(u)$ in direction e is the set of points within distance t of $F_{r,u}(e)$ and is denoted by $\partial_t F_{r,u}(e)$.

PROPOSITION 2.2. *Let $(\mathbb{Z}^d, \rho)$ be a rectangular metric space. Then there are $R = R(\rho) > 5n$, where $n \in \mathbb{N}$ satisfies $nf_i(1) \geq 1$ for all $i \in [\![1, d]\!]$, and $k \in \mathbb{N}$ with the following property: given $z_1, \ldots, z_k \in \mathbb{Z}^d$, $t(1), \ldots, t(k) \geq 1$ and a decreasing sequence $r(1), \ldots, r(k)$ with $r(k) \geq t(1) \cdots t(k)R$ such that $z_i \in \bigcap_{j<i} \partial_{t(j)} B_{r(j)}(z_j)$, then*

$$\bigcap_{i=1}^{k} \partial_{t(i)} B_{r(i)}(z_i) = \emptyset.$$

Proof. For notational clarity we write $r_i = r(i)$ and $t_i = t(i)$ in this proof.

We use induction on the d to prove that there is $k = k(d)$ with the required property. With d and the metric ρ fixed, we may then choose $R > 5n$ with $n \in \mathbb{N}$ chosen large enough for $nf_i(1) \geq 1$ for all $i \in [\![1, d]\!]$.

For $d = 1$ let $k = 2$. Let $f = f_1$. The set $\partial_{t(1)} B_{r(1)}(z_1)$ is a union of two closed intervals length $2\lfloor f(t_1) \rfloor + 1$ centred on $\pm\lfloor f(r_1) \rfloor$, respectively. These intervals are disjoint as $r(1) > t(1)$. We may assume z_2 lies in the interval centred on $-\lfloor f(r_1) \rfloor$. Now since $R > 5n$ we have

$$\lfloor f(r_2) \rfloor > f(r_2) - 1 \geq f(2t_1 + t_2 + 2n) - 1 \geq 2\lfloor f(t_1) \rfloor + \lfloor f(t_2) \rfloor + 1,$$

using superadditivity of f and the choice of n. In particular, $\partial_{t(2)} B_{r(2)}(x_2)$ does not intersect the interval centred on $-\lfloor f(r_1) \rfloor$.

Also

$$\lfloor f(r_2) \rfloor + \lfloor f(t_2) \rfloor < 2(\lfloor f(r_1) \rfloor - \lfloor f(t_1) \rfloor),$$

otherwise, using $R > 5n$ and the fact that the $r(i)$ are decreasing,

$$\begin{aligned} 2\lfloor f(t_1) \rfloor + \lfloor f(t_2) \rfloor &\geq 2\lfloor f(r_1) \rfloor - \lfloor f(r_2) \rfloor \\ &\geq \lfloor f(r_1) \rfloor \geq f(2t_1 + t_2 + 2n) - 1 > 2\lfloor f(t_1) \rfloor + \lfloor f(t_2) \rfloor. \end{aligned}$$

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This means that $\partial_{t(2)} B_{r(2)}(z_2)$ also does not intersect the interval centred on $+\lfloor f(r_1)\rfloor$, and the claim follows.

Now assume we have proved $k(d-1)$ exists. Suppose $k \geq 2dk(d-1)+2$. By the pigeonhole principle the thickening of some face $F(e)$ of $B_{r(1)}(z_1)$ contains $k(d-1)+1$ of the points $z_2, \ldots, z_{k(d)}$. As these are the only points used, we may henceforth assume they are $z_2, \ldots, z_{k(d-1)+2}$. Using essentially the argument from the initial step, the thickened faces in directions $\pm e$ of each $\{\partial_{t(i)} B_{r(i)}(z_i)\}_{i=2}^{2k(d-1)+2}$ cannot intersect the thickened faces $F(\pm e)$ of $\partial_{t(1)} B_{r(1)}(z_1)$. Therefore the $\partial_{t(i)} B_{r(i)}(z_i)$ intersect in $\partial_t F(e)$ only if the projections of $\partial_{t(i)} B_{r(i)}(z_i) \cap \partial_{t(1)} F(e)$ along e onto $F(e)$ intersect. These projections are exactly thick boxes for projection of our rectangular metric in direction e, so we may apply the previous case to deduce that

$$\partial_{t(1)} F(e) \cap \bigcap_{i=2}^{k(d-1)+1} \partial_{t(i)} B_{r(i)}(z_i) = \emptyset,$$

but by assumption $z_{k(d-1)+2}$ lies in that intersection. Hence $k < 2dk(d-1)+2$ and so $k(d) \leq 2dk(d-1)+1$. $\square$

Using the above we are able to prove the following claim.

