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ISOMETRIES OF HILBERT SPACE VALUED FUNCTION SPACES

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Abstract

Let X be a (real or complex) rearrangement-invariant function space on Ω (where $\Omega = [0, 1]$ or $\Omega \subseteq \mathbb{N}$) whose norm is not proportional to the L_2-norm. Let H be a separable Hilbert space. We characterize surjective isometries of $X(H)$. We prove that if T is such an isometry then there exist Borel maps $a : \Omega \to \mathbb{K}$ and $\sigma : \Omega \to \Omega$ and a strongly measurable operator map S of Ω into $\mathscr{B}(H)$ so that for almost all ω, $S(\omega)$ is a surjective isometry of H, and for any $f \in X(H)$, $Tf(\omega) = a(\omega)S(\omega)(f(\sigma(\omega)))$ a.e. As a consequence we obtain a new proof of the characterization of surjective isometries in complex rearrangement-invariant function spaces.

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1. Introduction

We study isometries of Hilbert space-valued rearrangement-invariant function spaces $X(H)$, where $\dim H \geq 2$ and H is separable. Our results are valid for both symmetric sequence spaces and non-atomic rearrangement-invariant function spaces on [0, 1] with norm not proportional to L_2 but they are new only in the non-atomic case. If X is a sequence space, not necessarily even symmetric, Theorem 11 is a special case of a much more general result of Rosenthal [14] about isometries of Functional Hilbertian Sums. We include here the case of X being a symmetric sequence space since the proof is essentially the same as when X is a non-atomic rearrangement-invariant function space, and also our techniques are much simpler than those developed in [14].

Spaces of the form $X(H)$ appear naturally in the theory of Banach spaces (see [10, Chapter 2.d]). In particular, if X is rearrangement-invariant (with Boyd indices $1 < p_X \leq q_X < \infty$) then $X(L_2)$ is isomorphic to X ([10, Proposition 2.d.4]) and this plays an important role in the study of the uniqueness of unconditional bases in X.

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Isometries of Hilbert space-valued function spaces have been studied by many authors. In 1974, Cambern [2] characterized isometries of $L_p(L_2)$ in the complex case (see also an alternative proof of Fleming and Jamison [5]). Isometries of $L_p(L_2)$ in both real and complex cases are described (among other spaces) in the general paper of Greim [7] in 1983. In 1981 Cambern [3] described isometries of both real and complex, $L_\infty(L_2)$. In 1986 Jamison and Loomis [8] gave the characterization of isometries in complex Hilbert space-valued non-atomic Orlicz spaces $X(L_2)$. Also there have been a number of studies of various L_2-valued analytic function spaces. For a fuller discussion of the literature we refer the reader to the forthcoming survey of Fleming and Jamison [4].

We use a method of proof which is designed for spaces over $\mathbb{R}$, but clearly complex linear operators $T : X(H) \to X(H)$ can be always considered as real linear operators acting on $X(H)(\ell_2^2)$ and therefore our results are valid also in the complex case.

Moreover, Theorem 11 with $H = \ell_2^2$ may be viewed as a statement about the form of isometries of complex rearrangement-invariant spaces. Thus we give a new proof of the fact that all surjective isometries on X can be represented as weighted composition operators, that is, if T is such an isometry, then there are Borel maps a, σ such that $Tf = af \circ \sigma$ for all f in X (cf. [17], [18] for non-atomic spaces, and [16] for sequence spaces).

2. Preliminaries

We follow standard notations as in [10].

In the following, H denotes a separable Hilbert space with $\dim H \geq 2$. If we want to stress that we restrict our attention to the case when $\dim H = \infty$ we will write $H = \ell_2$.

If X is a Köthe function space ([10, Definition 1.b.17]) we denote by X' the *Köthe dual* of X; thus X' is the Köthe space of all g such that $\int |f||g|\, d\mu < \infty$ for every $f \in X$ equipped with the norm $\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int |f||g|\, d\mu$. Then X' can be regarded as a closed subspace of the dual X^* of X.

If X is a Köthe function space on (Ω_1, μ_1) and H is a separable Hilbert space on (Ω_2, μ_2), we will denote by **X(H)** the Köthe function space on $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ with the following norm:

$$\|f(\omega_1, \omega_2)\|_{X(H)} = \| \, \|f(\omega_1, \cdot)\|_2 \, \|_X.$$

This definition coincides with the notion of H-valued Bochner spaces.

