Probability in the Engineering and Informational Sciences, **32**, 2018, 567–579.
doi:10.1017/S0269964817000389

PRESERVATION OF LOG-CONCAVITY UNDER CONVOLUTION

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Log-concave random variables and their various properties play an increasingly important role in probability, statistics, and other fields. For a distribution F, denote by $\mathcal{D}_F$ the set of distributions G such that the convolution of F and G has a log-concave probability mass function or probability density function. In this paper, we investigate sufficient and necessary conditions under which $\mathcal{D}_F \subseteq \mathcal{D}_G$, where F and G belong to a parametric family of distributions. Both discrete and continuous settings are considered.

Keywords: exponential distribution, geometric distribution, negatively binomial distribution, normal distribution, poisson distribution

Mathematics Subject Classifications (2000): 60E05

1. INTRODUCTION

A sequence $\{h(n),\ n \in \mathbb{N}\}$ is said to be log-concave (LC), if $h(n) \geq 0$ for $n \in \mathbb{N} \equiv \{0, 1, \ldots\}$, and

$$h^2(n) \geq h(n+1)h(n-1), \quad \forall\ n \in \mathbb{N}_+ \equiv \{1, 2, \ldots\}.$$

A LC sequence $\{h(n)\}$ does not have internal zeros, that is, there does not exist $i < j < k$ such that $h(i)h(k) \neq 0$ and $h(j) = 0$. A random variable X taking values in $\mathbb{N}$ is said to be LC if its probability mass function (pmf), denoted by $\{f(n), n \in \mathbb{N}\}$, is LC.

A random variable X taking value at $\mathbb{R}$ is said to be LC if its probability density function (pdf), denoted by $f(x)$, $x \in \mathbb{R} \equiv (-\infty, \infty)$, is LC, that is,

$$f^2(x) \geq f(x+\delta)f(x-\delta), \quad \forall\ x \in \mathbb{R},\ \delta \in \mathbb{R}_+ \equiv (0, \infty).$$

A LC density $f(x)$ does not have internal zeros.

Log-concavity distributions and their appealing properties have wide applications in probability, statistics, combinatorics, econometrics, reliability, optimization, information, and other fields of applied probability. Saumard and Wellner [14] is a comprehensive review of log-concavity in the statistics literature, which also includes some connections between log-concavity and other areas of mathematics and statistics (including concentration of measure, MCMC algorithms, log-Sobolev inequalities, and machine learning). An [1] and

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Bagnoli and Bergstrom [2] are two other reviews of log-concavity in econometrics. For more on log-concavity, we refer the reader to Efron [5], Finner and Roters [7,8], Liggett [13], Sengupta and Nanda [15], Wang and Yeh [16], Kahn and Neiman [12], Yu [17], Bobkov and Madiman [3], Fradelizi, Madiman, and Wang [9], and references therein.

It is well known that the convolution of two independent random variables with LC pmfs or pdfs is still LC. To study preservation of log-concavity under convolution, for each fixed distribution F, define a class of distribution functions, denoted by $\mathcal{D}_F$, such that their convolutions with F have LC pmfs or pdfs. Specifically, $\mathcal{D}_F$ is given by

$$\mathcal{D}_F = \{G :\ Y + X \text{ is LC},\ Y \perp X,\ X \sim F,\ Y \sim G\}, \tag{1.1}$$

where "$Y \perp X$" represents that Y is independent of X, and "$X \sim F$" means that F is the distribution function of X. Throughout, in the discrete case, X and Y in (1.1) are assumed to take values in $\mathbb{N}$; while in the continuous case, X and Y are assumed to take values in $\mathbb{R}$. We call $\mathcal{D}_F$ the attraction domain of log-concavity of distribution F.

Intuitively, for two distribution functions F and G, if $\mathcal{D}_F \subset \mathcal{D}_G$, then G is more LC than F in some sense. The purpose of this paper is to investigate sufficient and necessary conditions under which $\mathcal{D}_F \subseteq \mathcal{D}_G$, where F and G belong to a parametric family of distributions. Johnson and Goldschmidt [11] considered the family of geometric distributions; see (2.1). The families considered in this paper are negatively binomial, Poisson, Bernoulli, discrete uniform, exponential, and normal distributions. The results for discrete and continuous distributions are given in Sections 2 and 3, respectively. Such a study will provide us a new sight on the properties of distribution functions.

2. DISCRETE DISTRIBUTIONS

2.1. Negatively Binomial Distribution

Let Geo(p) denote the distribution of a Geometric random variable X with parameter $p \in [0, 1]$, that is, $\mathbb{P}(X = k) = (1 - p)^k p$ for $k \in \mathbb{N}$. Johnson and Goldschmidt [11] showed that

$$\mathcal{D}_{\mathrm{Geo}(p_1)} \subseteq \mathcal{D}_{\mathrm{Geo}(p_2)} \Longleftrightarrow p_1 \le p_2. \tag{2.1}$$

From this equivalent characterization, we can give a similar characterization for the negatively binominal (NB) distribution. To state the result, write NB(r, p) for the NB distribution with pmf

$$\mathbb{P}(X = k) = \binom{k + r - 1}{k} p^r (1 - p)^k, \quad \forall\, k \in \mathbb{N},$$

where $p \in (0, 1)$ and $r \in \mathbb{R}_+$.

Proposition 2.1: *For each $r \in \mathbb{N}_+$, we have*

$$\mathcal{D}_{\mathrm{NB}(r,p_1)} \subseteq \mathcal{D}_{\mathrm{NB}(r,p_2)} \Longleftrightarrow p_1 \le p_2. \tag{2.2}$$

Proof: We show the sufficiency "$\Leftarrow$" by induction. Suppose that $p_1 \le p_2$. (2.2) reduces to (2.1) when $r = 1$. Assume that (2.2) holds for $r = k$. We aim to show that (2.2) holds for

