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Nonlinear distortion of travelling waves in variable-area ducts with entropy gradients

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This paper presents the effects of variable-area ducts and axial temperature gradients on the nonlinear distortion of a travelling wave. Quasi-one-dimensional continuity, momentum and energy equations for isentropic flow are solved using a wave front expansion technique. An evolution equation for the slope of the wave front is obtained. This is a nonlinear ordinary differential equation that can be integrated to obtain a solution in closed form for the slope of the wave front. The solution may admit a singularity for compression wave fronts. The analysis considers pure compression and pure expansion travelling waves. A general criterion is developed for the steepening of a compression wave front into a shock. A general formula is obtained for the location of shock formation. The effects of area variation and axial temperature gradient, and their combined effect on the nonlinear distortion of travelling waves are studied. A number of examples highlighting these effects are presented in the paper.

1. Introduction

In the standard linear theory of propagation of sound, a plane wave in a homentropic environment propagates with constant energy and uniform wave speed. If thermo-viscous effects are significant, the amplitude (energy) of a wave decreases due to damping and after some time the wave almost disappears. However, in the case of a finite-amplitude wave travelling in a region where the damping is not very high, nonlinear effects come into the picture. The compressive part of a wave pulse travels faster than the expansive part and hence wave crests tend to catch up with wave troughs. As a result of this effect, the local slope of the wave increases and steepening of the wave occurs. The steepening of the wave will eventually lead to the wave 'breaking', with multi-valued solutions of the wave equation being obtained (Whitham 1974). In the context of acoustic wave propagation, multi-valued solutions are meaningless. In fact, at this point, the solution of the wave equation becomes discontinuous (shock).

By using the method of characteristics, it is possible to analyse such a problem when a plane compression wave is propagating in a constant-area duct with uniform entropy (homentropic field). In this simple case, the characteristics have a constant slope, and the Riemann invariants are constant along the characteristics. These simplifications can be used to predict the time and location of shock formation (Liepmann & Roshko 1957). However, the shock formation in variable-area ducts in a non-homentropic flow is more complex.

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The propagation of a small-amplitude wave (linear theory) in variable-area duct and in the presence of a temperature gradient has not been solved for the general case. However using the high-frequency approximation or WKB Approximation (Crighton *et al.* 1992), one may show that the acoustic pressure and velocity are given by

$$p_{acoustic} \propto \frac{f\left(t - \int_0^x \mathrm{d}x/a_0(x)\right)}{\sqrt{A_0(x)a_0(x)}}, \quad u_{acoustic} \propto \frac{\sqrt{a_0(x)}\,f\left(t - \int_0^x \mathrm{d}x/a_0(x)\right)}{\sqrt{A(x)}}. \tag{1.1}$$

Here f is an arbitrary function that is determined from boundary or initial conditions. $A_0(x)$ and $a_0(x)$ are the cross-sectional area of the duct and the speed of sound respectively. The high-frequency approximation is valid only when the wavelength of the acoustic disturbance is small relative to the change in cross-sectional area or temperature. It is evident from the above expressions that a change in the cross-sectional area alone affects only the amplitude of the wave and the wave shape remains the same (the wavelength remains the same). However, in the case of temperature gradients, both the wave shape and amplitude are affected.

Subrahmanyam, Sujith & Lieuwen (2001) have shown that exact solutions (valid in the whole range of frequency) for the acoustic pressure and acoustic velocity of the form (1.1) exist for a family of temperature and area profiles that have following form:

$$\frac{T_0(x)}{T_0(0)} = (1+bx)^n, \quad \frac{A_0(x)}{A_0(0)} = (1+bx)^m. \tag{1.2}$$

Here b is an arbitrary constant. The indices n and m are related to each other, and the relation depends upon whether the acoustic pressure or acoustic velocity admit travelling wave solutions of the form obtained by the WKB approximation (1.1). In the case of acoustic pressure, n and m are related by either of the following relations:

$$n = -2m \quad \text{or} \quad n = -(2m-4)/3. \tag{1.3}$$

In the case when the acoustic velocity is of the form obtained by the WKB approximation, the indices n and m are related by either of the following relations:

$$n = -2m \quad \text{or} \quad n = 2m+4. \tag{1.4}$$

It can easily be shown that the acoustic velocity and pressure will simultaneously be of the form suggested by the WKB approximation if n is the negative of twice the value of m.

Hammerton & Crighton (1993) considered the model nonlinear equation $\partial u/\partial t + u\partial u/\partial x = 0$. They solved the problem by numerical means based on the use of intrinsic coordinates for plane waves. By expressing the wave profile in terms of intrinsic coordinates, all the difficulties associated with the overturning are circumvented. However, in this approach, exact solutions are limited. Moreover, the model equation used is weakly nonlinear, i.e. the nonlinear term is smaller than the linear ones by a factor of the order of u/a_0, and is therefore valid only when $u/a_0 \ll 1$.

Lin & Szeri (2001) investigated the nonlinear steepening of plane and spherical waves in the presence of an entropy gradient. In their analysis, the wave is considered as a discontinuity in the first derivative of thermodynamic variables (for example gas velocity) that propagates in a quiescent filed of varying entropy. They used the wave front expansion technique to obtain a global solution in closed form for the first derivative of gas velocity at the wave front. The analysis considered pure compression and expansion waves. The solution for the slope of the wave front may admit a singularity at a finite time, which is responsible for the steepening of the wave front into a shock. For a plane wave, the initial compressive disturbance is a necessary condition for the shock formation. A plane expansion wave never steepens into a shock. In the absence of entropy gradients (homentropic flow), every compression wave steepens into a shock while an expansion wave relaxes. In a linearly increasing entropy field, only sufficiently steep compression waves can steepen into a shock; otherwise they will relax. An expansion wave always relaxes in this situation. In a linearly decreasing entropy field, every compression wave steepens into shock while an expansion wave tends toward a constant value of slope. In Lin & Szeri's analysis of spherical waves, a wave propagating inwardly to the centre of coordinate system was considered. In the homentropic flow, a compression wave moving inwardly always blows up before it reaches the centre. An expansion wave front continuously steepens as it moves toward the origin and an infinitesimal expansion shock forms at the centre. However, it was argued that due the mechanical instability of the expansion shock and the importance of diffusion near the origin, such a situation never occurs in reality.

In this paper, the waveform distortion of a travelling wave is analysed in a variable-area duct with entropy gradients. A wave pulse, which is discontinuous in its first derivative, is launched into a quiescent field. The wave pulse is either pure compressive or expansive. An entropy gradient is caused by a change in the temperature of the quiescent field. The continuity and momentum equations along with the isentropic condition for the unsteady disturbance are solved using the wave front expansion technique. An evolution equation for the slope of the wave front is obtained. The steepening of the wave front (change in the slope) is analysed separately for compression and expansion waves in variable-area ducts and in a field with an entropy gradient. A general criterion for the steepening of a compression wave into a shock is developed. The location of shock formation is calculated for a variety of ducts and for different entropy gradients.

2. Governing equations

Assuming a perfect, inviscid and non-heat-conducting gas, the quasi-one-dimensional continuity, momentum and energy equations for isentropic flow can be written as (Thompson 1972) follows:

continuity

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{\rho u}{A} \frac{\mathrm{d}A}{\mathrm{d}x} = 0, \tag{2.1}$$

momentum

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \tag{2.2}$$

energy

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0. \tag{2.3}$$

Using the thermodynamic relation $T\,\mathrm{d}s = \mathrm{d}h - (1/\rho)\,\mathrm{d}p$ and the equation of state for an ideal gas $p = \rho R T$, the following relation can be derived:

$$\frac{p}{\rho^{\gamma}} = \exp\left(\frac{s}{c_v}\right), \tag{2.4}$$

where γ is the ratio of specific heats of the gas, c_p/c_v.

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Using (2.4) one variable (for example p) can be eliminated from (2.2). Thus there are three equations (2.1), (2.2) and (2.3) with three dependent variables. One can show that these equations form a hyperbolic system (Whitham 1974).