It is well-known that $(X(H))^* = X^*(H)$, and that the space $(X(H))' \subset X^*(H)$ can be identified with the space of functions $\varphi : \Omega_1 \to H$ such that for every $y \in H$ the map $\omega_1 \longmapsto \langle \varphi(\omega_1), y \rangle$ is measurable and the map $\varphi_\# : \omega_1 \longmapsto \|\varphi(\omega_1)\|_H$ belongs

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to X'. The operation of φ on $X(H)$ is given by

$$\varphi(f) = \int_{\Omega_1} \langle \varphi(\omega_1), f(\omega_1)\rangle \, d\mu_1(\omega_1)$$

for any $f \in X(H)$. Thus $(X(H))' = X'(H)$.

For any function $f \in X(H)$ we define the map $f_\# : \omega_1 \to \mathbb{R}$ by $f_\#(\omega) = \|f(\omega)\|_H$. Then $f_\# \in X$. We say that functions $f, g \in X(H)$ are *disjoint in a vector sense* if $f_\#$ and $g_\#$ are disjointly supported, that is, $f_\#(\omega) \cdot g_\#(\omega) = 0$ for a.e. $\omega \in \Omega_1$. We say that an operator $T : X(H) \to X(H)$ is *disjointness preserving in a vector sense* if $(Tf)_\# \cdot (Tg)_\# = 0$ whenever $f_\# \cdot g_\# = 0$.

We will say that an operator $T : X(H) \to X(H)$ has *a canonical vector form* if there exists a non-vanishing Borel function a on Ω (where $\Omega = [0, 1]$ if X is non-atomic or $\Omega \subset \mathbb{N}$ if X is a sequence space) and an invertible Borel map $\sigma : \Omega \to \Omega$ such that, for any Borel set $B \subset \Omega$, we have $\mu(\sigma^{-1}B) = 0$ if and only if $\mu(B) = 0$ and a strongly measurable map S of Ω into $\mathscr{B}(H)$ (that is, for each $h \in H$ the mapping $\omega \mapsto S(\omega)h$ is measurable) so that $S(t)$ is an isometry of H onto itself for almost all t and $Tf(t) = a(t)S(t)(f(\sigma(t)))$ a.e. for any $f \in X(H)$.

Note that the name 'a canonical vector form' is introduced here only for the purpose of this paper – we do not know the standard name for this type of operator. We will need the following simple observation (cf. [9, Lemma 2.4]):

LEMMA 1. *Suppose that $T : X(H) \to X(H)$ is an invertible operator which has a canonical vector form. Then $T' : X'(H) \to X'(H)$ exists and has a canonical vector form.*

PROOF. Operator T has a representation $Tf(\omega_1) = a(\omega_1)S(\omega_1)(f(\sigma(\omega_1)))$ where a, S, σ satisfy the above conditions for canonical forms and moreover a is non-vanishing and σ is an invertible Borel map with $\mu(\sigma^{-1}B) = 0$ if and only if $\mu(B) = 0$. Let v be the Radon-Nikodym derivative of the σ-finite measure $\nu(B) = \mu(\sigma^{-1}B)$.

Then for $f \in X(H)$, $g \in X'(H)$ we have

$$\begin{aligned} g(Tf) &= \int_{\Omega_1} \langle g(\omega_1), Tf(\omega_1)\rangle \, d\mu(\omega_1) \\ &= \int_{\Omega_1} \langle g(\omega_1), a(\omega_1)S(\omega_1)(f(\sigma(\omega_1)))\rangle \, d\mu(\omega_1) \\ &= \int_{\Omega_1} \langle a(\omega_1)(S(\omega_1))'(g(\omega_1)), f(\sigma(\omega_1))\rangle \, d\mu(\omega_1) \\ &= \int_{\Omega_1} \langle a(\sigma^{-1}(\omega_1))(S(\sigma^{-1}(\omega_1)))'(g(\sigma^{-1}(\omega_1))), f(\omega_1)\rangle v(\omega_1) \, d\mu(\omega_1), \end{aligned}$$

since $(S(\omega))^* = (S(\omega))'$.