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$r = k+1$. For simplicity, we spoil the notation $X \perp Y \perp Z$, which represents that the three random variables X, Y, and Z are jointly independent. Note that

$$\begin{aligned}
\mathcal{D}_{\mathrm{NB}(k+1,p_1)} &= \{G : Y + X \text{ is LC},\ Y \perp X,\ X \sim \mathrm{NB}(k+1,p_1),\ Y \sim G\} \\
&= \{G : Y + X_1 + X_2 \text{ is LC},\ Y \perp X_1 \perp X_2,\ X_1 \sim \mathrm{NB}(k,p_1), \\
&\quad X_2 \sim \mathrm{Geo}(p_1),\ Y \sim G\} \\
&= \{G : Y + X_1 \in \mathcal{D}_{\mathrm{Geo}(p_1)},\ Y \perp X_1,\ X_1 \sim \mathrm{NB}(k,p_1),\ Y \sim G\} \\
&\subseteq \{G : Y + X_1 \in \mathcal{D}_{\mathrm{Geo}(p_2)},\ Y \perp X_1,\ X_1 \sim \mathrm{NB}(k,p_1)),\ Y \sim G\} \\
&= \{G : Y + X_1 + X_2 \text{ is LC},\ Y \perp X_1 \perp X_2,\ X_1 \sim \mathrm{NB}(k,p_1), \\
&\quad X_2 \sim \mathrm{Geo}(p_2),\ Y \sim G\} \\
&= \{G : Y + X_2 \in \mathcal{D}_{\mathrm{NB}(k,p_1)},\ Y \perp X_2,\ X_2 \sim \mathrm{Geo}(p_2),\ Y \sim G\} \\
&\subseteq \{G : Y + X_2 \in \mathcal{D}_{\mathrm{NB}(k,p_2)},\ Y \perp X_2,\ X_2 \sim \mathrm{Geo}(p_2),\ Y \sim G\} \\
&= \mathcal{D}_{\mathrm{NB}(k+1,p_2)},
\end{aligned}$$

where the second equality is due to that $X_1 + X_2 \sim \mathrm{NB}(k+1,p)$ for $X_1 \sim \mathrm{NB}(k,p)$, $X_2 \sim \mathrm{Geo}(p)$ such that $X_1 \perp X_2$, the first inclusion follows from that $\mathcal{D}_{\mathrm{Geo}(p_1)} \subseteq \mathcal{D}_{\mathrm{Geo}(p_2)}$ by (2.1), and the second inclusion follows from that $\mathcal{D}_{\mathrm{NB}(k,p_1)} \subseteq \mathcal{D}_{\mathrm{NB}(k,p_2)}$ by the induction assumption.

To show the necessity "$\Rightarrow$", assume that $p_1 > p_2$. Then it follows from the sufficiency that $\mathcal{D}_{\mathrm{NB}(r,p_2)} \subsetneq \mathcal{D}_{\mathrm{NB}(r,p_1)}$ (we can find a distribution in $\mathcal{D}_{\mathrm{NB}(r,p_1)} \setminus \mathcal{D}_{\mathrm{NB}(r,p_2)}$; see Remark 2.2), yielding a contradiction. Hence, $p_1 \le p_2$. This completes the proof of the proposition. ■

Remark 2.2: Let Y_η be a random variable with pmf

$$\mathbb{P}(Y_\eta = 0) = \mathbb{P}(Y_\eta = 1) = \frac{1}{\eta+3}, \quad \mathbb{P}(Y_\eta = 2) = \frac{\eta+1}{\eta+3}. \tag{2.3}$$

Here $\eta > 0$ and Y_η is not LC. Then

$$Y_\eta \in \mathcal{D}_{\mathrm{NB}(r,p)} \Longleftrightarrow p[(r-1)p - 2r] \ge \frac{2\eta}{r} - r - 1. \tag{2.4}$$

To prove the necessity of (2.4), let $X \sim \mathrm{NB}(r,p)$ and $X \perp Y_\eta$. Define $q_i = \mathbb{P}(X + Y_\eta = i)$ for $i \in \mathbb{N}$. It is easy to see that

$$q_0 = \frac{p^r}{\eta+3}, \quad q_1 = p^r \cdot \frac{r(1-p)+1}{\eta+3}, \quad q_2 = p^r \cdot \frac{r(r+1)(1-p)^2/2 + r(1-p) + \eta + 1}{\eta+3}.$$

Since $X + Y_\eta$ is LC, we have $q_1^2 \ge q_0 q_2$, which is equivalent to the right-hand side of (2.4).

To prove the sufficiency of (2.4), it requires to prove that $q_j^2 \ge q_{j-1}q_{j+1}$ for $j \ge 2$. Note that $q_j = (p_j + p_{j-1} + (\eta+1)p_{j-2})/(\eta+3)$. Then, for $j \ge 2$ and $r > 1$,

$$\begin{aligned}
&\frac{1}{r-1}(\eta+3)^2(j+r-2)^2(j+r-3)(j+1)j^2(1-p)^2 \cdot (q_j^2 - q_{j-1}q_{j+1}) \\
&\quad = (j+r-1)(j+r-2)^2(j+r-3)(1-p)^4 + 2j(j+r-1)(j+r-2)(j+r-3)(1-p)^3 \\
&\qquad + j(j+r-2)[(2\eta+3)(j+r)^2 - (2\eta+3)(r+2)(j+r) + 6\eta + 3r + 3](1-p)^2 \\
&\qquad + 2j(\eta+1)(j+r-2)(j^2-1)(1-p) + (\eta+1)^2 j^2 (j^2-1) > 0.
\end{aligned}$$

Also, for $j \ge 2$ and $r = 1$, $q_j^2 - q_{j-1}q_{j+1} = 0$. Therefore, $q_j^2 \ge q_{j-1}q_{j+1}$ for $j \ge 2$.

Now, choose $r = 1$. Then

$$Y_\eta \in \mathcal{D}_{\mathrm{Geo}(p)} \iff \eta \le 1 - p. \tag{2.5}$$

For any $\mathbb{N}$-valued random variable Z, denote its pmf by $p_Z(i)$. Lemma 4.3(1) in Johnson and Goldschmidt [11] states that $Z \in \mathcal{D}_{\mathrm{Geo}(p)}$ if $p_Z(i+1)/p_Z(i) \le p$ for all $i \ge 1$. However, the condition that $p_Z(i+1)/p_Z(i) \le p$ is not necessary, which can be seen from (2.5) with $\eta = 1 - p$ since $p_{Y_\eta}(2)/p_{Y_\eta}(1) = \eta + 1 > p$.

