On the Inextendibility of Space-Time

John Byron Manchak*†

It has been argued that space-time must be inextendible—that it must be "as large as it can be" in some sense. Here, we register some skepticism with respect to this position.

1. Introduction. A space-time is counted as inextendible if, intuitively, it is "as large as it can be." It has been argued that inextendibility is a "reasonable physical condition to be imposed on models of the universe" (Geroch 1970, 262) and that a space-time must be inextendible if it is "to be a serious candidate for describing actuality" (Earman 1995, 32). Here, in a variety of ways, we register some skepticism with respect to such positions.

2. Preliminaries. We begin with a few preliminaries concerning the relevant background formalism of general relativity.[1] An n-dimensional, relativistic *space-time* (for $n \geq 2$) is a pair of mathematical objects (M, g_{ab}), where M is a connected n-dimensional Hausdorff manifold (without boundary) that is smooth and g_{ab} is a smooth, nondegenerate, pseudo-Riemannian metric of Lorentz signature $(-, +, \ldots, +)$ defined on M. We say two space-times (M, g_{ab}) and (M', g'_{ab}) are *isometric* if there is a diffeomorphism $\varphi : M \to M'$ such that $\varphi_* g_{ab} = g'_{ab}$. Two space-times (M, g_{ab}) and (M', g'_{ab}) are *locally isometric* if, for each point $p \in M$, there is an open neighborhood

* To contact the author, please write to: Logic and Philosophy of Science, 3151 Social Science Plaza A, University of California, Irvine, Irvine, CA 92697; e-mail: manchak@gmail.com.

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1. The reader is encouraged to consult Hawking and Ellis (1973), Wald (1984), and Malament (2012) for details. An outstanding (and less technical) survey of the global structure of space-time is given by Geroch and Horowitz (1979).

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O of p and an open subset O' of M' such that O and O' are isometric, and, correspondingly, with the roles of (M, g_{ab}) and (M', g'_{ab}) interchanged.

A space-time (M, g_{ab}) is *extendible* if there exists a space-time (M, g_{ab}) and a proper isometric embedding $\varphi : M \to M'$. Here, the space-time (M', g'_{ab}) is an *extension* of (M, g_{ab}). A space-time is *inextendible* if it has no extension. A $\mathcal{P}$-*space-time* is a space-time with property $\mathcal{P}$. A $\mathcal{P}$-space-time (M', g'_{ab}) is a $\mathcal{P}$-*extension* of a $\mathcal{P}$-space-time (M, g_{ab}) if (M', g'_{ab}) is an extension of (M, g_{ab}). A $\mathcal{P}$-space-time is $\mathcal{P}$-*extendible* if it has a $\mathcal{P}$-extension and is $\mathcal{P}$-*inextendible* otherwise. We say (M, g_{ab}, F) is an n-dimensional *framed space-time* if (M, g_{ab}) is an n-dimensional space-time and F is an orthonormal n-ad of vectors $\{\xi_1, \dots, \xi_n\}$ at some point $p \in M$. We say an n-dimensional framed space-time (M', g'_{ab}, F') is a *framed extension* of the n-dimensional framed space-time (M, g_{ab}, F) if there is a proper isometric embedding $\varphi : M \to M'$, which takes F into F'.

For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *time-like* if $g_{ab}\xi^a\xi^b < 0$, *null* if $g_{ab}\xi^a\xi^b = 0$, or *space-like* if $g_{ab}\xi^a\xi^b > 0$. Null vectors create the cone structure; time-like vectors are inside the cone, while space-like vectors are outside. A *time orientable* space-time is one that has a continuous time-like vector field on M. In what follows, it is assumed that space-times are time orientable.

