Canad. J. Math. Vol. **68** (2), 2016 pp. 395–421
http://dx.doi.org/10.4153/CJM-2015-042-5

Bilinear and Quadratic Forms on Rational Modules of Split Reductive Groups

Skip Garibaldi and Daniel K. Nakano

Abstract. The representation theory of semisimple algebraic groups over the complex numbers (equivalently, semisimple complex Lie algebras or Lie groups, or real compact Lie groups) and the questions of whether a given complex representation is symplectic or orthogonal have been solved since at least the 1950s. Similar results for Weyl modules of split reductive groups over fields of characteristic different from 2 hold by using similar proofs. This paper considers analogues of these results for simple, induced, and tilting modules of split reductive groups over fields of prime characteristic as well as a complete answer for Weyl modules over fields of characteristic 2.

1 Introduction

The representation theory of semisimple algebraic groups over the complex numbers (equivalently, semisimple complex Lie algebras or Lie groups) is well known. The set of isomorphism classes of irreducible representations of the simply connected cover of a group G is in bijection with the cone of dominant weights of the root system of G.

Classifying representations of G over $\mathbb{C}$ is the same as classifying homomorphisms from G into a group of type A. One can equally well ask about homomorphisms into other groups, for which homomorphisms into groups of type B or D (orthogonal representations) and groups of type C (symplectic representations) play a distinguished role. The basics of this theory were laid out in [Mal], were known to Dynkin [Dy], and a complete solution is clearly described in [GW09, §3.2.4], [St, pp. 226, 227], or [Bou L7, §VIII.7.5]. One reduces the problem to studying irreducible representations, then giving an algorithm in terms of the dominant weight λ for determining whether the irreducible representation $L(\lambda)$ has a nonzero G-invariant symmetric or skew-symmetric bilinear form. The algorithm is proved via restricting the representation to a principal A_1 subgroup of G. This material is now a plank in the foundations of representation theory, where it has applications to determining the subalgebras of semisimple complex Lie algebras [Dy] and distinguishing real and quaternionic representations of compact real Lie groups [Bou L7, Ch. IX, App. II.2, Prop. 3]. For general k, invariant bilinear forms are used to provide information about the groups, for

Received by the editors March 25, 2015.
Published electronically January 27, 2016.
The first author was partially supported by NSF grant DMS-1201542 and the Charles T. Winship Fund at Emory University.
The second author was partially supported by NSF grant DMS-1402271.
AMS subject classification: **20G05**, 11E39, 11E88, 15A63, 20G15.
Keywords: orthogonal representations, symmetric tensors, alternating forms, characteristic 2, split reductive groups.

example throughout the book [KMRT], or for bounding the essential dimension of G as in [CS] or [BaC], or for controlling Lie subalgebras as in [SF, §5.8, Exercise 1].

This paper concerns the generalization of the above theory of symplectic and orthogonal representations to the case of a field k of prime characteristic. The situation is more complicated. There are four classes of representations (each of which is in bijection with the cone of dominant weights) that have a claim to being the natural generalization of the irreducible representation over $\mathbb{C}$ with highest weight λ: the irreducible representation $L(\lambda)$, the induced module $H^0(\lambda)$, the Weyl module $V(\lambda)$, and the tilting module $T(\lambda)$. The four representations are related by nonzero maps

$$\begin{tikzcd} V(\lambda) \arrow[r, "\text{surjective}"] \arrow[d, "\text{injective}"'] & L(\lambda) \arrow[d, "\text{injective}"] \\ T(\lambda) \arrow[r, "\text{surjective}"] & H^0(\lambda), \end{tikzcd} \tag{1.1}$$

where all four maps are unique up to multiplication by an element of $k^\times$ (and the diagram commutes up to multiplication by a scalar). The definitions make sense for every field k, and when char $k = 0$ or λ is minuscule, all the maps in (1.1) are isomorphisms. We refer to [Jan] for information on this subject. In extending the theory of orthogonal and symplectic representations to include the case where the characteristic of k is prime, two complications arise.

First, although the theory of bilinear forms on irreducible representations over $\mathbb{C}$ translates easily to irreducible representations and Weyl modules over any k (see Lemma 4.3), it does not directly translate to the representations $H^0(\lambda)$ and $T(\lambda)$. For example, when V is irreducible or Weyl, the space of G-invariant bilinear forms is at most 1-dimensional, but for $V = H^0(\lambda)$ or $T(\lambda)$ it can be larger. We give a formula for this dimension in the case of $T(\lambda)$ in Theorem 6.2 and relate it to B-cohomology in the case of $H^0(\lambda)$ in Theorem 5.5.

Second, when char $k = 2$, the theory of G-invariant bilinear forms is insufficient to treat homomorphisms of G into groups of type B and D, because such groups are related to the existence of G-invariant quadratic forms. The question of the existence of G-invariant quadratic forms on irreducible representations has previously been studied (*cf.* [W, GowW, SinW]), and there is no known, straightforward necessary and sufficient condition (*cf.* Section 10). In contrast to this, we completely solve the question of which Weyl modules have a G-invariant quadratic form; see Theorem 9.5.

Ad hoc constructions in the literature show that the half-spin representations of Spin_{4m} for m odd, the representation $\Lambda^r(k^{2r})$ of SL_{2r} for r odd, and the 56-dimensional minuscule representation of E_7 are all examples of groups G and irreducible representations V such that V has a G-invariant quadratic form if and only if char $k = 2$ (see [KMRT, p. 150 and 10.12], and in the case of E_7, one notes that the invariant quartic—in [Brown, p. 87], for example—is defined over $\mathbb{Z}$ and becomes a square after reduction mod 2). These irreducible representations are also Weyl modules, because their highest weight is minuscule, so our Theorem 9.5 may be viewed as a generalization of these examples, as it gives a necessary and sufficient condition for the existence of such a G-invariant quadratic form. Proposition 8.5 gives a new explicit example of such a G-invariant quadratic form that does not seem to be in the literature.

One major feature throughout our paper is the interplay with the admissibility of various G-invariant bilinear forms and the rational cohomology of G. In the final section (Section 11), as a byproduct of our results, we are able to calculate $\mathrm{H}^1(G, \Lambda^2(V))$ for many cases where V is $L(\lambda)$, $T(\lambda)$, or $H^0(\lambda)$.

This paper treats the case where G is split reductive. A sequel work will extend these results to the case where G need not be split.

2 Background: Representations of Split Reductive Groups

2.1 Notation

We will follow the notation from [Jan] and [Bou L4]. Throughout, we consider a field k of characteristic $p \geq 0$ and an algebraic group G over k (*i.e.,* a smooth affine group scheme of finite type over k) that is reductive and split. If k is separably closed, then every reductive algebraic k-group is split. We fix in G a pinning, which includes the following data:

- T: a k-split maximal torus in G.
- Φ: the root system of G with respect to T. When referring to short and long roots, when a root system has roots of only one length, all roots shall be considered as both short and long.
- $\Pi = \{\alpha_1, \dots, \alpha_n\}$: the set of simple roots. We adhere to the ordering of the simple roots as given in [Jan] (following Bourbaki); see (8.3) for the numbering for type C.
- $\widetilde{\alpha}$: the maximal root.
- B: a Borel subgroup containing T corresponding to the negative roots.
- $W = N_G(T)/T$: the Weyl group.
- w_0: longest element in W, relative to the choice of simple roots Π.
- $P := \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$: the weight lattice, where the fundamental dominant weights ω_i are defined by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}, 1 \leq i, j \leq n$.
- $X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \subseteq P$.
- $\leq$ on P: a partial ordering of weights, for $\lambda, \mu \in P$, $\mu \leq \lambda$ if and only if $\lambda - \mu$ is in $\mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_n$.
- $P_+ := \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$: the dominant weights.
- $X(T)_+ := X(T) \cap P_+$.
- $Q := \mathbb{Z}\Phi$: the root lattice.

We further define the following:

- $F\colon G \to G$: the Frobenius morphism.
- $G_r = \ker F^r$: the r-th Frobenius kernel of G.
- $M^{[r]}$: the module obtained by composing the underlying representation for a rational G-module M with F^r.
- $H^0(\lambda) := \mathrm{ind}_B^G \lambda$, $\lambda \in X(T)_+$: the induced module whose character is provided by Weyl's character formula.
- $V(\lambda) := H^0(-w_0\lambda)^*$: the Weyl module.

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

- $L(\lambda)$: the irreducible finite dimensional G-module with highest weight $\lambda \in X(T)_+$. Its dual is $L(-w_0\lambda)$.
- $T(\lambda)$: Ringel's indecomposable tilting module corresponding to $\lambda \in X(T)_+$, as defined in [Jan, Prop. E.6]. Its dual is $T(-w_0\lambda)$.

Example 2.1 In the case when G is a torus, $\Phi = \varnothing$, $Q = 0$, and $P = X(T) = X(T)_+$. For each $\lambda \in X(T)_+$, $L(\lambda) = H^0(\lambda) = V(\lambda) = T(\lambda)$ is one-dimensional.

2.2 Direct Products

Suppose $G = T_0 \times \prod_{i=1}^n G_i$ for T_0 a split torus and G_i simple and simply connected. Let T_i be a maximal split torus in G_i and take $T = \prod_{i=0}^n T_i$; a pinning of G relative to T is equivalent to fixing a pinning of each G_i relative to T_i. Then Φ is the union of the root systems of the G_i. Setting P_i to be the weight lattice of G_i, *i.e.*, $X(T_i)$, we find that $X(T)_+ = P_0 \oplus \bigoplus_i (P_i)_+ = P_+$.

Lemma 2.2 *With the notation of the previous paragraph, for $\lambda_i \in X(T_i)_+$, we have*

$$L\Big(\sum \lambda_i\Big) = \otimes L(\lambda_i), \quad V\Big(\sum \lambda_i\Big) = \otimes V(\lambda_i), \quad \textit{and} \quad H^0\Big(\sum \lambda_i\Big) = \otimes H^0(\lambda_i).$$

Proof The claim for H^0 is [Jan, Lemma I.3.8]. Dualizing gives the claim for V. For L, an irreducible representation of G is a tensor product of irreducible representations of T_0 and of the G_i, and inspecting highest weights yields the claim. ∎

For an arbitrary split reductive group G, there exists a split torus T_0 and split, simple, simply connected groups $G_1, \dots, G_n$ as above and a central isogeny $\pi\colon T_0 \times \prod G_i \to G$, all of which are in some sense unique. The quotient π relates the chosen pinning of G relative to T to a pinning of $T_0 \times \prod G_i$ relative to the split maximal torus $\pi^{-1}(T)^\circ$ such that $\pi^* X(T)_+ = \prod X(T_i)_+$, and representations of G induce representations of $T_0 \times \prod G_i$. In this way, when proving the results in this paper, it is harmless to assume that G is a direct product $T_0 \times \prod G_i$ as at the beginning of this subsection.

In later parts of the paper, G_1 will be used to denote the first Frobenius kernel of G; the difference will be clear from context.

3 Symmetric Tensors and Symmetric Powers

3.1 Symmetric Tensors

Let V be a k-vector space.

Definition 3.1 The symmetric group Σ_n on n letters acts on $\otimes^n V$ by permuting the indices of a tensor $v_1 \otimes \cdots \otimes v_n$. Define $S_n'(V) \subseteq \otimes^n V$ via

$$S_n'(V) = \{x \in \bigotimes^n V \mid \sigma x = x \text{ for all } \sigma \in \Sigma_n\};$$

it is the space of *symmetric tensors*. The symmetrization map $s\colon \otimes^n V \to \otimes^n V$ is defined by $s(x) = \sum_{\sigma \in \Sigma_n} \sigma x$ and

$$S_n''(V) := \operatorname{im} s \subseteq S_n'(V);$$

elements of $S''_n(V)$ are *symmetrized tensors*. Evidently, if $n!$ is not zero in k, then $S''_n(V) = S'_n(V)$.