The definition for the isentropic speed of sound is

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_{s=\text{constant}} \tag{2.5}$$

The derivatives of ρ in the continuity equation (2.1) can then be eliminated using (2.5) to yield

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + \rho\frac{\partial u}{\partial x} + \frac{u}{a^2}\frac{\partial p}{\partial x} + \frac{\rho u}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0. \tag{2.6}$$

With a little manipulation (2.2) and (2.6) can be rewritten as

$$\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x} + \frac{1}{a\rho}\left(\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right) + \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0, \tag{2.7}$$

$$\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x} - \frac{1}{a\rho}\left(\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right) - \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0. \tag{2.8}$$

Letting $\partial/\partial t + (u+a)\partial/\partial x = \mathrm{d}^+/\mathrm{d}t$ and $\partial/\partial t + (u-a)\partial/\partial x = \mathrm{d}^-/\mathrm{d}t$, where $\mathrm{d}^+/\mathrm{d}t$ and $\mathrm{d}^-/\mathrm{d}t$ are derivatives along the curves $\mathrm{d}x/\mathrm{d}t = u+a$ and $\mathrm{d}x/\mathrm{d}t = u-a$ respectively, equations (2.7) and (2.8) are equivalent to

$$\frac{\mathrm{d}^+u}{\mathrm{d}t} + \frac{1}{a\rho}\frac{\mathrm{d}^+p}{\mathrm{d}t} + \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \tag{2.9}$$

and

$$\frac{\mathrm{d}^-u}{\mathrm{d}t} - \frac{1}{a\rho}\frac{\mathrm{d}^-p}{\mathrm{d}t} - \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = 0. \tag{2.10}$$

In order to eliminate pressure p from (2.9) and (2.10), (2.3) and (2.4) are manipulated to yield

$$\frac{1}{a\rho}\frac{\mathrm{d}^+p}{\mathrm{d}t} = \frac{2}{\gamma-1}\frac{\mathrm{d}^+a}{\mathrm{d}t} - \frac{a^2}{\gamma R}\frac{\partial s}{\partial x}, \quad \frac{1}{a\rho}\frac{\mathrm{d}^-p}{\mathrm{d}t} = \frac{2}{\gamma-1}\frac{\mathrm{d}^-a}{\mathrm{d}t} + \frac{a^2}{\gamma R}\frac{\partial s}{\partial x}. \tag{2.11}$$

Equations (2.9) and (2.10) can then be rewritten as

$$\frac{\mathrm{d}^+u}{\mathrm{d}t} + \frac{2}{\gamma-1}\frac{\mathrm{d}^+a}{\mathrm{d}t} + \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{a^2}{\gamma R}\frac{\partial s}{\partial x}, \tag{2.12}$$

$$\frac{\mathrm{d}^-u}{\mathrm{d}t} - \frac{2}{\gamma-1}\frac{\mathrm{d}^-a}{\mathrm{d}t} - \frac{ua}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{a^2}{\gamma R}\frac{\partial s}{\partial x}. \tag{2.13}$$

The system of equations (2.12), (2.13) and (2.3) is equivalent to the system (2.1), (2.2) and (2.3). However, in this system all the equations are in the characteristic form (Whitham 1974). The Riemann variants of the system are $u + 2a/(\gamma-1)$, $u - 2a/(\gamma-1)$ and s along the characteristic velocities $u+a$, $u-a$ and u respectively.

Since there are three characteristic velocities, if a disturbance occurs at some point (say $x = 0$), three waves will evolve with the velocities $u+a$, $u-a$ and u. These waves are known as compound waves. In the case of a constant-area duct with a uniform state (no entropy gradients), the Riemann variants are constant along the characteristics and after travelling a sufficient distance, the compound wave separates into two waves travelling in opposite directions with the velocities $u+a$ and $u-a$. These waves are called simple waves.

In a simple wave region, one can easily show that variables like u, a, p_e (excess pressure, $p - p_0$) etc. are constant along the characteristic velocities (Lighthill 1978). Therefore for a right-running simple wave:

$$p_e = \text{constant along } \mathrm{d}x/\mathrm{d}t = u+a. \tag{2.14}$$

In the case of a variable-area duct with entropy gradients, a compound wave does not separate into simple waves. Therefore the above simplifications are not valid. However, for relatively weak waves Lighthill (1978) argued that (2.14) is still valid if p_e is replaced by $p_e\sqrt{Y}$, where the admittance Y varies gradually with x. Using the high-frequency approximation for the admittance $Y(x) = A_0(x)/(a_0(x)\rho_0(x))$, a relation similar to (2.14) can be written for a variable-area duct with entropy gradients:

$$p_e[A_0(x)/(a_0(x)\rho_0(x))]^{1/2} = \text{constant along } \mathrm{d}x/\mathrm{d}t = u+a. \tag{2.15}$$

Thus for a weak pulse, Lighthill first approximated a compound wave as a two simple waves moving in opposite directions and then used the linear theory to account the effect of change in area and temperature. Exploiting these approximations, he derived the following quasi-linear hyperbolic differential equation:

$$\frac{\partial V_1}{\partial T_1} + V_1\frac{\partial V_1}{\partial X_1} = 0, \tag{2.16}$$

where

$$V_1 = \frac{(\gamma+1)p_e}{2\rho_0(x)a_0^3(x)V_0(x)}, \quad X_1 = \int_0^x \frac{\mathrm{d}x}{a_0(x)} - t, \quad T_1 = \int_0^x V_0(x)\,\mathrm{d}x, \tag{2.17}$$

and

$$V_0(x) = \frac{1}{\sqrt{A_0(x)\rho_0(x)a_0^5(x)}}. \tag{2.18}$$

Equation (2.16) can be solved to find the location of shock formation as

$$\int_0^{x_s} \frac{V_0(x)}{V_0(0)}\,\mathrm{d}x = \frac{2\rho_0(0)a_0^3(0)}{(\gamma+1)\mathrm{Max}[\partial p_e/\partial t]_{x=0}}. \tag{2.19}$$

The condition for $\mathrm{Max}[\partial p_e/\partial t]_{x=0}$ to form a shock is

$$\mathrm{Max}[\partial p_e/\partial t]_{x=0} > \frac{2\rho_0(0)a_0^3(0)}{(\gamma+1)\displaystyle\int_0^{\infty}(V_0(x)/V_0(0))\,\mathrm{d}x}. \tag{2.20}$$

The present paper investigates the nonlinear steepening of plane finite-amplitude waves in variable-area ducts with temperature gradients. There is no restriction on the amplitude and frequency of the wave. Since the analysis is valid only for a plane wave front, changes in cross-sectional area of the duct should be gradual so that the wave front always remains planar. However there is no restriction on the temperature gradient as long as it is smooth.

3. Expansion near a wave front

The wave front expansion is an easy way to deal with the propagation of a discontinuity along the characteristics. Whitham (1974) has shown that for a hyperbolic system the discontinuity in the first derivative can only occur on the characteristics. Hence, if the wave is treated as a discontinuity in the first derivative of the gas velocity, the leading edge of the wave will always be on one of the characteristics. In general, it is difficult to deal with a complex wave having both a compressive and an expansive part. Therefore, pure compression and expansion waves will be treated separately in this paper.

The entire analysis is based on the assumption that the maximum slope occurs at the wave front and the shock forms when the slope of the wave front becomes infinite. Therefore it is sufficient to know the slope of wave front to find the time and location of the shock formation. Since there is no flow ($u=0$) in the quiescent field, two wave fronts in the compound wave will be moving in opposite directions with velocities a and $-a$. Note that the value of a at the wave front is known in the quiescent field. The wave front corresponding to the third wave (moving with the velocity u) does not move. In the present paper, the behaviour of a right-running wave front will be investigated. Due to the symmetry of the problem, a left-running wave front will behave in the same way.

Although the behaviour of such a particular waveform is not exactly the same as the compressive or expansive part of a complex wave, it is quite useful to obtain a rough estimate of time and location of wave breaking. Morse & Ingard (1968) calculated the time of shock formation for a compression wave by taking appropriate limits in the nonlinear wave equation of plane travelling waves. In the present case, the propagation of a travelling wave is governed by the three equations (2.12), (2.13) and (2.3). Therefore, instead of evaluating the limits of the derivatives in these equations, a wave front expansion will be used to obtain an evolution equation in terms of the local slope (first derivative) of the leading edge of the wave front. For convenience, equations (2.12) and (2.13) can be written as

$$A\frac{\partial a}{\partial t}+Au\frac{\partial a}{\partial x}+\frac{\gamma-1}{2}Aa\frac{\partial u}{\partial x}+\frac{\gamma-1}{2}ua\frac{\mathrm{d}A}{\mathrm{d}x}=0, \tag{3.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2}{\gamma-1}a\frac{\partial a}{\partial x}-\frac{a^2}{\gamma R}\frac{\partial s}{\partial x}=0, \tag{3.2}$$

with the equation for isentropic flow

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}=0. \tag{3.3}$$

Figure 1 (a,b) shows the velocity–distance curve of a right-running wave front. The slope of the wave front is negative for a compression wave front and positive for an expansion wave front. If $x=X(t)$ is the position of the wave front, there is an undisturbed quiescent flow for $x>X(t)$ and an unsteady flow for $x<X(t)$. The velocity of wave front will be

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\dot{X}(t)=a(x,t)|_{x=X(t)}. \tag{3.4}$$

In the wave front expansion technique, the frame of reference is fixed to the wave front. The wave front is moving at velocity $\dot{X}(t)$ with respect to the ground. Let ξ