Remark 2.3: Relation (2.2) does not hold when r is not an integer, as shown by the following counterexample. Let $X_1 \sim \mathrm{NB}(1/2, 1/3)$, $X_2 \sim \mathrm{NB}(1/2, 2/3)$ and $Y \sim \mathrm{NB}(1/2, 1/3)$ such that $X_1 \perp X_2 \perp Y$. Since $X_1 + Y \sim \mathrm{NB}(1, 1/3) = \mathrm{Geo}(1/3)$, it follows that $Y \in \mathcal{D}_{\mathrm{NB}(1/2,1/3)}$. However, $Y \notin \mathcal{D}_{\mathrm{NB}(1/2,2/3)}$. To see it, denote $q_n = \mathbb{P}(X_2 + Y = n)$. It can be checked that

$$q_0 = \frac{\sqrt{2}}{3}, \quad q_1 = \frac{\sqrt{2}}{3} \cdot \frac{1}{2}, \quad q_1 = \frac{\sqrt{2}}{3} \cdot \frac{19}{72},$$

and that $q_1^2 < q_0 q_2$. This means that q_n is not LC on $\mathbb{N}$. Therefore, $\mathcal{D}_{\mathrm{NB}(1/2,1/3)} \not\subseteq \mathcal{D}_{\mathrm{NB}(1/2,2/3)}$. Also, noting that $X_2 \notin \mathcal{D}_{\mathrm{NB}(1/2,1/3)}$, we have $\mathcal{D}_{\mathrm{NB}(1/2,2/3)} \not\subseteq \mathcal{D}_{\mathrm{NB}(1/2,1/3)}$.

Remark 2.4: In fact, for $r_1, r_2 \in \mathbb{N}_+$, if $r_1 \le r_2$ and $p_1 \le p_2$, we have that $\mathcal{D}_{\mathrm{NB}(r_1,p_1)} \subseteq \mathcal{D}_{\mathrm{NB}(r_2,p_2)}$. To see it, it suffices to note that $\mathcal{D}_{\mathrm{NB}(r_1,p)} \subseteq \mathcal{D}_{\mathrm{NB}(r_2,p)}$ for $r_1 < r_2$ since $\mathrm{NB}(r_2 - r_1, p)$ is LC and $X + Y \sim \mathrm{NB}(r_2, p)$ for any $X \sim \mathrm{NB}(r_1, p)$ and $Y \sim \mathrm{NB}(r_2 - r_1, p)$ such that $X \perp Y$.

2.2. Poisson Distribution

A random variable X is said to have a Poisson distribution with parameter $\lambda > 0$, denoted by $X \sim \mathrm{Poi}(\lambda)$, if $\mathbb{P}(X = k) = e^{-\lambda}\lambda^k/k!$ for $k \in \mathbb{N}$. It is well known that the Poisson distribution can be viewed as the limit of some sequence of NB distributions. Specifically, we have that the pmf of $\mathrm{NB}(r, r/(r+\lambda))$ converges to that of $\mathrm{Poi}(\lambda)$ as $r \to \infty$. Hence, it is reasonable to conjecture that a parallel result of (2.2) holds for the Poisson distribution.

Proposition 2.5: *For $\lambda_1, \lambda_2 > 0$, we have*

$$\mathcal{D}_{\mathrm{Poi}(\lambda_1)} \subseteq \mathcal{D}_{\mathrm{Poi}(\lambda_2)} \iff \lambda_1 \le \lambda_2. \tag{2.6}$$

Proof: *Sufficiency*: For $\lambda_1 < \lambda_2$, let $G \in \mathcal{D}_{\mathrm{Poi}(\lambda_1)}$, that is, there exist $X_1 \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim G$ such that $Y \perp X_1$ and $Y + X_1$ is LC. Take $Z \sim \mathrm{Poi}(\lambda_2 - \lambda_1)$ such that Z is independent of (X_1, Y). Then $X_2 := X_1 + Z \sim \mathrm{Poi}(\lambda_2)$. Observing that Z is LC and that the convolution of two LC random variables is still LC, we have that $Y + X_2 = (Y + X_1) + Z$ is LC. This means $G \sim \mathcal{D}_{\mathrm{Poi}(\lambda_2)}$ and, thus, $\mathcal{D}_{\mathrm{Poi}(\lambda_1)} \subseteq \mathcal{D}_{\mathrm{Poi}(\lambda_2)}$.

Necessity: It suffices to prove that if $\lambda_1 > \lambda_2$ then $\mathcal{D}_{\mathrm{Poi}(\lambda_1)} \not\subseteq \mathcal{D}_{\mathrm{Poi}(\lambda_2)}$. Let $X_1 \sim \mathrm{Poi}(\lambda_1)$ and $X_2 \sim \mathrm{Poi}(\lambda_2)$, and assume that $X_1 \perp Y_\eta$ and $X_2 \perp Y_\eta$, where Y_η is a random variable with pmf given be (2.3). Here η is to be determined later. Define $q_i = \mathbb{P}(X_1 + Y_\eta = i)$ for

$i \in \mathbb{N}$. Then

$$q_i = \frac{1}{\eta+3}[p_i + p_{i-1} + (\eta+1)p_{i-2}], \quad i \in \mathbb{N},$$

where p_i is the pmf of X_1. Observe that $p_{i-1} = \frac{i}{\lambda_1}p_i$, $p_{i-2} = (i(i-1)/\lambda_1^2)p_i$, $p_{i-3} = (i(i-1)(i-2)/\lambda_1^2)p_i$ and $p_{i+1} = (\lambda_1/i+1)p_i$. Then, for $i \in \mathbb{N}$,

$$\frac{(\lambda_1+3)^2\lambda_1^4(i+1)}{p_i^2}\left(q_i^2 - q_{i-1}q_{i+1}\right) = \lambda_1^4 + 2i(i-1)\lambda_1^3 + i(i^3+2i^2+2i-5)\lambda_1^2 + 2i(i^2-1)(i+1) + i^2(i^2-1). \quad \mathbf{(2.7)}$$

Clearly, $q_i^2 - q_{i-1}q_{i+1} > 0$ for $i \in \mathbb{N}_+$. For $i = 1$, (2.7) reduces to

$$q_1^2 - q_0q_2 \overset{\text{sgn}}{=} \frac{1}{2}\lambda_1^2 + \lambda_1 - \eta,$$

where $\overset{\text{sgn}}{=}$ means equality in sign. Now, choose $\eta = \lambda_1^2/2 + \lambda_1$. Then $X_1 + Y_\eta$ is LC. However, $X_2 + Y_\eta$ is not LC. ■

Remark 2.6:

(i) Define the set of all LC random variables generated by convolution with Poi(λ)-distributed random variable as follows:

$$\mathcal{H}_{\text{Poi}(\lambda)} = \{X+Y : X \sim \text{Poi}(\lambda), Y \sim G, G \in \mathcal{D}_{\text{Poi}(\lambda)}, X \perp Y\}.$$

It is easy to verify that

$$\lambda_1 \le \lambda_2 \Longrightarrow \mathcal{H}_{\text{P}(\lambda_2)} \subseteq \mathcal{H}_{\text{P}(\lambda_1)}.$$