Example 3.2 (Bilinear forms) The space $\operatorname{Bil} V := V^* \otimes V^*$ is the vector space of bilinear forms on V, and $S'_2(V^*)$ is the subspace of symmetric bilinear forms. If $\operatorname{char} k = 2$, then $S''_2(V^*)$ is the span of elements $x \otimes y - y \otimes x$ for $x, y \in V^*$, *i.e.,* $S''_2(V^*)$ is the space of alternating bilinear forms.

Regardless of the characteristic of k, we identify $\bigotimes^2 V^* \xrightarrow{\sim} \operatorname{Hom}(V, V^*)$ denoted $b \mapsto \widehat{b}$, where $\widehat{b}$ is defined by $\widehat{b}_v := b(v, \cdot)$. Note that, as our vector spaces are finite-dimensional, the canonical inclusion $\bigotimes^n(V^*) \subseteq (\bigotimes^n V)^*$ is an equality by dimension count, so we omit the unnecessary parentheses from this expression. The (left) radical of b is $\operatorname{rad} b := \ker \widehat{b}$.

3.2 Symmetric Powers

The *n-th symmetric power* $S^n(V)$ of V is the image of $\bigotimes^n V$ in the quotient of $\oplus_{i \geq 0} \bigotimes^i V$ by the two-sided ideal generated by all elements of the form $v \otimes v' - v' \otimes v$ for $v, v' \in V$; we write $\rho\colon \bigotimes^n V \twoheadrightarrow S^n(V)$ for the quotient map. One typically omits the symbol $\otimes$ when writing elements of $S^n(V)$. If we fix a basis $v_1, \dots, v_d$ of V, then $S^n(V)$ has basis $\{v_1^{e_1} v_2^{e_2} \cdots v_d^{e_d} \mid \sum e_i = n\}$.

We obtain a commutative diagram (*cf.* [Bou A4, §IV.5.8]) of linear maps

$$\begin{tikzcd}
\bigotimes^n V \arrow[d, "\rho"] \arrow[dr, "s"] & & & & \bigotimes^n V \arrow[d, "\rho"] \\
S^n(V) \arrow[r, "\phi"] & S''_n(V) & \subseteq & S'_n(V) \arrow[ur] \arrow[r, "\psi"] & S^n(V).
\end{tikzcd}$$

The map ϕ is the multilinearization map $\phi(v_1 \cdots v_n) = \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. The compositions $\phi\psi\colon S''_n(V) \to S''_n(V)$ and $\psi\phi\colon S^n(V) \to S^n(V)$ are multiplications by $n!$.

Example 3.3 (Quadratic forms) The elements of the vector space

$$\operatorname{Quad} V := S^2(V^*)$$

are the quadratic forms on V. The multilinearization map ϕ sends a quadratic form q to its *polar bilinear form* $b_q := \phi(q)$ given by

$$b_q(v, v') = q(v + v') - q(v) - q(v').$$

If $b_q = 0$, then q is said to be *totally singular*. The *radical* of q is

$$\operatorname{rad} q := \{v \in V \mid v \in \operatorname{rad} b_q \text{ and } q(v) = 0\}.$$

Evidently $\operatorname{rad} q$ is a vector space, and for each extension K of k,

$$\operatorname{rad}(q \otimes K) \supseteq (\operatorname{rad} q) \otimes K.$$

If $\operatorname{char} k \neq 2$, then $\operatorname{rad} q = \operatorname{rad} b_q$, and the only totally singular quadratic form is the zero form.

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

We refer to [EKM] for background concerning quadratic forms over a field, including the case $\operatorname{char} k = 2$.

3.3 Alternating Tensors

Definition 3.4 We define subspaces $A''_n(V) \subseteq A'_n(V) \subseteq \bigotimes^n V$ via setting

$$A'_n(V) := \left\{ x \in \bigotimes^n V \mid \sigma x = (\operatorname{sign} \sigma) x \text{ for all } \sigma \in \Sigma_n \right\}$$

and $A''_n(V)$ to be the image of the skew-symmetrization map $\alpha\colon \bigotimes^n V \to A'_n(V)$ defined via $\alpha(x) = \sum_{\sigma\in\Sigma_n} (\operatorname{sign}\sigma) \cdot \sigma x$. These subspaces are analogous to the symmetric tensors in Definition 3.1.

Evidently, the restriction of α to $A'_n(V)$ acts as multiplication by $n!$. We can identify $\Lambda^2(V)$ with $A''_2(V)$ and $\Lambda^2(V^*)$ with the space of alternating bilinear forms on V (as $\Lambda^2(V^*) = \Lambda^2(V)^*$), *i.e.,* the forms $b \in V^* \otimes V^*$ such that $b(v, v) = 0$ for all $v \in V$. If $\operatorname{char} k = 2$, then $A'_2(V^*) = S'_2(V^*)$ is the space of symmetric bilinear forms on V and $A''_2(V^*)$ is a proper subspace of $A'_2(V^*)$ for nonzero V.

The sequences

$$0 \longrightarrow S'_2(V) \longrightarrow V \otimes V \xrightarrow{\alpha} \Lambda^2(V) \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow \Lambda^2(V) \longrightarrow V \otimes V \xrightarrow{\rho} S^2(V) \longrightarrow 0 \tag{3.2}$$

are exact. If $\operatorname{char} k \neq 2$, then both sequences split and encode the direct sum decomposition: $V \otimes V \cong \Lambda^2(V) \oplus S^2(V)$.

3.4 Representations

If V is a representation of a group G, then so are the vector spaces deduced from V such as $S''_2(V^*)$.

If $\operatorname{char} k \neq 2$, then replacing V with V^* in (3.2) gives a direct sum decomposition $(\operatorname{Bil} V)^G \cong (\Lambda^2 V^*)^G \oplus (\operatorname{Quad} V)^G$. Furthermore, if $(\operatorname{Bil} V)^G = kb$ for some nonzero b, then b is either symmetric or alternating.

However, when $\operatorname{char} k = 2$, the situation is slightly different.

Lemma 3.5 *Suppose* $\operatorname{char} k = 2$ *and V is a representation of a group G such that* $(\operatorname{Bil} V)^G = kb$ *for some nonzero b. Then b is symmetric and the quadratic form $\psi(b)$ is totally singular.*

Proof The exact sequence (3.1) yields an exact sequence

$$0 \longrightarrow S'_2(V^*)^G \longrightarrow (\operatorname{Bil} V)^G \xrightarrow{\alpha} \Lambda^2(V^*)^G$$

and $\alpha(b) = mb$ for some $m \in k$. Writing a Gram matrix M for b with respect to some basis of V, the equation $\alpha(b) = mb$ says that $M - M^t = mM$, hence $m = 0$, *i.e.,* b is symmetric.

For the second claim, suppose $\phi\psi(b) \neq 0$. Then the alternating form $\phi\psi(b)$ equals nb for some $n \in k^\times$, thus b is alternating and $\psi(b) = 0$. This is a contradiction, hence $\phi\psi(b) = 0$. ■

3.5 Symmetric p-linear Forms Versus Symmetric Powers of Degree p

We now return to the notions introduced in this section in the special case where k has characteristic $p \neq 0$, and compare the symmetrized p-linear tensors, $S_p''(V)$, and the symmetric powers of degree p, $S^p(V)$.

Consider, for example, the case where V has basis v_1, v_2, and $p = 3$. Then $S_3''(V)$ is 2-dimensional with basis $v_i \otimes v_i \otimes v_{3-i} + v_i \otimes v_{3-i} \otimes v_i + v_{3-i} \otimes v_i \otimes v_i = \phi(v_i^2 v_{3-i})/2$ for $i = 1, 2$. This contradicts the (erroneous) formula for $\dim S_p''(V)$ given in [Bou A1, Exercise 5b, Chapter III, §6]; a correct formula is implicit in the following result.

Proposition 3.6 *Suppose* char $k = p$. *Then the sequence*

$$0 \longrightarrow V^{[1]} \longrightarrow S^p(V) \xrightarrow{\phi} S_p''(V) \longrightarrow 0 \tag{3.3}$$

is exact, and $S_p'(V)$ is a direct sum of $S_p''(V)$ and the k-span of $\{\otimes^p v \mid v \in V\}$.

Proof For exactness, the only thing to check is exactness at the middle term $S^p(V)$. Fix a basis $v_1, \dots, v_d$ of V and form the corresponding basis

$$\mathscr{B} = \left\{ v_1^{e_1} v_2^{e_2} \cdots v_d^{e_d} \mid \sum e_i = p \right\}$$

of $S^p(V)$. We partition $\mathscr{B}$ as $X = \{v_i^p \mid 1 \leq i \leq d\}$ and $Y = \mathscr{B} \setminus X$. We may identify $V^{[1]}$ with the k-span of X, because, for any $v = \sum c_i v_i \in V$, we have $v^p = \sum c_i^p v_i^p$.

For an element $v_1^{e_1} v_2^{e_2} \cdots v_d^{e_d}$, we define

$$h := \Big(\bigotimes^{e_1} v_1\Big) \otimes \Big(\bigotimes^{e_2} v_2\Big) \otimes \cdots \otimes \Big(\bigotimes^{e_d} v_d\Big) \in \bigotimes^p V$$

and H to be the stabilizer in Σ_p of h. Clearly,

$$\phi(v_1^{e_1} v_2^{e_2} \cdots v_d^{e_d}) = |H| \cdot \sum_{\sigma \in \Sigma_p/H} \sigma h.$$

Therefore, for $v_i^p \in X$, we have $H = \Sigma_p$, so $\phi(v_i^p) = p! \cdot h = 0$. For $h = v_1^{e_1} \cdots v_d^{e_d} \in Y$, the size of H is $e_1! e_2! \cdots e_d!$, which is not divisible by p, hence $\phi(h)$ is in $k^\times \cdot \sum_{\sigma \in \Sigma_p/H} \sigma h$. For distinct cosets σH and $\sigma' H$ of H, the elements $\sigma H \cdot h$ and $\sigma' H \cdot h$ are linearly independent in $\otimes^p V$, so it follows that $\ker \phi = \operatorname{span}(X) = \operatorname{im} V^{[1]}$.

From this argument, the final claim is clear. ■

Replacing V with V^*, we may view sequence (3.3) as relating homogeneous polynomials of degree p and symmetric p-linear forms.

Definition 3.7 For k a field of characteristic p, a symmetric p-linear form f is *characteristic* if $f(v, v, \dots, v) = 0$ for all $v \in V$. For $p = 2$, a characteristic symmetric 2-linear form is nothing but an alternating bilinear form.

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

Corollary 3.8 *For every vector space V over a field of characteristic p, $S''_p(V^*)$ is the space of symmetric p-linear forms on V that are characteristic.*

Proof For $f \in S'_p(V^*)$, f is not characteristic if and only if $f(v_1, v_1, \dots, v_1) \neq 0$ for some $v_1 \in V$ if and only if with respect to some basis $v_1, \dots, v_n$ of V with dual basis $x_1, \dots, x_n$, when we write f in terms of the dual basis, we find a nonzero multiple of $\otimes^p x_1$. Proposition 3.6 gives the claim. ■

For a different view on Proposition 3.6 and Corollary 3.8, see [DV, Section 3, esp. Theorem 3.4].

Corollary 3.9 *Suppose* char $k = p$ *and V is a representation of an algebraic group G.*

(i) *If V has no codimension-1 G-submodules, then $\phi\colon S^p(V^*)^G \to S''_p(V^*)^G$ is injective.*
(ii) *If $\mathrm{H}^1(G, V^{*[1]}) = 0$, then $\phi\colon S^p(V^*)^G \to S''_p(V^*)^G$ is surjective.*

Part (ii), in the special case $p = 2$, can be found in [W, Satz 2.5].