Thus $T^*g \in X'(H)$ and

$$T'g(\omega_1) = a(\sigma^{-1}(\omega_1))v(\omega_1)(S(\sigma^{-1}(\omega_1)))'g(\sigma^{-1}(\omega_1)) \text{ a.e.}$$

Clearly the map $\omega_1 \mapsto S(\sigma^{-1}(\omega_1))'$ is strongly measurable and thus T' has a canonical vector form.

A *rearrangement-invariant function space* (*r.i. space*) [10, Definition 2.a.1] is a Köthe function space on (Ω, μ) which satisfies the conditions:

(1) X' is a norming subspace of X^*.
(2) If $\tau : \Omega \to \Omega$ is any measure-preserving invertible Borel automorphism then $f \in X$ if and only if $f \circ \tau \in X$ and $\|f\|_X = \|f \circ \tau\|_X$.
(3) $\|\chi_B\|_X = 1$ if $\mu(B) = 1$.

Next we will quickly state a definition of Flinn elements. For fuller description and proofs we refer to [9, 12].

We say that an element u of a Köthe space X is *Flinn* if there exists an $f \in X^*$ such that $f \neq 0$ and for every $x \in X$ and $x^* \in X^*$ with $x^*(x) = \|x\|_X \cdot \|x^*\|_{X^*}$ we have $f(x) \cdot x^*(u) \geq 0$. We say that (u, f) is a *Flinn pair*. We denote by $\mathscr{F}(X)$ the set of Flinn elements in X. We will need the following facts:

PROPOSITION 2 ([9, Proposition 3.2]). *Suppose $U : X \to Y$ is a surjective isometry. Then $U(\mathscr{F}(X)) = \mathscr{F}(Y)$; furthermore if (u, f) is a Flinn pair then $(U(u), (U^*)^{-1}f)$ is a Flinn pair.*

THEOREM 3 (Flinn, [13, Theorem 1.1], [9, Theorem 3.3]). *Let X be a Banach space and π a contractive projection on X with range Y. Suppose (u, f) is a Flinn pair in X. Suppose $f \notin Y^\perp$. Then $\pi(u) \in \mathscr{F}(Y)$.*

THEOREM 4 ([9, Theorem 4.3]). *Suppose μ is non-atomic and suppose X is an order-continuous Köthe function space on (Ω, μ). Then $u \in X$ is a Flinn element if and only if there is a non-negative function $w \in L_0(\mu)$ with* $\operatorname{supp} w = \operatorname{supp} u = B$, *so that:*

(a) *If $x \in X(B)$ then $\|x\| = \left(\int |x|^2 w \, d\mu\right)^{1/2}$, and*
(b) *If $v \in X(\Omega \setminus B)$ and $x, y \in X(B)$ satisfy $\|x\| = \|y\|$, then $\|v + x\| = \|v + y\|$.*

The last fact about Flinn elements that we will need is a reformulation of Calvert and Fitzpatrick's characterization of ℓ_p-spaces [1]:

THEOREM 5. *Suppose that X is a sequence space with* $\dim X = d < \infty$, $d \geq 3$, *and basis* $\{e_i\}_{i=1}^d$. *Suppose that every element u of X with support on at most two coordinates is Flinn in X, that is,*

$$\{u \in X \ : \ u = a_i e_i + a_j e_j \text{ for some } i, j \leq d, \ a_i, a_j \in \mathbb{R}\} \subset \mathscr{F}(X).$$

Then $X = \ell_p^d$ *for some* $1 \leq p \leq \infty$.

PROOF. By [14, Lemma 1.4] (u, f) is a Flinn pair in X if and only if the projection P defined by $P(x) = x - f(x)u$ has norm 1 in X. Hence, if (u, f) is a Flinn pair in X then there is a projection of norm 1 onto the hyperplane $\ker f \subset X$.

It is also clear from the definition that if (u, f) is a Flinn pair in X then (f, u) is a Flinn pair in X'. Therefore there exists a projection of norm 1 onto $\ker u \subset X'$ for every u with support on at most two coordinates. But then [1, Theorem 1] asserts that if $d \geq 3$ then $X' = \ell_q^d$ for some $1 \leq q \leq \infty$. Thus $X = \ell_p^d$.

Finally let us introduce the following notation.