To see it, denote by $\mathcal{H}_{\text{LC}}$ the set of all $\mathbb{N}$-valued random variables with LC pmfs. Note that for $\lambda_1 < \lambda_2$,

$$\begin{aligned}
\mathcal{H}_{\text{Poi}(\lambda_2)} &= \{X+Y : X \sim \text{Poi}(\lambda_2), Y \sim G, G \in \mathcal{D}_{\text{Poi}(\lambda_2)}, X \perp Y\} \\
&= \{X+Y : X \sim \text{Poi}(\lambda_2), X \perp Y\} \cap \mathcal{H}_{\text{LC}} \\
&= \{X_1+X_2+Y : X_1 \sim \text{Poi}(\lambda_1), X_2 \sim \text{Poi}(\lambda_2-\lambda_1), X_1 \perp X_2 \perp Y\} \cap \mathcal{H}_{\text{LC}} \\
&\subseteq \{X_1+Z : X_1 \sim \text{Poi}(\lambda_1), X_1 \perp Z\} \cap \mathcal{H}_{\text{LC}} \\
&= \mathcal{H}_{\text{Poi}(\lambda_1)}.
\end{aligned}$$

This means that although the set of distributions whose convolutions with Poi(λ) are LC is non-decreasing in $\lambda \in \mathbb{R}_+$, the set of all LC distributions generated by convolution with Poi(λ) is non-increasing in $\lambda \in \mathbb{R}_+$.

(ii) If we define a partial order $\preceq_{\text{Poi}}$ on $\mathbb{R}_+$ by

$$\lambda_1 \preceq_{\text{Poi}} \lambda_2 \Longleftrightarrow \mathcal{D}_{\text{Poi}(\lambda_1)} \subseteq \mathcal{D}_{\text{Poi}(\lambda_2)}.$$

Then, by Proposition 2.5, we have that $\preceq_{\text{Poi}}$ is a total order on $\mathbb{R}_+$. Similar result holds for the negative binomial distribution.

2.3. Bernoulli Distribution

Let $\mathrm{Ber}(p)$ denote the distribution of a Bernoulli random variable I_p with $\mathbb{P}(I_p = 1) = p \in [0, 1]$. It is natural to consider the sufficient and necessary conditions under which $\mathcal{D}_{\mathrm{Ber}(p_1)} \subseteq \mathcal{D}_{\mathrm{Ber}(p_2)}$. However, we could not find out the general condition. In fact, we can show that $\mathcal{D}_{\mathrm{Ber}(p_1)}$ and $\mathcal{D}_{\mathrm{Ber}(p_2)}$ do not contain each other for any distinct $p_1 < p_2$.

To see it, let Y_η be a random variable with pmf given by (2.3). Throughout, assume that $I_p \perp Y_\eta$ for all $p \in (0, 1)$ and $\eta > 0$. Define $q_i = \mathbb{P}(I_p + Y_\eta = i)$. Then

$$q_1^2 \geq q_0 q_2 \Longleftrightarrow \eta \leq \frac{p}{(1-p)^2}. \tag{2.8}$$

Choose $\eta = p/(1-p)^2$. It is easy to see that $q_2^2 \geq q_1 q_3$. Then $I_p + Y_\eta$ is LC, that is, $Y_\eta \in \mathcal{D}_{\mathrm{Ber}(p)}$.

For $p_1 < p_2$, set $\eta_2 = p_2/(1-p_2)^2$. Then $Y_{\eta_2} \in \mathcal{D}_{\mathrm{Ber}(p_2)}$. However, $Y_{\eta_2} \notin \mathcal{D}_{\mathrm{Ber}(p_1)}$ in view of (2.8) and the fact $\eta_2 > p_1/(1-p_1)^2$. Thus, $\mathcal{D}_{\mathrm{Ber}(p_2)} \nsubseteq \mathcal{D}_{\mathrm{Ber}(p_1)}$.

On the other hand, note that $I_{p_2} + Y_{\eta_2}$ is LC. Then $3 - (I_{p_2} + Y_{\eta_2}) = (1 - I_{p_2}) + (2 - Y_{\eta_2})$ is LC. Since $1 - I_p \sim \mathrm{Ber}(1-p)$, it follows that $2 - Y_{\eta_2} \in \mathcal{D}_{\mathrm{Ber}(1-p_2)}$. Similarly, $2 - Y_{\eta_2} \notin \mathcal{D}_{\mathrm{Ber}(1-p_1)}$. Thus, $\mathcal{D}_{\mathrm{Ber}(1-p_2)} \nsubseteq \mathcal{D}_{\mathrm{Ber}(1-p_1)}$. Since p_1 and p_2 are arbitrary, we have $\mathcal{D}_{\mathrm{Ber}(p_1)} \nsubseteq \mathcal{D}_{\mathrm{Ber}(p_2)}$.

2.4. Discrete Uniform Distribution

Let $\mathrm{Unif}(n)$ denote the distribution of a random variable X_n uniformly distributed on the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}_+$, that is, $\mathbb{P}(X_n = k) = 1/n$ for $k = 1, \ldots, n$. In general, $\mathcal{D}_{\mathrm{Unif}(m)}$ and $\mathcal{D}_{\mathrm{Unif}(n)}$ do not contain each other for any distinct $1 \leq m < n$, as shown by the following counterexample ($m = 2$, $n = 3$).

To see it, let Y_η be a random variable with pmf given by (2.3) where $\eta \geq 0$. Throughout, assume that $X_n \perp Y_\eta$ for all $n \in \mathbb{N}_+$ and $\eta \geq 0$. Denote by $q_n(i) = \mathbb{P}(X_n + Y_\eta = i)$ for $i \in \mathbb{N}$. It can be checked that

$X_2 + Y_\eta$	1	2	3	4
prob.	$\frac{1}{2(\eta+3)}$	$\frac{2}{2(\eta+3)}$	$\frac{\eta+2}{2(\eta+3)}$	$\frac{\eta+1}{2(\eta+3)}$

and

$X_3 + Y_\eta$	1	2	3	4	5
prob.	$\frac{1}{3(\eta+3)}$	$\frac{2}{3(\eta+3)}$	$\frac{\eta+3}{3(\eta+3)}$	$\frac{\eta+2}{3(\eta+3)}$	$\frac{\eta+1}{3(\eta+3)}$