Proof Taking the exact sequence (3.3), replacing V with V^*, and taking fixed submodules gives the exact sequence

$$(V^{*[1]})^G \longrightarrow S^p(V^*)^G \xrightarrow{\phi} S''_p(V^*)^G \longrightarrow \mathrm{H}^1(G, V^{*[1]})$$

From this, (ii) is clear. For (i), if $(V^*)^G = 0$, then $(V^{*[1]})^G = 0$. ■

4 Bilinear Forms on Irreducible and Weyl Modules

4.1 Bilinear Forms on Irreducible Modules

The proof in the case $k = \mathbb{C}$ given in [GW09, Th. 3.2.13, 3.2.14] shows that for every field k and every $\lambda \in X(T)_+$, $(\operatorname{Bil} L(\lambda))^G$ *is nonzero if and only if $\lambda = -w_0\lambda$, if and only if $(\operatorname{Bil} L(\lambda))^G = kb$ for some nondegenerate b.*

Suppose these conditions hold. If char $k \neq 2$, then the splitting of sequence (3.2) shows that b is symmetric or skew-symmetric. *If* char $k = 2$ *and $\lambda \neq 0$, then b is alternating.* Indeed, in that case b is symmetric with $\psi(b)$ totally singular by Lemma 3.5, but $\psi(b)$ is the zero quadratic form by Corollary 3.9(i), *i.e.,* b is alternating, as claimed.

4.2 Reducible Modules

The material in the preceding subsection is enough to determine $(\operatorname{Bil} V)^G$ when V is semisimple. We now consider arbitrary (finite-dimensional) representations V. Recall that the *socle*, denoted by soc V, is the largest semisimple submodule of V. The *head* of V, denoted by head V, is the maximal semisimple quotient; the kernel of the map $V \to \operatorname{head} V$ is the *radical* of V, denoted by rad V. The following observation has many applications.

Lemma 4.1 *Let $U \subseteq \operatorname{rad} V$. If U and $(\operatorname{head} V)^*$ have no common composition factors, then the pullback $(\operatorname{Bil} V/U)^G \to (\operatorname{Bil} V)^G$ is an isomorphism.*

Proof Suppose first that U is simple. The induced map $\operatorname{Bil}(V/U) \to \operatorname{Bil} V$ is obviously injective and G-equivariant. For every $\widehat{b} \in \operatorname{Hom}_G(V, V^*)$, we have

$$\widehat{b}(U) \subseteq \operatorname{soc}(V^*) = (\operatorname{head} V)^*,$$

so by hypothesis $\widehat{b}$ vanishes on U. Furthermore, $\widehat{b}$ induces a homomorphism $\operatorname{head} V \to \operatorname{head}(V^*) \to U^*$, which must also vanish by the hypothesis, hence every element of $\widehat{b}(V)$ vanishes on U, and $\widehat{b}$ is in the image of $\operatorname{Hom}_G(V/U, (V/U)^*) \to \operatorname{Hom}_G(V, V^*)$, as claimed.

The general case follows by induction on the length of a composition series for U, because for simple $U_0 \subseteq \operatorname{rad} V$ we have $\operatorname{rad}(V)/U_0 = \operatorname{rad}(V/U_0)$ and $\operatorname{head}(V/U_0) = \operatorname{head}(V)$. ∎

4.3 Weyl Modules

For each $\lambda \in X(T)_+$, the Weyl module $V(\lambda)$ has head $L(\lambda)$, and therefore $\operatorname{rad} V(\lambda)$ is the kernel of the map $V(\lambda) \to L(\lambda)$ from (1.1). Note that $\operatorname{rad} V(\lambda) = 0$ if and only if $V(\lambda)$ is irreducible if and only if all maps in (1.1) are isomorphisms.

Example 4.2 Let G be an adjoint group of type B_n over a field k, hence it is $\mathrm{SO}(V, q)$ for a quadratic form q on a $(2n+1)$-dimensional vector space V by [KMRT, p. 364]. The G-module V can be identified with the Weyl module $V(\omega_1)$. If $\operatorname{char} k = 2$, the radical of the bilinear form $\operatorname{rad} b_q$ (whose definition is recalled in Example 3.2) is 1-dimensional, the irreducible representation $L(\omega_1)$ is $V(\omega_1)/\operatorname{rad} b_q$, and $H^0(\lambda) = V^*$ has a unique proper submodule, $L(\omega_1)$.

Lemma 4.3 *For $\lambda \in X(T)_+$, the surjection $V(\lambda) \to L(\lambda)$ induces an isomorphism $(\operatorname{Bil} L(\lambda))^G \xrightarrow{\sim} (\operatorname{Bil} V(\lambda))^G$.*

Proof #1 $(\operatorname{Bil} V(\lambda))^G = \operatorname{Hom}_G(V(\lambda), H^0(-w_0\lambda))$, which is k (if and only if $\lambda = -w_0\lambda$) or 0. Thus, the induced map $(\operatorname{Bil} L(\lambda))^G \to (\operatorname{Bil} V(\lambda))^G$ is onto. ∎

Proof #2 Apply Lemma 4.1 with $U = \operatorname{rad} V(\lambda)$, using Lemma 5.1 to see that $L(-w_0\lambda)$ is not a composition factor of U. ∎

Definition 4.4 We say a representation V of G is *symplectic* if there is a nonzero G-invariant alternating bilinear form on V (*i.e.,* $\Lambda^2(V^*)^G \neq 0$) and V is *orthogonal* if $(\operatorname{Quad} V)^G \neq 0$. Clearly, if $\operatorname{char} k \neq 2$, a representation $V(\lambda)$ or $L(\lambda)$ can be symplectic or orthogonal or neither, but not both. If $\operatorname{char} k = 2$ and $\lambda \neq 0$, it can be symplectic, both orthogonal and symplectic, or neither.

4.4 Integral Models

As G is a split reductive group, there is a split reductive group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ such that $G_{\mathbb{Z}} \times k$ is isomorphic to G [Gro11]. Moreover, for each $\lambda \in X(T)_+$ there is a representation $V(\lambda, \mathbb{Z})$ of $G_{\mathbb{Z}}$ such that base change identifies $V(\lambda, \mathbb{Z}) \times k$ with the Weyl module $V(\lambda)$ of G over k. Consequently, it makes sense to write $V(\lambda, K)$ for the Weyl module $V(\lambda, \mathbb{Z}) \times K$ of $G_{\mathbb{Z}} \times K$, for any field K. We use this convention when we want to emphasize the field of definition.

Suppose that $\lambda = -w_0\lambda$, so $V(\lambda, \mathbb{C})$ is orthogonal or symplectic; the recipe described in [GW09] will tell which. Because $V(\lambda, \mathbb{Q})$ is also self-dual, there is a nonzero $(G_{\mathbb{Z}} \times \mathbb{Q})$-invariant bilinear form on $V(\lambda, \mathbb{Q})$, and by clearing denominators we assume it is indivisible and defined on $V(\lambda, \mathbb{Z})$ with values in $\mathbb{Z}$. From this we find that when char $k \neq 2$, $V(\lambda, k)$ is orthogonal (resp., symplectic) if and only if $V(\lambda, \mathbb{C})$ is orthogonal (resp., symplectic).

For any k, if $V(\lambda, \mathbb{C})$ is orthogonal, then we can similarly use the symmetric bilinear form on $V(\lambda, \mathbb{Q})$ to construct a $(G_{\mathbb{Z}} \times \mathbb{Q})$-invariant quadratic form that is nonzero and indivisible on $V(\lambda, \mathbb{Z})$ and so conclude that $V(\lambda, k)$ is orthogonal. The converse of this is false; see, for example, Proposition 8.5.

Entirely parallel remarks hold for the induced module, $H^0(\lambda)$.

4.5 Reduced Killing Form

Let G be a split *quasi-simple* group defined over $\mathbb{Z}$. The highest root $\widetilde{\alpha}$ is in $X(T)_+$, and the Weyl module $V(\widetilde{\alpha}, \mathbb{Z})$ is the Lie algebra $\widetilde{\mathfrak{g}}$ of the simply connected cover $\widetilde{G}$ of G [Ga, 2.5]. Dividing the Killing form of $\widetilde{\mathfrak{g}}$ by twice the dual Coxeter number $h^\vee$ gives an even and indivisible symmetric bilinear form (*cf.* [GN, p. 633] or [SpSt, pp. 180–181]) so there exists a unique indivisible quadratic form s so that $2h^\vee \cdot b_s$ is the Killing form κ. It is called the *reduced Killing quadratic form* on $\widetilde{\mathfrak{g}}$.

We now sketch how to determine the isomorphism class of s over any field k. It is harmless to assume that G is simply connected. The roots Φ and simple roots Π index a basis $\{h_\delta \mid \delta \in \Pi\} \cup \{x_\alpha, x_{-\alpha} \mid \alpha \in \Phi\}$ for Lie(G). As s is invariant under T, it vanishes on each $x_{\pm\alpha}$, and it quickly follows that s is an orthogonal sum of its restrictions to Lie(T) and $\mathbb{Z}x_\alpha + \mathbb{Z}x_{-\alpha}$ for each $\alpha \in \Phi$. Put r for the square-length ratio of long roots to short roots, so $r \in \{1, 2, 3\}$. The calculations in [SpSt, pp. 180, 181] show that $\mathbb{Z}x_\alpha + \mathbb{Z}x_{-\alpha}$ contributes a zero form to $s \otimes k$ if α is short and r is zero in k; otherwise, it contributes a hyperbolic plane.

As for the restriction to Lie(T), recall that there is a unique positive-definite quadratic form $q^\vee$ on the coroot lattice $Q^\vee$ (for the simple root system Φ of G) that takes the value 1 on short coroots and r on long coroots. Since s restricts to a Weyl-invariant form on Lie(T), the formulas in [SpSt] show that the restriction of s to Lie(T) is $q^\vee$. In summary, $s \otimes k$ is an orthogonal sum of hyperbolic planes, a zero form (if char $k \mid r$) and $q^\vee \otimes k$.

To calculate $q^\vee \otimes k$, fix a basis $\alpha_1^\vee, \dots, \alpha_n^\vee$ of simple coroots and set C to be the Cartan matrix with respect to the basis $\alpha_1, \dots, \alpha_n$ of simple roots. For D the diagonal matrix whose i-th diagonal entry is the square-length $q^\vee(\alpha_i^\vee)$ of $\alpha_i^\vee$, the product DC is a symmetric integer matrix with even entries on the diagonal, and

$$q^\vee(v) = \tfrac{1}{2} v^T DC v \text{ for } v \in Q^\vee. \tag{4.1}$$

In case char $k = 2$, formulas for the isometry class of s can be found in [BaC, §3].

4.6 A Uniserial Example

Let $\lambda, \mu \in X(T)_+$ such that $\mu = -w_0\mu$, and let V be a uniserial G-module with composition factors $L(\mu)$, $L(\lambda)$, $L(\mu)$. Note that V does not satisfy the hypotheses of Lemma 4.1.

Example 4.5 If $\lambda = -w_0\lambda$ and $\operatorname{Ext}^1_G(L(\lambda), L(\mu)) = k$, then $\dim(\operatorname{Bil} V)^G = 2$. To see why this is so, note that V^* is also uniserial with the same composition factors, and that the corresponding module diagrams in the sense of [BC] are "rigid" by the hypothesis on Ext, so by Proposition 6.5 of ibid. we can read off the elements of $\operatorname{Hom}_G(V, V^*)$ from the diagrams; clearly $\operatorname{Hom}_G(V, V^*)$ is 2-dimensional with a basis consisting of an isomorphism and a map that sends V onto $\operatorname{soc}(V^*)$.

The following provides a tool to check the Ext hypothesis in the example.

