106.39 Visual proofs of sums of powers of positive integers

In 1992 Elizabeth M. Markham gave a beautiful visual proof of 'Sums of fourth powers' [1]. In this Note I generalise the result for any even number in Case I and for any odd number in Case II.

Case I: When $m = 2k$ is even:

$$\sum_{i=1}^{n} i^m = \sum_{i=1}^{n} i^{2k} = \left(\sum_{i=1}^{n} i^k\right)^2 - 2\left[\sum_{j=2}^{n}\left(j^k \sum_{i=1}^{j-1} i^k\right)\right]$$

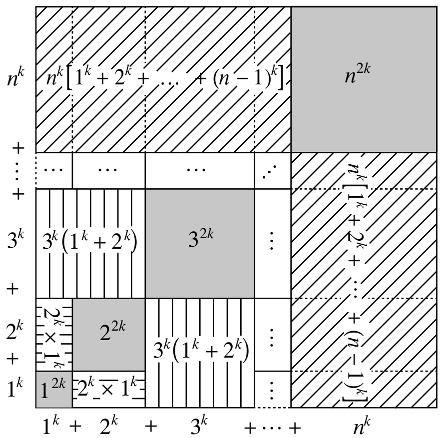

FIGURE 1

Case II: When $m = 2k + 1$ is odd:

$$\sum_{i=1}^{n} i^m = \sum_{i=1}^{n} i^{2k+1} = \left(\sum_{i=1}^{n} i^k\right)\left(\sum_{i=1}^{n} i^{k+1}\right) - \left[\sum_{j=2}^{n}\left(j^k \sum_{i=1}^{j-1} i^{k+1}\right) + \sum_{j=2}^{n}\left(j^{k+1} \sum_{i=1}^{j-1} i^k\right)\right]$$

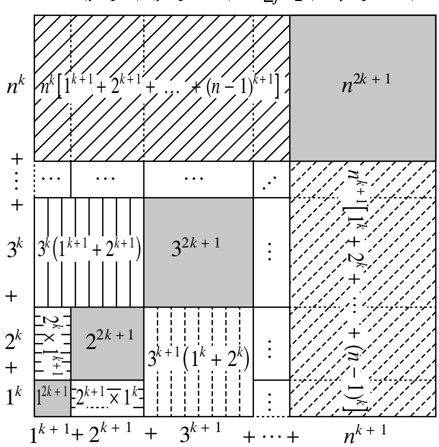

FIGURE 2

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Reference

1. E. M. Markham, Proof without words, *Mathematics Magazine*, **65**(1), (1992) p. 55.

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106.40 The law of tangents and the formulae of Mollweide and Newton

Analysing the following picture, in which the angles of the triangle *ABC* are α, β, γ and *AG* is parallel to *BE*, we obtain a proof of the law of tangents

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}.$$

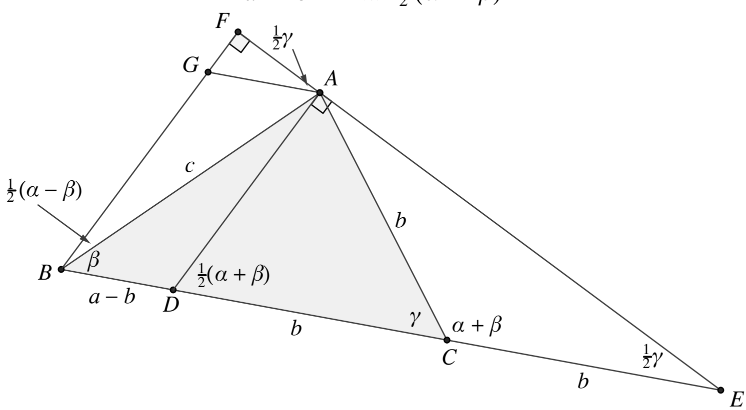

FIGURE 1: $BE : GA = FE : FA = \dfrac{FE}{FB} : \dfrac{FA}{FB}$

Moreover, we have $AF = c\sin\left(\frac{1}{2}(\alpha-\beta)\right)$ and $BF = c\cos\left(\frac{1}{2}(\alpha-\beta)\right)$, then

$$c\sin\left(\tfrac{1}{2}(\alpha-\beta)\right) = (a-b)\cos\tfrac{1}{2}\gamma, \qquad c\cos\left(\tfrac{1}{2}(\alpha-\beta)\right) = (a+b)\sin\tfrac{1}{2}\gamma$$

i.e. the Mollweide's and the Newton's formula.

We can also note that the division of Newton's formula by Mollweide's gives another proof of the law of tangents. An alternative proof of the formulae of Mollweide and Newton, which just uses a portion *ABDCE* of the figure, is to use the sine rule twice. In fact, in $\triangle ABD$ we have

$$\frac{a-b}{\sin\left(\frac{1}{2}(\alpha-\beta)\right)} = \frac{c}{\sin\left(\frac{1}{2}(\alpha+\beta)\right)} = \frac{c}{\cos\frac{1}{2}\gamma}.$$