For some connected interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma : I \to M$ is time-like if the tangent vector ξ^a at each point in $\gamma[I]$ is time-like. Similarly, a curve is null (respectively, space-like) if its tangent vector at each point is null (respectively, space-like). A curve is *causal* if its tangent vector at each point is either null or time-like. A causal curve is *future directed* if its tangent vector at each point falls in or on the future lobe of the light cone. We say a time-like curve $\gamma : [s, s'] \to M$ is *closed* if $\gamma(s) = \gamma(s')$. A space-time (M, g_{ab}) satisfies *chronology* if it does not contain a closed time-like curve. For any two points $p, q \in M$, we write $p \ll q$ if there exists a future-directed time-like curve from p to q. This relation allows us to define the *time-like past* of a point p: $I^-(p) = \{q : q \ll p\}$. We say a space-time (M, g_{ab}) satisfies *past distinguishability* if there do not exist distinct points $p, q \in M$ such that $I^-(p) = I^-(q)$. We say a set $S \subset M$ is *achronal* if there do not exist $p, q \in S$ such that $p \in I^-(q)$.

An *extension* of a curve $\gamma : I \to M$ is a curve $\gamma' : I' \to M$ such that I is a proper subset of I' and $\gamma(s) = \gamma'(s)$ for all $s \in I$. A curve is *maximal* if it has no extension. A curve $\gamma : I \to M$ in a space-time (M, g_{ab}) is a *geodesic* if $\xi^a \nabla_a \xi^b = \mathbf{0}$, where ξ^a is the tangent vector and ∇_a is the unique derivative operator compatible with g_{ab}. A point $p \in M$ is a *future endpoint* of a future-directed causal curve $\gamma : I \to M$ if, for every neighborhood O of p, there exists a point $t_0 \in I$ such that $\gamma(t) \in O$ for all $t > t_0$. A *past endpoint* is defined similarly. A causal curve is *inextendible* if it has no future or past endpoint.

A causal geodesic $\gamma : I \to M$ in a space-time (M, g) is *past incomplete* if it is maximal and there is an $r \in \mathbb{R}$ such that $r < s$ for all $s \in I$.

For any set $S \subseteq M$, we define the *past domain of dependence of S*, written $D^{-}(S)$, to be the set of points $p \in M$ such that every causal curve with past endpoint p and no future endpoint intersects S. The *future domain of dependence of S*, written $D^{+}(S)$, is defined analogously. The entire *domain of dependence of S*, written $D(S)$, is just the set $D^{-}(S) \cup D^{+}(S)$. The *edge* of an achronal set $S \subset M$ is the collection of points $p \in S$ such that every open neighborhood O of p contains a point $q \in I^{+}(p)$, a point $r \in I^{-}(p)$, and a time-like curve from r to q that does not intersect S. A set $S \subset M$ is a *slice* if it is closed, achronal, and without edge. A space-time (M, g_{ab}) that contains a slice S such that $D(S) = M$ is said to be *globally hyperbolic*.

Given a space-time (M, g_{ab}), let T_{ab} be defined by $(1/8\pi)(R_{ab} - \frac{1}{2}Rg_{ab})$, where R_{ab} is the Ricci tensor and R the scalar curvature associated with g_{ab}. We say that (M, g_{ab}) satisfies the *weak energy condition* if, for each time-like vector ξ^{a}, we have $T_{ab}\xi^{a}\xi^{b} \geq 0$. We say that (M, g_{ab}) is a *vacuum solution* if $T_{ab} = \mathbf{0}$.

Let S be a set. A relation $\leq$ on S is a *partial order* if, for all $a, b, c \in S$, (i) $a \leq a$; (ii) if $a \leq b$ and $b \leq c$, then $a \leq c$; and (iii) if $a \leq b$ and $b \leq a$, then $a = b$. If $\leq$ is a partial ordering on a set S, we say a subset $T \subseteq S$ is *totally ordered* if, for all $a, b \in T$, either $a \leq b$ or $b \leq a$. Let $\leq$ be a partial ordering on S, and let $T \subseteq S$ be totally ordered. An *upper bound* for T is an element $u \in S$ such that for all $a \in T$, $a \leq u$. A *maximal element* of S is an element $m \in S$ such that for all $c \in S$, if $m \leq c$, then $c = m$. Zorn's lemma (which is equivalent to the axiom of choice) is as follows: Let $\leq$ be a partial order on S. If each totally ordered subset $T \subseteq S$ has an upper bound, there is a maximal element of S.

