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Robust optimal attitude control of a laboratory helicopter without angular velocity feedback

Hao Liu†‡, Jianxiang Xi§* and Yisheng Zhong‡

†*School of Astronautics, Beihang University, Beijing 100191, P.R. China*
‡*Department of Automation, TNList, Tsinghua University, Beijing 100084, P.R. China*
§*High-Tech Institute of Xi'an, Xi'an 710025, P.R. China*

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SUMMARY
In this paper, the robust, optimal, output control problem is dealt with for a 3-degree-of-freedom laboratory helicopter. The control goal is to achieve the practical tracking of the desired elevation and pitch angles without the angular velocity feedback. A nominal linear time-invariant system is introduced and the real system is considered as the nominal one with uncertainties, including parameter perturbations, nonlinear time-varying uncertainties, and external disturbances. An observer is first used to estimate angular velocity. Then a nominal controller based on the optimal control method is designed for the nominal system to achieve the desired tracking properties. Lastly, a robust output compensator is added to restrain the effects of uncertainties in the real system. It is shown that asymptotic tracking properties and robust stability can be achieved. Experimental results on the laboratory helicopter are shown to verify the effectiveness of the proposed control method.

KEYWORDS: Robust control; Nonlinear control; Optimal control; Laboratory helicopter; Angular velocity.

1. Introduction
Unmanned helicopters have received much attention in the last 20 years because of their versatile civilian and military applications (see, e.g., refs. [1–6]). The motion control system design for the helicopters is a challenge because of the uncertainties in their dynamical models. First, there exist parameter pertubations and unmodeled uncertainties in the helicopter models. Second, the helicopters are nonlinear systems with strong inter-axis couplings attributed to the generation process of the forces and torques of flying robots. In addition, the flight qualities of the unmanned helicopters can be influenced by external disturbances such as wind gusts.

The optimal control approach can find a control law to satisfy a certain optimality criterion for a given system. It has been widely used in aerospace, marine, and defense industries because of its simple structure in practical applications.[7] But the helicopter models, based on which dynamical characteristics are analyzed and controllers are designed, are generally the approximate descriptions of the helicopter systems due to the uncertainties mentioned above. Therefore, it is almost impossible for a helicopter system to respond exactly similar to the true system by the optimal control approach, which needs an accurate model of the controlled plant. Then the uncertainty rejection problem has been presented for the optimal control, and the robust optimal control method, which could achieve the desired tracking properties and restrain the effects of the uncertainties, gains much attention.

Among early works on the robust optimal control problem, Petersen[8] studied a class of linear systems with uncertain parameters, and a feedback control law based on a quadratic Lyapunov function was designed to achieve asymptotic stability of the closed-loop control system. Based on Petersen's[8] Riccati equation approach, Douglas and Athans[9] derived a linear quadratic regulation (LQR) controller, which was robust against real parameter uncertainties. In addition, multi-objective feedback control approaches were discussed by taking the optimal theory and the Riccati equation

* Corresponding author. E-mail: xijx07@mails.tsinghua.edu.cn

Fig. 1. The laboratory helicopter.

into account together. A mixed H_2 and H_∞ performance analysis under the structured uncertainty was presented in Zhou *et al*.[10] A control law based on the vector optimization technique was designed to minimize multiple performance objectives for an uncertain linear plant in Dorato *et al*.[11] In Trentini and Pieper,[12] the mixed-norm control methodology was applied to attenuate the effects of high-order disturbances for a Bell helicopter in simulations. Most of the previous studies on the robust optimal control problem mainly focused on one or several kinds of uncertainties such as parametric uncertainties and external disturbances. But in this paper constant variations, nonlinear time-varying uncertainties, and external disturbances are investigated together for the helicopter to achieve robust stability and the desired dynamical and steady-state tracking performances simultaneously in actual flight tests.