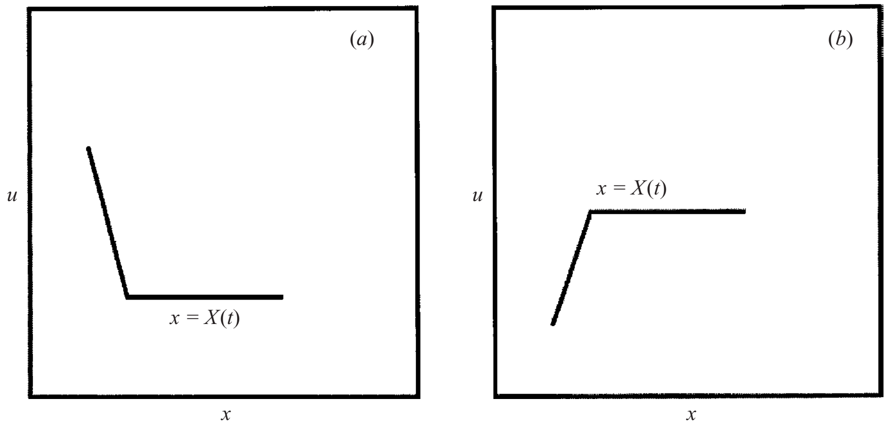

FIGURE 1. Plots of particle velocity versus distance at a particular instant of time near the wave front. Here $x = X(t)$ is the position of the wave front. (*a*) Compression wave front. (*b*) Expansion wave front.

be the position of a particle on the wave in this new frame of reference. Then $\xi = 0$, where $\xi = x - X(t)$ denotes the position of the wave front. All the dependent variables are expanded in powers of ξ about the wave front for $\xi > 0$ and $\xi < 0$. If the first derivatives are discontinuous, the appropriate expansions are (Whitham 1974)

$$\left.\begin{aligned} a(\xi, t) &= a_0(X(t)) + \xi a_0'(X(t)) + \frac{\xi^2}{2} a_0''(X(t)) + \dots \\ u(\xi, t) &= 0 \\ A(\xi, t) &= A_0(X(t)) + \xi A_0'(X(t)) + \frac{\xi^2}{2} A_0''(X(t)) + \dots \\ s(\xi, t) &= s_0(X(t)) + \xi s_0'(X(t)) + \frac{\xi^2}{2} s_0''(X(t)) + \dots \end{aligned}\right\} \text{for} \quad \xi > 0, \tag{3.5}$$

$$\left.\begin{aligned} a(\xi, t) &= a_0(X(t)) + \xi a_1(t) + \frac{\xi^2}{2} a_2(t) + \dots \\ u(\xi, t) &= \xi u_1(t) + \frac{\xi^2}{2} u_2(t) + \dots \\ A(\xi, t) &= A_0(X(t)) + \xi A_1(t) + \frac{\xi^2}{2} A_2(t) + \dots, \\ s(\xi, t) &= s_0(X(t)) + \xi s_1(t) + \frac{\xi^2}{2} s_2(t) + \dots \end{aligned}\right\} \text{for} \quad \xi < 0. \tag{3.6}$$

Note that $a_0(x)$ and $s_0(x)$ are the known sound speed and entropy in the quiescent field. $A_0(x)$ is the cross-sectional area of duct. In a quiescent field $u_0(x)$ is zero. For $\xi > 0$, the coefficients of all the powers of ξ are known. For $\xi < 0$, only the coefficients of ξ^0 are known and others have to be found. Also, for $A_0(x)$, all the derivatives are known on both sides of the wave front, i.e. $A_1(t) = A_0'(X(t))$, $A_2(t) = A_0''(X(t))$ and so on.

Now evaluating the required derivatives for $\xi < 0$:

$$\left.\begin{aligned}
\frac{\partial a(\xi,t)}{\partial t} &= [a_0'(X(t)) - a_1(t)]\dot{X}(t) + \xi[a_1'(t) - a_2(t)\dot{X}(t)] + \dots,\\
\frac{\partial u(\xi,t)}{\partial t} &= [-u_1(t)]\dot{X}(t) + \xi[u_1'(t) - u_2(t)\dot{X}(t)] + \dots,\\
\frac{\partial A(\xi,t)}{\partial t} &= [A_0'(X(t)) - A_1(t)]\dot{X}(t) + \xi[A_1'(t) - A_2(t)\dot{X}(t)] + \dots = 0,\\
\frac{\partial s(\xi,t)}{\partial t} &= [s_0'(X(t)) - s_1(t)]\dot{X}(t) + \xi[s_1'(t) - s_2(t)\dot{X}(t)] + \dots;
\end{aligned}\right\} \tag{3.7}$$

$$\left.\begin{aligned}
\frac{\partial a(\xi,t)}{\partial x} &= a_1(t) + \xi a_2(t) + \dots,\\
\frac{\partial u(\xi,t)}{\partial x} &= u_1(t) + \xi u_2(t) + \dots,\\
\frac{\partial s(\xi,t)}{\partial x} &= a_1(t) + \xi a_2(t) + \dots,\\
\frac{\partial A(\xi,t)}{\partial x} &= A_1(t) + \xi A_2(t) + \dots.
\end{aligned}\right\} \tag{3.8}$$

Substituting the above power series expansions into (3.1), (3.2) and (3.3) and equating the coefficients of ξ^0, ξ^1 etc, gives

ξ^0:

$$a_0' - a_1 + \frac{\gamma-1}{2}u_1 = 0, \tag{3.9}$$

$$-u_1 + \frac{2}{\gamma-1}a_1 - \frac{a_0}{\gamma R}s_1 = 0, \tag{3.10}$$

$$s_0' - s_1 = 0, \tag{3.11}$$

ξ^1:

$$a_1' - a_2a_0 + (a_0'a_0 - a_1a_0)\frac{A_1}{A_0} + a_1u_1 + \frac{\gamma-1}{2}\left[a_0u_2 + a_1u_1 + a_0u_1\frac{A_1}{A_0}\right] + \frac{\gamma-1}{2}u_1a_0\frac{A_1}{A_0} = 0, \tag{3.12}$$

$$-a_0u_2 + \frac{2}{\gamma-1}a_0a_2 + u_1' + u_1^2 + \frac{2}{\gamma-1}a_1^2 - \frac{1}{\gamma R}\left[a_0^2s_2 + 2a_0a_1s_1\right] = 0, \tag{3.13}$$

$$s_1' - a_0s_2 + u_1s_1 = 0. \tag{3.14}$$

The system of equations (3.9), (3.10) and (3.11) is singular with respect to variables a_1, u_1 and s_1. Therefore it cannot be solved for them. Similarly the system of equations (3.12), (3.13) and (3.14) cannot be solved for the variables a_2, u_2 and s_2. Elimination of a_1, u_1 and s_1 from (3.9), (3.10) and (3.11) yields

$$s_0' = \frac{2\gamma R a_0'}{(\gamma-1)a_0}. \tag{3.15}$$

In the same fashion, the variables a_2, u_2 and s_2 areeliminated from (3.12), (3.13) and

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(3.14), which gives the following equation:

$$\frac{2}{\gamma-1}\left[a_1'+(a_0'a_0-a_1a_0)\frac{A_1}{A_0}+a_1u_1+\frac{\gamma-1}{2}a_0u_1\frac{A_1}{A_0}+a_1^2\right]+a_1u_1+u_1a_0\frac{A_1}{A_0}$$
$$+u_1'+u_1^2-\frac{a_0}{\gamma R}[s_0'+u_1s_1+2a_1s_1]=0. \quad (3.16)$$

Now, (3.16) and any two equations from system (3.9), (3.10) and (3.11) can be used to solve for the variables a_1, u_1 and s_1. Solving for u_1 and knowing that $A_1(t)=A_0'(X(t))$, one obtains the following first-order nonlinear differential equation (Riccati equation):

$$u_1'(t)+\frac{1}{2}\left(\frac{a_0(X(t))A_0'(t)}{A_0(X(t))}+a_0'(X(t))\right)u_1(t)+\frac{\gamma+1}{2}u_1^2(t)=0. \quad (3.17)$$

Before proceeding to a detailed analysis of the problem, some discussion about the physical significance of (3.17) is necessary. The first term ($u_1'(t)$) is the rate of change of the slope of the wave front. The second term is due to the change in the cross-sectional area and the axial variation in the mean temperature, and is linear in $u_1(t)$. The third term is nonlinear in $u_1(t)$. For a plane wave moving in a constant-area duct with homentropic flow, (3.17) becomes

$$u_1'(t)+\frac{\gamma+1}{2}u_1^2(t)=0. \quad (3.18)$$

In the linear acoustic theory of plane wave propagation, the nonlinear terms are neglected, and a wave moves without any nonlinear distortion. If one neglects the nonlinear term in (3.18), the slope of the wave front will remain constant. However, for a finite-amplitude wave, the slope of the wave front changes in a nonlinear fashion.