Lemma 4.6 *If $V(\lambda_2)$ has two composition factors $L(\lambda_2)$ and $L(\lambda_1)$ (with $L(\lambda_1)$ as the socle) and $H^0(\lambda_1) = L(\lambda_1)$, then $\operatorname{Ext}^1_G(L(\lambda_1), L(\lambda_2)) \cong k$.*

Proof Apply $\operatorname{Hom}_G(\cdot, L(\lambda_1))$ to the exact sequence

$$0 \longrightarrow L(\lambda_1) \longrightarrow V(\lambda_2) \longrightarrow L(\lambda_2) \longrightarrow 0$$

to get

$$\operatorname{Hom}_G\big(V(\lambda_2), L(\lambda_1)\big) \longrightarrow \operatorname{Hom}_G\big(L(\lambda_1), L(\lambda_1)\big) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_G\big(L(\lambda_2), L(\lambda_1)\big) \longrightarrow \operatorname{Ext}^1_G\big(V(\lambda_2), L(\lambda_1)\big).$$

The first term is zero because $\lambda_1 \neq \lambda_2$. The last term is zero by [Jan, Prop. II.4.16] because $L(\lambda_1)$ has a good filtration. ■

5 Bilinear Forms on Induced Modules

5.1 Induced Modules

The theory of bilinear forms on an induced module $H^0(\lambda)$ is notably different from that for irreducible and Weyl modules, and in general the forms on $H^0(\lambda)$ need not have much to do with $L(\lambda)$. We start with the following basic lemma.

Lemma 5.1 *If $\lambda \in P_+$ and $-w_0\lambda$ are comparable in the partial ordering on P, then $\lambda = -w_0\lambda$.*

Proof We have that $-w_0\lambda = \lambda + \sigma$ for σ a sum of positive roots or a sum of negative roots. Applying $-w_0$ to both sides and subtracting, we find that $\sigma = w_0\sigma$, but w_0 interchanges positive and negative roots, hence $\sigma = 0$, *i.e.*, $\lambda = -w_0\lambda$. ■

Then we find the following lemma.

Lemma 5.2 *If $H^0(\lambda)$ is reducible, then the pullback*

$$\big(\operatorname{Bil} H^0(\lambda)/L(\lambda)\big)^G \to \big(\operatorname{Bil} H^0(\lambda)\big)^G$$

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

is an isomorphism.

Proof The dual of head $H^0(\lambda)$ is the socle of $H^0(\lambda)^* = V(-w_0\lambda)$. By Lemma 5.1, λ cannot be less than $-w_0\lambda$, and therefore $L(\lambda)$ cannot be a component of the socle. The conclusion follows by Lemma 4.1. ∎

Example 5.3 Let G be a quasi-simple group. Then

$$\dim(\operatorname{Bil} H^0(\widetilde{\alpha}))^G = \begin{cases} 4 & \text{if } \operatorname{char} k = 2 \text{ and } G \text{ has type } D_n \text{ for } n \geq 4 \text{ and even,} \\ 2 & \text{if } \operatorname{char} k = 2 \text{ and } G \text{ has type } B_n \text{ or } C_n \text{ with } n \geq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Note that the dimension can be bigger than 1, unlike for irreducible and Weyl modules. To see the claim, we combine Lemma 5.2 with the preceding discussion and with the G-module structure of $V(\widetilde{\alpha}) = H^0(\widetilde{\alpha})^*$ given in [Hiss]. Put α_0 for the highest short root.

Suppose G has type B_n or C_n for $n \geq 2$ and $\operatorname{char} k = 2$. Then $H^0(\widetilde{\alpha})/L(\widetilde{\alpha})$ is either $k \oplus L(\alpha_0)$ or is uniserial with composition factors k, $L(\alpha_0)$, k as in Section 4.6. In the latter case, $V(\alpha_0)$ has socle k by [Hiss], so Lemma 4.6 applies, and in both cases we find $\dim(\operatorname{Bil} H^0(\widetilde{\alpha}))^G = 2$, as claimed.

If G has type F_4 and $\operatorname{char} k = 2$, or if G has type G_2 and $\operatorname{char} k = 3$, then $H^0(\widetilde{\alpha})/L(\widetilde{\alpha})$ is $L(\alpha_0)$, so $\dim(\operatorname{Bil} H^0(\widetilde{\alpha}))^G = 1$.

In the remaining cases, writing Z for the scheme-theoretic center of the simply connected cover of G, $H^0(\widetilde{\alpha})/L(\widetilde{\alpha}) \cong \operatorname{Lie}(Z)^*$, on which G acts trivially. If Z is not étale, then $\dim(\operatorname{Bil} H^0(\widetilde{\alpha}))^G = (\dim \operatorname{Lie}(Z))^2$, whence the claim.

Note that, in calculating the dimension of $(\operatorname{Bil} H^0(\widetilde{\alpha}))^G$, we implicitly gave formulas for all of the G-invariant bilinear forms.

We remark that for G of type A, D, or E, $H^0(\widetilde{\alpha})$ is the Lie algebra of the adjoint group [Ga, 3.5(2)].

5.2 A Necessary Condition

Lemma 5.4 *If there is a nonzero G-invariant bilinear form on $H^0(\lambda)$ or $T(\lambda)$, then 2λ is in the root lattice Q.*

Proof On the one hand, the action of the torus T on the representation $V = H^0(\lambda)$ or $T(\lambda)$ turns $V^* \otimes V^* = \operatorname{Bil}(V)$ into a graded vector space with grade group $X(T)$, and the hypothesis gives that 0 is a weight. On the other hand, all weights of V are congruent to λ mod the root lattice Q, hence all weights of $\operatorname{Bil}(V)$ are congruent to $-2w_0\lambda$ mod Q, so $-2w_0\lambda \in Q$. As $-w_0$ normalizes Q, the conclusion follows. ∎

Note that $-w_0$ acts on P/Q as -1, hence the condition $\lambda = -w_0\lambda$ (for the existence of a nonzero G-invariant bilinear form on $V(\lambda)$ or $L(\lambda)$) implies that $2\lambda \in Q$.

5.3 Connections with Cohomology

The following result demonstrates that the number of G-invariant bilinear forms on the induced module $H^0(\lambda)$ is related to the rational B-cohomology. In [HN], it was shown that similar B-cohomology calculations for $\mathrm{GL}_n(k)$ are related to the cohomology of Specht modules for the symmetric group on n letters.

Theorem 5.5 *Let $\lambda \in X(T)_+$. Then for $N = |\Phi|/2$,*

$$\begin{aligned}\dim(\mathrm{Bil}\, H^0(\lambda))^G &= \dim \mathrm{Ext}_B^N(H^0(\lambda), -\lambda - 2\rho)\\ &= \dim \mathrm{H}^N(B, V(-w_0\lambda) \otimes (-\lambda - 2\rho)).\end{aligned}$$

Proof Recall that $H^0(\lambda)^* = V(-w_0\lambda)$. Furthermore, by using Serre duality [Jan, II 4.2(9)],

$$V(-w_0\lambda) \cong H^N\big(w_0 \cdot (-w_0\lambda)\big) \cong H^N(-\lambda - 2\rho).$$

Consider the following spectral sequence [Jan, I.4.5]

$$E_2^{i,j} = \mathrm{Ext}_G^i\big(H^0(\lambda), R^j \mathrm{ind}_B^G\, (w_0 \cdot (-w_0\lambda))\big) \Rightarrow \mathrm{Ext}_B^{i+j}\big(H^0(\lambda), w_0 \cdot (-w_0\lambda)\big).$$

According to Serre duality [Jan, II.4.2(9)],

$$\begin{aligned}R^i\, \mathrm{ind}_B^G\big(w_0 \cdot (-w_0\lambda)\big) &\cong \big[R^{N-i}\, \mathrm{ind}_B^G\, (-(w_0 \cdot (-w_0\lambda) + 2\rho))\big]^*\\ &\cong \big[R^{N-i}\, \mathrm{ind}_B^G\, (\lambda)\big]^*.\end{aligned}$$

By assumption, $\lambda \in X(T)_+$, so by Kempf's vanishing theorem, $R^{N-i}\, \mathrm{ind}_B^G\, (\lambda) = 0$ when $N - i > 0$ (or $N > i$).

This shows that there is only one non-zero row in the spectral sequence; thus, the spectral sequence collapses and for all $i > 0$:

$$\begin{aligned}\mathrm{Ext}_G^i(H^0(\lambda), R^N\, \mathrm{ind}_B^G\, (w_0 \cdot (-w_0\lambda))) &\cong \mathrm{Ext}_B^{i+N}\big(H^0(\lambda), w_0 \cdot (-w_0\lambda)\big)\\ &\cong \mathrm{Ext}_B^{i+N}\big(H^0(\lambda), \lambda - 2\rho\big).\end{aligned}$$

The result now follows by specializing to the case when $i = 0$. ∎

5.4 SL_n Examples; Symmetric Powers

Let $G = \mathrm{SL}_n$ and consider $H^0(d\omega_1)$, where ω_1 is the first fundamental weight. We have $H^0(d\omega_1) \cong S^d(V)$ where V is the natural n-dimensional representation.

Example 5.6 For $G = \mathrm{SL}_2$, $H^0(d\omega_1)^* \cong V(d\omega_1)$ has a simple socle [Jan, II.5.16 Corollary] and $H^0(d\omega_1)$ is multiplicity-free as a G-module (note the weight spaces are all one-dimensional). So taking $U = \mathrm{rad}\, H^0(d\omega_1)$ in Lemma 4.1 gives

$$\big(\mathrm{Bil}\, H^0(d\omega_1)\big)^G = \big(\mathrm{Bil}\, \mathrm{head}\, H^0(d\omega_1)\big)^G \cong k.$$

That is, base change from the integral model as in Section 4.4 provides a nonzero SL_2-invariant bilinear form on $S^d(k^2)$, and it is the only one up to a factor in $k^\times$.

Example 5.7 Let $G = \mathrm{SL}_n$ where $n \geq 3$. The G-modules $H^0(d\omega_1)$ can have complicated submodule structures. However, one can employ the weight criterion in Corollary 5.4 to deduce the following: If $n \nmid 2d$, then $2d\omega_1 \notin \mathbb{Z}\Phi$, and

$$\dim(\operatorname{Bil} H^0(d\omega_1))^G = 0.$$

5.5 SL_3 Examples

Example 5.8 Let $G = \mathrm{SL}_3$ and let λ be a generic weight in the lowest p^2-alcove (see [DS]). Then the following are true: (i) the composition factors of $H^0(\lambda)$ are multiplicity free and (ii) the head of $H^0(\lambda)$ is a single irreducible representation. Moreover, if $\lambda \neq -w_0\lambda$, then the head of $H^0(\lambda)$ is not a composition factor of $V(-w_0\lambda)$. Therefore, in this case, $\dim(\operatorname{Bil} H^0(\lambda))^G = 0$.

Suppose $p = \operatorname{char} k \geq 3$ and consider the following weights of $G = \mathrm{SL}_3$ expressed in terms of the fundamental weights:

$$\begin{aligned}
\lambda_1 &= (0,0),\\
\lambda_2 &= s_{\alpha_1+\alpha_2, p(\alpha_1+\alpha_2)} \cdot \lambda_1 = (p-2, p-2),\\
\lambda_3 &= s_{\alpha_1, p\alpha_1} \cdot \lambda_2 = (p, p-3),\\
\lambda_4 &= s_{\alpha_2, p\alpha_2} \cdot \lambda_2 = (p-3, p) = -w_0\lambda_3.
\end{aligned}$$

We have indicated how λ_j, for $j = 1,2,3$ are linked to $(0,0)$ under the dot action of the affine Weyl group W_p.

By using the standard translation functor arguments [Jan, II.7.19, II.7.20] or employing [DS] we can deduce the following facts. The representation $H^0(\lambda_2)$ is uniserial with two composition factors (from head to socle): $L(0,0)$, $L(\lambda_2)$ For $j = 3,4$, $H^0(\lambda_j)$ is uniserial with two composition factors (from head to socle): $L(\lambda_2)$, $L(\lambda_j)$.

Example 5.9 For $j = 2,3,4$,

$$\left(\operatorname{Bil} H^0(\lambda_j)\right)^G = \left(\operatorname{Bil} \operatorname{head} H^0(\lambda_j)\right)^G \cong k$$

by Lemma 4.1. Pulling back along the surjection $T(\lambda_3) \to H^0(\lambda_3)$ from (1.1) gives a nonzero G-invariant bilinear form on the tilting module $T(\lambda_3)$ also. (In fact it generates $(\operatorname{Bil} T(\lambda_3))^G$ by Example 6.4 below.) That is, $H^0(\lambda_3)$ and $T(\lambda_3)$ each have a nonzero G-invariant bilinear form, yet $\lambda_3 \neq -w_0\lambda_3$, in contrast with the situation for simple and Weyl modules described in Section 4.

