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RELIABILITY EVALUATION OF A LINEAR k-WITHIN-(r, s)-OUT-OF-(m, n):F LATTICE SYSTEM

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The linear k-within-(r,s)-out-of-(m,n):F lattice system consists of mn components arranged in m rows and n columns and fails whenever there is at least one rectangle of dimension $r \times s$ which contains k or more failed components. We propose recursive formulas for the calculation of the linear k-within-(r,s)-out-of-(m,n):F lattice system. The computing complexity of the system reliability using the recursive formulas is $O((n-s)m^{s+1} + m^{2s})$ for $r = m$ and $O((r+1)r^{s(m-r+1)}[(n-s)m + 2^{r(s-1)}m^s])$ for $r < m$.

1. INTRODUCTION

Generalizations of the consecutive-k-out-of-n:F system have been reported in a considerable number of papers [2]. One of the generalizations is the linear connected-(r,s)-out-of-(m,n):F lattice system. It consists of mn components arranged in m rows and n columns. The system fails whenever there is at least one rectangle of dimension $r \times s$ which contains all failed components. Bounds, reliability evaluation, and invariant optimal design of the linear connected-(r,s)-out-of-(m,n):F lattice system are studied in [2,6,7]. Zuo, Lin, and Wu [8] propose combined k-out-of-

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n:F, consecutive-k-out-of-n:F, and linear connected-(r, s)-out-of-(m, n):F lattice system structures and provide recursive formulas for the reliability of the combined system structures. Koutras [1] uses the Markov chain approach for reliability evaluation of Markov chain imbeddable systems (MIS). He shows that the tool is very useful to a great variety of well-known one-dimensional reliability structures. However, it is very difficult or impossible to apply the idea to two-dimensional reliability structures such as the linear connected-(r, s)-out-of-(m, n):F lattice system and the above-mentioned combined system structures.

A further generalization of the linear connected-(r, s)-out-of-(m, n):F lattice system is the linear k-within-(r, s)-out-of-(m, n):F lattice system. It consists of mn components arranged in m rows and n columns. The system fails whenever there is at least one cluster of size $r \times s$, which contains k or more failed components. It becomes a linear connected-(r, s)-out-of-(m, n):F lattice system when $k = rs$. The bounds for the reliability of the linear k-within-(r, s)-out-of-(m, n):F lattice system is given in [3] for a special case when the components of the system are identical. To the authors' best knowledge, however, no recursive formula for the reliability of the linear k-within-(r, s)-out-of-(m, n):F lattice system is given in the literature.

In the sections to follow, we provide recursive formulas for system reliability evaluation of the linear k-within-(r, s)-out-of-(m, n):F lattice system. The proofs of the formulas in Sections 2 and 3 are given in the Appendix. The following assumptions will be used:

1. Each component, and the system, is either working or failed.
2. The failures of the components are s independent (statistically independent).
3. The reliabilities of the components are not necessarily identical.
4. $1 \le k < rs$, $1 \le r \le m$, and $1 \le s \le n$.

2. THE LINEAR k-WITHIN-(r, s)-OUT-OF-(m, n):F LATTICE SYSTEM WITH $r = m$

The linear k-within-(m, s)-out-of-(m, n):F lattice system consists of mn components which are arranged in m rows and n columns. This is a special case of the general linear k-within-(r, s)-out-of-(m, n):F lattice system in which we let $r = m$. The system fails whenever there is at least one cluster of size $m \times s$ such that the number of the failed components within this cluster is at least k. The component in the ith row and jth column is called component (i, j) throughout this paper. The notation used is as follows:

$p_{ij}(q_{ij})$	Reliability (unreliability) of component (i, j).
x_{ij}	Binary variable; 1 if component (i, j) fails, 0 otherwise.
$\mathbf{g}_j$	An s-dimensional nonnegative integer vector $(g_{j-s+1}, \ldots, g_j)$.
$A(m, j, s, k)$	The event that the linear k-within-(m, s)-out-of-(m, j):F lattice subsystem works. This subsystem consists of the components in the first j columns of the original system.