4. Nonlinear distortion of a wave front

In this section, the nonlinear distortion of a travelling wave front in a duct with varying cross-section area and varying temperature will be analysed. Equation (3.17) can be solved to obtain the first derivative of the wave front at time t. It is easier to deal with the position of wave front, $X(t)$, as an independent variable instead of time t. Therefore, writing

$$\frac{\mathrm{d}u_1}{\mathrm{d}t}=\frac{\mathrm{d}u_1}{\mathrm{d}X(t)}\frac{\mathrm{d}X(t)}{\mathrm{d}t}=\frac{\mathrm{d}u_1}{\mathrm{d}X(t)}a_0(X(t)), \quad (4.1)$$

equation (3.17) becomes

$$\frac{\mathrm{d}u_1}{\mathrm{d}y}+\frac{1}{2}\left(\frac{A_0'(y)}{A_0(y)}+\frac{a_0'(y)}{a_0(y)}\right)u_1+\frac{\gamma+1}{2a_0(y)}u_1^2=0. \quad (4.2)$$

Here, $y=X(t)$ is the position of the wave front. The time t corresponding to the position y can then be evaluated from the following integral:

$$t=\int_0^y\frac{\mathrm{d}y}{a_0(y)}. \quad (4.3)$$

Though (4.2) is a nonlinear ordinary differential equation (ODE) in $u_1(y)$, it can easily be transformed to a linear ODE in $1/u_1$ by dividing with u_1^2:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{u_1}\right)-\frac{1}{2}\left(\frac{A_0'(y)}{A_0(y)}+\frac{a_0'(y)}{a_0(y)}\right)\frac{1}{u_1}=\frac{\gamma+1}{2a_0(y)}. \quad (4.4)$$

Equation (4.4) is a first-order linear ODE whose solution is given by

$$\frac{1}{u_1(y)} = \frac{1}{u_1(0)} \frac{IF(0)}{IF(y)} + \frac{\gamma+1}{2IF(y)} \int_0^y \frac{IF(y)}{a_0(y)}\,\mathrm{d}y. \tag{4.5}$$

where the integrating factor

$$IF(y) = \exp\left[-\int \frac{1}{2}\left(\frac{A_0'(y)}{A_0(y)} + \frac{a_0'(y)}{a_0(y)}\right)\mathrm{d}y\right] = \frac{1}{\sqrt{A_0(y)a_0(y)}}, \tag{4.6}$$

and $u_1(0)$ is the initial slope of the wave front.

4.1. *Compression wave*

As stated earlier, in our analysis pure compression and pure expansion waves are treated separately. In this section, the nonlinear distortion of a compression wave front will be discussed. For a compression wave front, the initial slope is negative, i.e. $u_1(0) < 0$. For convenience, write

$$u_1(0) = -|u_1(0)|, \tag{4.7}$$

then (4.5) becomes

$$\frac{1}{u_1(y)} = -\frac{IF(0)}{IF(y)}\left[\frac{1}{|u_1(0)|} - \frac{\gamma+1}{2IF(0)} \int_0^y \frac{IF(y)}{a_0(y)}\,\mathrm{d}y\right]. \tag{4.8}$$

Equation (4.8) gives the slope of the wave front at a position y. There is a possibility that the right hand side of (4.8) could become zero at some finite value of $y = y_s$. This will occur when

$$|u_1(0)| = \frac{1}{\dfrac{\gamma+1}{2IF(0)} \displaystyle\int_0^{y_s} \frac{IF(y)}{a_0(y)}\,\mathrm{d}y}. \tag{4.9}$$

At this stage, the first derivative of the wave front becomes infinite. This phenomenon is referred to as a shock. The steepening of a compression wave front into shock is greatly influenced by variations in the cross-section of the duct and entropy. In some cases it is also possible that for a given compression wave front, the right-hand side of (4.8) always remains negative. Equivalently

$$\left[\frac{1}{|u_1(0)|} - \frac{\gamma+1}{2IF(0)} \int_0^y \frac{IF(y)}{a_0(y)}\,\mathrm{d}y\right]_{\max} > 0. \tag{4.10}$$

Now $\int_0^y (IF(y)/a_0(y))\,\mathrm{d}y$ is an increasing function of y that has a maximum value of $\int_0^\infty (IF(y)/a_0(y))\,\mathrm{d}y$. Therefore, condition (4.10) can be rewritten as

$$|u_1(0)| < \frac{1}{\dfrac{\gamma+1}{2IF(0)} \displaystyle\int_0^{\infty} \frac{IF(y)}{a_0(y)}\,\mathrm{d}y}. \tag{4.11}$$

Inequality (4.11) gives the maximum value of the initial slope of a wave front that will not steepen into a shock. Thus, only sufficiently steep compression wave fronts steepen into a shock; they must have initial slope

$$|u_1(0)| > \frac{1}{\dfrac{\gamma+1}{2IF(0)} \displaystyle\int_0^{\infty} \frac{IF(y)}{a_0(y)}\,\mathrm{d}y}. \tag{4.12}$$

It is clear from (4.12) that if the integral $\int_0^\infty (IF(y)/a_0(y))\,\mathrm{d}y$ does not converge, all the compression waves fronts will steepen into shocks at some finite distance given by (4.9). If it converges, only those compression wave fronts having initial slope given by (4.12) will steepen into shocks.

It can be easily shown that the condition of shock formation (4.12) and the relation for the location of the shock (4.9) are same as obtained by Lighthill (1978) in the limit of weakly nonlinear waves at high frequencies, (2.19) and (2.20). Similar to the expansions (3.6) in § 3, the power series expansion for p is

$$p(\xi, t) = p_0 + \xi p_1(t) + \frac{\xi^2}{2} p_2(t) + \dots ; \tag{4.13}$$

p_1 is related to u_1 through the relation $p_1 = \rho_0(x) a_0(x) u_1$. Also $[\partial p/\partial t]_{x=X(t)} = -a_0(X(t)) p_1(t)$. Since, in our case, the maximum slope occurs at the wave front, $\mathrm{Max}[\partial p_e/\partial t]_{x=0} = -a_0(0) p_1(t)$. Substituting these relations into (2.19) and (2.20) results in (4.9) and (4.12).

It is clear from the above analysis that a change in the slope of the wave front determines the nonlinear steepening of a wave front. If one takes the nonlinearity to be weak (Lighthill's theory, explained in § 2), the maximum value of the initial slope in (2.19) and (2.20) could be related to the amplitude and frequency of a harmonic wave. Consider a source of plane harmonic waves at the position $x = 0$ emitting a wave of amplitude u_0 and frequency ω, i.e.

$$u(0, t) = u_0 \sin(\omega t). \tag{4.14}$$

The maximum slope of the wave $\mathrm{Max}[\partial u/\partial t]_{x=0}$ will be $u_0\omega$, i.e. the maximum slope of a harmonic wave is the product of its amplitude and frequency. As in the present analysis it is assumed that the maximum slope of a wave occurs at the wave front, $|u_1(0)|$ should be equal to $u_0\omega$. Since nonlinear distortion is greater (shock formation is favoured) for a steep wave front, a wave of high frequency and amplitude will steepen fast. Therefore, for a wave of given frequency, a shock will form only if the amplitude is larger than a threshold value.

4.1.1. *Compression wave travelling in a varying-cross-section duct in a homentropic environment*

In the previous subsection, it was shown that wave distortion and steepening of a wave front into a shock are affected very much by changes in the environment. Here, the term 'environment' refers to the cross-sectional area of the duct under consideration and the temperature in the duct. It is possible to prevent the steepening of a wave front into a shock by changing the properties of the environment. In this subsection, the effect of variation in the area of cross-section alone on the waveform distortion will be discussed. Although (4.8) is applicable for any smooth duct, in this analysis only ducts with monotonically increasing (diverging) or decreasing (converging) cross-section will be considered. In fact any smooth duct consists of many converging and diverging ducts, each of which can be analysed separately.

In a homentropic environment $a_0(x)$ is constant. Hence, (4.8) becomes

$$\frac{1}{u_1(y)} = -\sqrt{\frac{A_0(y)}{A_0(0)}}\left[\frac{1}{|u_1(0)|} - \frac{\gamma+1}{2a_0(0)}\int_0^y \sqrt{\frac{A_0(0)}{A_0(y)}}\,\mathrm{d}y\right], \tag{4.15}$$

and (4.9) reduces to

$$\int_0^{y_s} \sqrt{\frac{A_0(0)}{A_0(y)}}\,\mathrm{d}y = \frac{2a_0(0)}{|u_1(0)|(\gamma+1)} = \beta. \tag{4.16}$$

Hereβ is the shock formation distance for a plane wave front in a constant-area duct in homentropic conditions.