Then

$$[q_2(2)]^2 - q_2(1)q_2(3) = \frac{2-\eta}{4(\eta+3)^2}, \quad [q_2(3)]^2 - q_2(2)q_2(4) = \frac{\eta^2+2\eta+2}{4(\eta+3)^2} > 0;$$

and

$$[q_3(2)]^2 - q_3(1)q_3(3) = \frac{1-\eta}{9(\eta+3)^2}, \quad [q_3(3)]^2 - q_3(2)q_3(4) = \frac{\eta^2+4\eta+5}{9(\eta+3)^2} > 0,$$

$$[q_3(4)]^2 - q_3(3)q_3(5) = \frac{1}{9(\eta+3)^2} > 0.$$

Thus, for $\eta \in (1, 2)$, $Y_\eta + X_2$ is LC, while $Y_\eta + X_3$ is not LC. This means that $\mathcal{D}_{\mathrm{Unif}(2)} \subsetneq \mathcal{D}_{\mathrm{Unif}(3)}$.

https://doi.org/10.1017/S0269964817000389 Published online by Cambridge University Press

On the other hand, let Z_η be a random variable, independent of (X_2, X_3), with pmf

Z_η	0	1	2	3
prob.	$\frac{1}{\eta+4}$	$\frac{1}{\eta+4}$	$\frac{1}{\eta+4}$	$\frac{\eta+1}{\eta+4}$

Here $\eta > 0$ and Z_η is not LC. Then

$X_2 + Z_\eta$	1	2	3	4	5
prob.	$\frac{1}{2(\eta+4)}$	$\frac{2}{2(\eta+4)}$	$\frac{2}{2(\eta+4)}$	$\frac{\eta+2}{2(\eta+4)}$	$\frac{\eta+1}{2(\eta+4)}$

and

$X_3 + Z_\eta$	1	2	3	4	5	6
prob.	$\frac{1}{3(\eta+4)}$	$\frac{2}{3(\eta+4)}$	$\frac{3}{3(\eta+4)}$	$\frac{\eta+3}{3(\eta+4)}$	$\frac{\eta+2}{3(\eta+4)}$	$\frac{\eta+1}{3(\eta+4)}$

It can be checked that $X_2 + Z_\eta$ is not LC for any $\eta > 0$, and that $X_3 + Z_\eta$ is LC for $\eta \in (0, 3/2]$. Therefore, $\mathcal{D}_{\mathrm{Unif}(3)} \subsetneq \mathcal{D}_{\mathrm{Unif}(2)}$.

Similar examples can be given to shown that $\mathcal{D}_{\mathrm{Unif}(m)}$ and $\mathcal{D}_{\mathrm{Unif}(n)}$ do not contain each other for any $m < n$.

3. CONTINUOUS DISTRIBUTIONS

3.1. Exponential Distribution

A random variable X is said to have an exponential distribution, if it has density function $f(x, \lambda) = \lambda e^{-\lambda x}$ for $x \in \mathbb{R}_+$. We write $X \sim \mathrm{Exp}(\lambda)$. To state the following proposition, we define $\mathcal{D}_{\mathrm{Exp}(\lambda)}$ by (1.1) with F replaced by the exponential distribution with parameter λ.

PROPOSITION 3.1: *For $\lambda_1, \lambda_2 > 0$, we have that*

$$\mathcal{D}_{\mathrm{Exp}(\lambda_1)} \subseteq \mathcal{D}_{\mathrm{Exp}(\lambda_2)} \iff \lambda_1 \geq \lambda_2. \tag{3.1}$$

PROOF: *Sufficiency.* For $\lambda_1 > \lambda_2$, let f be a density function in $\mathcal{D}_{\mathrm{Exp}(\lambda_1)}$, that is,

$$q_1(x) = \int_0^\infty f(x-s)\lambda_1 e^{-\lambda_1 s} \mathrm{d}s = \lambda_1 e^{-\lambda_1 x} \int_{-\infty}^x e^{\lambda_1 s} f(s) \mathrm{d}s$$

is LC. Note that $q_1(x)$ is LC if and only if

$$p_1(x) := \int_{-\infty}^x e^{\lambda_1 s} f(s) \mathrm{d}s$$

is LC in $x \in \mathbb{R}$. We aim to show that

$$p_2(x) := \int_{-\infty}^x e^{\lambda_2 s} f(s) \mathrm{d}s$$

is LC in $x \in \mathbb{R}$. Note that $f(x) = p_1'(x) e^{-\lambda_1 x}$ and denote $\Delta = \lambda_1 - \lambda_2$. It follows that

$$p_2(x) = \int_{-\infty}^x e^{-\Delta s} \mathrm{d}p_1(s) = e^{-\Delta x} p_1(x) + \int_{-\infty}^x \Delta e^{-\Delta s} p_1(s) \mathrm{d}s. \tag{3.2}$$

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On the other hand, note that for any $\delta > 0$,

$$p_2(x) = p_2(x-\delta) + \int_{x-\delta}^{x} e^{\lambda_2 s} f(s)\mathrm{d}s$$

with

$$\begin{aligned}\int_{x-\delta}^{x} e^{\lambda_2 s} f(s)\mathrm{d}s &= \int_{x-\delta}^{x} e^{\lambda_2 s} e^{-\lambda_1 s}\mathrm{d}p_1(s)\\ &= e^{-\Delta x}p_1(x) - e^{-\Delta(x-\delta)}p_1(x-\delta) + \Delta\int_{x-\delta}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s.\end{aligned}$$

It follows that

$$\begin{aligned}&p_2^2(x) - p_2(x+\delta)p_2(x-\delta)\\ &\quad= p_2(x)\int_{x-\delta}^{x} e^{\lambda_2 s} f(s)\mathrm{d}s - p_2(x-\delta)\int_{x}^{x+\delta} e^{\lambda_2 s} f(s)\mathrm{d}s\\ &\quad= \left(e^{-\Delta x}p_1(x) + \int_{-\infty}^{x}\Delta e^{-\Delta s}p_1(s)\mathrm{d}s\right)\int_{x-\delta}^{x} e^{\lambda_2 s} f(s)\mathrm{d}s\\ &\qquad - \left(e^{-\Delta(x-\delta)}p_1(x-\delta) + \int_{-\infty}^{x-\delta}\Delta e^{-\Delta s}p_1(s)\mathrm{d}s\right)\int_{x}^{x+\delta} e^{\lambda_2 s} f(s)\mathrm{d}s\\ &\quad= \left(e^{-\Delta x}p_1(x) + \int_{-\infty}^{x}\Delta e^{-\Delta s}p_1(s)\mathrm{d}s\right)\\ &\qquad \times\left(\left[e^{-\Delta x}p_1(x) - e^{-\Delta(x-\delta)}p_1(x-\delta)\right] + \Delta\int_{x-\delta}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s\right)\\ &\qquad - \left(e^{-\Delta(x-\delta)}p_1(x-\delta) + \int_{-\infty}^{x-\delta}\Delta e^{-\Delta s}p_1(s)\mathrm{d}s\right)\\ &\qquad \times\left(\left[e^{-\Delta(x+\delta)}p_1(x+\delta) - e^{-\Delta x}p_1(x)\right] + \Delta\int_{x}^{x+\delta} e^{-\Delta s}p_1(s)\mathrm{d}s\right)\\ &\quad\overset{\text{def}}{=} (I_1+I_2)(J_1+J_2) - (I_1^*+I_2^*)(J_1^*+J_2^*).\end{aligned} \tag{3.3}$$