Suppose that X is a non-atomic r.i. space on $[0, 1]$ and n is a natural number. Let $e_i^n = \chi_{[(i-1)2^{-n}, i2^{-n}]}$ for $1 \leq i \leq 2^n$. Denote $X_n = [e_i^n : 1 \leq i \leq 2^n]$. If $\dim X < \infty$ then, for the uniformity of notation, we will use $X_n = X$ for any $n \in \mathbb{N}$. Notice that X_n^* can be identified naturally with X_n'.

We now need to introduce a technical definition. We will say that an r.i. space X has *property* (P) if for every $t > 0$,

(a) $\|\chi_{[0,\frac{1}{2}]}\|_X < \|\chi_{[0,\frac{1}{2}]} + t\chi_{[\frac{1}{2},1]}\|_X$ if X is a non-atomic function space on $[0, 1]$; or
(b) $\|e_1\|_X < \|e_1 + te_2\|_X$ if X is a sequence space with basis $\{e_i\}_{i=1}^{\dim X}$.

We say that X has *property* (P') if X' has property (P).

Notice that, clearly, if X has property (P) (respectively (P')) then for every $n \in \mathbb{N}$, X_n has property (P) (respectively (P')).

LEMMA 6. ([9, Lemma 5.2]) *Any r.i. space X has at least one of the properties (P) or (P').*

The reason for introducing property (P) is the following fact which will be important for our applications.

If $v \in X_n(H)$ then $v = (v_i)_{i=1}^{2^n}$, where $v_i \in H$ for all i and $v_i = (v_{i,j})_{j=1}^{\dim H}$. Similarly for $f \in X_n'(H)$, $f = (f_i)_{i=1}^{2^n}$, and $f_i = (f_{i,j})_{j=1}^{\dim H} \in H$. In this notation we have:

LEMMA 7. *Suppose that X has property (P') and $v \otimes f$ is a Flinn pair in $X_n(H)$. If $\|v_1\|_2 = |v_{11}|$ then $f_{11} \neq 0$.*

PROOF. Assume that $f_{11} = 0$. Then, since $v \otimes f \not\equiv 0$ there exist $i > 1$ and $j \geq 1$ such that $f_{ij} \neq 0$ and $v_{ij} \neq 0$. In fact $v_{ij} f_{ij} > 0$ since $f(e_{ij}) \cdot e_{ij}^*(v) \geq 0$.

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Consider $e_{11}^* + te_{ij}^* \in X_n'(\ell_2^d)$. Then

$$\|e_{11}^* + te_{ij}^*\|_{X_n'(H)} = \|e_1^* + te_i^*\|_{X_n'} > \|e_1^*\|$$

for all $t \neq 0$ since X has property (P'). Hence for any $t \neq 0$, if an element $(a_t e_{11} + b_t e_{ij})$ in $X_n(H)$ is norming for $(e_{11}^* + te_{ij}^*)$, then $b_t \neq 0$. In fact $b_t \cdot t > 0$. Let us take $t = -v_{11}/(2v_{ij})$. Then $\operatorname{sgn} b_t = \operatorname{sgn} t = -\operatorname{sgn}(v_{11} \cdot v_{ij}) = -\operatorname{sgn}(v_{11} f_{ij})$. Furthermore,

$$\begin{aligned} f(a_t e_{11} + b_t e_{ij}) &\cdot \left(e_{11}^* - \frac{v_{11}}{2v_{ij}} e_{ij}^*\right)(v) \\ &= b_t f_{ij} \cdot \left(v_{11} - \frac{v_{11}}{2v_{ij}} v_{ij}\right) = \frac{1}{2} b_t \cdot f_{ij} \cdot v_{11} < 0, \end{aligned}$$

and the resulting contradiction with numerical positivity of $v \otimes f$ proves the lemma.

3. Main results

We start with with an important (for us) proposition about the form of Flinn elements in $X_n(H)$. In the case when $\dim H < \infty$ our proof requires a certain technical restriction on the space X, which is irrelevant in the case when $H = \ell_2$. We present here proofs for both cases since they are quite different. However, for the application to Theorem 11 we need only to know the validity of Proposition 8.