3. Definition.

Recall the standard definition of space-time inextendibility.

> **Definition.** A space-time (M, g_{ab}) is *inextendible* if there does not exist a space-time (M', g'_{ab}) such that there is a proper isometric embedding $\varphi : M \to M'$.

The definition requires that an inextendible space-time be "as large as it can be" in the sense that one compares it to a background class of all "possible" space-times. Standardly, one takes this class to be the set of all (smooth, Hausdorff) Lorentzian manifolds as defined in the previous section. But what should this class be? The answer is unclear.

Consider Misner space-time (Hawking and Ellis 1973). Let Misner* be the globally hyperbolic "bottom half" of Misner space-time. By the standard definition of inextendibility, Misner* is extendible and cannot be extended and remain globally hyperbolic (see below). But suppose that a ver-

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sion of the cosmic censorship conjecture is correct and all physically reasonable space-times are globally hyperbolic (Penrose 1979). Then should Misner* not be considered "as large as it can be"? It follows that whether Misner* space-time should count as inextendible depends crucially on the outcome of this version of the cosmic censorship conjecture—a conjecture that is far from settled (Earman 1995; Penrose 1999) and perhaps may never be settled (Manchak 2011).

Because of examples like these, one is tempted to revise the definition of inextendibility.[2] But Geroch (1970, 278) has argued that a revision is less urgent if one can show, for a variety of physically reasonable properties $\mathcal{P}$, that the following statement is true.

(*) Every $\mathcal{P}$-inextendible $\mathcal{P}$-space-time is inextendible.

The significance of (*) is this: if a property $\mathcal{P}$ satisfies (*), then any $\mathcal{P}$-space-time is inextendible if and only if it is $\mathcal{P}$-inextendible. Effectively, it makes no difference in such cases whether one defines inextendibility relative to the standard class of all space-times or a revised class of all $\mathcal{P}$-space-times. Accordingly, one would like to investigate (*) with respect to a variety of properties $\mathcal{P}$. Already from the Misner* example above, we have the following proposition (a proof is provided in the appendix).

Proposition 1. If $\mathcal{P}$ is global hyperbolicity, (*) is false.

Are there physically reasonable properties $\mathcal{P}$ that render (*) true? Geroch (1970, 289) has suggested a number of good candidates, including being a vacuum solution, satisfying chronology, and satisfying an energy condition. The first two cases are still open. Here, we settle the case in which $\mathcal{P}$ is the weak energy condition (a proof is provided in the appendix).

Proposition 2. If $\mathcal{P}$ is the weak energy condition, (*) is false.

We see that the prospect of avoiding the need to revise to the definition of inextendibility does not look good. In the meantime, we may conclude that it is not yet clear that the standard definition captures the intuitive idea that an inextendible space-time is "as large as it can be."

4. Metaphysics. A number of experts in general relativity (Penrose 1969; Geroch 1970; Clarke 1976) seem to be committed to the following state-

2. See Manchak (2016a) for an extended discussion.

ment. “Metaphysical considerations suggest that to be a serious candidate for describing actuality, a spacetime should be [inextendible]. For example, for the Creative Force to actualize a proper subpart of a larger spacetime would seem to be a violation of Leibniz’s principles of sufficient reason and plenitude. If one adopts the image of spacetime as being generated or built up as time passes then the dynamical version of the principle of sufficient reason would ask why the Creative Force would stop building if it is possible to continue” (Earman 1995, 32).

These metaphysical views are underpinned by an important result due to Geroch (1970).

Proposition 3. Every extendible space-time has an inextendible extension.

The result (which makes use of Zorn’s lemma) seems to show that the Creative Force can always build space-time until it is no longer possible to build. But of course, this interpretation presupposes that we have been working with the proper definition of inextendibility. And as we have noted, it is not yet clear that we are. Accordingly, one would like to know, for a variety of physically reasonable properties $\mathcal{P}$, whether the following version of the Geroch (1970) result is true.