PROPOSITION 2.3. *$\mathbb{Z}^d$ has finite coarse dimension with respect to any rectangular metric.*

Proof. As before, we write $r_i = r(i)$ and $t_i = t(i)$ in this proof.

Let $R = R(\rho)$ and $k' = k$ from the previous proposition. Let $k'' \in \mathbb{N}$, to be determined, and $k = k'k'' + 1$. In order to show $\mathbb{Z}^d$ has finite coarse dimension it suffices to show that if we are given

(1) $t(1), \ldots, t(k) \geq 1$,
(2) $r(1), \ldots, r(k)$ such that $r(i) \geq t(1) \cdots t(k) R$, and
(3) points $z_1, \ldots, z_k \in \mathbb{Z}^d$ such that $z_i \in \bigcap_{j<i} \partial_{t(j)} B_{r(j)}(z_j)$ for $j < i$,

then $\bigcap_{i=1}^{k} \partial_{t(i)} B_{r(i)}(z_i) = \emptyset$.

By the previous proposition it suffices to find a subsequence of length k' for which the radii are decreasing. Consider the points $z_2, \ldots, z_l$ $(l \geq 2)$ and suppose $r(j) > r(1)$ for each $2 \leq j \leq l$. Each of these points lies inside $\partial_{t(1)} B_{r(1)}(z_1)$, by assumption. Moreover, if $i > j$ then

$$\begin{aligned} z_j &\notin z_i + \prod_{m=1}^{d} (-\lfloor f_m(r_i)\rfloor + \lfloor f_m(t_i)\rfloor, \lfloor f_m(r_i)\rfloor - \lfloor f_m(t_i)\rfloor) \\ &\supseteq z_i + \prod_{m=1}^{d} (-\lfloor f_m(r_1)\rfloor + \lfloor f_m(r_1/R)\rfloor, \lfloor f_m(r_1)\rfloor - \lfloor f_m(r_1/R)\rfloor). \end{aligned}$$

Let $A = \mathbb{Z}^d \cap \prod_{m=1}^{d} (-\lfloor f_m(r_1)\rfloor + \lfloor f_m(r_1/R)\rfloor, \lfloor f_m(r_1)\rfloor - \lfloor f_m(r_1/R)\rfloor)$. The final line implies that we also have $z_i \notin z_j + A$. Now, $z_2, \ldots, z_l$ is a collection of points contained by $B = \partial_{t(1)} B_{r(1)}(z_1) \cup B_{r(1)}(z_1)$ such that $z_i \notin z_j + A$ for all $i \neq j$. Then the sets $z_j + \frac{1}{2}A$ are disjoint and each $B \cap (z_j + \frac{1}{2}A)$ contains at least one orthant of $z_j + \frac{1}{2}A$,

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and hence at least

$$\prod_{m=1}^{d} \left\lfloor \frac{1}{2}(\lfloor f_m(r_1)\rfloor - \lfloor f_m(r_1/R)\rfloor - 1) \right\rfloor$$

points. By the disjointness we must have

$$(l-1)\prod_{m=1}^{d} \left\lfloor \frac{1}{2}(\lfloor f_m(r_1)\rfloor - \lfloor f_m(r_1/R)\rfloor - 1) \right\rfloor \leq \prod_{m=1}^{d}(2(\lfloor f_m(r_1)\rfloor + \lfloor f_m(t_1)\rfloor) + 1),$$

that is,

$$l \leq 1 + 2^d \prod_{m=1}^{d} \frac{2(\lfloor f_m(r_1)\rfloor + \lfloor f_m(r_1/R)\rfloor) + 1}{\lfloor f_m(r_1)\rfloor - \lfloor f_m(r_1/R)\rfloor - 3}.$$

Dividing through each fraction by $\lfloor f_m(r_1)\rfloor$ and recalling from the previous proposition that $R = R(\rho) > 5n$, where $n \in \mathbb{N}$ satisfies $nf_l(1) \geq 1$ for all $l \in [\![1, d]\!]$, we see that $f_m(r_1) \geq f_m(5n) \geq 5$ (from superadditivity). By using this in addition to the fact that the f_m are increasing and superadditive we see that

$$\frac{\lfloor f_m(r_1/R)\rfloor}{\lfloor f_m(r_1)\rfloor} \leq \frac{\lfloor f_m(r_1/5)\rfloor}{5\lfloor f_m(r_1/5)\rfloor - 1} \leq \frac{1}{4}$$

and so

$$l \leq 1 + 2^d \prod_{m=1}^{d} \frac{2(1 + 1/4) + 1/5}{1 - 1/4 - 3/5} \leq 36^d + 1.$$