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$B(i,l,u)$	The event that there are u failed components in components $(1,l)$, $\ldots,(i,l)$ of the subsystem.
$X(m,j,s,k,\mathbf{g}_j)$	Conditional event: $A(m,j,s,k)\mid\bigcap_{l=j-s+1}^{j}B(m,l,g_l)$.
$R(m,j,s,k)$	Probability that event $A(m,j,s,k)$ occurs.
$S(m,j,s,k,\mathbf{g}_j)$	Probability that event $X(m,j,s,k,\mathbf{g}_j)$ occurs.
$Q(i,l,u)$	Probability that event $B(i,l,u)$ occurs.
Φ_{mks}	The set of all s-dimensional nonnegative integer vectors such that all elements of each vector are between 0 and m and the sum of all elements of each vector is less than k.

When $n=s$, the linear k-within-(m,s)-out-of-(m,n):F lattice system becomes a k-out-of-ms system and the recursive formula for calculating the reliability of this system is already available [4]. For $n>s$, we provide the following formulas which are proved in the Appendix:

$$R(m,n,s,k)=\sum_{\mathbf{g}_n\in\Phi_{mks}}S(m,n,s,k,\mathbf{g}_n)\prod_{l=n-s+1}^{n}Q(m,l,g_l). \quad \textbf{(1)}$$

The conditional probability $S(m,n,s,k,\mathbf{g}_n)$ in Eq. (1) can be computed using the following recursive formula:

$$S(m,j,s,k,\mathbf{g}_j)=\sum_{\varepsilon_{j-s}=0}^{m}S(m,j-1,s,k,\varepsilon_{j-1})Q(m,j-s,\varepsilon_{j-s})\quad\text{if }\mathbf{g}_j\in\Phi_{mks} \quad \textbf{(2)}$$

for $j=s+1,\ldots,n$, where

$$\varepsilon_{j-1}=(\varepsilon_{j-s},g_{j-s+1},\ldots,g_{j-1}).$$

It is noted that the boundary conditions for the above recursive formula is

$$S(m,j,s,k,\mathbf{g}_j)=0\quad\text{if }\mathbf{g}_j\notin\Phi_{mks}\quad\text{for }j=s,s+1,\ldots,n,$$
$$S(m,s,s,k,\mathbf{g}_s)=1\quad\text{if }\mathbf{g}_s\in\Phi_{mks}.$$

The probability $Q(m,l,g_l)$ in Eq. (1) and the probability $Q(m,j-s,\varepsilon_{j-s})$ in Eq. (2) can be calculated by the recursive formula

$$Q(i,l,u)=p_{il}Q(i-1,l,u)+q_{il}Q(i-1,l,u-1),$$
$$i=1,2,\ldots,m,\qquad 1\le l\le j, \quad \textbf{(3)}$$

with boundary conditions

$$Q(i,l,u)=1\quad\text{if }i=0\text{ and }u=0,$$
$$Q(i,l,u)=0\quad\text{if }i<u\text{ or }u<0.$$

TABLE 1. System Reliability and Computing Time (in Seconds) for the Linear 4-Within-$(3, s)$-out-of-$(3, n)$:F Lattice System

$s = 2$			$s = 3$			$s = 5$		
n	$R(3, n, s, 4)$	Time	n	$R(3, n, s, 4)$	Time	n	$R(3, n, s, 4)$	Time
5	0.999999	0.21	5	0.999994	0.39	5	0.999977	0.05
10	0.999968	0.41	10	0.999783	0.80	10	0.998769	3.44
20	0.999033	0.83	20	0.993500	1.67	20	0.965114	7.37
50	0.922598	2.09	50	0.651598	4.12	50	0.200798	19.54
100	0.142103	4.19	100	0.000661	8.36	100	8×10^{-9}	39.78

The complexity for computing $Q(m, l, g_l)$ using recursive formula (3) is $O(m)$. The cardinality of Φ_{mks} does not exceed $(m + 1)^s$. Thus, the computing time of $R(m, n, s, k)$ using formulas (1)–(3) is

$$O((n - s)m(m + 1)^s + m^s(m + 1)^s) = O((n - s)m^{s+1} + m^{2s}),$$

when s is fixed. In other words, the computing time of $R(m, n, s, k)$ is polynomial in m and linear in n.