For a diverging duct $A_0(y)$ is always greater than $A_0(0)$, i.e. $\sqrt{A(0)/A(y)} < \sqrt{A(0)/A(0)} \Rightarrow y_s > \beta$, indicating that shock formation is delayed in a diverging duct compared to a duct with constant cross-section area. In the case of a converging duct $A_0(y)$ always is less than $A_0(0)$ and hence the shock will form before β. For a homentropic environment, the shock condition (4.12) reduces to

$$|u_1(0)| > \frac{1}{\dfrac{\gamma+1}{2a_0(0)} \displaystyle\int_0^{\infty} \sqrt{\frac{A_0(0)}{A_0(y)}}\,\mathrm{d}y}. \tag{4.17}$$

It can be easily shown by using the properties of improper integrals that for a converging duct, the integral $\int_0^\infty \sqrt{A(0)/A(y)}\,\mathrm{d}y$ is divergent. Consequently any compression wave front travelling in a converging duct will eventually steepen into a shock. If the area of cross-section of a converging duct becomes zero at some finite point y^*, the integral $\int_0^\infty \sqrt{A(0)/A(y)}\,\mathrm{d}y$ must be replaced by $\int_0^{y^*} \sqrt{A(0)/A(y)}\,\mathrm{d}y$. In this case, every compression wave front will steepen into shock before y^*. This situation is similar to a spherical compression wave travelling towards the centre of a sphere that always steepens into a shock before reaching the centre (Lin & Szeri 2001).

In the case of diverging ducts, there is a possibility that $\int_0^\infty \sqrt{A(0)/A(y)}\,\mathrm{d}y$ may converge. Usually this occurs when a duct is diverging rapidly (for example, an exponential horn). When a compression wave front travels in such a duct, the magnitude of its initial slope must exceed a minimum value (4.17) in order to steepen into shock.

In order to provide a better understanding, the above analysis is applied to some common ducts.

Example (*a*)

Consider a plane compression wave front travelling in a duct, whose area of cross-section varies as $A_0(x) = A_0(0)(1+\alpha x)^n$, $\alpha > 0, n \in R$. Such a function may represent four different types of ducts depending upon the value of n (see figure 2*a*–*d*). It will be seen that the behaviour of the compression wave in the four ducts will be different. Our integral of interest is

$$\int_0^{\infty} \sqrt{\frac{A_0(0)}{A_0(y)}}\,\mathrm{d}y = \int_0^{\infty} \frac{1}{(1+\alpha y)^{n/2}}\,\mathrm{d}y \tag{4.18}$$

$$\begin{aligned} \int_0^{\infty} \frac{1}{(1+\alpha y)^{n/2}}\,\mathrm{d}y &= \infty, \qquad n \leqslant 2 \\ &= \frac{2}{\alpha(n-2)}, \qquad n > 2. \end{aligned} \tag{4.19}$$

From these values, it can be concluded that a compression wave front travelling in a duct for which $n \leqslant 2$ will always steepen into shock. This case includes converging ducts ($n < 0$) as well (for example see figure 2*d*). Figure 2(*b*, *c*) shows a diverging

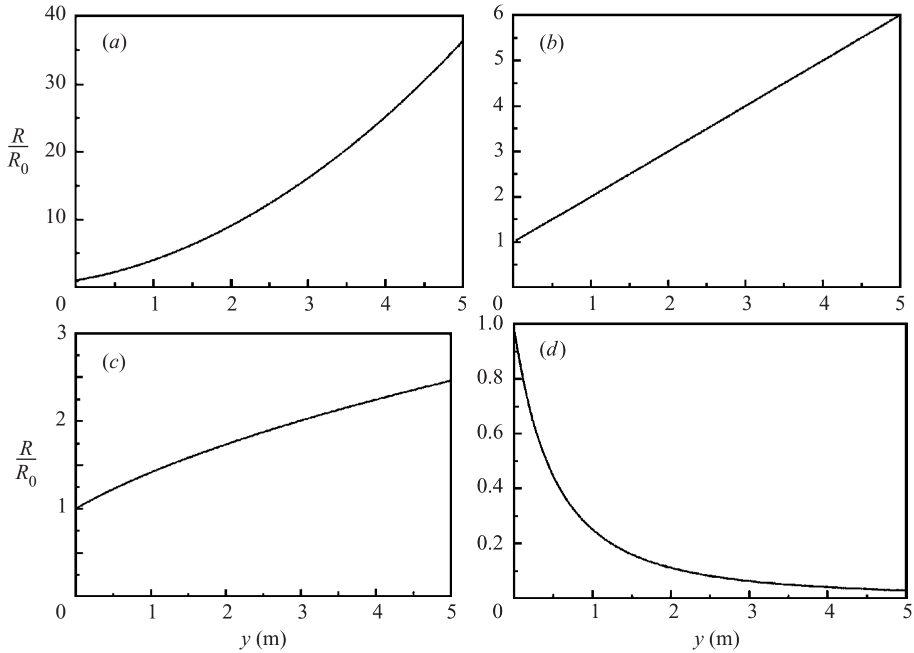

FIGURE 2. Plots of the cross-section radii for polynomial area variation. (*a*) $n = 4$ ($n > 2$), (*b*) $n = 2$, (*c*) $n = 1$ ($n < 2$) and (*d*) $n = -4$ ($n < 0$). Here the value of α is taken to be unity.

conical duct ($n = 2$) and a diverging parabolic duct ($n = 1$) respectively. These two ducts are examples of diverging ducts in which every compression wave evolves into a shock. For $n > 2$ (for example see figure 2*a*) a shock will form only if the initial slope of the wave front satisfies the following condition:

$$|u_1(0)| > \frac{(n-2)\alpha a_0(0)}{\gamma + 1}. \tag{4.20}$$

In all the above cases, if a shock forms the location of the shock can be evaluated from (4.16) as

$$\begin{aligned} y_s &= \frac{1}{\alpha}\left[\left(1 + \alpha\beta - \frac{1}{2}n\alpha\beta\right)^{2/(2-n)} - 1\right], \quad n \neq 2 \\ &= \frac{e^{\alpha\beta} - 1}{\alpha}, \quad n = 2. \end{aligned} \tag{4.21}$$

Here β is the shock formation distance for a plane wave in a homentropic environment.

Figure 3(*a*) shows the change in the slope of a compression wave front as it propagates in a diverging duct of area variation $A_0(y) = A_0(0)(1 + y)^4$ for different values of initial slope. In this example the compression waves steepen into shock for $|u_1(0)| > 250\,s^{-1}$. Figure 3(*b*) shows the shock formation distance as a function of n in ducts of polynomial area variation $A_0(y) = A_0(0)(1 + y)^n$. The value of $|u_1(0)|$ is $10\,s^{-1}$. It can be seen that as the value of n increases, the shock formation distance increases. Figure 3(*c*) shows the shock formation distance as a function of the slope, α, in a conical duct $A_0(y) = A_0(0)(1 + \alpha y)^2$. The value of $|u_1(0)|$ is $10\,s^{-1}$. It is clear that the shock formation distance increases as α increases.

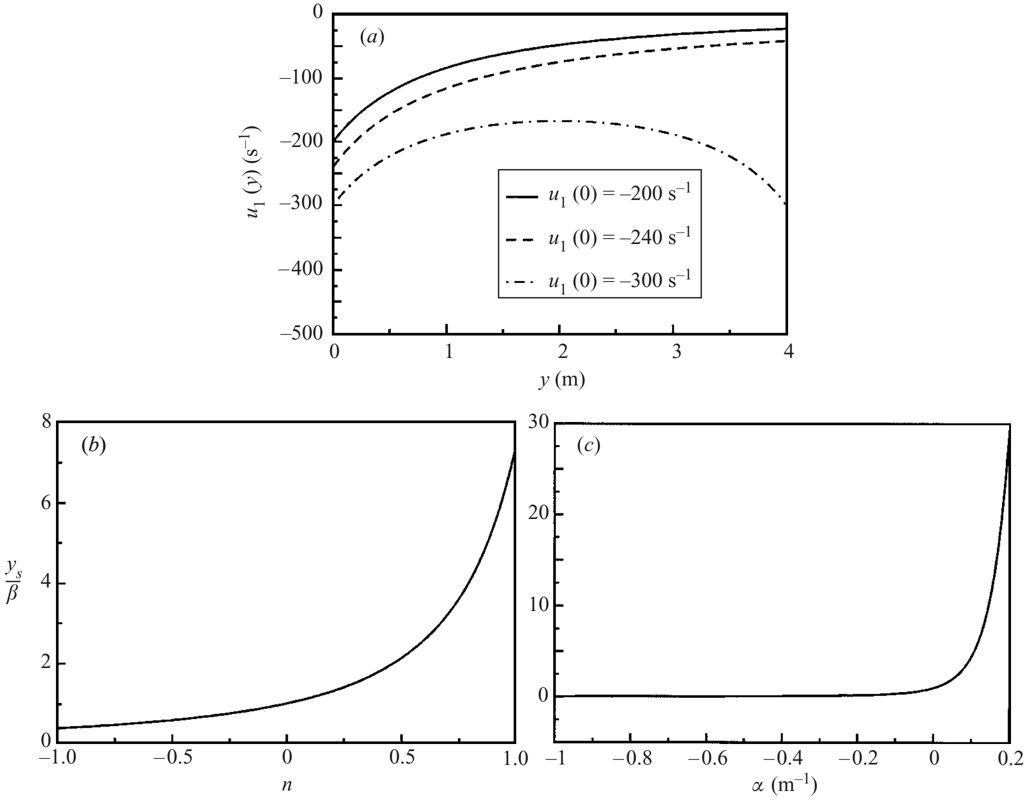

FIGURE 3. (*a*) The evolution of the slope of a compressive wave front for different values of initial slope in a duct of area variation $A_0(y) = A_0(0)(1+y)^4$. (*b*) Plot of shock formation distance y_s/β as a function of n for polynomial area variation $A_0(y) = A_0(0)(1+y)^n$. (*c*) Plot of shock formation distance y_s/β in a conical duct as a function of slope α of the cone, $A_0(y) = A_0(0)(1+\alpha y)^2$. Here the speed of sound $a_0(0) = 300\,\mathrm{m\,s^{-1}}$, and the ratio of specific heat of gas $\gamma = 1.4$. The value of initial slope taken in (*b*, *c*) is $10\,\mathrm{s^{-1}}$.