Therefore if we take $k'' > 36^d + 1$, then $r(j) \leq r(1)$ for some $2 \leq j \leq k''$. We can then repeat this process with $r(j)$ and so on to find a subsequence with decreasing radii satisfying the conditions, which will have length at least k' by our choice of k. □

The proof of this proposition concludes the summary of the more significant changes which it is necessary to make to Hochman's work [7]. The remainder of the argument can be concluded as in that paper from Theorem 4.4 onwards, as outlined earlier in this section.

3. *Critical dimension for $\mathbb{Z}^d$-actions*

We now have a varied collection of summing sequences in $\mathbb{Z}^d$ for which the ergodic theorem holds, and hence for which the critical dimensions are invariants of metric isomorphism. In this section we restrict attention to these sequences in order to address the first question raised in the introduction: how do α and β depend on the choice of summing sequence?

3.1. *Critical dimension for symmetric summing sets in $\mathbb{Z}$.*

The integer theory predominantly sums over the sets $[\![1, n]\!]$. It will be useful to examine what the critical dimension of a $\mathbb{Z}$-action with respect to $[\![1, n]\!]$ says about the critical dimension with respect to $[\![-n, n]\!]$.

Let $T : X \to X$ be a non-singular transformation describing a $\mathbb{Z}$-action. We shall refer to the critical dimensions of T with the summing sets $[\![1, n]\!]$ as *standard* and denote the

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lower and upper standard critical dimensions by α_+ and β_+, respectively. We will denote the lower and upper standard critical dimensions of T^{-1} by α_- and β_-. Let L_t^+, L_t^- denote L_t for T and T^{-1} respectively, with the standard summing sets, and similarly with U_t.

LEMMA 3.1. *Let $T : X \to X$ determine a non-singular $\mathbb{Z}$-action. Let α and β be the critical dimensions with respect to $[\![-n, n]\!]$. Then*

$$\max(\alpha_+, \alpha_-) \leq \alpha \leq \beta \leq \max(\beta_+, \beta_-).$$

Proof. We first prove the result for the lower critical dimension. Observe that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{(2n+1)^t} \sum_{i=-n}^{n} \omega_i(x) &= \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x) \\ &\geq \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{2^t} \liminf_{n\to\infty} \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x).\end{aligned}$$

Hence $L_t \supseteq L_t^+ \cup L_t^-$ and the result follows. In the other case we get

$$\limsup_{n\to\infty} \frac{1}{(2n+1)^t} \sum_{i=-n}^{n} \omega_i(x) \leq \frac{1}{2^t} \limsup_{n\to\infty} \frac{1}{n^t} \sum_{i=-n}^{-1} \omega_i(x) + \frac{1}{2^t} \limsup_{n\to\infty} \frac{1}{n^t} \sum_{i=1}^{n} \omega_i(x).$$

Therefore $U_t \supseteq U_t^+ \cap U_t^-$ and we are done. □

In particular, if the standard upper and lower critical dimensions of T agree and those of T^{-1} do also then $\alpha = \max(\alpha_+, \alpha_-) = \beta$.

The following theorem of Mortiss and Dooley provides a number of situations where the upper and lower critical dimensions with respect to $[\![1, n]\!]$ of a transformation T, and those of its inverse, agree.

THEOREM 3.2. (See [5]) *Let T denote the odometer transformation on the space $(\prod_{i=1}^{\infty} \mathbb{Z}_2, \prod_{i=1}^{\infty} \mu_i)$. Then the lower and upper critical dimensions are given by*

$$\alpha = \liminf_{n\to\infty} -\frac{1}{n} \sum_{i=1}^{n} \log_2 \mu_i(x_i) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(\mu_i)$$

and

$$\beta = \limsup_{n\to\infty} -\frac{1}{n} \sum_{i=1}^{n} \log_2 \mu_i(x_i) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(\mu_i)$$

almost everywhere, where $H(\mu_i) = -\sum_{j=0}^{1} \mu_i(j) \log_2(\mu_i(j))$, the entropy of the measure μ_i.