Example 1: Consider a linear k-within-(m, s)-out-of-(m, n):F lattice system with $m = r = 3$, $k = 4$, and the following component reliabilities:

$$p_{ij} = 1 - 0.0015[(i - 1)n + j] \quad \text{for } i = 1,2,3 \text{ and } j = 1,2,\ldots,n.$$

We compute the system reliability $R(3, n, s, 4)$ using formulas (1)–(3) for various n and s. Matlab [5] is used for the computation work. The reliability $R(3, n, s, 4)$ and the computing time for various n and s are listed in Table 1.

3. THE GENERAL LINEAR k-WITHIN-(r, s)-OUT-OF-(m, n):F LATTICE SYSTEM

The linear k-within-(r, s)-out-of-(m, n):F lattice system fails whenever there is at least one cluster of size $r \times s$ such that the number of the failed components within this cluster is at least k. In this section, we assume $r < m$. The notation used is as follows:

$p_{ij}(q_{ij})$	Reliability (unreliability) of component (i, j).
x_{ij}	Binary variable; 1 if component (i, j) fails, 0 otherwise.
$\mathbf{x}_{ij}$	An r-dimensional binary vector $(x_{i-r+1,j}, \ldots, x_{i,j})$.
$\mathbf{g}_{ij}$	An $(i - r + 1)$-dimensional nonnegative integer vector $(g_{1j}, \ldots, g_{i-r+1,j})$, $0 \leq g_{ij} \leq r$, $i = r, r + 1, \ldots, m$, $j = 1, 2, \ldots, n$.

$\mathbf{G}_{mj}$	A sequence of $(m-r+1)$-dimensional nonnegative integer vectors $(\mathbf{g}_{m,j-s+1},\ldots,\mathbf{g}_{mj})$.
$A'(m,j,r,s,k)$	The event that the linear k-within-(r,s)-out-of-(m,j):F lattice subsystem works. The subsystem consists of the components in the first j columns of the original system.
$B'(u,i,j)$	The event that there are exactly u failures from component (i,j) to $(i+r-1,j)$ in the jth column.
$C(\mathbf{x}_{ij})$	The event that a specific $\mathbf{x}_{ij}=(x_{i-r+1,j},\ldots,x_{ij})$ occurs.
$D(i,j,r,\mathbf{g}_{ij})$	Event: $\bigcap_{l=1}^{i-r+1} B'(g_{lj},l,j)$.
$E(i,j,r,\mathbf{g}_{ij},\mathbf{x}_{ij})$	Conditional event: $D(i,j,r,\mathbf{g}_{ij})\|C(\mathbf{x}_{ij})$.
$X'(m,j,r,s,k,\mathbf{G}_{mj})$	Conditional event: $A'(m,j,r,s,k)\|\bigcap_{l=j-s+1}^{j} D(m,l,r,\mathbf{g}_{ml})$.
$R'(m,j,r,s,k)$	Probability that event $A'(m,j,r,s,k)$ occurs.
$Q'(i,j,r,\mathbf{g}_{ij})$	Probability that event $D(i,j,r,\mathbf{g}_{ij})$ occurs.
$Q''(i,j,r,\mathbf{g}_{ij},\mathbf{x}_{ij})$	Probability that event $E(i,j,r,\mathbf{g}_{ij},\mathbf{x}_{ij})$ occurs.
$S'(m,j,r,s,k,\mathbf{G}_{mj})$	Probability that event $X'(m,j,r,s,k,\mathbf{G}_{mj})$ occurs.
Λ_{ru}	The set of all r-dimensional binary vectors such that the sum of its components equals u.
Ψ_{mr}	The set of all $(m-r+1)$-dimensional nonnegative integer vector whose elements are all between 0 and r.
Φ_{mrsk}	The set of all sequences of s $(m-r+1)$-dimensional nonnegative integer vectors whose elements are all between 0 and r, and each element in the sum vector of the s vectors is between 0 and $k-1$.