A laboratory-scale 3-degree-of-freedom (DOF) helicopter is used here, as shown in Fig. 1. Its particular features, such as parameter perturbations and nonlinear time-varying uncertainties, guarantee it as an ideal testbed to examine the effectiveness of control approaches, as witnessed by many contributions in the last 10 years (see, e.g., refs. [13–17]). In Kutay *et al.*,[13] a single-input–single-output (SISO) controller was designed for the pitch angle, but the flight control problem under coupling condition was not discussed fully. Adaptive feedback control laws were proposed in Andrievsky *et al*.[14] and Ishitobi *et al.*,[16] but dynamical tracking performances of the closed-loop system cannot be specified by these control approaches. In Kiefer *et al.*,[15] the trajectory tracking control was achieved by an optimal controller without discussing the influences of uncertainties on the closed-loop system. The output regulation problem was studied in Zheng and Zhong,[17] but an exogenous system was required to generate references for elevation and pitch angles.

In this paper, a nominal linear time-invariant helicopter model is obtained by the linearized approximation whereas the real model is regarded as the nominal one with equivalent disturbances, which contain parameter uncertainties, nonlinearities, coupling, and external disturbances. A new robust optimal control strategy combining the optimal control method with the robust output compensation technique is proposed for the attitude control of the laboratory helicopter without angular velocity measurements. The designed controller consists of three parts: an observer, a nominal controller, and a robust output compensator. The observer is applied to estimate the angular velocity values of elevation and pitch angles; the nominal controller by the optimal control method is designed for the nominal linear system to get the desired tracking performances; and the robust output compensator is introduced to restrain the influences of equivalent disturbances. This paper is different from refs. [17–19], where the designed robust controllers depend on the angular velocity measurements. Actually, the angle positions are measured from the encoders and the angle speeds cannot be obtained directly. In the current paper, the designed output controller does not depend on the angular velocity feedback. Furthermore, the stability of the whole closed-loop control system,

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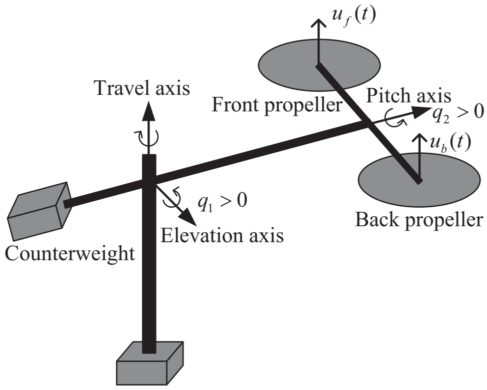

Fig. 2. A 3-DOF laboratory helicopter.

including the helicopter dynamical system and the robust optimal controller, is analyzed by the Lyapunov method, but the discussions on the controller stability were ignored in our previously published papers.

One advantage of this robust optimal control strategy is that the resulted controller does not require the feedback of angular velocities. Moreover, the desired tracking properties and the robust stability can be achieved simultaneously under the effects of various uncertainties. The desired tracking performances can be specified by the nominal LQR controller, and the robust stability can be guaranteed by the robust compensation technique. It is proven that the attitude tracking errors of the closed-loop system can asymptotically converge to the neighborhood of the origin with the given boundary in a finite time for the given initial conditions. In addition, the proposed robust optimal controller is realized with a linear time-invariant structure, which guarantees its easy implementation in practical applications.

The paper is organized as follows. Section 2 presents the system and uncertainty description of a 3-DOF laboratory helicopter. Section 3 formulates the design procedure of the robust optimal controller. The asymptotical tracking properties and robust stability are proven in Section 4. Experimental results are shown in Section 5, and conclusions are drawn in Section 6.

Notations: For $p \in \mathbb{R}^{n\times 1}$ and $D \in \mathbb{R}^{m\times n}$, we denote that $\|p\| = \sqrt{p^T p}$ and $\|D\| = \sqrt{\lambda_{\max}(D^T D)}$.