Next, we investigate the sign of $I_iJ_j - I_i^*J_j^*$ for $i,j\in\{1,2\}$. Note that

$$\begin{aligned}I_1J_1 - I_1^*J_1^* &= e^{-2\Delta x}\Big\{p_1(x)\left[p_1(x) - e^{\Delta\delta}p_1(x-\delta)\right] - p_1(x-\delta)\left[p_1(x+\delta) - e^{\Delta\delta}p_1(x)\right]\Big\}\\ &\overset{\text{sgn}}{=} p_1^2(x) - p_1(x-\delta)p_1(x+\delta) \ge 0,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}I_1J_2 - I_1^*J_2^* &= e^{-\Delta x}\Delta\Big(p_1(x)\int_{x-\delta}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s - e^{\Delta\delta}p_1(x-\delta)\int_{x}^{x+\delta} e^{-\Delta s}p_1(s)\mathrm{d}s\Big)\\ &\overset{\text{sgn}}{=} \int_{x-\delta}^{x} e^{-\Delta s}p_1(s)p_1(x)\mathrm{d}s - \int_{x-\delta}^{x} e^{-\Delta s}p_1(x-\delta)p_1(s+\delta)\mathrm{d}s\\ &= \int_{x-\delta}^{x} e^{-\Delta s}\Big[p_1(s)p_1(x) - p_1(x-\delta)p_1(s+\delta)\Big]\mathrm{d}s \ge 0,\end{aligned} \tag{3.5}$$

where the last inequality follows from that $p_1(s)p_1(x) - p_1(x-\delta)p_1(s+\delta) \geq 0$ for $s \in (x-\delta, x)$ by the log-concavity of p_1. For $I_2J_1 - I_2^*J_1^*$, we have that

$$\begin{aligned} I_2J_1 - I_2^*J_1^* &= e^{-\Delta x}\int_{-\infty}^{x} \Delta e^{-\Delta s}p_1(s)\mathrm{d}s \cdot \left[p_1(x) - e^{\Delta\delta}p_1(x-\delta)\right] \\ &\quad - e^{-\Delta x}\int_{-\infty}^{x-\delta} \Delta e^{-\Delta s}p_1(s)\mathrm{d}s \cdot \left[e^{-\Delta\delta}p_1(x+\delta) - p_1(x)\right] \\ &\overset{\text{sgn}}{=} \int_{-\infty}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s \cdot \left[p_1(x) - e^{\Delta\delta}p_1(x-\delta)\right] \\ &\quad - \int_{-\infty}^{x} e^{-\Delta s}p_1(s-\delta)\mathrm{d}s \cdot \left[p_1(x+\delta) - e^{\Delta\delta}p_1(x)\right] \\ &= \int_{-\infty}^{x} e^{-\Delta s}\Big(p_1(s)\Big[p_1(x) - e^{\Delta\delta}p_1(x-\delta)\Big] \\ &\quad - p_1(s-\delta)\Big[p_1(x+\delta) - e^{\Delta\delta}p_1(x)\Big]\Big)\mathrm{d}s \geq 0, \end{aligned} \tag{3.6}$$

where the last inequality follows from that for $s \leq x$ and $\delta > 0$,

$$\frac{p_1(s)}{p_1(s-\delta)} \geq \frac{p_1(x)}{p_1(x-\delta)} \geq \frac{p_1(x+\delta) - e^{\Delta\delta}p_1(x)}{p_1(x) - e^{\Delta\delta}p_1(x-\delta)}.$$

Similarly, we have that

$$\begin{aligned} I_2J_2 - I_2^*J_2^* &\overset{\text{sgn}}{=} \int_{-\infty}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s \int_{x-\delta}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s \\ &\quad - \int_{-\infty}^{x-\delta} e^{-\Delta s}p_1(s)\mathrm{d}s \int_{x}^{x+\delta} e^{-\Delta s}p_1(s)\mathrm{d}s \\ &= \int_{-\infty}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s \int_{x-\delta}^{x} e^{-\Delta s}p_1(s)\mathrm{d}s \\ &\quad - \int_{-\infty}^{x} e^{-\Delta s}p_1(s-\delta)\mathrm{d}s \int_{x-\delta}^{x} e^{-\Delta s}p_1(s+\delta)\mathrm{d}s \\ &= \int_{-\infty}^{x}\int_{x-\delta}^{x} e^{-\Delta(s+t)}\Big[p_1(s)p_1(t) - p_1(s-\delta)p_1(t+\delta)\Big]\mathrm{d}s\mathrm{d}t \geq 0, \end{aligned} \tag{3.7}$$

where the last inequality follows from the log-concavity of p_1, and that $t+\delta = \max\{s, t, s-\delta, t+\delta\}$, $s+t = (s-\delta)+(t+\delta)$. Substituting (3.4)–(3.7) into (3.3) yields that p_2 is also LC. This implies that

$$q_2(x) = \int_{0}^{\infty} f(x-s)\lambda_2 e^{-\lambda_2 s}\mathrm{d}s = \lambda_2 e^{-\lambda_2 x}p_2(x)$$

is LC in $x \in \mathbb{R}$, that is, $f \in \mathcal{D}_{\mathrm{Exp}(\lambda_2)}$.