The entropy $H(\mu)$ of the measure μ on $\{0, 1\}$ can be chosen to take any value between 0 and 1, by varying $p \in (0, 1)$ where $\mu(0) = p$. It is clear that for many choices of product measure $\prod_{i=1}^{\infty} \mu_i$ the sequence $(1/n) \sum_{i=1}^{n} H(\mu_i)$ converges as $n \to \infty$. In this case the upper and lower critical dimensions are equal. Moreover, any value in (0, 1) can be achieved by the dimensions.

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Another consequence of this theorem is that for an odometer action T on $(\prod_{i=1}^{\infty}\mathbb{Z}_2, \prod_{i=1}^{\infty}\mu_i)$ the inverse T^{-1} has the same upper and lower critical dimensions as T. This follows from how T^{-1} can also be considered as an odometer on the same space, with the roles of 0 and 1 reversed, and the fact that $H(\mu_i) = H(\nu_i)$ where $\nu_i(0) = 1 - \mu_i(0)$.

These observations, combined with Lemma 3.1, ensure we can produce examples of transformations with a single critical dimension $\alpha = \beta = \gamma$ with respect to $[\![-n, n]\!]$ for any $\gamma \in (0, 1)$.

3.2. *Critical dimension for balls of norms.* In this part we show that the critical dimensions for balls of a norm are independent of the choice of norm.

Let $B_r = B_r(0) \cap \mathbb{Z}^d$ where $B_r(0)$ is the closed ball of radius r with respect to a given norm $\|\cdot\|$, and let B_r' denote the corresponding set for another norm $\|\cdot\|'$. We consider the summing sequences (B_n) and (B_n'). The proof relies on essentially two properties of these sequences, which we will make precise below. The first is that any two sequences of balls are intertwined, in the sense that each ball is contained by a sufficiently large ball in the other sequence. The second property is that each ball is somewhat well approximated from above and below by balls in the other sequence.

The ideas used here make sense in a general countable group G, as in the introduction, so we temporarily return to that setting.

Let each of $\{A_n\}_{n=1}^{\infty}$ and $\{A_n'\}_{n=1}^{\infty}$ be an increasing sequence of subsets of G. We say that $\{A_n\}_{n=1}^{\infty}$ *overlays* $\{A_n'\}_{n=1}^{\infty}$ if for all $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $A_n' \subseteq A_N$. We say that $\{A_n\}_{n=1}^{\infty}$ and $\{A_n'\}_{n=1}^{\infty}$ are *interweaving* if both $\{A_n\}_{n=1}^{\infty}$ overlays $\{A_n'\}_{n=1}^{\infty}$ and vice versa. In particular, this is the case if $\bigcup_n A_n = G = \bigcup_n A_n'$, as is the case for the sequences of balls in $\mathbb{Z}^d$ described above.

Suppose $\{A_n\}_{n=1}^{\infty}$ overlays $\{A_n'\}_{n=1}^{\infty}$. Let

$$m(n) = \max(k \geq 0 : A_k' \subseteq A_n) \quad \text{and} \quad M(n) = \min(k \geq 0 : A_n \subseteq A_k')$$

where for technical reasons we take $A_0' = \emptyset$. Then both $m(n)$ and $M(n)$ are increasing with n and diverge as $n \to \infty$. We say $\{A_n\}_{n=1}^{\infty}$ *closely overlays* $\{A_n'\}_{n=1}^{\infty}$ if there exists $\delta \in (0, 1)$ such that for all n sufficiently large,

$$\min\left\{\frac{|A_{m(n)}'|}{|A_n|}, \frac{|A_n|}{|A_{M(n)}'|}\right\} \geq \delta.$$

Similarly, we say two interweaving sequences $\{A_n\}_{n=1}^{\infty}$ and $\{A_n'\}_{n=1}^{\infty}$ are *closely interweaving* if $\{A_n\}_{n=1}^{\infty}$ closely overlays $\{A_n'\}_{n=1}^{\infty}$ and vice versa. This defines an equivalence relation between these sequences of subsets of G.