(**) Every $\mathcal{P}$-extendible $\mathcal{P}$-space-time has a $\mathcal{P}$-inextendible $\mathcal{P}$-extension.

With a bit of work (and Zorn’s lemma), one can show the following proposition (a proof is provided in the appendix).

Proposition 4. If $\mathcal{P}$ is chronology, (**) is true.

We see that if we revise the definition of inextendibility to be relative to the class of all chronological space-times (rather than the standard class of all space-times), we have an analogue of the Geroch (1970) result. This is certainly good news for those committed to the metaphysical views expressed above. But are there physically reasonable properties $\mathcal{P}$ that render (**) false? There are.

Of course, space-time properties may be considered physically reasonable in various senses. Let us conservatively restrict attention to a property usually taken to be satisfied by models of our own universe: the property of having every inextendible time-like geodesic be past incomplete. Let us call this the *big bang property*, given that it is satisfied by all of the standard “big bang” cosmological models. We are now in a position to state the following proposition (Manchak 2016b).

Proposition 5. If $\mathcal{P}$ is the big bang property, (**) is false.

We see that if we revise the definition of inextendibility to be relative to the class of all space-times with the big bang property (rather than the standard class of all space-times), we do not have an analogue of the Geroch (1970) result. It is not yet clear that the Creative Force always has the option of building space-time to be "as large as it can be."

5. Epistemology. What observational evidence is there (or could there be) in support of the position that space-time is "as large as it can be"? Following Malament (1977), let us say that a space-time (M, g_{ab}) is *observationally indistinguishable* from another space-time (M', g'_{ab}) if, for every point $p \in M$, there is a point $p' \in M'$ such that $I^-(p)$ and $I^-(p')$ are isometric. One can show the following proposition (Manchak 2011).

Proposition 6. Every chronological space-time is observationally indistinguishable from some other (nonisometric) space-time that is extendible.

Under the standard definition of inextendibility, it seems that any observer in a chronological space-time is not in a position to know that her space-time is "as large as it can be." But this interpretation presupposes that we have been working with the proper definition of inextendibility. And as we have noted, it is not yet clear that we are. Accordingly, one would like to know, for a variety of physically reasonable properties $\mathcal{P}$, whether the following version of the Manchak (2011) result is true.

(***) Every chronological $\mathcal{P}$-space-time is observationally indistinguishable from some other (nonisometric) $\mathcal{P}$-space-time that is $\mathcal{P}$-extendible.

It turns out that a large class of physically reasonable properties render (***) true. Following Manchak (2011), let us say that a property $\mathcal{P}$ on a space-time is *local* if, given any two locally isometric space-times (M, g_{ab}) and (M', g'_{ab}), (M, g_{ab}) has $\mathcal{P}$ if and only if (M', g'_{ab}) has $\mathcal{P}$. Local properties include being a vacuum solution and satisfying the weak energy condition. We are now in a position to state the following proposition (a proof is provided in the appendix).

Proposition 7. If $\mathcal{P}$ is a local property, (***) is true.

We see that, whenever $\mathcal{P}$ is a local property, if we revise the definition of inextendibility to be relative to the class of all $\mathcal{P}$-space-times (rather than the standard class of all space-times), we have an analogue of the Manchak

(2011) result. It is not yet clear that we can ever have observational evidence that space-time is "as large as it can be."

6. Conclusion. We have registered some skepticism with respect to the position that space-time must be inextendible—that it must be "as large as it can be" in some sense. We have done this in a variety of ways. First we have shown that it is not yet clear that the standard definition of inextendibility captures the intuitive idea that an inextendible space-time is "as large as it can be." Second, we have shown, by exploring some plausible revisions to the definition of inextendibility, that it is not yet clear that a space-time can always be extended to be "as large as it can be." Finally we have shown, by exploring a class of plausible revisions to the definition of inextendibility, that it is not yet clear that we can ever know that space-time is "as large as it can be."

Appendix

Proposition 1. If $\mathcal{P}$ is global hyperbolicity, (*) is false.