PROPOSITION 8. *Suppose that X is an r.i. space with property (P'),* $\dim X \geq 3$ *and such that the norm of X is not proportional to the L_p-norm for any $1 \leq p \leq \infty$. Then there exists $N \in \mathbb{N}$, such that if $n \geq N$ and $u = (u_i)_{i=1}^{2^n} \in \mathscr{F}(X_n(H))$, then there exists $1 \leq i_0 \leq 2^n$ such that $\|u_i\|_2 = 0$ for all $i \neq i_0$.*

REMARK. Proposition 8 can be also understood as a statement about the form of 1-codimensional hyperplanes in $X_n(H)$ which are ranges of a norm-1 projection.

PROOF. Let n be big enough so that $X_n \neq \ell_p^{2^n}$, $1 \leq p \leq \infty$. Let $u \in \mathscr{F}(X_n(H))$. Then $u = (u_i)_{i=1}^{2^n}$, $u_i \in H$. Let $m = \operatorname{card}\{i : u_i \not\equiv 0\}$. We want to prove that $m = 1$.

By Proposition 2 we can assume without loss of generality that $u_i \not\equiv 0$ for $i = 1, \dots, m$, $u_i = 0$ for $i > m$ and $\alpha_1 = \|u_1\|_2 = \min\{\|u_i\|_2 : i = 1, \dots, m\}$. Now, for any numbers $\alpha_2, \dots, \alpha_m \in \mathbb{R}$ with $|\alpha_1|, \dots, |\alpha_m| \leq \alpha_1$ there exist isometries $\{U_i\}_{i=1}^m$ of H such that $(U_i(u_i))_1 = \alpha_i$ for $i = 1, \dots, m$. Hence by Proposition 2 the element v with

$$v_i = \begin{cases} U_i(u_i) & \text{if } i \leq m \\ 0 & \text{if } i > m \end{cases}$$

is Flinn in $X_n(H)$. By Theorem 3 and Lemma 7, $\bar{v} = (v_{i,1})_{i=1}^{2^n} \in \mathscr{F}(X_n)$. Since the sequence $\{\alpha_2, \dots, \alpha_m\}$ is arbitrary, this implies that every element with support of cardinality less than or equal to m is Flinn in X_n. But if $m \geq 2$, Theorem 5 implies that $X_n = \ell_p^{2^n}$ for some $1 \leq p \leq \infty$, contrary to our assumption. So $m = 1$.

As mentioned above, in the case when $H = \ell_2$, Proposition 8 is valid for any r.i. space X. Namely we have:

PROPOSITION 9. *Let X_n be a n-dimensional r.i. space not isometric to ℓ_2^n ($n \geq 2$). If $u = (u_i)_{i=1}^n \in \mathscr{F}(X_n(L_2))$, then there exists $1 \leq i_0 \leq n$ such that $\|u_i\|_2 = 0$ for all $i \neq i_0$.*

REMARK. We use here the notation L_2 for the separable Hilbert space to stress the fact that it is non-atomic. Clearly L_2 is isometric to ℓ_2 and $X_n(L_2)$ is isometric to $X_n(\ell_2)$ via a surjective isometry which preserves disjointness in a vector sense and hence our result is valid also in $X_n(\ell_2)$.

PROOF. Let $u \in \mathscr{F}(X_n(L_2))$ be such that $m = \operatorname{card}\{i : u_i \not\equiv 0\}$ is maximal. By Proposition 2 we can assume without loss of generality that $u_i \equiv 0$ for $i = m+1, \dots, n$ and $\operatorname{supp} u_i = [0, 1]$ for $i = 1, \dots, m$.

If we consider $X_n(L_2)$ as a function space on $\{1, \dots, n\} \times [0, 1]$, then $\operatorname{supp} u_i = \{1, \dots, m\} \times [0, 1] = B$. Since $X_n(L_2)$ is non-atomic, we can apply Theorem 4 to conclude that there exists a measurable function w such that $\operatorname{supp} w = B$ and for every $x \in X_n(L_2)(B)$,

$$\|x\| = \left(\int |x|^2 w \, d\mu\right)^{1/2}. \tag{1}$$

Since X_n and L_2 are r.i., w is constant, say $w \equiv k$. We need to show that $m = 1$.