To see that two sequences of norm balls are closely interweaving, take $\|\cdot\|$ to be the supremum norm and observe that, by equivalence of norms, for some $k \in \mathbb{N}$ we have $B_{r/k} \subseteq B_r' \subseteq B_{kr}$ for all $r > 0$. It follows that

$$\frac{|B_{m(n)}'|}{|B_n|} \geq \frac{|B_{\lfloor n/k \rfloor}|}{|B_n|} = \left(\frac{2n/k + 1}{2n + 1}\right)^d \to k^{-d}$$

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and

$$\frac{|B_{m'(n)}|}{|B'_n|} \geq \frac{|B_{\lfloor n/k \rfloor}|}{|B_{nk}|} \geq \left(\frac{2n/k+1}{2nk+1}\right)^d \to k^{-2d},$$

which deals with the conditions on $m(n)$ and its counterpart. A similar argument applies for $M(n)$, ensuring that every sequence of balls closely interweaves with those of the supremum norm, which suffices due to transitivity.

PROPOSITION 3.3. *Let G be a countable group with a non-singular ergodic action on a standard finite measure space (X, μ). Suppose that $\{A_n\}_{n=1}^{\infty}$ closely overlays $\{A'_n\}_{n=1}^{\infty}$. Then $L'_t \subseteq L_t$ and $U'_t \subseteq U_t$. Hence $\alpha' \leq \alpha \leq \beta \leq \beta'$ and, in particular, when the two sequences are closely interweaving they have the same upper and lower critical dimensions.*

Proof. We just tackle the lower case as the upper case is a similar argument involving the function $M(n)$ and $M'(n)$. Observe that with N taken sufficiently large, for all $n \geq N$,

$$\begin{aligned}\frac{1}{|A_n|^t}\sum_{g\in A_n}\omega_g(x) &\geq \left(\frac{|A'_{m(n)}|}{|A'_n|}\right)^t \frac{1}{|A'_{m(n)}|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\\ &\geq \delta^{|t|}\frac{1}{|A'_{m(n)}|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\end{aligned}$$

and hence

$$\begin{aligned}\inf_{n\geq N}\frac{1}{|A_n|^t}\sum_{g\in A_n}\omega_g(x) &\geq \delta^{|t|}\inf_{n\geq N}\frac{1}{|A'_{m(n)}|^t}\sum_{g\in A'_{m(n)}}\omega_g(x)\\ &\geq \delta^{|t|}\inf_{n\geq m(N)}\frac{1}{|A'_n|^t}\sum_{g\in A'_n}\omega_g(x).\end{aligned}$$

By letting $N \to \infty$, and recalling that $m(N) \to \infty$ as $n \to \infty$, it follows that

$$\liminf_{n\to\infty}\frac{1}{|A_n|^t}\sum_{g\in A_n}\omega_g(x) \geq \delta^{|t|}\liminf_{n\to\infty}\frac{1}{|A'_n|^t}\sum_{g\in A'_n}\omega_g(x)$$

and hence $L'_t \subseteq L_t$. The same argument holds with the sequences exchanged. □

It is an immediate consequence that every sequence of balls of norms produces the same critical dimension.

As one might expect, it is not difficult to see that the sequences *closely* interweaving is necessary to the above argument. Consider, for example, the sequences $A'_n = [\![-n, n]\!]^2$ and $A_n = [\![-\lfloor e^n - 1\rfloor, \lfloor e^n - 1\rfloor]\!] \times [\![-n, n]\!]$ in $\mathbb{Z}^2$. We have $m(n) = n$ and hence

$$\frac{|A'_{m(n)}|}{|A_n|} = \frac{(2n+1)^2}{(2n+1)(2\lfloor e^n - 1\rfloor + 1)} \to 0.$$

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This means that the argument used in the above proof fails if one attempts to compare balls of arbitrary rectangular metrics to those of norms. Next we show that these sequences give rise to different critical dimensions for numerous actions.

3.3. *Critical dimension for product measure spaces.* We examine non-singular product actions, which are constructed as follows. Suppose that for each $1 \le i \le d$ we are given a non-singular transformation $T_i : X_i \to X_i$ on a probability space (X_i, μ_i), the factors of the product. We can define a non-singular $\mathbb{Z}^d$-action on the product measure space $X = X_1 \times \cdots \times X_d$ with measure $\mu = \mu_1 \times \cdots \times \mu_d$ via

$$(u_1, \ldots, u_d) \cdot (x_1, \ldots, x_n) = (T_1^{u_1} x_1, \ldots, T_d^{u_d} x_d).$$

This action is ergodic if and only if every T_i is ergodic, where the reverse implication can be deduced using Fubini's theorem.