When $n=s$, it becomes the case studied in Section 2 by taking the transpose of the $m\times n$ component matrix of the original system. In this case, the system structure becomes a linear k-within-(r,n)-out-of-(m,n):F lattice system. When $n>s$, we have the following formulas, proofs of which are also given in the Appendix:

$$R'(m,n,r,s,k)=\sum_{\mathbf{G}_{mn}\in\Phi_{mrsk}} S'(m,n,r,s,k,\mathbf{G}_{mn})\prod_{l=n-s+1}^{n} Q'(m,l,r,\mathbf{g}_{ml}). \tag{4}$$

The conditional probability $S'(m,n,r,s,k,\mathbf{G}_{mj})$ in Eq. (4) can be calculated using the recursive formula

$$S'(m,j,r,s,k,\mathbf{G}_{mj})=\sum_{\mathbf{g}_{m,j-s}\in\Psi_{mr}} S'(m,j-1,r,s,k,\mathbf{G}_{m,j-1})Q'(m,j-s,r,\mathbf{g}_{m,j-s}), \tag{5}$$

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if $\mathbf{G}_{mj} \in \Phi_{mrsk}$, for $j = s+1, \ldots, n$, with boundary conditions

$$S'(m, j, r, s, k, \mathbf{G}_{mj}) = 0 \quad \text{if } \mathbf{G}_{mj} \notin \Phi_{mrsk} \text{ for } j = s, s+1, \ldots, n,$$
$$S'(m, s, r, s, k, \mathbf{G}_{ms}) = 1 \quad \text{if } \mathbf{G}_{ms} \in \Phi_{mrsk}.$$

The probability $Q'(m, l, r, \mathbf{g}_{ml})$ in Eq. (4) and the probability $Q'(m, j-s, r, \mathbf{g}_{m,j-s})$ in Eq. (5) can be computed by using

$$Q'(m, j, r, \mathbf{g}_{mj}) = \sum_{\mathbf{x}_{mj} \in \Lambda_{r, g_{m-r+1,j}}} Q''(m, j, r, \mathbf{g}_{mj}, \mathbf{x}_{mj}) \prod_{i=m-r+1}^{m} [p_{ij}(1 - x_{ij}) + q_{ij} x_{ij}] \tag{6}$$

for $j = 1, 2, \ldots, n$. Whereas the conditional probability $Q''(m, j, r, \mathbf{g}_{mj}, \mathbf{x}_{mj})$ in Eq. (6) can be calculated using the recursive formula

$$Q''(i, j, r, \mathbf{g}_{ij}, \mathbf{x}_{ij}) = \sum_{x_{i-r,j}=0}^{1} Q''(i-1, j, r, \mathbf{g}_{i-1,j}, \mathbf{x}_{i-1,j}) \times [p_{i-r,j}(1 - x_{i-r,j}) + q_{i-r,j} x_{i-r,j}] \tag{7}$$

if $\mathbf{x}_{ij} \in \Lambda_{r, g_{i-r+1,j}}$, for $i = r+1, \ldots, m$, $1 \leq j \leq n$, with boundary conditions

$$Q''(i, j, r, \mathbf{g}_{ij}, \mathbf{x}_{ij}) = 0 \quad \text{if } \mathbf{x}_{ij} \notin \Lambda_{r, g_{i-r+1,j}} \text{ for } i = r, r+1, \ldots, m,$$
$$Q''(r, j, r, \mathbf{g}_{rj}, \mathbf{x}_{rj}) = 1 \quad \text{if } \mathbf{x}_{rj} \in \Lambda_{r, g_{1j}}.$$

The complexity for computing $Q'(m, j, r, \mathbf{g}_{mj})$ is $O(2^r m)$. The cardinality of Φ_{mrsk} does not exceed $(r+1)^{s(m-r+1)}$. Hence, the computing time of $R'(m, n, r, s, k)$ using formulas (4)–(7) would be

$$O((r+1)^{s(m-r+1)}[(n-s)2^r m + (2^r m)^s])$$
$$= O((r+1)r^{s(m-r+1)}[(n-s)m + 2^{r(s-1)} m^s])$$

when r and s are fixed; that is, the complexity of the algorithm for computing the system reliability is exponential in m and linear in n.