Firstly, notice that $m < n$ since X_n is not isometric to ℓ_2^n and (1). Assume, to obtain a contradiction, that $m \geq 2$, and consider any element $z = (z_i)_{i=1}^n \in X_n(L_2)$ such that $z_i \equiv 0$ for $i = m+2, \dots, n$. Define $v, x, y \in X_n(L_2)$ by

$$v_i = \begin{cases} 0 & \text{if } i \neq m+1, \\ z_{m+1} & \text{if } i = m+1 \end{cases}; \quad x_i = \begin{cases} z_i & \text{if } i \leq m, \\ 0 & \text{if } i > m \end{cases}; \quad y_i = \begin{cases} \|x\|_2 & \text{if } i = 1, \\ 0 & \text{if } i > 1 \end{cases}$$

respectively. Then $\operatorname{supp} v \cap B = \emptyset$, $x, y \in X_n(L_2)(B)$ and $\|x\| = \|y\|$, so by Theorem 4(b), $\|v + x\| = \|v + y\|$, that is, $\|z\| = \|v + y\|$. Since X_n is r.i.,

$$\|v + y\| = k(\|z_{m+1}\|_2^2 + \|x\|_2^2)^{1/2} = k\|z\|_2.$$

Hence $\|z\| = k\|z\|_2$ for every $z \in X_n(L_2)(\{1, \dots, m+1\} \times [0, 1])$ and Theorem 4 quickly leads to a contradiction with maximality of m.

PROPOSITION 10. *Suppose that H is a separable Hilbert space and X is a rearrangement-invariant function space with norm not proportional to the L_2-norm. Suppose further that either X is non-atomic on* $[0, 1]$ *or is a sequence space* ($\dim X \leq \infty$), *and*

(a) $H = \ell_2$; *or*
(b) $H = \ell_2^d$, *X has a norm not proportional to an L_p-norm for any $1 \leq p \leq \infty$, X satisfies property* (P') *and* $\dim X \geq 3$.

Then every surjective isometry $T : X(H) \to X(H)$ preserves disjointness in a vector sense.

PROOF. We will present the proof in the case when X is non-atomic. If X is a sequence space the proof is almost identical and slightly simpler.

Let us denote $e_{i,j}^n = e_i^n \otimes e_j \in X_n(H)$ (e_j denotes elements of the natural basis of H) and $f_{i,j}^n = Te_{i,j}^n$ for $j, n \in \mathbb{N}$, $i \leq 2^n$.

Define for any $\omega \in [0, 1] \times \mathbb{N}$ (or $\omega \in [0, 1] \times \{1, \ldots, d\}$ in case (b))

$$F_n(\omega) = \sum_{i=1}^{2^n} \sum_{j=1}^{\infty} f_{i,j}^n(\omega) e_{i,j}^n.$$

Following the same argument as in [9, Theorem 6.1] we see that for almost every ω, $F_n(\omega) \in \mathscr{F}(X_n'(H))$.

For the sake of completness we present this argument here.

Denote by $\Pi(X(H))$ the set of pairs (x, x^*) where $x \in X(H)$, $x^* \in X'(H)$ and $1 = \|x\| = \|x^*\| = x^*(x)$.

We note first that by [9, Proposition 2.5], T^{-1} is $\sigma(X(H), X'(H))$-continuous and so has an adjoint $S = (T^{-1})' : X'(H) \to X'(H)$. We define $g_i^n = Se_i^n$. Suppose $(x, x^*) \in \Pi(X_n(H))$ where $x = \sum a_{i,j} e_{i,j}^n$ and $x^* = \sum a_{i,j}^* e_{i,j}^n$. Then $(Tx, Sx^*) \in \Pi(X(H))$ and this implies that

$$\left(\sum_{i=1}^{2^n} \sum_{j=1}^{\infty} a_{i,j} f_{i,j}^n(\omega) \right) \left(\sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{i,j}^* g_{i,j}^n(\omega) \right) \geq 0 \tag{2}$$

for μ-a.e. $\omega \in \Omega$.

Using the fact that $\Pi(X_n(H))$ is separable it follows that there is a set Ω_0^n of measure zero so that if $\omega \notin \Omega_0^n$, (2) holds for every $(x, x^*) \in \Pi(X_n(H))$. Let $\Omega_0 = \bigcup_{n \geq 1} \Omega_0^n$.