2. Problem Statement

As shown in Fig. 2, the laboratory helicopter from Quanser Consulting Inc. has 3 rotational DOFs: to elevate, travel, and pitch. Two DC motors are installed at two ends of the helicopter frame to drive two propellers. This helicopter is an underactuated mechanical system, that is, a system possessing more degrees of freedom than independent control inputs. Therefore, one can select two of the three angles as outputs. In this paper, the attitude controller focuses on tracking the references of elevation and pitch angles.

As shown in ref. [15], the motions of the elevation and pitch channels can be described by the following equations:

$$\begin{aligned}\ddot{q}_1(t) &= \tilde{a}_1 \sin q_1(t) + \tilde{a}_3 \cos q_1(t) + \tilde{b}_1 \cos q_2(t)(u_f(t) + u_b(t)) + d_1(t),\\ \ddot{q}_2(t) &= \tilde{a}_2 \cos q_1(t) \sin q_2(t) + \tilde{b}_2(u_f(t) - u_b(t)) + d_2(t),\end{aligned}$$

where $q_1(t)$ and $q_2(t)$ are the elevation angle and the pitch angle respectively, $u_f(t)$ and $u_b(t)$ are the control voltages of the front motor and the back motor respectively, $d_i(t)$ $(i = 1, 2)$ is the external disturbance, and $\tilde{a}_i$ $(i = 1, 2,\ 3)$ and $\tilde{b}_i$ $(i = 1, 2)$ are the helicopter parameters. The nominal values of the helicopter parameters are denoted by a_i $(i = 1, 2,\ 3)$ and b_i $(i = 1, 2)$, and the dynamical model of the elevation and pitch angles can be rewritten as

$$\begin{aligned}\ddot{q}_1(t) &= a_1 q_1(t) + a_3 + b_1 u_1(t) + \Delta_{12}(t),\\ \ddot{q}_2(t) &= a_2 q_2(t) + b_2 u_2(t) + \Delta_{22}(t),\end{aligned} \tag{1}$$

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where $u_1(t) = u_f(t) + u_b(t)$, $u_2(t) = u_f(t) - u_b(t)$, and $\Delta_{i2}(t)$ $(i = 1, 2)$ are called the equivalent disturbances and take the following forms:

$$\begin{aligned} \Delta_{12}(t) &= \tilde{a}_1 \sin q_1(t) - a_1 q_1(t) + \tilde{a}_3 \cos q_1(t) - a_3 + (\tilde{b}_1 \cos q_2(t) - b_1) u_1(t) + d_1(t), \\ \Delta_{22}(t) &= \tilde{a}_2 \cos q_1(t) \sin q_2(t) - a_2 q_2(t) + (\tilde{b}_2 - b_2) u_2(t) + d_2(t). \end{aligned} \tag{2}$$

This paper will investigate the problem of designing a robust optimal controller to achieve the practical tracking of references $r_1(t)$ and $r_2(t)$ for elevation and pitch channels respectively.

Let $x_i(t) = [\,x_{i1}(t)\ x_{i2}(t)\,]^T$ and $\Delta_i(t) = [\,0\ \Delta_{i2}(t)\,]^T$ $(i = 1, 2)$, then the dynamical model can be described in a state-space form as

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i\,(u_i(t) - z_i(t)) + \Delta_i(t), \\ y_i(t) &= C_i x_i(t), \quad i = 1, 2, \end{aligned} \tag{3}$$

where $x_{i1}(t) = q_i(t) - r_i(t)$, $x_{i2}(t) = \dot{x}_{i1}(t)$, and

$$z_1(t) = (\ddot{r}_1(t) - a_1 r_1(t) \ - a_3)\,/b_1,\ z_2(t) = (\ddot{r}_2(t) - a_2 r_2(t))\,/b_2,$$

$$A_i = \begin{bmatrix} 0 & 1 \\ a_i & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T.$$

Assumption 2.1 *The uncertain parameters $\tilde{a}_i$ $(i = 1, 2, 3)$ are bounded, and $\tilde{b}_i$ $(i = 1, 2)$ satisfy that $|\tilde{b}_i - b_i| < b_i$, where the nominal parameters b_i $(i = 1, 2)$ are positive.*

Assumption 2.2 *There exists a mechanical limit on the pitch angle so that $q_2(t) \in [-\pi/2 + \delta_{q2}, \pi/2 - \delta_{q2}]$ with δ_{q2} a positive constant.*

Define $\rho_1 = \max|\tilde{b}_1 \cos q_2(t) - b_1|/b_1$ and $\rho_2 = |\tilde{b}_2 - b_2|/b_2$.