Proof. Let (N, g_{ab}) be Misner space-time. So, $N = \mathbb{R} \times \mathbb{S}$ and $g_{ab} = 2\,\nabla_{(a}\, t\, \nabla_{b)}\, \varphi - t\, \nabla_a\, \varphi\, \nabla_b\, \varphi$, where the points (t, φ) are identified with the points $(t, \varphi + 2\pi n)$ for all integers n. Now, let $M = \{(t, \varphi) \in N\colon t < 0\}$ and consider the space-time (M, g_{ab}). Clearly, it is extendible. It is also globally hyperbolic since the slice $S = \{(t, \varphi) \in M\colon t = -1\}$ is such that $D(S) = M$. We need only show that any extension to (M, g_{ab}) fails to be globally hyperbolic.

Let (M', g'_{ab}) be any extension of (M, g_{ab}), and let p be a point in $\partial M \cap M'$. In any neighborhood of p, there will be a point $q \in \partial M \cap M'$ such that $q \neq p$. One can verify that $I^-(p) = M = I^-(q)$. Thus, (M', g'_{ab}) is not past distinguishing and therefore not globally hyperbolic (Hawking and Ellis 1973). QED

Proposition 2. If $\mathcal{P}$ is the weak energy condition, (*) is false.

Proof. Consider Minkowski space-time $(\mathbb{R}^4, \eta_{ab})$ in standard (t, x, y, z) coordinates, where $\eta_{ab} = -\,\nabla_a\, t\, \nabla_b\, t + \nabla_a x\, \nabla_b\, x + \nabla_a y\, \nabla_b\, y + \nabla_a z\, \nabla_b\, z$. Let $\Omega : \mathbb{R}^4 \to \mathbb{R}$ be the function defined by $\Omega(t, x, y, z) = \exp(t^3)$. Consider the space-time $(\mathbb{R}^4, g_{ab})$, where $g_{ab} = \Omega^2 \eta_{ab}$. Associated with g_{ab} we have (Wald 1984, 446)

$$R_{ab} = -2\,\nabla_a\,\nabla_b t^3 - \eta_{ab}\eta^{cd}\,\nabla_c\,\nabla_d t^3 + 2(\nabla_a t^3)(\nabla_b t^3)$$
$$-2\eta_{ab}\eta^{cd}(\nabla_c t^3)(\nabla_d t^3)$$
$$R = \frac{1}{\Omega^2}\left[-6\eta^{ab}\,\nabla_a\,\nabla_b t^3 - 6\eta^{ab}(\nabla_a t^3)(\nabla_b t^3)\right].$$

We note that $\nabla_a t^3 = 3t^2\,\nabla_a t$ and $\nabla_a\,\nabla_b t^3 = 6t(\nabla_a t)(\nabla_b t)$. Of course, $\eta^{ab}(\nabla_a t)(\nabla_b t) = -1$. Simplifying, we have

$$R_{ab} = (18t^4 - 12t)(\nabla_a t)(\nabla_b t) + (18t^4 + 6t)\eta_{ab}$$
$$R = \frac{1}{\Omega^2}[36t + 54t^4].$$

Einstein's equation $R_{ab} - (1/2)Rg_{ab} = 8\pi T_{ab}$ requires that

$$T_{ab} = \frac{1}{8\pi}[(18t^4 - 12t)(\nabla_a t)(\nabla_b t) - (9t^4 + 12t)\eta_{ab}].$$

In $(\mathbb{R}^4, g_{ab})$, consider any time-like vector $\xi^a = k_0(\partial/\partial t)^a + k_1(\partial/\partial x)^a + k_2(\partial/\partial y)^a + k_3(\partial/\partial z)^a$, where $k_0, k_1, k_2, k_3 \in \mathbb{R}$ and $k_0^2 > k_1^2 + k_2^2 + k_3^2$. We have

$$T_{ab}\xi^a\xi^b = \frac{1}{8\pi}[(18t^4 - 12t)k_0^2 + (9t^4 + 12t)(k_0^2 - k_1^2 - k_2^2 - k_3^2)]$$
$$= \frac{1}{8\pi}[t^4(27k_0^2 - 9(k_1^2 + k_2^2 + k_3^2)) - 12t(k_1^2 + k_2^2 + k_3^2)].$$

Because $k_0^2 > k_1^2 + k_2^2 + k_3^2$, we know $t^4(27k_0^2 - 9(k_1^2 + k_2^2 + k_3^2)) \geq 0$. And for $t \leq 0$, we know $-12t(k_1^2 + k_2^2 + k_3^2) \geq 0$. It follows that $T_{ab}\xi^a\xi^b \geq 0$ for $t \leq 0$.