Now define $G_n(\omega) = \sum_{i=1}^{2^n} \sum_{j=1}^{\infty} g_{i,j}^n(\omega) e_{i,j}^n \in X_n(H)$. The above remarks show that if $\omega \notin \Omega_0$ then $x^*(G_n(\omega)) \cdot F_n(\omega)(x) \geq 0$ for all $(x, x^*) \in \Pi(X_n'(H))$, that is, $F_n(\omega) \in \mathscr{F}(X_n'(H))$ provided that $G_n(\omega) \neq 0$ and $\omega \notin \Omega_0$. We will show that this happens for a.e. $\omega \in [0, 1]$.

Let $B_n = \{\omega : G_n(\omega) = 0\}$. Clearly (B_n) is a descending sequence of Borel sets. Let $B = \bigcap B_n$. If $\mu(B) > 0$ then there exists a non-zero $h \in X(H)$ supported on B

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with $\langle h, Sx' \rangle = 0$ for every $x' \in X'(H)$. Thus $T^{-1}h = 0$, which contradicts the fact that T is an isometry.

Let $D_n = \Omega \setminus (\Omega_0 \cup B_n)$. Then $\mu(D_n) = 0$, and if $\omega \in D_n$, then $G_n(\omega) \neq 0$, and so it follows that $F_n(\omega) \in \mathscr{F}(X_n')$. Hence, by Proposition 8,

$$\text{(3)} \qquad \text{for a.e. } \omega \quad \exists i_\omega \quad \text{so that } f^n_{i,j}(\omega) = 0 \quad \forall i \neq i_\omega,\ j \in \mathbb{N}.$$

Let ν_1, ν_2 be any natural numbers. Consider the isometry V of H defined by

$$V(e_j) = \begin{cases} e_j & \text{if } j \neq \nu_1, \nu_2, \\ \frac{1}{\sqrt{2}}(e_{\nu_1} + e_{\nu_2}) & \text{if } j = \nu_1, \\ \frac{1}{\sqrt{2}}(e_{\nu_1} - e_{\nu_2}) & \text{if } j = \nu_2, \end{cases}$$

and the induced isometry $\overline{V}$ of $X(H)$ defined by V on each fiber.

$$\overline{V}Te^n_{i,j}(t, \nu) = \begin{cases} f^n_{i,j}(t, \nu) & \text{if } \nu \neq \nu_1, \nu_2, \\ \frac{1}{\sqrt{2}}(f^n_{i,j}(t, \nu_1) + f^n_{i,j}(t, \nu_2)) & \text{if } \nu = \nu_1, \\ \frac{1}{\sqrt{2}}(f^n_{i,j}(t, \nu_1) - f^n_{i,j}(t, \nu_2)) & \text{if } \nu = \nu_2. \end{cases}$$

Similarly as in 3 we conclude that for almost every t there exists $\bar{\imath}_{(t,\nu_1)}$ such that $\overline{V}Te^n_{i,j}(t, \nu_1) = 0$ for all $i \neq \bar{\imath}_{(t,\nu_1)}$. Therefore, for a.e. t,

$$f^n_{i,j}(t, \nu_1) + f^n_{i,j}(t, \nu_2) = 0 \quad \forall i \neq \bar{\imath}_{(t,\nu_1)},\ \forall\, j.$$

Combining this with (3) we get that for almost every $t \in [0, 1]$ and any $\nu_1, \nu_2 \in \mathbb{N}$, $\bar{\imath}_{(t,\nu_1)} = i_{t,\nu_1} = i_{t,\nu_2}$. It follows easily that T preserves disjointness of functions supported in disjoint dyadic intervals.

We are now ready to present the main result of this paper.

THEOREM 11. *Suppose that X is a rearrangement-invariant function space with norm not proportional to the L_2-norm. Suppose further that either X is non-atomic on* $[0, 1]$ *or it is a sequence space* ($\dim X \leq \infty$), *and let H be a separable Hilbert space.*

Suppose that $T : X(H) \to X(H)$ is a surjective isometry. Then there exists a non-vanishing Borel function a on Ω (where $\Omega = [0, 1]$ if X is non-atomic or $\Omega \subset \mathbb{N}$ if X is a sequence space) and an invertible Borel map $\sigma : \Omega \to \Omega$ such that, for any Borel set $B \subset \Omega$, we have $\mu(\sigma^{-1}B) = 0$ if and only if $\mu(B) = 0$, and a strongly measurable map S of Ω into $\mathscr{B}(H)$ so that $S(t)$ is an isometry of H onto itself for almost all t and

$$Tf(t) = a(t)S(t)(f(\sigma(t))) \quad a.e.$$

for any $f \in X(H)$.