Example 2: Consider a linear k-within-(r, s)-out-of-(m, n):F lattice system with $m = 3$, $r = 2$, $k = 3$, and the following component reliabilities:

$$p_{ij} = 1 - 0.0015[(i-1)n + j] \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, \ldots, n.$$

We compute the system reliability $R'(3, n, 2, s, 3)$ using formulas (4)–(7) for various n and s. Matlab is used for the computation work. The reliability $R'(3, n, 2, s, 3)$ and the computing time for various n and s are listed in Table 2.

4. SUMMARY AND CONCLUDING REMARKS

In this paper, we propose recursive formulas for computing the reliability of the linear k-within-(r, s)-out-of-(m, n):F lattice system. When $r = m$, the computing

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Table 2. System Reliability and Computing Time (in Seconds) for the Linear 3-Within-$(2, s)$-out-of-$(3, n)$:F Lattice System

$s = 2$			$s = 3$		
n	$R'(3, n, 2, s, 3)$	Time	n	$R'(3, n, 2, s, 3)$	Time
5	0.999933	18.32	5	0.999789	41.20
10	0.998896	49.49	10	0.996240	124.79
20	0.983139	100.03	20	0.995192	296.33
30	0.921616	159.16	30	0.774805	457.46
50	0.568857	268.25	50	0.207783	788.97

time for the system reliability using the provided recursive formulas is polynomial in m and linear in n. For the general case when $r < m$, the computing complexity for the system reliability using the derived recursive formulas is exponential in m and linear in n. Practically, the computing time for the system reliability is of order $O((n-s)m^{s+1} + m^{2s})$ for $r = m$ and $O((r+1)r^{s(m-r+1)}[(n-s)m + 2^{r(s-1)}m^s])$ for $r < m$, because the cardinality of Φ_{mks} is usually less than $(m+1)^s$ and that of Φ_{mrsk} is less than $(r+1)^{s(m-r+1)}$. The programs in Matlab of the algorithms are available upon request from the authors.

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APPENDIX

The boundary conditions for recursive formulas in Sections 2 and 3 can be derived by observing the corresponding events which would be null events or sure events. Their proofs are omitted here.

Proof of formulas(1)–(3) in Section 2

Event $A(m,n,s,k)$ can be decomposed as

$$A(m,n,s,k) = \bigcup_{\mathbf{g}_n \in \Phi_{mks}} \left[A(m,n,s,k) \cap \left(\bigcap_{l=n-s+1}^{n} B(m,l,g_l) \right) \right].$$

The terms within brackets for all $\mathbf{g}_n$ are mutually exclusive, and $B(m,l,g_l)$, $l = n-s+1,\ldots,n$, are independent. Then, from the above decomposition of $A(m,n,s,k)$, it is very straightforward that formula (1) holds.

If $\mathbf{g}_j \in \Phi_{mks}$, we have the following decomposition:

$$\begin{aligned} X(m,j,s,k,\mathbf{g}_j) &= A(m,j,s,k) \Bigg| \bigcap_{l=j-s+1}^{j} B(m,l,g_l) \\ &= A(m,j-1,s,k) \Bigg| \bigcap_{l=j-s+1}^{j} B(m,l,g_l) \\ &= A(m,j-1,s,k) \Bigg| \bigcap_{l=j-s+1}^{j-1} B(m,l,g_l) \\ &= \bigcup_{\varepsilon_{j-s}=0}^{m} \Big(A(m,j-1,s,k) \bigcap B(m,j-s,\varepsilon_{j-s}) \Big) \Bigg| \bigcap_{l=j-s+1}^{j-1} B(m,l,g_l), \end{aligned}$$

for $j = s+1,\ldots,n$. The terms in the large parentheses are mutually exclusive for all ε_{j-s}. From the above decomposition, we immediately obtain formula (2).