Let $M = \{(t, x, y, z) \in \mathbb{R}^4 : t < 0\}$. We have shown that the space-time $(M, g_{ab|M})$ is such that it satisfies the weak energy condition and is extendible. It remains for us to show that any extension to $(M, g_{ab|M})$ fails to satisfy the weak energy condition.

Let (M', g'_{ab}) be any extension to $(M, g_{ab|M})$. Let p be a point in $\partial M \cap M'$. Let (O, φ) be a chart with $p \in O$ such that we can extend the coordinates (t, x, y, z) on M to $M \cup O \subset M'$. So, for some $p_1, p_2, p_3 \in \mathbb{R}$ we have $p = (0, p_1, p_2, p_3)$. Find some $\delta > 0$ such that $(\delta, p_1, p_2, p_3) \in O$. For $t \in (-\delta, \delta)$, let $p(t) = (t, p_1, p_2, p_3) \in M'$.

Consider the smooth function $f : (-\delta, \delta) \to \mathbb{R}$ given by $f(t) = g'_{ab}\zeta^a\zeta^b_{|p(t)}$, where $\zeta^a = \sqrt{2}(\partial/\partial t)^a + (\partial/\partial x)^a$. Of course, for all $t < 0$, we have

$$g'_{ab}\zeta^a\zeta^b = g_{ab}\zeta^a\zeta^b = \Omega^2\eta_{ab}\zeta^a\zeta^b = -\Omega^2 = -\exp(2t^3).$$

Smoothness requires that $f(0) = -1$. This allows us to find an $\varepsilon \in (0, \delta)$ such that $f(t) < 0$ for all $t \in (-\varepsilon, \varepsilon)$. So ζ^a is time-like at $p(t)$ for all $t \in (-\varepsilon, \varepsilon)$.

Consider the smooth function $g: (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by $g(t) = T'_{ab}\zeta^a\zeta^b_{|p(t)}$, where T'_{ab} is defined on M' in the natural way (using the metric g'_{ab} and Einstein's equation). Of course, for all $t < 0$, we have

$$T'_{ab} = T_{ab} = \frac{1}{8\pi}[(18t^4 - 12t)(\nabla_a t)(\nabla_b t) - (9t^4 + 12t)\eta_{ab}].$$

Because $\nabla_a t\, \nabla_b t \zeta^a \zeta^b = 2$ and $\eta_{ab}\zeta^a\zeta^b = -1$, we have for all $t < 0$

$$T'_{ab}\zeta^a\zeta^b = T_{ab}\zeta^a\zeta^b = \frac{1}{8\pi}[(36t^4 - 24t) + (9t^4 + 12t)] = \frac{1}{8\pi}[45t^4 - 12t].$$

Smoothness requires that $g(0) = 0$ and $(d/dt)g(0) = -(3/2\pi)$. This allows us to find a $\gamma \in (0, \varepsilon)$ such that $g(t) < 0$ for $t \in (0, \gamma)$. Thus, the weak energy condition is violated at $p(t)$ for all $t \in (0, \gamma)$. QED

Definition. Let $\mathfrak{F}$ denote the set of framed space-times. Let $\leq$ denote the relation on $\mathfrak{F}$ such that $(M, g_{ab}, F) \leq (M', g'_{ab}, F')$ if and only if (M', g'_{ab}, F') is a framed extension of (M, g_{ab}, F).

Lemma 1. The relation $\leq$ is a partial ordering on $\mathfrak{F}$.