We consider the upper and lower critical dimensions with respect to sequences of rectangles $B_n = B_n^1 \times \cdots \times B_n^d$ where each $B_n^i = [\![-s_i(n), s_i(n)]\!]$ for some increasing functions $s_i : \mathbb{N}_0 \to \mathbb{N}_0$. This set-up includes rectangular summing sequences. For each $1 \le i \le d$ we write α_i and β_i for the lower and upper critical dimensions of T_i with respect to $[\![-n, n]\!]$, taken in the space (X_i, μ_i).

Given two increasing functions $s, s' : \mathbb{N} \to \mathbb{N}_{>1}$, we write $s \lesssim s'$ and say that s is *controlled by* s' if

$$\liminf_{n\to\infty} \frac{\log s'(n)}{\log s(n)} > 0.$$

$\lesssim$ defines a preorder on the space such functions, and this preorder is total. We can use $\lesssim$ to define an equivalence relation by declaring that s and s' have *equivalent growth*, denoted $s \approx s'$, if both $s \lesssim s'$ and $s \lesssim s'$, that is, if

$$0 < \liminf_{n\to\infty} \frac{\log s'(n)}{\log s(n)} \le \limsup_{n\to\infty} \frac{\log s'(n)}{\log s(n)} < \infty.$$

This definition ensures that all the functions $\lfloor n^t \rfloor$ for $t > 0$ are in the same equivalence class, but $\lfloor e^n - 1 \rfloor$ is strictly greater.

Using the axiom of choice, we may fix a representative of each equivalence class. Suppose that $\bar{s}$ is the representative of the equivalence class of s. Then we set

$$a(s) = \liminf_{n\to\infty} \frac{\log s(n)}{\log \bar{s}(n)} \quad \text{and} \quad b(s) = \limsup_{n\to\infty} \frac{\log s(n)}{\log \bar{s}(n)}.$$

When referring to rectangles B_n as above, let us write $a_i = a(s_i)$ and $b_i = b(s_i)$ wherever there is no ambiguity.

Our first, foundational, result of this part provides bounds for the critical dimensions with respect to the rectangles B_n in terms of the critical dimensions of the product transformations and the growth rates of the rectangle sides.

THEOREM 3.4. *Let $\mathbb{Z}^d$ act on a product space (X, μ) via a non-singular and ergodic product action, as described above. Let $D \subseteq [\![1, d]\!]$ such that for each $i \in D$ the function*

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s_i is a greatest element in $\{s_1, \ldots, s_d\}$ with respect to $\lesssim$. Then

$$\frac{\sum_{i\in D} a_i\alpha_i}{\sum_{i\in D} b_i} \le \alpha \le \beta \le \frac{\sum_{i\in D} b_i\beta_i}{\sum_{i\in D} a_i}.$$

Note that these bounds may depend on the choice of representative $\bar{s}$, but the inequalities remain the same if $\bar{s}$ is replaced by any s for which the limit $\lim_{n\to\infty}(\log s(n)/\log \bar{s}(n))$ exists and is non-zero. One usually chooses functions s_i which are related to one another in this way, and then in addition the representative can be chosen such that $a_i = b_i$ for all i. The benefit of the above more general formulation of the theorem is that it allows for some sides of the rectangles to grow rather slowly for periods of time but then 'catch up' later.

The inner bound is true by definition. The two outer bounds have slightly different proofs but both rely on two key ideas. The first is that a small portion of the growth from the fastest-growing sides can be used to dominate and hence neglect the behaviour from the slower-growing sides. The second idea is that the rates of growth from the fastest-growing sides can be compared using the representative of their equivalence class, resulting in the weighted average of critical dimensions seen above.

We first prove the lower bound, where growth from the slow-growing sides is absorbed by the faster-growing sides.

LEMMA 3.5. *Let $\mathbb{Z}^d$ act on a product space X via a non-singular and ergodic product action, as described above. Let $D \subseteq [\![1, d]\!]$ such that for each $i \in D$ the function s_i is a greatest element in $\{s_1, \ldots, s_d\}$ with respect to $\lesssim$. Then*

$$\alpha \ge \frac{\sum_{i\in D} a_i\alpha_i}{\sum_{i\in D} b_i}.$$

Proof. Suppose

$$t = \frac{\sum_{i\in D}(a_i - \epsilon)(\alpha_i - 2\epsilon)}{\sum_{i\in D} b_i}$$

for some $\epsilon > 0$. It follows from considering cylinder sets and applying Fubini's theorem that for $u \in \mathbb{Z}^d$ we have $\omega_u(x) = \prod_{i=1}^d \omega^i_{u_i}(x)$ where

$$\omega^i_j(x) = \frac{d\mu_i \circ T_i^j}{d\mu_i}(x_i).$$

Then

$$\frac{1}{|B_n|^t}\sum_{u\in B_n} \omega_u = \frac{1}{2^{dt}}\frac{1}{(\prod_{i=1}^d s_i(n))^t}\prod_{i=1}^d \sum_{j\in B_n^i} \omega^i_j. \tag{3.1}$$