6 Bilinear Forms on Tilting Modules

6.1 Tilting Modules

The tilting module $T(\lambda)$ has both a good and Weyl filtration, and the composition factor of highest weight in $T(\lambda)$ is $L(\lambda)$. We briefly discuss the maps induced by applying the functor $V \mapsto (\operatorname{Bil} V)^G$ to (1.1).

Lemma 6.1 *For all $\lambda \in X(T)_+$, the pullback map $(\mathrm{Bil}\, T(\lambda))^G \to (\mathrm{Bil}\, V(\lambda))^G$ is surjective. If $T(\lambda)$ is reducible, then the composition*

$$(\mathrm{Bil}\, H^0(\lambda))^G \longrightarrow (\mathrm{Bil}\, T(\lambda))^G \longrightarrow (\mathrm{Bil}\, V(\lambda))^G$$

is zero.

Proof Assume $(\mathrm{Bil}\, V(\lambda))^G \neq 0$, so $\lambda = -w_0\lambda$ and there exists an isomorphism $f: T(\lambda) \xrightarrow{\sim} T(-w_0\lambda) = T(\lambda)^*$. As λ and $-w_0\lambda$ are weights of $T(\lambda)$ (and $V(\lambda)$) of multiplicity 1, the pullback of f to $V(\lambda)$ is nonzero, proving the first claim. Commutativity of (1.1) and Lemma 5.2 give the second claim. ■

The lemma shows that when $T(\lambda)$ is reducible:

(6.1) $$\dim(\mathrm{Bil}\, T(\lambda))^G \geq \dim(\mathrm{Bil}\, H^0(\lambda))^G + \dim(\mathrm{Bil}\, V(\lambda))^G.$$

6.2 Dimensions and Filtration Multiplicities

We now compute the dimension of the space of G-invariant bilinear forms on $T(\lambda)$ in terms of the good filtration multiplicities. Define $[T(\lambda):H^0(\mu)]$ to be the number of times $H^0(\mu)$ appears in a good filtration for $T(\lambda)$; it equals $\mathrm{Hom}_G(V(\mu), T(\lambda))$ by [Jan, Prop. II.4.16(a)], so it is independent of the choice of good filtration.

Theorem 6.2 *Let G be a reductive algebraic group. Then for each $\lambda \in X(T)_+$,*

(i) $\dim(\mathrm{Bil}\, T(\lambda))^G = \sum_{\mu \in X(T)_+} [T(-w_0\lambda):H^0(\mu)][T(\lambda):H^0(\mu)]$;

(ii) *if $\lambda = -w_0\lambda$, then* $\dim(\mathrm{Bil}\, T(\lambda))^G = \sum_{\mu \in X(T)_+} [T(\lambda):H^0(\mu)]^2$.

Proof (i) According to [Jan, II 4.13 Proposition], $\mathrm{Ext}^1_G(V(\sigma_1), H^0(\sigma_2)) = 0$ for all $\sigma_1, \sigma_2 \in X(T)_+$. Since $T(\lambda)$ has a Weyl filtration, it follows that

$$\mathrm{Ext}^1_G(T(\lambda), H^0(\sigma)) = 0 \text{ for all } \lambda, \sigma \in X(T)_+.$$

Therefore, the functor $\mathrm{Hom}_G(T(\lambda), -)$ is exact on short exact sequences of modules that admit good filtrations. Now $T(-w_0\lambda)$ admits a good filtration, thus

$$\begin{aligned}\dim(\mathrm{Bil}\, T(\lambda))^G &= \dim \mathrm{Hom}_G\big(T(\lambda), T(-w_0\lambda)\big)\\ &= \sum_{\mu \in X(T)_+} \big[T(-w_0\lambda):H^0(\mu)\big] \dim \mathrm{Hom}_G\big(T(\lambda), H^0(\mu)\big).\end{aligned}$$

Because $T(\lambda) = T(\lambda)^\tau$ and $H^0(\lambda) = V(\lambda)^\tau$ under the duality τ defined in [Jan, II 2.12, 2.13], $\mathrm{Hom}_G(T(\lambda), H^0(\mu)) = \mathrm{Hom}_G(V(\mu), T(\lambda))$ and (i) follows. Part (ii) follows immediately from (i). ■

In the sums in Theorem 6.2, the $\mu = \lambda$ term contributes 0 if $\lambda \neq -w_0\lambda$ (by Lemma 5.1) and 1 if $\lambda = -w_0\lambda$. In either case, the sum restricted to $\mu \neq \lambda$ gives the dimension of the kernel of the pullback $(\mathrm{Bil}\, T(\lambda))^G \to (\mathrm{Bil}\, V(\lambda))^G$.

6.3 SL_3 Examples

Let $G = \mathrm{SL}_3$ with $p \geq 3$ and let λ_j for $j = 1, 2, 3, 4$ be as in Section 5.5.

Example 6.3 The tilting module $T(\lambda_2)$ is uniserial with composition factors (from the head to the socle):

$$L(0,0),\ L(\lambda_2),\ L(0,0).$$

As in Section 4.6, we find that $\dim(\mathrm{Bil}\, T(\lambda_2))^G = 2$, which agrees with the $1^2 + 1^2 = 2$ provided by Theorem 6.2(ii), and we find equality in (6.1).

Example 6.4 The structure of the tilting module $T(\lambda_j)$ for $j = 3, 4$ is given by the following diagrams in the style of [BC], with the head on top and the socle on the bottom:

$$\begin{tikzcd}[arrows=dash] & L(\lambda_2) \arrow[dl] \arrow[dr] & \\ L(\lambda_j) \arrow[dr] & & L(0,0) \arrow[dl] \\ & L(\lambda_2) & \end{tikzcd}$$

Therefore, $\dim(\mathrm{Bil}\, T(\lambda_j))^G = 1$, which agrees with Theorem 6.2(i) by looking at the structure of the tilting module above with its good filtration factors $H^0(\lambda_j)$, $H^0(\lambda_2)$.

Remark 6.5 Andersen, Stoppel, and Tubbenhaur [AST, Th. 4.11] have recently proved that $A_\lambda := \mathrm{End}_G(T(\lambda))$ for $\lambda \in X(T)_+$ is a cellular algebra. Under composition, $\mathrm{Bil}\, T(\lambda)$ becomes a A_λ-$A_{-w_0\lambda}$ bimodule. As a left A_λ-module, $\mathrm{Bil}\, T(\lambda)$ admits a filtration of cell modules, and as a right $A_{-w_0\lambda}$-module, it admits a filtration of dual cell modules. The reader is referred to [AST, Def. 5.1] for the definitions of these modules.

7 Quadratic Forms on Tilting Modules

7.1 Modules with a Good Filtration

We characterize symmetric G-invariant p-linear forms for tilting modules and Weyl modules via cohomological vanishing.

Proposition 7.1 *Let G be a simple split algebraic group and let V be a finite-dimensional G-module such that V^* admits a good filtration. Assume further that if $p = 2$ and G is of type C then $[V^* : H^0(\omega_1)] = 0$. Then every G-invariant characteristic symmetric p-linear form on V is the polarization of a G-invariant homogeneous polynomial of degree p on V.*

Proof By Corollary 3.9(ii), it suffices to prove that $E_1 := \mathrm{H}^1(G, (V^*)^{[1]}) = 0$. Apply the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{i,j} = \mathrm{H}^i(G/G_1, \mathrm{H}^j(G_1, k) \otimes (V^*)^{[1]}) \Rightarrow \mathrm{H}^{i+j}(G, (V^*)^{[1]}).$$

with the five term exact sequence $0 \to E_2^{1,0} \to E_1 \to E_2^{0,1} \to E_2^{2,0} \to E_2$.

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

Since V^* admits a good filtration, it follows that $E_2^{i,0} = 0$ for $i \geq 1$. Therefore,

$$\begin{aligned}\mathrm{H}^1(G,(V^*)^{[1]}) &\cong \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(k,k) \otimes (V^*)^{[1]}) \\ &\cong \mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(k,k)^{(-1)} \otimes V^*).\end{aligned}$$

Now by [BNP, Theorem 3.1(C)(f)],

$$\mathrm{Ext}^1_{G_1}(k,k)^{(-1)} \cong \begin{cases} H^0(\omega_1) & p = 2 \text{ and } \Phi = C_n \\ 0 & \text{else,}\end{cases}$$

hence the claim holds apart from the exceptional case. In the exceptional case,

$$(H^0(\omega_1) \otimes V^*)^G = \mathrm{Hom}_G(V, H^0(\omega_1)) = \mathrm{Hom}_G(V(\omega_1), V^*),$$

whose dimension equals $[V^* : H^0(\omega_1)]$, so again the claim follows. ■

7.2 Tilting Modules

Taking $p = 2$ in the proposition and specializing to tilting modules gives the following corollary.

Corollary 7.2 *Let G be a simple split algebraic group over a field k of characteristic* 2*, and let $\lambda \in X(T)_+$. Assume further, in the case that G has type C, that*

$$[\,T(\lambda)^* : H^0(\omega_1)\,] = 0.$$

Then every G-invariant alternating bilinear form on $T(\lambda)$ is the polarization of a G-invariant quadratic form on V.

Proof $T(\lambda)^* = T(-w_0\lambda)$ has a good filtration, so Proposition 7.1 yields the result. ■

8 Exterior Powers of Alternating Forms

8.1 Existence of Quadratic Forms

In this section, b denotes a nondegenerate alternating form on a vector space V of finite dimension $2n$ over a field k. We analyze quadratic forms on $\Lambda^r(V)$ induced by b for various r. This is related to quadratic forms on the fundamental representations of the group $\mathrm{Sp}(V,b)$ of type C.

It is well known that there is a bilinear form $b_{(r)}$ on $\Lambda^r(V)$ defined by

$$b_{(r)}(x_1 \wedge \cdots \wedge x_r, y_1 \wedge \cdots \wedge y_r) := \det(b(x_i, y_j)_{1 \leq i,j \leq r}), \tag{8.1}$$

and that $b_{(r)}$ is non-degenerate, since b is [Bou A9, Prop. IX.1.9.10]. If k has characteristic different from 2, then evidently $b_{(r)}$ is symmetric for even r and skew-symmetric for odd r. If k has characteristic 2, we consider quadratic forms.

Proposition 8.1 *Let* char k = 2 *and let r be odd. For a fixed symplectic basis $\mathscr{B}$ of V, there is a quadratic form $q_{(r)}$ on $\Lambda^r(V)$ with polar bilinear form $b_{(r)}$ such that $q_{(r)}(v_1 \wedge \cdots \wedge v_r) = 0$ for $v_1, \ldots, v_r \in \mathscr{B}$.*

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

Proof As char $k = 2$, we can write out $\mathscr{B}$ as $e_1, \dots, e_n, f_1, \dots, f_n$ such that

$$b(e_i, e_j) = b(f_i, f_j) = 0 \text{ for all } i, j \text{ and } b(e_i, f_j) = \delta_{ij}. \tag{8.2}$$

Write $V_{\mathbb{Z}}$ for a free $\mathbb{Z}$-module of rank $2n$ whose basis we denote also by $\mathscr{B}$ by abuse of notation; we may equally define a *symmetric* bilinear form $b_{\mathbb{Z}}$ on $V_{\mathbb{Z}}$ by (8.2) so that we may identify b with $b_{\mathbb{Z}} \otimes k$. Then $(b_{\mathbb{Z}})_{(r)}$ is a symmetric bilinear form on $\Lambda^r(V_{\mathbb{Z}})$ and the map $f: x \mapsto (b_{\mathbb{Z}})_{(r)}(x, x)$ is a quadratic form on $\Lambda^r(V_{\mathbb{Z}})$.