Formula (3) can be easily derived by the decomposition of event $B(i,l,u)$:

$$B(i,l,u) = [\{x_{il}=0\} \cap B(i-1,l,u)] \cup [\{x_{il}=1\} \cap B(i-1,l,u-1)],$$

noting that the two events in brackets are mutually exclusive.

Proof of formulas (4)–(7) in Section 3

We decompose event $A'(m,n,r,s,k)$ as

$$A'(m,n,r,s,k) = \bigcup_{\mathbf{G}_{mn} \in \Phi_{mrsk}} \left[A'(m,n,r,s,k) \cap \left(\bigcap_{l=n-s+1}^{n} D(m,l,r,\mathbf{g}_{ml}) \right) \right].$$

Noting that the terms in brackets for all $\mathbf{G}_{mn}$ are mutually exclusive and $D(m,l,r,\mathbf{g}_{ml})$, $l = n - s+1,\ldots,n$, are independent, we get formula (4).

If $\mathbf{G}_{mn} \in \Phi_{mrsk}$, we have

$$\begin{aligned}
X'(m,j,r,s,k,\mathbf{G}_{mj}) &= A'(m,j,r,s,k)\Bigg|\bigcap_{l=j-s+1}^{j} D(m,l,r,\mathbf{g}_{ml}) \\
&= A'(m,j-1,r,s,k)\Bigg|\bigcap_{l=j-s+1}^{j} D(m,l,r,\mathbf{g}_{ml}) \\
&= A'(m,j-1,r,s,k)\Bigg|\bigcap_{l=j-s+1}^{j-1} D(m,l,r,\mathbf{g}_{ml}) \\
&= \bigcup_{\mathbf{g}_{m,j-s}\in\Psi_{mr}} [(A'(m,j-1,r,s,k) \cap D(m,j-s,r,\mathbf{g}_{m,j-s})]\Bigg|\bigcap_{l=j-s+1}^{j-1} \\
&\quad \times D(m,l,r,\mathbf{g}_{ml})
\end{aligned}$$

for $j = s+1,\ldots,n$. The events in brackets are mutually exclusive for all $\mathbf{g}_{m,j-s}$ given that event $\bigcap_{l=j-s+1}^{j-1} D(m,l,r,\mathbf{g}_{ml})$ occurs. From this decomposition, we derive formula (5).

Similarly, formulas (6) and (7) can be derived by the following event decompositions respectively:

$$D(m,j,r,\mathbf{g}_{mj}) = \bigcup_{\mathbf{x}_{mj}\in\Lambda_{r,g_{m-r+1,j}}} (D(m,j,r,\mathbf{g}_{mj}) \cap C(\mathbf{x}_{mj}))$$

for $j = 1,2,\ldots,n$, and

$$\begin{aligned}
E(i,j,r,\mathbf{g}_{ij},\mathbf{x}_{ij}) &= D(i,j,r,\mathbf{g}_{ij})|C(\mathbf{x}_{ij}) \\
&= D(i-1,j,r,\mathbf{g}_{i-1,j})|C(\mathbf{x}_{ij}) \\
&= D(i-1,j,r,\mathbf{g}_{i-1,j})|C'(x_{i-r+1},\ldots,x_{i-1}) \\
&= \bigcup_{x_{i-r,j}=0}^{1} [D(i-1,j,r,\mathbf{g}_{i-1,j}) \cap C''(x_{i-r,j})]|C'(x_{i-r+1},\ldots,x_{i-1})
\end{aligned}$$

if $\mathbf{x}_{ij} \in \Lambda_{r,g_{i-r+1,j}}$ for $i = r+1,\ldots,m$, $1 \le j \le n$, where $C'(x_{i-r+1},\ldots,x_{i-1})$ represents the event that a specific $(x_{i-r+1},\ldots,x_{i-1})$ occurs and $C''(x_{i-r,j})$ represents the event that a specific $(x_{i-r,j})$ occurs.