Let $\bar{s}$ be the representative of the growth equivalence class of the s_i with $i \in D$ and fix a positive real number δ. For $i \notin D$ we have

$$\liminf_{n\to\infty} \frac{\log s_i(n)}{\log \bar{s}(n)} = 0.$$

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Hence for $i \notin D$ for all n sufficiently large, $s_i(n) \leq \bar{s}(n)^{\delta}$. By definition, for $i \in D$ for large n we must have $\bar{s}(n)^{a_i-\epsilon} \leq s_i(n) \leq \bar{s}(n)^{b_i+\delta}$. Therefore, for all sufficiently large n we have

$$\prod_{i=1}^{d} s_i(n) \leq (\bar{s}(n))^{d\delta+\sum_{i\in D} b_i},$$

and so for some $\eta = O(\delta)$ we have

$$\bigg(\prod_{i=1}^{d} s_i(n)\bigg)^t \leq (\bar{s}(n))^{\sum_{i\in D}(a_i-\epsilon)(\alpha_i+\eta-2\epsilon)} \leq \prod_{i\in D}(s_i(n))^{\alpha_i+\eta-2\epsilon}.$$

As we retain the freedom to shrink δ we can assume that each $\eta < \epsilon$, to deduce that for large enough n,

$$\frac{1}{|B_n|^t}\sum_{u\in B_n}\omega_u \geq \frac{1}{2^{dt}}\bigg(\prod_{i\notin D}\sum_{j\in B_n^i}\omega_j^i\bigg)\bigg(\prod_{i\in D}\frac{1}{s_i(n)^{\alpha_i-\epsilon}}\sum_{j\in B_n^i}\omega_j^i\bigg).$$

The first bracket is always at least 1 and each term of the latter product diverges to infinity. Hence we see that $\alpha \geq t$, but since $\epsilon > 0$ was arbitrary the lemma follows. □

For the upper bound a little of the growth from the fast-growing sides is used to dominate the slower-growing sides.

LEMMA 3.6. *Let $\mathbb{Z}^d$ act on a product space X via a non-singular and ergodic product action, as described above. Let $D \subseteq [[1, d]]$ such that for each $i \in D$ the function s_i is a greatest element in $\{s_1, \ldots, s_d\}$ with respect to $\lesssim$. Then*

$$\beta \leq \frac{\sum_{i\in D} b_i\beta_i}{\sum_{i\in D} a_i}.$$

Proof. The result is trivial if any $\beta_i = \infty$, so assume not. Suppose

$$t = \frac{\sum_{i\in D}(b_i+\epsilon)(\beta_i+2\epsilon)}{\sum_{i\in D} a_i}$$

for some $\epsilon > 0$. Let $\bar{s}$ be the representative of the s_i with $i \in D$ and fix $\delta > 0$. By definition, for $i \in D$ and n sufficiently large, $\bar{s}(n)^{a_i-\delta} \leq s(n) \leq \bar{s}(n)^{b_i+\epsilon}$. Hence for these n,

$$\prod_{i=1}^{d} s_i(n) \geq \bar{s}(n)^{-|D|\delta+\sum_{i\in D} a_i},$$

and so for some $\eta = O(\delta)$ we have

$$\bigg(\prod_{i=1}^{d} s_i(n)\bigg)^t \geq \bar{s}(n)^{-\eta+\sum_{i\in D}(b_i+\epsilon)(\beta_i+2\epsilon)} \geq \bar{s}(n)^{-\eta+\epsilon\sum_{i\in D} b_i}\bigg(\prod_{i\in D} s_i(n)^{\beta_i+\epsilon}\bigg).$$

By shrinking δ we can assume that $c = 1/(d-|D|)(\epsilon\sum_{i\in D} b_i - \eta) > 0$ and use (3.1) to deduce that for large n,

$$\frac{1}{|B_n|^t}\sum_{u\in B_n}\omega_u \leq \frac{1}{2^{dt}}\bigg(\prod_{i\notin D}\frac{1}{\bar{s}(n)^c}\sum_{j\in B_n^i}\omega_j^i\bigg)\bigg(\prod_{i\in D}\frac{1}{s_i(n)^{\beta_i+\epsilon}}\sum_{j\in B_n^i}\omega_j^i\bigg).$$

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For each $i \notin D$ eventually $\bar{s}(n)^c \geq s_i(n)^{\beta_i+\delta}$ and so each term in the first product tends to 0. Similarly with each of the terms in the second product. Hence we see that $\beta < t$, but since $\epsilon > 0$ was arbitrary the lemma follows. □

This completes the proof of Theorem 3.4. We can combine it with the integer theory to start to answer our question about the dependence on the summing sequence.