Example (*b*)

Now consider an exponential horn duct, i.e. $A_0(x) = A_0(0)\mathrm{e}^{\alpha x}$ where α is the flare constant of the horn. The horn is diverging for $\alpha > 0$ and converging for $\alpha < 0$. In this case, the value of the required integral is

$$\int_0^\infty \sqrt{\frac{A_0(0)}{A_0(y)}}\,\mathrm{d}y = \int_0^\infty \frac{1}{\mathrm{e}^{\alpha\tilde{y}/2}}\,\mathrm{d}\tilde{y}, \tag{4.22}$$

$$\begin{aligned}\int_0^\infty \frac{1}{\mathrm{e}^{\alpha y/2}}\,\mathrm{d}y &= \infty, \quad \alpha < 0 \\ &= \frac{2}{\alpha}, \quad \alpha \geqslant 0\end{aligned} \tag{4.23}$$

Therefore, every compression wave front travelling in a converging horn ($\alpha \leqslant 0$) will steepen into a shock. In a diverging horn, a shock will form if

$$|u_1(0)| > \frac{\alpha a_0(0)}{\gamma+1}. \tag{4.24}$$

The location of the shock in an exponential horn is

$$y_s = \frac{2}{\alpha} \ln \left[1 \Big/ \left(1 - \frac{\alpha\beta}{2} \right) \right]. \tag{4.25}$$

4.1.2. *Compression wave travelling in the presence of an entropy gradient in a uniform-cross-section duct*

In this subsection, the effect of a change in the entropy of the environment on waveform distortion will be investigated. In the present analysis, the entropy is altered by varying the speed of sound (3.15), which is in turn related to the temperature. Hence, an entropy gradient is equivalent to a temperature gradient along the direction of wave propagation. In order to understand the effect of an entropy gradient on the waveform distortion, a duct with a constant cross-section area is considered. In this case, the condition for shock formation, (4.12), for a compression wave becomes

$$|u_1(0)| > \frac{1}{\dfrac{(\gamma+1)}{2a_0(0)} \displaystyle\int_0^\infty \left(\frac{a_0(0)}{a_0(y)} \right)^{3/2} \mathrm{d}y}. \tag{4.26}$$

The shock formation distance (4.9) is given by

$$\int_0^{y_s} \left(\frac{a_0(0)}{a_0(y)} \right)^{3/2} \mathrm{d}y = \beta. \tag{4.27}$$

When a compression wave moves in an environment where the temperature is decreasing, it can be shown that $\int_0^\infty (a_0(0)/a_0(y))^{3/2}\,\mathrm{d}y$ diverges. Consequently every compression wave travelling in such an environment will blow up at some finite distance. On the other hand, in the presence of a positive temperature gradient the integral $\int_0^\infty (a_0(0)/a_0(y))^{3/2}\,\mathrm{d}y$ may diverge or converge. If it diverges, as usually happens for low temperature gradients, every compression wave front will steepen into shock at some finite distance. However in the presence of a very high temperature gradient, it may converge. Under such a condition, the initial slope of the wave front must exceed a minimum value in order to steepen into a shock.

Lin & Szeri (2001) predict that when a compression wave travels into a field with a constant positive entropy gradient, the initial slope of the wave front must exceed a critical value in order to steepen into a shock. However, as mentioned above, there can be a field with a positive entropy gradient in which every compression wave will evolve into a shock. The following example highlights such a field.

Example (*a*)

Consider a compression wave travelling in an environment where the speed of sound has a polynomial variation: $a_0(x) = a_0(0)(1+\alpha x)^n$, $\alpha > 0, n \in R$. Note that $n > 0$ corresponds to a positive entropy gradient and $n < 0$ corresponds to a negative entropy gradient. In this case, the value of the integral of interest is

$$\int_0^\infty \left(\frac{a_0(0)}{a_0(y)} \right)^{3/2} \mathrm{d}y = \int_0^\infty \frac{1}{(1+\alpha y)^{3n/2}} \mathrm{d}y, \tag{4.28}$$

$$\begin{aligned} \int_0^\infty \frac{1}{(1+\alpha y)^{3n/2}} \mathrm{d}y &= \infty, & n \leqslant \frac{2}{3} \\ &= \frac{2}{\alpha(3n-2)}, & n > \frac{2}{3}. \end{aligned} \tag{4.29}$$

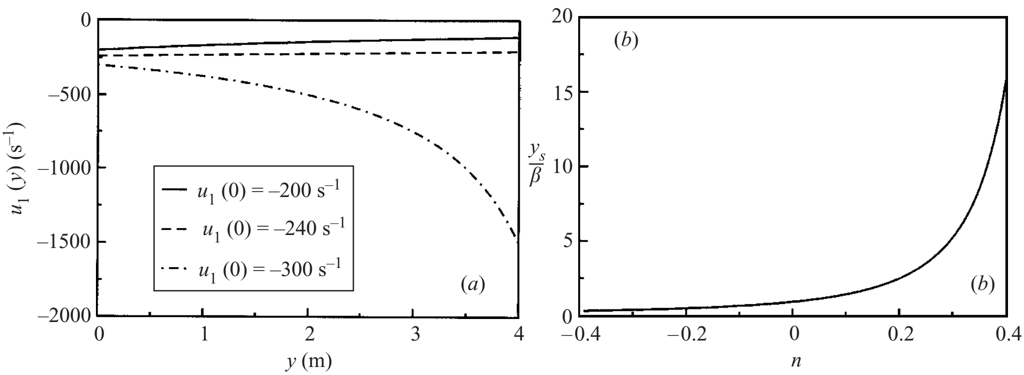

FIGURE 4. (*a*) The evolution of the slope of a compression wave front for different values of initial slope in an increasing temperature field, $a_0(y) = a_0(0)(1 + y)^{4/3}$. (*b*) Plot of shock formation distance y_s/β as a function of n in a polynomial temperature variation, $a_0(y) = a_0(0)(1 + y)^n$. Here the speed of sound $a_0(0) = 300\,\mathrm{m\,s^{-1}}$, and the ratio of specific heat of gas $\gamma = 1.4$. The value of the initial slope taken in (*b*) is $10\,\mathrm{s^{-1}}$.

Thus every compression wave travelling in an environment for which $n \leqslant 2/3$ (this also includes a negative gradient) will blow up. For $n > 2/3$ a shock will form if

$$|u_1(0)| > \frac{(3n-2)\alpha a_0(0)}{\gamma+1}. \tag{4.30}$$

The shock formation distance y_s can be found from (4.27) as

$$\begin{aligned} y_s &= \frac{1}{\alpha}\left[\left(1+\alpha\beta-\frac{3}{2}n\alpha\beta\right)^{2/(2-3n)}-1\right], \quad && n \neq 2/3 \\ &= \frac{e^{\alpha\beta}-1}{\alpha}, && n = 2/3. \end{aligned} \tag{4.31}$$

Figure 4(*a*) shows the change in the slope of a compression wave front as it propagates in an increasing temperature field given by $a_0(y) = a_0(0)(1 + y)^{4/3}$ for different values of the initial slope. In this example, compression waves steepen into a shock for $|u_1(0)| > 250\,\mathrm{s^{-1}}$. Figure 4(*b*) shows the shock formation distance as function of n for the polynomial temperature variation $a_0(y) = a_0(0)(1 + y)^n$. The value of $|u_1(0)|$ is $10\,\mathrm{s^{-1}}$. It can be seen that as the value of n increases, the shock formation distance increases. The expression for shock formation distance in example (*a*) in §4.1.1 for $n = 2$ is the same as in this example for $n = 2/3$. Therefore for the case of a temperature variation that leads to a sound speed profile of the form $a_0(y) = a_0(1 + \alpha y)^{2/3}$, figure 3(*c*) shows the plot of y_s as a function of α for the initial slope $|u_1(0)| = 10\,\mathrm{s^{-1}}$.