Proof. See Geroch (1969, 188–89). QED

Lemma 2. Let $\mathfrak{C}$ denote the set of framed space-times that satisfy chronology. $\mathfrak{C}$ is partially ordered by $\leq$. Every subset $\mathfrak{T} \subset \mathfrak{C}$ that is totally ordered by $\leq$ has an upper bound in $\mathfrak{C}$.

Proof. Since $\mathfrak{C} \subset \mathfrak{F}$, it follows from lemma 1 that $\mathfrak{C}$ is partially ordered by $\leq$. Let $\mathfrak{T} = \{(M_i, g_i, F_i)\}$ be a subset of $\mathfrak{C}$ that is totally ordered by $\leq$. Following Hawking and Ellis (1973, 249), let M be the union of all the M_i, where, for $(M_i, g_i, F_i) \leq (M_j, g_j, F_j)$, each $p_i \in M_i$ is identified with $\varphi_{ij}(p_i)$, where $\varphi_{ij}: M_i \to M_j$ is the unique isometric embedding that takes F_i into F_j. The manifold M will have an induced metric g equal to $\varphi_{i*}g_i$ on each $\varphi_i[M_i]$, where $\varphi_i: M_i \to M$ is the natural isometric embedding. Finally, take F to be the result of carrying along a chosen F_i using $\varphi_i: M_i \to M$. Consider the framed space-time (M, g, F). We claim it is an upper bound for $\mathfrak{T}$. Clearly, for all i, we have $(M_i, g_i, F_i) \leq (M, g, F)$. We need only show that $(M, g, F) \in \mathfrak{C}$.

Suppose $(M, g, F) \notin \mathfrak{C}$, and let $\gamma \subset M$ be (the image of) a closed time-like curve. As a topological space (with induced topology from M), γ is compact. For all i, let γ_i be $\gamma \cap M_i$. So, $A = \{\gamma_i\}$ is an open cover of γ.

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By compactness, there must be a finite subset $A' \subset A$ that is also a cover of γ. One can use the relation $\leq$ on $\mathfrak{T}$ to order the finite number of elements in A' into a nested sequence of subsets $\gamma_j \subseteq \ldots \subseteq \gamma_k$. It follows that $\gamma_k = \gamma$. So, $(M_k, g_k, F_k) \notin \mathfrak{C}$: a contradiction. QED

Proposition 4. If $\mathcal{P}$ is chronology, (**) is true.

Proof. Let $\mathcal{P}$ be chronology, and let (M, g_{ab}) be a $\mathcal{P}$-space-time that is $\mathcal{P}$-extendible. Let F be an orthonormal n-ad at some point $p \in M$. So, $(M, g_{ab}, F) \in \mathfrak{C}$, where $\mathfrak{C}$ is the set of framed space-times that satisfy chronology. By lemma 2 and Zorn's lemma, there is a maximal element $(M', g'_{ab}, F') \in \mathfrak{C}$ such that $(M, g_{ab}, F) \leq (M', g'_{ab}, F')$. It follows that (M', g'_{ab}) is a $\mathcal{P}$-inextendible $\mathcal{P}$-extension of (M, g_{ab}). QED

Proposition 7. If $\mathcal{P}$ is a local property, (***) is true.

Proof. Let $\mathcal{P}$ be any local property, and let (M, g_{ab}) be any chronological $\mathcal{P}$-space-time. Now construct (M', g'_{ab}) according to the method outlined in Manchak (2009). Note that (M', g'_{ab}) is a $\mathcal{P}$-space-time by construction. Next, remove any point in the $M(1, b)$ portion of the manifold M', and call the resulting space-time (M'', g''_{ab}). One can verify that (i) (M, g_{ab}) is observationally indistinguishable from (M'', g''_{ab}), (ii) (M'', g''_{ab}) is a $\mathcal{P}$-space-time, and (iii) (M, g_{ab}) is not isometric to (M'', g''_{ab}). Since (M', g'_{ab}) is a $\mathcal{P}$-extension to (M'', g''_{ab}), the latter is $\mathcal{P}$-extendible. QED

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