Remark 1. *Under Assumptions 2.1 and 2.2, one can obtain $0 \le \rho_i < 1$ $(i = 1, 2)$.*

Assumption 2.3 *The external disturbances $d_i(t)$ $(i = 1, 2)$ are bounded.*

Assumption 2.4 *The references of elevation and pitch angles and their derivatives $r_1^{(k)}$ and $r_2^{(k)}$ $(k = 0,\ 1, 2)$ are piecewise uniformly bounded.*

3. Robust Optimal Controller Design

Based on the optimal feedback control method and the output signal compensation technique,[20] the robust optimal control inputs are constructed by two parts: the nominal control inputs $u_i^N(t)$ $(i = 1, 2)$ and the robust compensating inputs $v_i(t)$ $(i = 1, 2)$; that is, the control inputs $u_i(t)$ $(i = 1, 2)$ have the following forms:

$$u_i(t) = u_i^N(t) + v_i(t), \quad i = 1, 2. \tag{4}$$

Step 1: Consider the following nominal systems,

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i\,(u_i(t) - z_i(t)), \\ y_i(t) &= C_i x_i(t), \quad i = 1, 2. \end{aligned}$$

The nominal control inputs for this system are designed by the LQR method for the following cost function:

$$J_i = \int_0^\infty e^{2\theta_i t}\left[\sigma_i y_i^2(t) + (u_i(t) - z_i(t))^2\right] dt \quad (i = 1, 2),$$

where σ_i $(i = 1, 2)$ are positive constants and θ_i $(i = 1, 2)$ are non-negative constants, which determine the minimum decaying rates of the outputs $y_i(t)$ $(i = 1, 2)$. Let $\eta_i = \text{diag}(\theta_i, 0)$ $(i = 1, 2)$.

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By obtaining the positive definite solutions P_i $(i = 1, 2)$ to the following Riccati equation,

$$P_i(A_i + \eta_i) + (A_i + \eta_i)^T P_i - P_i B_i B_i^T P_i + \sigma_i C_i^T C_i = 0, \quad i = 1, 2,$$

one can obtain the feedback gains $K_i = [\, k_{i1} \; k_{i2} \,]$ $(i = 1, 2)$ by $K_i = B_i^T P_i$. Thus, the nominal controller can be given as

$$u_i^N(t) = -K_i x_i(t) + z_i(t), \quad i = 1, 2. \tag{5}$$

However, since the angular velocities $\dot{q}_i(t)$ $(i = 1, 2)$ cannot be measured directly, redesign the nominal control laws as

$$u_i^N(t) = -\left(k_{i1}x_{i1}(t) + k_{i2}\hat{x}_{i2}(t)\right) + z_i(t), \quad i = 1, 2, \tag{6}$$

where $\hat{x}_{i2}(t)$ $(i = 1, 2)$ are the estimation values of $x_{i2}(t)$ $(i = 1, 2)$ respectively. $\hat{x}_{i2}(t)$ $(i = 1, 2)$ can be obtained by the reduced-order observers as

$$\begin{aligned} \dot{w}_{xi}(t) &= -l_{xi} w_{xi}(t) + b_i \left(u_i^N(t) - z_i(t)\right) - \left(l_{xi}^2 - a_i\right) x_{i1}(t), \\ \hat{x}_{i2}(t) &= w_{xi}(t) + l_{xi} x_{i1}(t), \quad i = 1, 2, \end{aligned} \tag{7}$$

where l_{xi} is a positive constant. It follows that

$$\dot{\hat{x}}_{i2}(t) = l_{xi}x_{i2}(t) - l_{xi}\hat{x}_{i2}(t) + a_i x_{i1}(t) - b_i k_{i1} x_{i1}(t) - b_i k_{i2}\hat{x}_{i2}(t), \quad i = 1, \quad 2. \tag{8}$$