We claim that f always takes even values. As $\Lambda^r(V_{\mathbb{Z}})$ is generated as an abelian group by symbols whose entries are taken from the symplectic basis, it suffices to verify that $f(x)$ is even when x is such a symbol. But $(b_{\mathbb{Z}})_{(r)}(x, x)$ can only be nonzero for such an x if for every e_i in x there is also a corresponding f_i and vice versa. As r is odd, this is impossible and the claim is proved.

As f is a homogeneous polynomial of degree 2 (with integer coefficients) in the basis dual to the bases of $\wedge^r V_{\mathbb{Z}}$ consisting of symbols with entries from $\mathscr{B}$, it follows that f is divisible by 2. The desired quadratic form on $\Lambda^r(V) = \Lambda^r(V_{\mathbb{Z}}) \otimes k$ is then $q_{(r)} := (\frac{1}{2}f) \otimes k$. By construction, $q_{(r)}$ has polar bilinear form $(b_{\mathbb{Z}})_{(r)} \otimes k = b_{(r)}$, and by the preceding paragraph $q_{(r)}$ vanishes on symbols with entries from $\mathscr{B}$ as desired. ∎

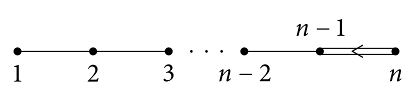
Remark 8.2 In the case where r is even and $2 \le r \le \dim V$, the bilinear form $b_{(r)}$ is symmetric, but it is not alternating because for $s = r/2$ and $x = e_1 \wedge f_1 \wedge e_2 \wedge f_2 \wedge \cdots \wedge e_s \wedge f_s$ we have $b_{(r)}(x, x) = \pm 1$. Therefore $x \mapsto b_{(r)}(x, x)$ is a nonzero quadratic form on $\Lambda^r(V)$ which is obviously invariant under $\mathrm{Sp}(V, b)$.

Remark 8.3 When char $k = 2$, q is a quadratic form on V, and r is odd, [MR, Th. 1.6.2] defines a quadratic form $\Lambda^r q$ on $\Lambda^r V$. Taking b to be the polar form of q, we find that $\Lambda^r q$ equals the form $q_{(r)}$ from Proposition (8.1). In particular, the isomorphism class of $\Lambda^r q$ depends only on the bilinear form b and not on the choice of q such that $b = \phi(q)$.

8.2 Fundamental Weyl Modules for Type C

As in [Bou L4], we write ω_r for the fundamental dominant weight such that $\langle \omega_r, \alpha_j \rangle = \delta_{rj}$, where α_j is the simple root numbered j in the diagram

(8.3)



We can see explicitly which of the fundamental Weyl modules $V(\omega_r)$ of $\mathrm{Sp}(V, b)$ are orthogonal. Some of these are easy: for r even, $V(\omega_r, \mathbb{C})$ is orthogonal, hence so is $V(\omega_r, k)$ for every k.

Example 8.4 $V(\omega_1)$ is the tautological representation V of $\mathrm{Sp}(V, b)$, and $\mathrm{Sp}(V, b)$ acts transitively on the nonzero vectors. It follows that the only $\mathrm{Sp}(V, b)$-invariant polynomials are constant, hence $V(\omega_1, k)$ is not orthogonal for any k.

For the remaining cases, where r is odd and $3 \le r \le n$, we identify $V(\omega_r)$ with the subspace of $\Lambda^r(V)$ generated by symbols $v_1 \wedge \cdots \wedge v_r$ such that $v_1, \dots, v_r$ generate a totally isotropic subspace of V as in [GowK, §1] or [De B]. We call such symbols *of generator type*.

Proposition 8.5 *If* $\operatorname{char} k = 2$, *$r$ is odd, and* $3 \le r \le n$, *then the quadratic form* $q_{(r)}$ *defined in Lemma 8.1 restricts to be* $\mathrm{Sp}(V, b)$*-invariant and nonzero on the Weyl module* $V(\omega_r)$.

Proof Let x be a symbol of generator type, where each entry in the symbol belongs to $\mathscr{B}$. The basis $\mathscr{B}$ defines a pinning of $\mathrm{Sp}(V, b)$ as in [Bou L7, §VIII.13.3] of a root subgroup relative to the pinning, so we can decompose V as an orthogonal sum $V = U \perp U'$ relative to b where $g(U) = U$, g is the identity on U', and U consists of $s = 1$ or 2 of the hyperbolic planes defined by $\mathscr{B}$. We write $x = y \wedge y'$, where $y \in \Lambda^t(U)$ and $y' \in \Lambda^{r-t}(U')$ are symbols of generator type and $t \le s$, because x has generator type. Writing $gy = \sum y_i$ where the y_i are symbols in $\Lambda^t(U)$ with entries from $\mathscr{B}$, we find

$$q_{(r)}(gx) = q_{(r)}\Big(\sum y_i \wedge y'\Big) = \sum_{i<j} b_{(r)}(y_i \wedge y', y_j \wedge y').$$

As $r \ge 3$, $r - t \ge 1$, and (8.1) shows that $b_{(r)}(y_i \wedge y', y_j \wedge y') = 0$. That is, $q(gx) = 0$. Furthermore, if g is instead taken to be in the maximal torus of $\mathrm{Sp}(V, b)$ defined by the pinning, then it scales x and again $q(gx) = 0$. It follows from these two calculations that $q(gx) = 0$ for all $g \in \mathrm{Sp}(V, b)$, hence $q_{(r)}$ vanishes on symbols of generator type, regardless of whether their entries are drawn from $\mathscr{B}$.

Any element z of $V(\omega_r)$ can be written as $z = \sum z_i$ for z_i symbols of generator type. For any $g \in \mathrm{Sp}(V, b)$, we have

$$q_{(r)}(gz) = \sum_i q_{(r)}(gz_i) + \sum_{i<j} b_{(r)}(gz_i, gz_j).$$

As gz_i also has generator type, $q_{(r)}(gz_i) = 0$. Furthermore, $b_{(r)}$ is canonically determined by b and so is $\mathrm{Sp}(V, b)$-invariant, so it follows that

$$q_{(r)}(gz) = \sum_{i<j} b_{(r)}(z_i, z_j) = q_{(r)}(z),$$

as desired. ■

9 Which Weyl Modules are Orthogonal when $\operatorname{char} k = 2$?

The goal of this section is to prove Theorem 9.5, which determines, in case $\operatorname{char} k = 2$, which Weyl modules have nonzero G-invariant quadratic forms. In case $\operatorname{char} k \neq 2$, quadratic forms are equivalent to symmetric bilinear forms and the answer is given by the material in Section 4 and the recipe described in [GW09] or [Bou L7].

9.1 Alternating Forms

Composing linear maps defined in Section 3, we obtain, for vector spaces V_1, V_2, a linear map,

$$\varpi\colon \Big(\bigotimes^2 V_1\Big) \otimes \Big(\bigotimes^2 V_2\Big) \xrightarrow{\mathrm{Id}\otimes\alpha} \Big(\bigotimes^2 V_1\Big) \otimes \Big(\bigotimes^2 V_2\Big) \xrightarrow{\sim} \bigotimes^2 (V_1 \otimes V_2) \xrightarrow{\rho} S^2(V_1 \otimes V_2),$$

where the middle isomorphism is $v_1 \otimes v_1' \otimes v_2 \otimes v_2' \mapsto v_1 \otimes v_2 \otimes v_1' \otimes v_2'$. Replacing the first map $\mathrm{Id}\otimes\alpha$ with $\alpha \otimes \mathrm{Id}$ does not change ϖ. Put $e_i\colon \bigotimes^2 V_i \twoheadrightarrow \Lambda^2(V_i)$ for the quotient map.

Lemma 9.1 (Tignol's product) *The map ϖ vanishes on $\ker(e_1 \otimes e_2)$ and induces a linear map $\Lambda^2(V_1) \otimes \Lambda^2(V_2) \to S^2(V_1 \otimes V_2)$.*

Proof For $i = 1, 2$, the map α vanishes on the subspace $\mathscr{I}_2''(V_i)$ of $\bigotimes^2 V_i$ spanned by elements $v \otimes v$ for $v \in V_i$, so ϖ vanishes on $(\bigotimes^2 V_1) \otimes \mathscr{I}_2''(V_2) = \ker(\mathrm{Id} \otimes e_2)$. Because replacing the first map in the definition of ϖ does not change ϖ, ϖ also vanishes on $\mathscr{I}_2''(V_1) \otimes (\bigotimes^2 V_2)$, and we conclude that ϖ vanishes on the kernel of $e_1 \otimes e_2$. ∎

The following proposition in the case char $k = 2$ can be found in [SinW, §3], or see [KMRT, p. 67, Ex. 21]. Our proof invokes Tignol's product from Lemma 9.1.

Proposition 9.2 *If V_1, V_2 are k-vector spaces with alternating bilinear forms b_1, b_2, then there is a unique quadratic form q on $V_1 \otimes V_2$ so that*

$$q\Big(\sum_i v_{1i} \otimes v_{2i}\Big) = \sum_{i<j} b_1(v_{1i}, v_{1j})\, b_2(v_{2i}, v_{2j}) \quad \text{for } v_{\ell i} \in V_\ell. \tag{9.1}$$

If b_1 and b_2 are nondegenerate, then so is q. If b_i is invariant under a group G_i, then q is invariant under $G_1 \times G_2$.

Proof The alternating form b_i determines an element of $\Lambda^2(V_i^*)$. Plugging $b_1 \otimes b_2$ into Tignol's product gives an element $q \in S^2(V_1^* \otimes V_2^*) = S^2((V_1 \otimes V_2)^*)$, *i.e.*, with a quadratic form on $V_1 \otimes V_2$. ∎

Corollary 9.3 *Let $\lambda, \mu \in X(T)_+$. If the Weyl modules $V(\lambda), V(\mu)$ have nonzero G-invariant alternating bilinear forms, then $V(\lambda + \mu)$ is orthogonal.*

Proof The tensor product $V(\lambda) \otimes V(\mu)$ has $\lambda + \mu$ as an extreme weight, and so there is a nonzero G-equivariant map $\pi\colon V(\lambda + \mu) \to V(\lambda) \otimes V(\mu)$. The hypothesis on $V(\lambda)$ and $V(\mu)$ gives a G-invariant quadratic form q on the tensor product by Proposition 9.2, hence $q\pi$ is a G-invariant quadratic form on $V(\lambda+\mu)$, and it suffices to check that $q\pi$ is nonzero.

Fix highest weight vectors x^+, y^+ and lowest weight vectors x^-, y^- of $V(\lambda), V(\mu)$ respectively; note that the bilinear forms are nonzero on the pairs (x^+, x^-) and

(y^+, y^-) as in Lemma 4.3. The sum of the highest and lowest weight spaces of $V(\lambda+\mu)$ is identified via π with $k(x^+ \otimes y^+) + k(x^- \otimes y^-)$, and $q(x^+ \otimes y^+ + x^- \otimes y^-)$ is nonzero by (9.1). ∎

9.2 Orthogonality and Weyl Modules

We continue by providing below some examples for symplectic groups.

Example 9.4 Let G be a quotient of $T_0 \times \prod_i G_i$ as in 2.2 such that $G_j \cong \mathrm{Sp}_{2n}$ for some $n \geq 1$ and some j, and $\lambda \in X(T)_+$ such that λ restricts to be zero on T_0 and G_i for $i \neq j$, but on G_j it is ω_1. Then by Lemma 2.2, the restriction of $V(\lambda)$ to G_j is the tautological representation as in Example 8.4 and the restriction to T_0 and to G_i for $i \neq j$ is a trivial representation. Therefore, $k[V(\lambda)]^G = k[V(\omega_1)]^{\mathrm{Sp}_{2n}} = k$ and $V(\lambda)$ is symplectic (*i.e.,* $\Lambda^2(V(\lambda)^*)^G \neq 0$) and not orthogonal (*i.e.,* $(\mathrm{Quad}\, V(\lambda))^G = 0$).