Example (*b*)

Consider an exponentially varying sound speed profile in a quiescent field: $a_0(x) = a_0(0)e^{\alpha x}$, $\alpha \in R$. Note that $\alpha > 0$ for positive entropy gradients and $\alpha < 0$ for negative entropy gradients. The value of the integral of interest is

$$\int_0^\infty \left(\frac{a_0(0)}{a_0(y)}\right)^{3/2} dy = \int_0^\infty \frac{1}{e^{3\alpha\tilde{y}/2}}\, d\tilde{y}, \tag{4.32}$$

$$\int_0^\infty \frac{1}{e^{3\alpha\tilde{y}/2}}\,d\tilde{y} = \infty, \qquad \alpha < 0$$

$$= \frac{2}{3\alpha}, \qquad \alpha \geqslant 0. \tag{4.33}$$

Thus every compression wave front travelling in a negative entropy gradient ($\alpha \leqslant 0$) will steepen into a shock. In a positive entropy gradient ($\alpha > 0$), the condition for shock formation is given by

$$|u_1(0)| > \frac{3\alpha a_0(0)}{\gamma+1}. \tag{4.34}$$

The shock formation distance will be

$$y_s = \frac{2}{3\alpha}\ln\left[1\Big/\left(1-\frac{3}{2}\alpha\beta\right)\right]. \tag{4.35}$$

4.2. *Expansion wave*

The behaviour of an expansion wave is quite different from that of a compression wave. For example, in a constant-area duct with uniform entropy, a compression wave has the tendency to become discontinuous whereas an expansion wave tends to relax.

In this section, the behaviour of an expansion wave travelling in a varying-area duct with entropy gradients will be investigated. For an expansion wave, the slope of the wave front is positive, i.e. $u_1(0) > 0$. Therefore (4.5) becomes

$$\frac{1}{u_1(y)} = \frac{1}{u_1(0)}\frac{IF(0)}{IF(y)} + \frac{\gamma+1}{2IF(y)}\int_0^y \frac{IF(y)}{a_0(y)}\,dy. \tag{4.36}$$

As $u_1(0)$ is positive for the expansion wave front, it is self-evident that the right-hand side of (4.36) is always positive. Now a shock can form only if the right-hand side of (4.36) reduces to zero. When areas $A_0(y)$ and $a_0(y)$ are non-zero, which is generally the case, for finite value of y, the right-hand side of (4.36) cannot vanish at finite distance, and therefore an expansion wave will not form a shock. From (4.36), it is also clear that an expansion wave front moving in a constant-area duct in a homentropic environment will relax at infinity. In the rest of this section, the effect of variations in entropy and cross-section of the duct on the shape of the expansion wave front at infinity will be investigated.

The value of the slope of the wave front at infinity will depend upon the following limit:

$$\lim_{y\to\infty} u_1(y) = \frac{1}{\lim_{y\to\infty}\left[\dfrac{1}{u_1(0)}\dfrac{IF(0)}{IF(y)} + \dfrac{\gamma+1}{2IF(y)}\displaystyle\int_0^y \frac{IF(y)}{a_0(y)}\,dy\right]}. \tag{4.37}$$

However, it should be noted that if the area of the duct vanishes at some finite point (for example a wave travelling in a converging cone), the slope is to be calculated at the zero-area location instead of infinity.

4.2.1. *Expansion wave travelling in a varying-cross-section duct in a homentropic environment*

Diverging duct

In this subsection, the behaviour of an expansion wave travelling in a diverging duct is investigated. In a homentropic environment (4.37) reduces to

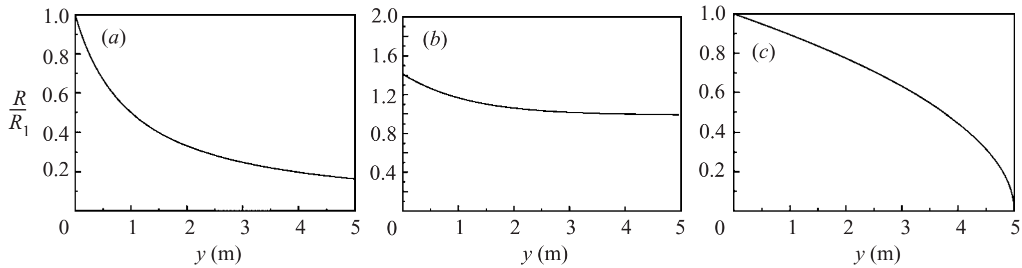

FIGURE 5. Plots of the cross-section radii for the different classes of ducts considered in §4.2.1. (*a*) Class 1: area variation $A(y)=A(0)(1+x)^{-2}$. (*b*) Class 2: area variation $A(y)=A(0)(1+e^{-x})$. (*c*) Class 3: area variation $A(y)=A(0)(1-y/y^*)$.

$$\lim_{y\to\infty} u_1(y)=\frac{1}{\left[\dfrac{1}{u_1(0)}\lim_{y\to\infty}\sqrt{\dfrac{A_0(y)}{A_0(0)}}+\dfrac{\gamma+1}{2a_0(0)}\lim_{y\to\infty}\sqrt{A_0(y)}\displaystyle\int_0^y\frac{1}{\sqrt{A_0(y)}}\,dy\right]}. \tag{4.38}$$

For a diverging duct $A_0(y)$ is an increasing function; it may be of two types: either monotonically increasing, i.e. $\lim_{y\to\infty}A_0(y)=\infty$ or having an asymptote parallel to the y-axis, i.e. $\lim_{y\to\infty}A_0(y)=\Gamma$ (a finite number). In the former case, it is obvious that the right-hand side of (4.38) vanishes at infinity. In the latter case, the denominator of the right-hand side becomes

$$\frac{1}{u_1(0)}\frac{\sqrt{\Gamma}}{\sqrt{A_0(0)}}+\frac{\gamma+1}{2a_0(0)}\lim_{y\to\infty}\sqrt{A_0(y)}\int_0^y\frac{1}{\sqrt{A_0(y)}}\,dy.$$

The value of the limit is

$$\lim_{y\to\infty}\sqrt{A_0(y)}\int_0^y\frac{1}{\sqrt{A_0(y)}}\,dy=\sqrt{\Gamma}\lim_{y\to\infty}\int_0^y\frac{1}{\sqrt{A_0(y)}}\,dy=\sqrt{\Gamma}\int_0^\infty\frac{1}{\sqrt{A_0(y)}}\,dy. \tag{4.39}$$

Since $A_0(y)<\Gamma\Rightarrow 1/\sqrt{A_0(y)}>1/\sqrt{\Gamma}$, the integral $\int_0^\infty 1/\sqrt{A(y)}\,dy$ is divergent. Consequently, the right-hand side of (4.38) vanishes in this case as well. Thus, in a diverging duct, the slope of an expansion wave approaches zero as it moves.

It should be noted that a duct will have infinite cross-sectional area as $y\to\infty$ and the pressure amplitude goes to zero at infinity. However, at this point the wave front will no longer be planar. Nevertheless, the above analysis gives the trend of an expansion wave front as the wave moves in a diverging duct.

Converging duct

The behaviour of an expansion wave in a converging duct is not unique as in a diverging duct. For simplicity, converging ducts can be divided in three classes (see figure 5*a*–*c*). Each class of duct will be investigated separately. In the first class, the function $A_0(y)$ starts from some finite initial value $A_0(0)$ and approaches zero at infinity (with the y-axis as an asymptote), i.e. $\lim_{y\to\infty}A_0(y)=0$. In real situations, any converging duct falls into this class. In the second class of ducts, $A_0(y)$ starts from a finite initial value $A_0(0)$ and attains a finite value l (less than $A_0(0)$) at infinity, i.e. $\lim_{y\to\infty}A_0(y)=l$ (asymptote parallel to the y-axis). Physically this class may represent a duct system in which a converging duct is attached to a duct of constant cross-sectional area. In the third class of ducts, $A_0(y)$ starts from some finite initial value $A_0(0)$ and crosses the y-axis at some point y^*, i.e. $\lim_{y\to y^*}A_0(y)=0$. A converging cone is a common example of this class. In this type of duct, the wave cannot go beyond y^*.

Class 1

It is clear that for ducts of class 1 the integral $\int_0^\infty 1/\sqrt{A(y)}\,\mathrm{d}y$ is divergent. Therefore the denominator of the right-hand side of (4.38) becomes

$$\frac{\gamma+1}{2a_0(0)} \lim_{y\to\infty} \sqrt{A_0(y)} \int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y.$$

This is an indeterminate form of the type $0\times\infty$. Evaluating the limit using L'Hopital's rule gives

$$\lim_{y\to\infty} \sqrt{A_0(y)} \int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y = \lim_{y\to\infty} \frac{\displaystyle\int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y}{\displaystyle\frac{1}{\sqrt{A_0(y)}}} = \lim_{y\to\infty} \frac{-2A_0(y)}{A_0'(y)}. \tag{4.40}$$

Now the limit

$$\frac{2a_0(0)}{(\gamma+1)\lim\limits_{y\to\infty}[-2A_0(y)/A_0'(y)]}$$

will give the slope of the expansion wave front at infinity. It should also be noted that the slope of the expansion wave front at infinity is independent of the initial slope of the wave.