From (3) and (6), one can obtain that

$$\dot{x}_{i2}(t) = a_i x_{i1}(t) - b_i k_{i1} x_{i1}(t) - b_i k_{i2}\hat{x}_{i2}(t) + b_i v_i(t) + \Delta_{i2}(t), \quad i = 1, 2. \tag{9}$$

If one defines the estimation errors $\tilde{x}_{i2}(t)$ $(i = 1, 2)$ as $\tilde{x}_{i2}(t) = x_{i2}(t) - \hat{x}_{i2}(t)$ $(i = 1, 2)$, then one can have

$$\dot{\tilde{x}}_{i2}(t) = -l_{xi}\tilde{x}_{i2}(t) + b_i v_i(t) + \Delta_{i2}(t), \quad i = 1, 2. \tag{10}$$

Define $\tilde{x}_i(t) = [\, x_{i1}(t) \; x_{i2}(t) \; \tilde{x}_{i2}(t) \,]^T (i = 1, 2)$, and

$$\tilde{A}_{ic} = \begin{bmatrix} 0 & 1 & 0 \\ a_i - b_i k_{i1} & -b_i k_{i2} & b_i k_{i2} \\ 0 & 0 & -l_{xi} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ b_i \\ b_i \end{bmatrix}, \quad \tilde{C}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T,$$

where $\tilde{A}_{ic}(t)$ $(i = 1, 2)$ are Hurwitz matrices. Then, from (3), (4), (6), and (10), one can obtain

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= \tilde{A}_{ic}\tilde{x}_i(t) + \tilde{B}_i v_i(t) + \tilde{\Delta}_i(t), \\ y_i(t) &= \tilde{C}_i \tilde{x}_i(t), \quad i = 1, 2, \end{aligned} \tag{11}$$

where $\tilde{\Delta}_i(t) = [\, 0 \; \Delta_{i2}(t) \; \Delta_{i2}(t) \,]^T$.

Step 2: In order to restrain the influences of the equivalent disturbances $\tilde{\Delta}_i(t)$ $(i = 1, 2)$, a robust compensator is designed. Define

$$\tilde{G}_i(s) \triangleq \tilde{C}_i(sI - \tilde{A}_{ic})^{-1}\tilde{B}_i = \tilde{M}_i^{-1}(s)\tilde{N}_i(s), \quad i = 1, 2,$$

where s is the Laplace operator, I is a unit matrix, and $\tilde{M}_i^{-1}(s)\tilde{N}_i(s)$ is the left matrix fraction descriptions of $\tilde{G}_i(s)$, which is irreducible. It follows that

$$y_i(s) = \tilde{M}_i^{-1}(s)\tilde{N}_i(s)v_i(s) + \tilde{C}_i(sI - \tilde{A}_{ic})^{-1}\tilde{\Delta}_i(s), \quad i = 1, 2.$$

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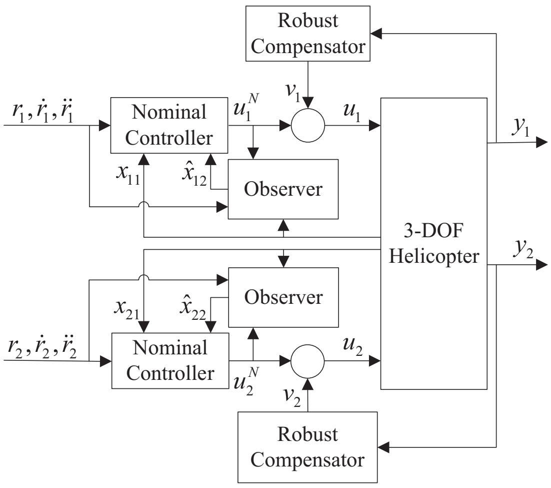

Fig. 3. The block diagram of a robust optimal control system.