Theorem 9.5 *Let G be a split reductive group over a field k of characteristic* 2 *and let $\lambda \in X(T)_+$ be nonzero. Then exactly one of the following holds:*

(i) *$V(\lambda)$ is orthogonal (*i.e., $(\mathrm{Quad}\, V(\lambda))^G \neq 0$*), or*
(ii) *G and $V(\lambda)$ are as in Example 9.4, or*
(iii) *$\lambda \neq -w_0\lambda$.*

Furthermore, if (i) *occurs, then $(\mathrm{Quad}\, V(\lambda))^G = kq$ for a quadratic from q with $\mathrm{rad}\, b_q = \mathrm{rad}\, V(\lambda)$ and one of the following holds:*

(a) *$\mathrm{rad}\, q = \mathrm{rad}\, V(\lambda)$, $(\mathrm{Quad}\, L(\lambda))^G$ is* 1*-dimensional and spanned by the quadratic form $\overline{q}$ induced by q, and $\mathrm{rad}\, b_{\overline{q}} = 0$; or*
(b) *$(\mathrm{Quad}\, L(\lambda))^G = 0$, and $\mathrm{rad}\, q$ is a codimension-*1 *G-submodule of $\mathrm{rad}\, V(\lambda)$.*

Proof Suppose first that (ii) and (iii) are false; we prove (i). As in Section 2.2, without loss of generality we can assume that G is of the form $T_0 \times \prod_{i=1}^n G_i$ where T_0 is a split torus and the G_i are simple, split, simply connected algebraic groups over k, and write $\lambda = \sum_{i=0}^n \lambda_i$. As $\lambda = -w_0\lambda$ and w_0 is the product of the longest element in the Weyl groups for each of the G_i, it follows that λ_i has the same property for all i and in particular that $\lambda_0 = 0$. If $\lambda_i \neq 0$, then $V(\lambda_i)$ has a nonzero alternating bilinear form by Lemma 4.3. Hence, if two or more λ_i's are nonzero, Corollary 9.3 and Lemma 2.2 combine to give (i).

So assume $\lambda = \lambda_1$. As T_0 and G_i for $i \neq 1$ act trivially on $V(\lambda)$, we may assume that $G = G_1$, *i.e.,* that G is simple. By hypothesis, $(\mathrm{Bil}\, V(\lambda))^G \neq 0$, and Proposition 7.1 completes the proof of (i).

Now suppose that (i) holds. Because $\lambda \neq 0$, for $V = V(\lambda)$ or $L(\lambda)$, V lacks a codimension-1 submodule, hence polarization gives an injection $(\mathrm{Quad}\, V)^G \hookrightarrow \Lambda^2(V^*)^G$ by Corollary 3.9(i), thus (i) implies not (iii) completing the proof of the first claim, and $\dim(\mathrm{Quad}\, V)^G \leq 1$. Let $q \in (\mathrm{Quad}\, V(\lambda))^G$ be nonzero. If $\mathrm{rad}\, b_q = \mathrm{rad}\, q$, then the remaining claims in (a) are obvious, so suppose $q|_{\mathrm{rad}\, V(\lambda)}$ is not identically zero. Recall from Section 4.4 that G and $V(\lambda)$ are defined over $\mathbb{F}_2$. The natural map $\mathbb{F}_2[V(\lambda, \mathbb{F}_2)]^G \otimes k \to k[V(\lambda, k)]^G$ is an isomorphism [Se], so q is the base change from $\mathbb{F}_2$ of a G-invariant quadratic form q_0 on $V(\lambda, \mathbb{F}_2)$. We take X_0 to be the kernel of q_0 restricted to the radical of $V(\lambda, \mathbb{F}_2)$; it has codimension at most one because $\mathbb{F}_2$ is perfect. Then the radical X of q is a proper subspace of $\operatorname{rad} V(\lambda)$ and contains $X_0 \otimes k$, hence $\dim(\operatorname{rad} V(\lambda)/X) = 1$, and the claim is proved. ∎

9.3 Meaning of Orthogonality for Weyl Modules

If $L(\lambda)$ is orthogonal, then the action of G on $L(\lambda)$ is a homomorphism $G \to \mathrm{SO}(q)$ for some G-invariant quadratic form q, and $\mathrm{SO}(q)$ is a semisimple group of type B or D. We now describe the relevant group schemes in the case of a G-invariant quadratic form q on the Weyl module $V(\lambda)$.

Suppose for the moment that f is a polynomial function on a vector space V over an infinite field and $U < V$ is a subspace such that $f(u+v) = f(v)$ for all $u \in U$ and $v \in V$. Then f induces canonically a polynomial function $\overline{f}$ on $\overline{V} := V/U$, and we define $\mathrm{O}(\overline{f})$ to be the closed subgroup scheme of $\mathrm{GL}(\overline{V})$ stabilizing $\overline{f}$. Similarly, set $\mathrm{O}(f, U)$ to be the sub-group-scheme of $\mathrm{GL}(V)$ leaving both f and U invariant.

Lemma 9.6 *In the notation of the preceding paragraph, there is a short exact sequence of group schemes*

$$1 \longrightarrow \mathrm{Hom}(V/U, U) \longrightarrow \mathrm{O}(f, U) \longrightarrow \mathrm{O}(\overline{f}) \times \mathrm{GL}(U) \longrightarrow 1.$$

Proof Choosing any complement U' of U in V and writing linear transformations of V in terms of a basis adapted to the decomposition $V = U \oplus U'$, we find that for every k-algebra S,

$$\mathrm{O}(f, U)(S) = \begin{pmatrix} \mathrm{GL}(U)(S) & \mathrm{Hom}(U' \otimes S, U \otimes S) \\ 0 & \mathrm{O}(\overline{f})(S) \end{pmatrix}.$$ ∎

Now suppose that q is a quadratic form on V, and that $\dim(\operatorname{rad} b_q / \operatorname{rad} q) \le 1$. This holds if $\operatorname{char} k \neq 2$, or if k is perfect, or by Theorem 9.5 if V is a Weyl module for a split reductive group G and q is G-invariant. Then we take $U := \operatorname{rad} q$; the hypothesis on the dimension assures us that for every extension K of k, $\operatorname{rad}(q \otimes K) = (\operatorname{rad} q) \otimes K = U \otimes K$, and we find that $\mathrm{O}(q) = \mathrm{O}(q, U)$ as group schemes. Therefore, for $\overline{q}$ the induced quadratic from on V/U, we have an exact sequence

$$1 \longrightarrow \mathrm{Hom}(V/U, U) \longrightarrow \mathrm{O}(q) \longrightarrow \mathrm{O}(\overline{q}) \times \mathrm{GL}(U) \longrightarrow 1.$$

We define $\mathrm{SO}(q)$ to be the fiber of $\mathrm{SO}(\overline{q}) \times \mathrm{SL}(U)$ in $\mathrm{O}(q)$. Clearly, the action of G on V preserving q gives homomorphisms $G \to \mathrm{SO}(q) \to \mathrm{SO}(\overline{q})$, where $\mathrm{SO}(\overline{q})$ is semisimple of type B or D.

10 Quadratic Forms on Irreducible Representations

10.1 Orthogonality and Irreducible Representations

For what $\lambda \in X(T)_+$ is $L(\lambda)$ orthogonal? By Theorem 9.5, $L(\lambda)$ of G is orthogonal if and only if (a) the Weyl module $V(\lambda)$ is orthogonal and (b) a nonzero G-invariant quadratic form q on $V(\lambda)$ vanishes on $\operatorname{rad} V(\lambda)$. As the case where $\operatorname{char} k \neq 2$ is treated by Lemma 4.3, we assume here that char $k = 2$, and we know the answer to (a) by Theorem 9.5.

For question (b), we have the following sufficient conditions. Let $W_p = W \ltimes p\mathbb{Z}\Phi$ be the affine Weyl group, which acts on $X(T)$ via the dot action. For $w \in W_p$ and $\lambda \in X(T)$, the action is denoted by $w \cdot \lambda$. The following result provides sufficient conditions to guarantee that $L(\lambda)$ is orthogonal.

Proposition 10.1 *Suppose* char $k = 2$, $\lambda \neq 0$, *and* $V(\lambda)$ *is orthogonal.*

(i) *λ is not a sum of positive roots;*
(ii) $\lambda \notin W_2 \cdot 0$*;*
(iii) $\mathrm{H}^1(G, L(\lambda)) = 0$*;*
(iv) $L(\lambda)$ *is orthogonal,*

The following implications hold: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof Using [Jan, II.2.14, II.2.12(4)] we have

$$\mathrm{H}^1\big(G, L(\lambda)\big) = \mathrm{Ext}^1_G\big(k, L(\lambda)\big) = \mathrm{Hom}_G\big(\operatorname{rad} V(\lambda), k\big).$$

If (iv) fails, then by Theorem 9.5(b) $\mathrm{Hom}_G(\operatorname{rad} V(\lambda), k) \neq 0$, *i.e.,* (iii) fails. If (iii) fails, then $L(0)$ is a factor in the composition series for $V(\lambda)$, hence $\lambda \in W_2 \cdot 0$ and (ii) fails. If λ is in $W_2 \cdot 0$, then λ is in the root lattice; since λ is a dominant weight, it is a sum of positive roots, so (i) implies (ii). ∎

It is not hard to find $\lambda \in Q$ but $\lambda \notin W_p \cdot 0$, *i.e.,* an example to show that (ii) $\not\Rightarrow$ (i). The examples in Section 10.2 show that the converses of each of the other implications can fail. Also, note that Proposition 10.1(ii) can be replaced by the statement that "0 is not strongly linked to λ", for strong linkage as described in [Jan, II Chapter 6].

Example 10.2 Let ω be a dominant weight of Sp_{2n} that is neither a sum of positive roots nor ω_1. For every field of characteristic 2, the Weyl module $V(\omega)$ of Sp_{2n} is orthogonal by Theorem 9.5, and Proposition 10.1 says that the irreducible representation $L(\omega)$ is orthogonal.

10.2 Adjoint Representations

Suppose now that G is a split simple group over a field k of characteristic 2. The highest weight of the adjoint representation is $\widetilde{\alpha}$. We illustrate the proposition by determining whether the irreducible representation $L(\widetilde{\alpha})$ is orthogonal, *i.e.,* whether the reduced Killing form s defined in Example 4.5 vanishes on rad $V(\widetilde{\alpha})$. According to the description of s in that example, it suffices to first find the radical U of the polar bilinear form of $q^\vee \otimes k$, which is the kernel of the linear transformation DC, and then to check if $q^\vee \otimes k$ as given by (4.1) is identically zero on U. These steps involve linear algebra with explicit matrices over $\mathbb{F}_2$, and explicit formulas can be found in [BaC], so we merely summarize the results in Table 1.

For types other than C, the orthogonality of $L(\widetilde{\alpha})$ has been determined in [GowW, §3] by a somewhat different argument. Although for type C, we find that $L(\widetilde{\alpha})$ is never orthogonal, which appears to contradict the final sentence of [GowW].

Φ	restrictions	$L(\widetilde{\alpha})$ orthogonal?	$\dim \mathrm{H}^1(G, L(\widetilde{\alpha}))$
A_n	$n \equiv 0, 2 \bmod 4$	yes	0
	$n \equiv 1 \bmod 4$	no	1
	$n \equiv 3 \bmod 4$	yes	1
B_n $(n \geq 3)$	$n \equiv 0 \bmod 4$	yes	1
	$n \equiv 1, 3 \bmod 4$	yes	0
	$n \equiv 2 \bmod 4$	no	1
D_n $(n \geq 4)$	$n \equiv 0 \bmod 4$	yes	2
	$n \equiv 1, 3 \bmod 4$	yes	1
	$n \equiv 2 \bmod 4$	no	2
C_n or E_7		no	1
E_6, E_8, F_4, or G_2		yes	0

Table 1: Orthogonality of $L(\widetilde{\alpha})$ for a Simple Group G over a Field of Characteristic 2

In the table, for the convenience of the reader, we also give $\dim \mathrm{H}^1(G, L(\widetilde{\alpha}))$. This amounts to identifying the cohomology group with $\mathrm{Hom}_G(\mathrm{rad}\, V(\widetilde{\alpha}), k)$ and consulting the description of the G-module structure of $V(\widetilde{\alpha}, k)$ given in [Hiss]. Note that the case of A_n with $n \equiv 0 \bmod 4$ shows that (iii) $\not\Rightarrow$ (ii) in Proposition 10.1 and with $n \equiv 3 \bmod 4$ shows that (iv) $\not\Rightarrow$ (iii).