Example (a)

Consider an expansion wave travelling in a converging duct, whose cross-section varies as $A_0(y)=A_0(0)(1+\alpha y)^n$, $n<0, \alpha>0$. Clearly, such a duct belongs to class 1. The value of the limit is

$$\lim_{y\to\infty} \frac{-2A_0(y)}{A_0'(y)} = \frac{-2(1+\alpha y)}{\alpha n} = \infty.$$

Hence in such a duct every expansion wave will relax at infinity.

Example (b)

Consider another duct of class 1: a converging exponential horn $A_0(y)=A_0(0)\mathrm{e}^{-\alpha y}$ where $\alpha>0$. In this case, the value of the limit is

$$\lim_{y\to\infty} \frac{-2A_0(y)}{A_0'(y)} = \frac{2}{\alpha} \Rightarrow \lim_{y\to\infty} u_1(y) = \frac{a_0(0)\alpha}{\gamma+1}.$$

Thus, in a converging exponential horn every expansion wave tends to attain a fixed value of slope $a_0(0)\alpha/(\gamma+1)$.

The above two examples show that if a duct is converging rapidly (as in the case of an exponentially converging horn), all the expansion wave fronts may approach some finite value which is independent of the initial slope.

Class 2

For ducts of class 2, the denominator on the right-hand side of (4.38) becomes

$$\frac{1}{u_1(0)} \frac{\sqrt{l}}{\sqrt{A_0(0)}} + \frac{\gamma+1}{2a_0(0)} \lim_{y\to\infty} \sqrt{A_0(y)} \int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y.$$

In this case also $\int_0^\infty 1/\sqrt{A(y)}\,\mathrm{d}y$ is divergent and hence the limit contains an indeterminate form of the type $0\times\infty$. Evaluating the limit using L'Hopital's rule gives

$$\lim_{y\to\infty}\sqrt{A_0(y)}\int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y = \lim_{y\to\infty}\frac{\displaystyle\int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y}{\dfrac{1}{\sqrt{A_0(y)}}} = \lim_{y\to\infty}\frac{-2A_0(y)}{A_0'(y)} \quad = \frac{-2l}{0} = \infty.$$

Consequently, every expansion wave front in ducts of class 2 will relax at infinity.

Class 3

In ducts belonging to class 3, the wave front cannot travel beyond y^*. Therefore the slope of the wave is calculated as $y\to y^*$, i.e.

$$\lim_{y\to y^*} u_1(y) = \frac{1}{\left[\dfrac{1}{u_1(0)}\lim_{y\to\infty}\sqrt{\dfrac{A_0(y)}{A_0(0)}}+\dfrac{\gamma+1}{2a_0(0)}\lim_{y\to\infty}\sqrt{A_0(y)}\displaystyle\int_0^y\frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y\right]}. \tag{4.41}$$

It is clear that $\lim_{y\to y^*}\sqrt{A_0(y)} = 0$. Now there are two possible cases. If the integral $\int_0^{y^*} 1/\sqrt{A(y)}\,\mathrm{d}y$ converges, the denominator on the right-hand side of (4.41) will vanish. Consequently, the expansion wave will try to attain an infinite value of its slope as $y\to y^*$. But if $\int_0^{y^*} 1/\sqrt{A(y)}\,\mathrm{d}y$ diverges, the following limit is to be evaluated, using L'Hopital's rule:

$$\lim_{y\to y^*}\sqrt{A_0(y)}\int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y = \lim_{y\to y^*}\frac{\displaystyle\int_0^y \frac{1}{\sqrt{A_0(y)}}\,\mathrm{d}y}{\dfrac{1}{\sqrt{A_0(y)}}} = \lim_{y\to y^*}\frac{-2A_0(y)}{A_0'(y)} = 0.$$

Hence in this case also, an expansion wave will tend to blow up as $y\to y^*$. This situation is similar to the finding of Lin & Szeri (2001) that a spherical expansion wave moving toward the centre blows up at the centre. However, as stated earlier, this kind infinitesimal expansion shock is mechanically unstable and diffuses out very quickly.

4.2.2. *Expansion wave travelling in the presence of a temperature gradient in uniform-cross-section duct*

Positive entropy gradient

A positive entropy gradient has a similar effect as duct divergence. It can be easily proved that every expansion wave front with some finite initial value of its first derivative will finally disappear in an increasing entropy field. For a wave travelling in constant-cross-section duct (4.37) becomes

$$\lim_{y\to\infty} u_1(y) = \frac{1}{\left[\dfrac{1}{u_1(0)a_0(0)}\lim_{y\to\infty}\sqrt{a_0(y)}+\dfrac{\gamma+1}{2}\lim_{y\to\infty}\sqrt{a_0(y)}\displaystyle\int_0^y\frac{1}{a_0^{3/2}(y)}\,\mathrm{d}y\right]} = 0. \tag{4.42}$$

Negative entropy gradient

The negative entropy gradients can been classified into different classes similar to the diverging ducts. However, an entropy gradient corresponding to class 3 for diverging ducts, where the cross-sectional area reduces to zero, does not exist in the real world and therefore is not discussed here.

Class 1

In class 1, the speed of sound, $a_0(y)$, starts from some finite initial value $a_0(0)$ and becomes zero at infinity (y-axis as asymptote), i.e. $\lim_{y\to\infty} a_0(y) = 0$. Using the properties of improper integrals it can be shown that integral $\int_0^\infty (1/a_0^{3/2}(y))\,\mathrm{d}y$ diverges for this class of entropy gradient. Evaluating the limit, $\lim_{y\to\infty}\sqrt{a_0(y)}\int_0^y (1/a_0^{3/2}(y))\,\mathrm{d}y$ by L'Hopital's rule gives

$$\lim_{y\to\infty}\frac{\displaystyle\int_0^y \frac{1}{a_0^{3/2}(y)}\,\mathrm{d}y}{\dfrac{1}{\sqrt{a_0(y)}}} = \frac{-2}{\displaystyle\lim_{y\to\infty} a_0'(y)} = \infty$$

because the y-axis is an asymptote to the curve. Hence every expansion wave travelling in an entropy gradient of class 1 will relax at infinity.

Class 2

In entropy gradients of class 2, the function $a_0(y)$ starts from a finite initial value, $a_0(0)$ and attains a finite value l (less than $a_0(0)$) at infinity, i.e. $\lim_{y\to\infty} a_0(y) = l$. Such a curve will have an asymptote parallel to the y-axis. Now, $\lim_{y\to\infty}\sqrt{a_0(y)} = \sqrt{l}$. Again the integral $\int_0^\infty (1/a_0^{3/2}(y))\,\mathrm{d}y$ is divergent for this class. Therefore evaluating the limit, $\lim_{y\to\infty}\sqrt{a_0(y)}\int_0^y (1/a_0^{3/2}(y))\,\mathrm{d}y$ by L'Hopital's rule gives

$$\lim_{y\to\infty}\frac{\displaystyle\int_0^y \frac{1}{a_0^{3/2}(y)}\,\mathrm{d}y}{\dfrac{1}{\sqrt{a_0(y)}}} = \frac{-2}{\displaystyle\lim_{y\to\infty} a_0'(y)} = \infty.$$

Hence every expansion wave travelling in an entropy gradient of this type will relax at infinity.

5. Conclusions

In this paper, the wave front expansion is used to investigate the effect of varying cross-section and entropy gradient on a travelling wave. Pure compression and expansion waves have been considered. These waves have a discontinuity in the first derivative of the wave front. An evolution equation for the slope of the wave front is obtained. It is found that in converging ducts, all compression waves, irrespective of the values of their initial slope, steepen into a shock at some finite distance. If a converging duct reaches zero area at some point, a shock will form before this point. The shock formation distance in a converging duct is always less than the shock formation distance in a constant-area duct with homentropic flow. On the other hand, in the case of diverging ducts, compression waves may or may not become shocks depending upon the area variation of the duct. In some diverging ducts every compression wave front blows up and in other diverging ducts (highly diverging like exponential horns), only those compression wave fronts with initial slopes greater than a critical value will steepen into shocks. An expansion wave moving in a diverging duct always relaxes. In a converging duct, an expansion wave front either relaxes or tends toward a permanent form independent of its initial slope.

For a compression wave front, the effect of the entropy gradient is similar to that of area variation. Increasing entropy gradients oppose shock formation (similar to the diverging duct) and decreasing entropy gradients favour shock formation (similar to the converging duct). An expansion wave front always relaxes in entropy gradients whether it is increasing or decreasing.

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