Then the robust compensating inputs $v_i(s)$ $(i = 1, 2)$ can be given as

$$v_i(s) = -F_i(s)\tilde{N}_i^{-1}(s)\tilde{M}_i(s)\tilde{C}_i(sI - \tilde{A}_{ic})^{-1}\tilde{\Delta}_i(s), \quad i = 1, 2, \tag{12}$$

where $F_i(s)$ is the robust filter and has the form

$$F_i(s) = \frac{f_i}{s(p_i s + 1) + f_i},$$

with positive constants p_i and f_i $(i = 1, 2)$ to be determined. If p_i^{-1} and f_i $(i = 1, 2)$ are sufficiently large and satisfy $p_i^{-1} \gg f_i > 0$, the robust filter would have sufficiently large bandwidths, and thus the effects of $\tilde{\Delta}_i(s)$ $(i = 1, 2)$ could be reduced.

As one can obtain from (11),

$$\tilde{\Delta}_i(s) = (sI - \tilde{A}_{ic})\tilde{x}_i(s) - \tilde{B}_i v_i(s), \quad i = 1, 2,$$

it follows that

$$v_i(s) = -(1 - F_i(s))^{-1}F_i(s)\tilde{N}_i^{-1}(s)\tilde{M}_i(s)y_i(s), \quad i = 1, 2. \tag{13}$$

Therefore, the robust compensating inputs $v_i(s)$ $(i = 1, 2)$ can be implemented as follows:

$$v_i(s) = -\frac{f_i(l_{xi} + s)(s^2 + b_i k_{i2} s + b_i k_{i1} - a_i)}{b_i s(p_i s + 1)(s + b_i k_{i2} + l_{xi})} y_i(s), \quad i = 1, 2.$$

The configuration of this robust optimal control system is depicted in Fig. 3.

Remark 2. *One can see that the robust optimal controller given by (4), (6), and (13) is a linear time-invariant one. In addition, no information about equivalent disturbances is needed in the controller design, which guarantees that the proposed controller is easy to be realized in practical applications.*

4. Robust Optimal Control Properties

In this section, the asymptotical tracking properties and robust stability of the closed-loop control system will be proved.

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One can obtain from (12) that

$$v_i(s) = -\frac{f_i \Delta_{i2}(s)}{b_i(p_i s^2 + s + f_i)}, \quad i = 1, 2. \tag{14}$$

Then combining (11) with (14), one has

$$\tilde{x}_i(s) = (sI - \tilde{A}_{ic})^{-1} e_2 f_i^{-1} s g_i(s), \quad i = 1, 2,$$

where $e_2 = [0\ 1\ 1]^T$ and $g_i(s) = f_i(p_i s + 1)\Delta_{i2}(s)/(p_i s^2 + s + f_i)$. Consider the systems with inputs $\Delta_{i2}(t)$ $(i = 1, 2)$ and outputs $g_i(t)$ $(i = 1, 2)$, which can be realized as

$$\begin{aligned} \dot{x}_{iu1}(t) &= \left(-p_i^{-1} + f_i\right) x_{iu1}(t) - f_i x_{iu2}(t) + \sqrt{f_i}\,\Delta_{i2}(t), \\ \dot{x}_{iu2}(t) &= f_i x_{iu1}(t) - f_i x_{iu2}(t) + \sqrt{f_i}\,\Delta_{i2}(t), \\ g_i(t) &= \sqrt{f_i}\, x_{iu2}(t), \quad i = 1, 2. \end{aligned} \tag{15}$$

Define $x_{ic}(t) = [x_{ic1}(t)\ x_{ic2}(t)\ x_{ic3}(t)]^T = [x_{i1}(t)\ x_{i2}(t) - f_i^{-1} g_i(t)\ \tilde{x}_{i2}(t) - f_i^{-1} g_i(t)]^T$ and $\tilde{B}_{ic} = [1\ 0\ -l_{xi}]^T$, then one has

$$\begin{aligned} \dot{x}_{ic}(t) &= \tilde{A}_{ic} x_{ic}(t) + \tilde{B}_{ic} f_i^{-1} g_i(t), \\ y_i(t) &= \tilde{C}_i x_{ic}(t), \quad i = 1, 2. \end{aligned} \tag{16}$$