In §3.1 we saw it was possible to produce transformations with any (single) critical dimension in (0, 1). By constructing the product action using such T_i, and choosing the s_i to ensure $a_i = b_i$ for all $i \in D$, Theorem 3.4 ensures the resulting actions will have critical dimension

$$\gamma = \frac{\sum_{i \in D} a_i \gamma_i}{\sum_{i \in D} a_i}.$$

We are now equipped to examine some specific examples which answer our earlier question.

3.3.1. *Values taken by the critical dimension.* The simplest examples to consider are those where $s_1(n) = s_2(n) = \cdots = n$ which all satisfy $a(s_i) = 1$ with respect the natural choice of representative of their class, $\bar{s}(n) = n$. Then in the above circumstances there is a single critical dimension

$$\gamma = \frac{\gamma_1 + \cdots + \gamma_d}{d}.$$

This in turn means that for any d and $r \in (0, 1)$ we can produce a $\mathbb{Z}^d$-action with critical dimension r.

3.3.2. *Dependence on the choice of summing set.* Consider a $\mathbb{Z}^2$-action, constructed via the method above, and its critical dimension with respect to

$$[\![-n, n]\!] \times [\![-\lfloor e^n - 1 \rfloor, \lfloor e^n - 1 \rfloor]\!].$$

Here s_2 grows strictly faster than s_1 and, with the sensible choice representatives, the critical dimension is seen to be $\gamma = \gamma_2$. This, taken with the last example, shows that the critical dimension very much depends on the choice of summing sequence. It also shows that critical dimensions of the factors can be deduced from those of the product action and vice versa.

In fact, any desired weighting of the critical dimensions can be achieved. Suppose $t_i \in [0, 1]$ such that $t_1 + \cdots + t_d = 1$. We take $s_i(n) = n$ if $t_i = 0$ and $s_i(n) = \lfloor (e^n - 1)^{t_i} \rfloor$ otherwise. Then the critical dimension of the product action with respect to the corresponding summing sequence is given by $\gamma = t_1\gamma_1 + \cdots + t_d\gamma_d$. Moreover, each such summing sequence is rectangular, and so each of these weightings is an invariant of metric isomorphism.

4. *Further questions*

Underlying much of this paper is the question of how the choice of summing sequence affects not only the critical dimension but the ergodic theorem for $\mathbb{Z}^d$. On the one hand,

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for the sequences $[[0, n]]^d$ in $\mathbb{Z}^d$ with $d > 1$ there is the counterexample to the ratio ergodic Theorem [8], found by Brunel and Krengel. On the other, for balls of norms or for rectangular summing sequences the ergodic theorem holds. If the sets in a summing sequence have the Besicovitch property and the modified doubling condition then it seems likely that Hochman's method will work, so long as some analogue of the finite coarse dimension property can be found. It is in proving this latter condition that both cases make use of some natural structure of $\mathbb{Z}^d$. It would be interesting to know exactly what we require from a summing sequence in $\mathbb{Z}^d$ for the ergodic theorem to hold. The fact that large parts of Hochman's approach can be applied to rectangles suggests that the theorems for norms and rectangles may both be special cases of a wider phenomenon.

On the critical dimension, we have shown in the case of product actions on product spaces that the critical dimension for rectangles can be decomposed into a weighted average of the critical dimensions, for the projected measures, of maps corresponding to $e_1, \ldots, e_n$. It is an open question whether this extends more generally, for example the critical dimension of each e_i can be calculated on (X, μ) as a $\mathbb{Z}$-action regardless of whether the $\mathbb{Z}^d$-action is a product action. Therefore it is reasonable to ask how the critical dimension of the $\mathbb{Z}^d$-action is related to those of the generators.

Acknowledgement. K. Jarrett was supported by an EPSRC studentship for the duration of this work.

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