Moreover, if X *is not equal to* $L_p[0,1]$, *up to equivalent renorming, then* $|a| = 1$ *a.e. and* σ *is measure-preserving.*

PROOF. We prove the theorem under the assumption that either:

(a) $H = \ell_2$; or

(b) $H = \ell_2^d$, X has a norm not proportional to an L_p-norm for any $1 \leq p \leq \infty$, X satisfies property (P') and $\dim X \geq 3$.

If $\dim X = 2$ the theorem follows from [14, Theorem 3.12]. If $X = L_p[0,1]$, $p \neq 2$ the theorem was proved by Greim [6] and Cambern [3]. If X does not satisfy property (P') then X' does and the result follows by a duality argument. That is, [9, Proposition 2.5] says that the isometry T is $\sigma(X, X')$ continuous and thus T has an adjoint $T' : X'(H) \to X'(H)$ which is a surjective isometry. Since X' satisfies property (P'), T' has a canonical vector form. By Lemma 1, T'' and hence T, has a canonical vector form.

So in the following we assume that the assertion of Proposition 10 holds, that is, the isometry T preserves disjointness.

We follow almost exactly the argument of Sourour [15, Theorems 3.1 and 3.2].

Let $\{x_n\}$ be the countable linearly independent subset of H whose linear span $\mathscr{D}$ is dense in H and let $\mathscr{D}_0$ be the set of all linear combinations of $\{x_n\}$ with rational coefficients. For any measurable set E let $\Phi(E) = \bigcup_n \operatorname{supp}(T(\chi_E x_n))$. Then, since T is 1-1, Φ is a set-isomorphism.

Let $y_n = T(\underline{x}_n)$, where $\underline{x}_n$ denotes the function from $X(H)$ constantly equal to x_n. For every $t \in \Omega$ define $A(t)x_n = y_n(t)$ and extend $A(t)$ linearly to $\mathscr{D}$; thus for every $y \in \mathscr{D}$, $A(\cdot)y = T(\underline{y})$ a.e.

We will now extend $A(t)$ to a bounded operator on X. Let $E \subset \Omega$ be measurable and $y \in \mathscr{D}_0$; then

$$\begin{aligned}
(4) \qquad \|A(t)y\chi_{\Phi(E)}\|_{X(H)} &= \|T(\underline{y})(t)\chi_{\Phi(E)}\|_{X(H)} \\
&= \|T(\underline{y}\chi_E)\|_{X(H)} = \|\underline{y}\chi_E\|_{X(H)} \\
&= \|\chi_E\|_X \|y\|_2.
\end{aligned}$$

By absolute continuity we can define for almost every t:

$$a(t) = \lim_{\substack{\mu(E)\to 0 \\ t\in E}} \frac{\|\chi_{\Phi^{-1}(E)}\|_X}{\|\chi_E\|_X}$$

(notice that if $X = L_p$ then $a(\cdot)$ coincides with the function $h(\cdot)$ considered by Sourour).

By (4), $A(t) = a(t)S(t)$ a.e., where $S(t)$ is an isometry of H.

The strong measurability of S and surjectivity of almost all $S(t)$ follow as in the proof of Sourour without change.

Thus, similarly as in [15],

$$Tf(t) = a(t)S(t)(\Phi(f))(t) \quad \text{for a.e. } t \in \Omega,$$

for some set isomorphism Φ. But, by [11], every set isomorphism of [0, 1] is induced by a point isomorphism, that is, there exists an invertible Borel map $\sigma : \Omega \to \Omega$ such that, for any Borel set $B \subset \Omega$, we have $\mu(\sigma^{-1}B) = 0$ if and only if $\mu(B) = 0$ and $(\Phi(f))(t) = f(\sigma(t))$ for a.e. $t \in [0, 1]$. Clearly, if $\Omega \subset \mathbb{N}$ then every set isomorphism is a point isomorphism. Thus we have

$$Tf(t) = a(t)S(t)(f(\sigma(t))) \text{ a.e.}$$

for any $f \in X(H)$.

The final remark is now an immediate consequence of [9, Theorem 7.2].

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