Necessity: It suffices to prove that if $\lambda_1 < \lambda_2$ then there exists a distribution function G such that $G \in \mathcal{D}_{\mathrm{Exp}(\lambda_1)}$ but $G \notin \mathcal{D}_{\mathrm{Exp}(\lambda_2)}$. Let $X \sim \mathrm{Exp}(\lambda)$ and $X \perp Y_\eta$, where Y_η is a

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random variable with pmf given be (2.3). Here $\eta > 0$ is to be determined later. The pdf of $X + Y_\eta$ is given by

$$h(x,\lambda,\eta) = \frac{\lambda}{3+\eta}\left[e^{-\lambda x}1_{\{x>0\}} + e^{-\lambda(x-1)}1_{\{x>1\}} + (1+\eta)e^{-\lambda(x-2)}1_{\{x>2\}}\right]$$
$$= \frac{\lambda}{3+\eta}\Big[e^{-\lambda x}1_{\{x\in(0,1]\}} + (1+e^{\lambda})e^{-\lambda x}1_{\{x\in(1,2]\}}$$
$$+ (1+e^{\lambda}+(\eta+1)e^{2\lambda})e^{-\lambda x}1_{\{x\in(2,\infty)\}}\Big].$$

It is easy to see that $h(\cdot,\lambda,\eta)$ is LC on $\mathbb{R}$ if and only if $\eta e^{\lambda} \le 1$. Now, choose $\eta = e^{-\lambda_1}$ and denote $Y_\eta \sim G_\eta$. Then $G_\eta \in \mathcal{D}_{\mathrm{Exp}(\lambda_1)}$ but $G_\eta \notin \mathcal{D}_{\mathrm{Exp}(\lambda_2)}$. This completes the proof of the proposition. ■

Having Proposition 3.1, we can immediately obtain the following proposition for the Gamma distribution by the same arguments as in the proof of Proposition 2.1. To state the proposition, we write $\Gamma(\alpha,\lambda)$ for the Gamma distribution with shape and scale parameters $\alpha,\lambda \in \mathbb{R}_+$, that is, its density function is given by $f(x;\alpha,\lambda) = \lambda^\alpha x^{\alpha-1}e^{-\lambda x}/\Gamma(\alpha)$, $x \in \mathbb{R}_+$. Define $\mathcal{D}_{\Gamma(\alpha,\lambda)}$ by (1.1) with F replaced by $\Gamma(\alpha,\lambda)$.

Proposition 3.2: *For $r \in \mathbb{N}_+$ and $\lambda_1,\lambda_2 \in \mathbb{R}_+$, we have that*

$$\mathcal{D}_{\Gamma(r,\lambda_1)} \subseteq \mathcal{D}_{\Gamma(r,\lambda_2)} \Longleftrightarrow \lambda_1 \ge \lambda_2.$$

It is still unknown whether $\mathcal{D}_{\Gamma(r,\lambda)}$ is increasing in $r \in \mathbb{R}_+$ for any fixed λ, that is, $\mathcal{D}_{\Gamma(r_1,\lambda) \subseteq \mathcal{D}_{\Gamma(r_2,\lambda)}}$ whenever $0 < r_1 < r_2$ and $r_j \in \mathbb{R}_+$.

3.2. Normal Distribution

Let $\mathcal{D}_{N(\mu,\sigma^2)}$ be defined by (1.1) with F replaced by the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_+$.

Proposition 3.3: *For any $\mu_1,\mu_2 \in \mathbb{R}$, we have that*

$$\mathcal{D}_{N(\mu_1,\sigma_1^2)} \subseteq \mathcal{D}_{N(\mu_2,\sigma_2^2)} \Longleftrightarrow \sigma_1^2 \le \sigma_2^2. \tag{3.8}$$

Proof: *Sufficiency*. The sufficiency is trivial since the normal distribution is LC and $N(0,\sigma_2^2)$ distribution is the convolution of $N(0,\sigma_1^2)$ and $N(0,\sigma_2^2-\sigma_1^2)$ distributions for $\sigma_1^2 < \sigma_2^2$.

Necessity: Without loss of generality, assume that $\mu_1 = \mu_2 = 0$. It suffices to prove that if $\sigma_1^2 < \sigma_2^2$ then there exists a distribution function G such that $G \in \mathcal{D}_{N(0,\sigma_2^2)}$ but $G \notin \mathcal{D}_{N(0,\sigma_1^2)}$. Let $X \sim N(0,\sigma^2)$ and $X \perp I_p$, where $I_p \sim \mathrm{Ber}(p)$. We first show that

$$X + I_p \text{ is LC for all } p \in (0,1) \Longleftrightarrow \sigma^2 \ge \frac{1}{4}. \tag{3.9}$$

To this end, define $\psi(x) = \exp\{-x^2/(2\sigma^2)\}$ for all $x \in \mathbb{R}$. Then $X + I_p$ is LC if and only is the function $\ell_p(x) = \log(p\psi(x) + \bar{p}\psi(x-1))$ is concave, where $\bar{p} = 1-p$. Note that

$\psi'(x) = -(x/\sigma^2)\psi(x)$ for all x, and

$$\begin{aligned}
\ell_p'(x) &= -\frac{1}{\sigma^2} \cdot \frac{px\psi(x) + \bar{p}(x-1)\psi(x-1)}{p\psi(x) + \bar{p}\psi(x-1)} \\
&= -\frac{1}{\sigma^2}\left(x - \frac{\bar{p}\psi(x-1)}{p\psi(x) + \bar{p}\psi(x-1)}\right), \\
\sigma^2 \ell_p''(x) &= -1 + p\bar{p} \cdot \frac{\psi'(x-1)\psi(x) - \psi(x-1)\psi'(x)}{(p\psi(x) + \bar{p}\psi(x-1))^2} \\
&= -1 + \frac{p\bar{p}}{\sigma^2} \cdot \frac{\psi(x-1)\psi(x)}{(p\psi(x) + \bar{p}\psi(x-1))^2} \\
&= -1 + \frac{p\bar{p}}{\sigma^2} \cdot \left[\left(p\frac{\psi(x)}{\psi(x-1)} + \bar{p}\right)\left(\bar{p}\frac{\psi(x-1)}{\psi(x)} + p\right)\right]^{-1}.
\end{aligned}$$

Set $z = \psi(x)/\psi(x-1) = \exp\{-(2x-1)/(2\sigma^2)\} \in \mathbb{R}_+$. Then

$$\sigma^4 \ell_p''(x) = \frac{p\bar{p}}{(pz + \bar{p})(p + \bar{p}/z)} - \sigma^2 \stackrel{\text{def}}{=} \frac{1}{h_p(x)} - \sigma^2,$$

where

$$h_p(x) = \left(1 + \frac{pz}{\bar{p}}\right)\left(1 + \frac{\bar{p}}{pz}\right) = 2 + \frac{pz}{\bar{p}} + \frac{\bar{p}}{pz} \geq 4,$$

and the last equality holds for $z = \bar{p}/p$. Therefore, for $\sigma^2 \geq 1/4$, $\ell_p''(x) \leq 0$ for all $x \in \mathbb{R}$ and, hence, $X + I_p$ is LC; while for $\sigma^2 < 1/4$, there exists an $x_0 \in \mathbb{R}$ such that $\ell_p''(x_0) > 0$, which implies that $X + I_p$ is not LC. This proves (3.9).