11 Cohomology of Λ^2 of Irreducible and Weyl Modules

11.1 Vanishing of the First Cohomology

We begin with a lemma that gives conditions on V that guarantee $\mathrm{H}^1(G, V \otimes V) = 0$. Recall that G is assumed to be split reductive.

Lemma 11.1 *Let V be a module admitting a good filtration or $V = L(\lambda)$ with $\lambda \in X(T)_+$. Then $\mathrm{H}^1(G, V \otimes V) = 0$.*

Proof If V has a good filtration, then $V \otimes V$ has a good filtration and the result follows, because $\mathrm{H}^1(G, H^0(\sigma_1) \otimes H^0(\sigma_2)) = 0$ for all $\sigma_1, \sigma_2 \in X(T)_+$ by [Jan, II 4.13 Proposition].

Now suppose that $V = L(\lambda)$, and suppose that $\mathrm{H}^1(G, V \otimes V)$, *i.e.*,

$$\mathrm{Ext}^1_G\big(L(-w_0\lambda), L(\lambda)\big),$$

is not zero. According to [Jan, II 2.14 Remark], one has either $-w_0\lambda < \lambda$ or $\lambda < -w_0\lambda$, contradicting Lemma 5.1. ∎

11.2 First Cohomology with Coefficients in the Exterior Square

We now calculate $\mathrm{H}^1(G, \Lambda^2(V))$ for some of the modules V we have studied.

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press

Theorem 11.2 *Let G be a split reductive group and $\lambda \in X(T)_+$. Then*

$$\mathrm{H}^1(G, \Lambda^2(V)) = \begin{cases} 0 & \text{if } \operatorname{char} k \neq 2 \text{ and } V = T(\lambda), L(\lambda), \text{ or } H^0(\lambda), \\ (\operatorname{Quad} V^*)^G & \text{if } \operatorname{char} k = 2,\ V = L(\lambda) \text{ or } H^0(\lambda), \text{ and } \lambda \neq 0. \end{cases}$$

Evidently, if $\lambda = 0$, then for $V = L(\lambda) = H^0(\lambda) = k$, we have $(\operatorname{Quad} V)^G = k$ but $\Lambda^2(V) = 0$, so $\mathrm{H}^1(G, \Lambda^2(V)) = 0$.

Proof Taking G-fixed points in (3.2) gives an exact sequence

$$\Lambda^2(V)^G \xrightarrow{\alpha^G} \operatorname{Bil}(V^*)^G \xrightarrow{\rho^G} \operatorname{Quad}(V^*)^G \longrightarrow \mathrm{H}^1(G, \Lambda^2(V)) \longrightarrow \mathrm{H}^1(G, V \otimes V),$$

where the last term is zero for $V = H^0(\lambda)$, $T(\lambda)$, or $L(\lambda)$ by Lemma 11.1. If $\operatorname{char} k \neq 2$, then ρ^G is surjective and the claims follow.

So suppose $\operatorname{char} k = 2$ and $V = H^0(\lambda)$ or $L(\lambda)$ with $\lambda \neq 0$. If $(\operatorname{Quad} V^*)^G = 0$, we are done, so assume otherwise. Then Lemma 4.3 implies that ρ^G is the zero map, and again we are done. ∎

Example 11.3 Suppose $V = L(\lambda)$ for some $\lambda \in X(T)_+$; we claim that $(\operatorname{Quad} V^*)^G \cong (\operatorname{Quad} V)^G$. Indeed, if $\lambda \neq -w_0\lambda$, then $(\operatorname{Bil} V)^G = 0$, hence $(\operatorname{Quad} V)^G$ consists of totally singular quadratic forms, which are necessarily zero (by Theorem 9.5 because $\lambda \neq 0$); the same applies to $V^* = L(-w_0\lambda)$.

In the case where $\operatorname{char} k = 2$ and $V = L(\lambda)$, Sin and Willems [SinW, Proposition 2.7] showed that $\mathrm{H}^1(G, \Lambda^2(V)) = 0$ implies $(\operatorname{Quad} V)^G = 0$. We are able to say more in Theorem 11.2 thanks to Lemma 11.1.

11.3 Application to Induced Modules

By combining Theorem 11.2 with Theorem 9.5, we obtain the following first cohomology calculation for all induced modules.

Corollary 11.4 *Let G be a split reductive group and $\lambda \in X(T)_+$. Then*

$$\mathrm{H}^1(G, \Lambda^2(H^0(\lambda))) = \begin{cases} k & \text{if } \operatorname{char} k = 2 \text{ and } 0 \neq \lambda = -w_0\lambda, \text{ but} \\ & G \text{ and } V(\lambda) \text{ are not as in Example 9.4}, \\ 0 & \text{otherwise.} \end{cases}$$

Acknowledgments We wish to thank George McNinch, Anne Quéguiner-Mathieu, and Jean-Pierre Tignol for helpful discussions.

References

[AST] H. H. Andersen, C. Stroppel, and D. Tubbenhaur, *Cellular structures using U_q-tilting modules*. arxiv:1503.0022

[BaC] A. Babic and V. Chernousov, *Lower bounds for essential dimensions in characteristic 2 via orthogonal representations*. Pacific J. Math. **279**(2015), 37–63. doi=10.2140/pjm.2015.279.37

[BC] D. J. Benson and J. F. Carlson, *Diagrammatic methods for modular representations and cohomology*. Comm. Algebra **15**(1987), no. 1–2, 53–121. http://dx.doi.org/10.1080/00927878708823414

[BNP] C. P. Bendel, D. K. Nakano, and C. Pillen, *Extensions for Frobenius kernels*. J. Algebra **272**(2004), no. 2, 476–511. http://dx.doi.org/10.1016/j.jalgebra.2003.04.003

[Bor] A. Borel, *Linear algebraic groups*. Second ed., Graduate Texts in Mathematics, 126, Springer-Verlag, New York, 1991. http://dx.doi.org/10.1007/978-1-4612-0941-6

[Bou A1] N. Bourbaki, *Algebra I: Chapters 1–3*. Elements of Mathematics, Springer-Verlag, Berlin, 1989.

[Bou A4] ———, *Algebra II: Chapters 4–7*, Springer-Verlag, 1988.

[Bou A9] ———, *Algèbre IX*, Hermann, Paris, 1959.

[Bou L4] ———, *Lie groups and Lie algebras: Chapters 4–6*. Springer-Verlag, Berlin, 2002.

[Bou L7] ———, *Lie groups and Lie algebras: Chapters 7–9*. Springer-Verlag, Berlin, 2005.

[Brown] R. B. Brown, *Groups of type E_7*. J. Reine Angew. Math. **236**(1969), 79–102.

[CS] V. Chernousov and J-P. Serre, *Lower bounds for essential dimensions via orthogonal representations*. J. Algebra **305**(2006), 1055–1070. http://dx.doi.org/10.1016/j.jalgebra.2005.10.032

[De B] B. De Bruyn, *On the Grassmann modules for the symplectic groups*. J. Algebra **324**(2010), 218–230. http://dx.doi.org/10.1016/j.jalgebra.2010.03.033

[DS] S. R. Doty and J. B. Sullivan, *The submodule structure of Weyl modules for* SL_3. J. Algebra **96**(1985), no. 1, 78–93. http://dx.doi.org/10.1016/0021-8693(85)90040-7

[DV] A. Drápal and P. Vojtěchovský, *Symmetric multilinear forms and polarization of polynomials*. Linear Algebra Appl. **431**(2009), no. 5–7, 998–1012. http://dx.doi.org/10.1016/j.laa.2009.03.052

[Dy] E. B. Dynkin, *Maximal subgroups of the classical groups*. Amer. Math. Soc. Transl. (2) **6**(1957), 245–378 ; (Russian) Trudy Moskov. Mat. Obšč. **1**(1952), 39–166.

[EKM] R. S. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*. American Mathematical Society Colloquium Publications, 56, American Mathematical Society, Providence, RI, 2008.

[Ga] S. Garibaldi, *Vanishing of trace forms in low characteristic*. Algebra Number Theory **3**(2009), no. 5, 543–566. http://dx.doi.org/10.2140/ant.2009.3.543

[GN] B. H. Gross and G. Nebe, *Globally maximal arithmetic groups*. J. Algebra **272**(2004), no. 2, 625–642. http://dx.doi.org/10.1016/j.jalgebra.2003.09.033

[GowK] R. Gow and A. Kleshchev, *Connections between representations of the symmetric group and the symplectic group in characteristic 2*. J. Algebra **221**(1999), no. 1, 60–89. http://dx.doi.org/10.1006/jabr.1999.7943

[GowW] R. Gow and W. Willems, *Methods to decide if simple self-dual modules over fields of characteristic 2 are of quadratic type*. J. Algebra **175**(1995), 1067–1081. http://dx.doi.org/10.1006/jabr.1995.1227

[Gro11] A. Grothendieck, *Schémas en groupes III*. Société Mathématique de France, 2011.

[GW09] R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*. Graduate Texts in Mathematics, 255, Springer, Dordrecht, 2009. http://dx.doi.org/10.1007/978-0-387-79852-3

[Hiss] G. Hiss, *Die adjungierten Darstellungen der Chevalley-Gruppen*. Arch. Math. (Basel) **42**(1984), no. 5, 408–416. http://dx.doi.org/10.1007/BF01190689

[HN] D. J. Hemmer and D. K. Nakano, *On the cohomology of Specht modules*. J. Algebra **306**(2006), no. 1, 191–200. http://dx.doi.org/10.1016/j.jalgebra.2006.03.044

[Jan] J. C. Jantzen, *Representations of algebraic groups*. Second ed., Mathematical Surveys and Monographs, 107, American Mathematical Society, Providence, RI, 2003.

[KMRT] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*. American Mathematical Society Colloquium Publications, 44, American Mathematical Society, Providence, RI, 1998.

[Mal] A. I. Mal'cev, *On semi-simple subgroups of Lie groups*. Amer. Math. Soc. Translation **9**(1950), no. 33, 43; (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **8**(1944), 143–174.

[MR] A. Micali and Ph. Revoy, *Modules quadratiques*. Bull. Soc. Math. France Mém. (1979), no. 63, 144 pp.

[Se] C. S. Seshadri, *Geometric reductivity over arbitrary base*. Advances in Math. **26**(1977), no. 3, 225–274. http://dx.doi.org/10.1016/0001-8708(77)90041-X

[SF] H. Strade and R. Farnsteiner, *Modular Lie algebras and their representations*. Monographs and Textbooks in Pure and Applied Mathematics, 116, Marcel Dekker, New York, 1988.

[SpSt] T. A. Springer and R. Steinberg, *Conjugacy classes*. Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, 131, Springer, Berlin, 1970, pp. 167–266.

[St] R. Steinberg, *Lectures on Chevalley groups*. Yale University, New Haven, Conn., 1968.

[SinW] P. Sin and W. Willems, *G-invariant quadratic forms*. J. Reine Angew. Math. **420**(1991), 45–59. http://dx.doi.org/10.1515/crll.1991.420.45

[W] W. Willems, *Metrische G-Moduln über Körpern der Charakteristik 2*. Math. Z. **157**(1977), no. 2, 131–139. http://dx.doi.org/10.1007/BF01215147

Institute for Pure and Applied Mathematics, UCLA, 460 Portola Plaza, Box 957121, Los Angeles, CA 90095-7121, USA

and

Center for Communications Research, San Diego, CA 92121, USA
e-mail: skip@garibaldibros.com

Department of Mathematics, University of Georgia, Athens, Georgia 30602, USA
e-mail: nakano@uga.edu

https://doi.org/10.4153/CJM-2015-042-5 Published online by Cambridge University Press