For $\sigma_1 < \sigma_2$, let $X_1 \sim N(0, \sigma_1^2)$, $X_2 \sim N(0, \sigma_2^2)$, and $I_p \sim \mathrm{Ber}(p)$ with $I_p \perp X_i$ for $i = 1, 2$. Denote by G_p the distribution function of $2\sigma_2 I_p$. From (3.9), it follows that $X_2/(2\sigma_2) + I_p$ and hence $X_2 + 2\sigma_2 I_p$ is LC for all p. This means $G_p \in \mathcal{D}_{N(0,\sigma_2^2)}$ for all $p \in (0, 1)$. Also, from (3.9), it follows that $X_1/(2\sigma_2) + I_p$ and hence $X_1 + 2\sigma_2 I_p$ is not LC for any p. This means $G_p \notin \mathcal{D}_{N(0,\sigma_1^2)}$ for any $p \in (0, 1)$. This completes the proof of the proposition. ■

Remark 3.4: The implication (3.9) states that whether $X + I_p$ is LC does not depend upon p. This is interesting. LC functions are unimodal. Figure 1 plots the density functions of $X + I_p$ for $p = 1/2$ and different σ^2, which gives us some feeling about the critical value $1/4$ of σ^2.

Remark 3.5: It is well known that the convolution of two LC functions is also LC (see, for example, Dharmadhikari and Joag-dev [4], p. 17). However, when we state this assertion, we should pay more attention to the assumption that these two LC functions are both defined on the set of (positive) integers or on the set of (positive) real numbers. For example,

- The convolution of two LC pmfs defined on $\mathbb{N}$ is also LC on $\mathbb{N}$ (Fekete [6]);
- The convolution of two LC pdfs defined on $\mathbb{R}$ is also LC on $\mathbb{R}$ (Ibragimov [10]).

If f is a LC pmf on $\mathbb{N}$ and g is a LC pdf on $\mathbb{R}$, then we can not conclude that the convolution $f * g$ of f and g is also LC. A counterexample can obtained easily from (3.9).

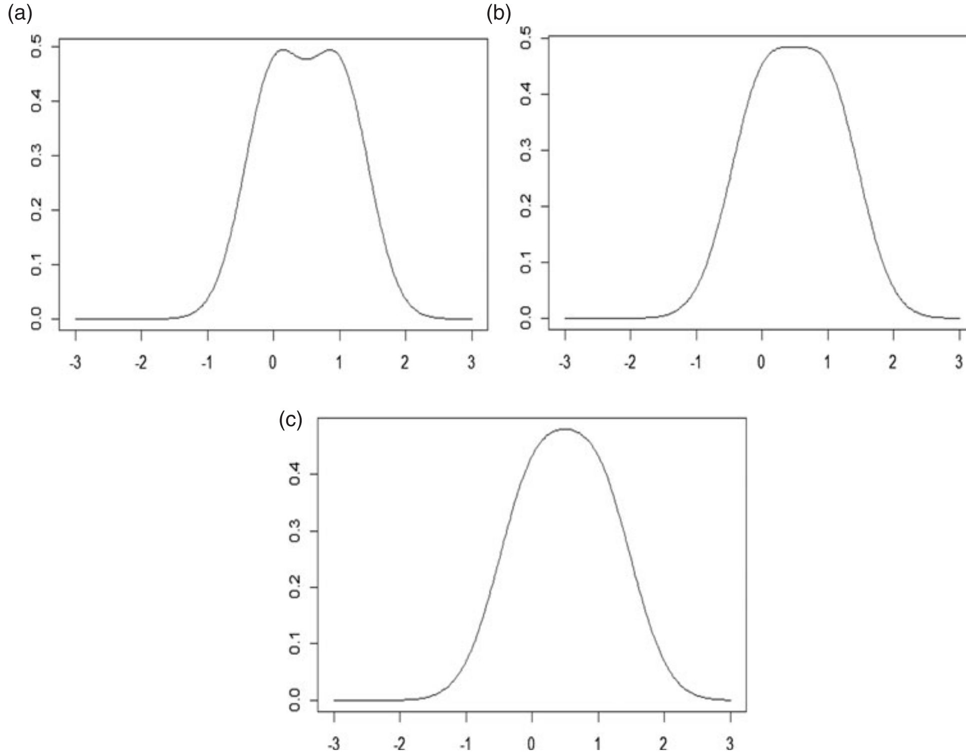

FIGURE **1.** Densities of $X + I_{0.5}$, where $X \sim N(0, \sigma^2)$, $I_{0.5} \sim \mathrm{Ber}(0.5)$ and $X \perp I_{0.5}$. (a) $\sigma^2 = 0.2$, (b) $\sigma^2 = 0.25$, (c) $\sigma^2 = 0.3$.

Remark 3.6: Let X be a random variable with distribution function F, and denote by F_θ the distribution function of θX, where $\theta \in \mathbb{R}_+$. Propositions 3.1–3.3 state that, for $F = \mathrm{Exp}(\lambda)$, $\Gamma(r, \lambda)$ with $r \geq 1$, or $N(\mu, \sigma^2)$,

$$\mathcal{D}_{F_a} \subseteq \mathcal{D}_{F_b}, \quad \text{whenever } 0 < a < b. \tag{3.10}$$

Note that $\mathrm{Exp}(\lambda)$, $\gamma(r, \lambda)$ with $r \geq 1$ and $N(\mu, \sigma^2)$ all have LC density functions. One may wonder whether (3.10) holds for the general case that X is LC. However, this is not true as shown by the following counterexample.

Let $X \sim U(0, 1)$, uniformly distributed over interval $(0, 1)$. Then X is v. Let $I_{0.5} \sim \mathrm{Ber}(0.5)$ and $I_{0.5} \perp X$. It is seen that $X + I_{0/5} \sim U(0, 2)$ is LC, that is, $\mathrm{Ber}(0.5) \in \mathcal{D}_{U(0,1)}$. However, $2X + I_{0.5}$ is not LC, that is, $\mathrm{Ber}(0.5) \notin \mathcal{D}_{U(0,2)}$. Therefore, (3.10) does not hold when F is a uniform distribution.

Acknowledgement

The authors were supported by the NNSF of China (grant numbers 71671176, 11371340, and 71631006). T. Hu is the corresponding author.

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