Theorem 1. *If the helicopter is controlled by the robust optimal controller designed in the previous section with Assumptions 2.1, 2.2, 2.3, and 2.4 satisfied, then for the given bounded initial conditions and the given constant* $\varepsilon > 0$, *there exist sufficiently large controller parameters* f_i *and* p_i^{-1} $(i = 1, 2)$ *satisfying* $p_i^{-1} \gg f_i > 0$ *and a positive constant* T *such that the states* $\tilde{x}_i(t)$, $x_{iu1}(t)$, *and* $x_{iu2}(t)$ $(i = 1, 2)$ *are bounded and* $|y_i(t)| < \varepsilon$ $(i = 1, 2)$, $\forall t > T$.

Proof. From (4), (6), (14), and (15), one can obtain that

$$|u_i| \le K_{\max} \|\tilde{x}_i\| + |z_i| + b_i^{-1}\sqrt{f_i}\,(|x_{iu1}| + |x_{iu2}|), \quad i = 1, 2,$$

where $K_{\max} = |k_{i1}| + 2\,|k_{i2}|$. Substituting the above expression into (2), one can find positive constants ζ_{xi} and ζ_{ci} $(i = 1, 2)$ such that

$$|\Delta_{i2}| \le \zeta_{xi} \|\tilde{x}_i\| + \rho_i \sqrt{f_i}\,(|x_{iu1}| + |x_{iu2}|) + \zeta_{ci}, \quad i = 1, 2. \tag{17}$$

Since $\tilde{A}_{ic}$ $(i = 1, 2)$ are stable matrices, one can find positive definite matrices P_{ic} $(i = 1, 2)$ such that $P_{ic}\tilde{A}_{ic} + \tilde{A}_{ic}^T P_{ic} = -I$. Consider the following Lynapunov function candidate:

$$V = \sum_{i=1}^{2} \left(x_{ic}^T P_{ic} x_{ic} + 0.5\left(x_{iu1}^2 + x_{iu2}^2\right) / f_i\right).$$

Its derivative can be given by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{2} \left(-x_{ic}^T x_{ic} + 2x_{ic}^T P_{ic}\tilde{B}_{ic} f_i^{-1} g_i + \left(1 - p_i^{-1} f_i^{-1}\right) x_{iu1}^2 - x_{iu2}^2 + (x_{iu1} + x_{iu2})\Delta_{i2}/\sqrt{f_i}\right) \\ &\le \sum_{i=1}^{2} \left(-\|x_{ic}\|^2 - \left(p_i^{-1} f_i^{-1} - 1 - \rho_i\right) |x_{iu1}|^2 - (1 - \rho_i)|x_{iu2}|^2 + 2\rho_i\,|x_{iu1}|\,|x_{iu2}|\right) \end{aligned}$$

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$$+\sum_{i=1}^{2}\left(2\left\|P_{ic}\tilde{B}_{ic}\right\|\|x_{ic}\|\,|x_{iu2}|+\zeta_{xi}\|\tilde{x}_i\|\left(|x_{iu1}|+|x_{iu2}|\right)+\zeta_{ci}\left(|x_{iu1}|+|x_{iu2}|\right)\right)/\sqrt{f_i}$$

$$\leq\sum_{i=1}^{2}\left(-\|x_{ic}\|^2-\left(\frac{p_i^{-1}}{f_i}-1-\rho_i-\alpha_{\rho i}\right)|x_{iu1}|^2-\left(1-\rho_i-\alpha_{\rho i}-\frac{\zeta_{xi}}{f_i}\right)|x_{iu2}|^2+\frac{\zeta_{ci}^2 f_i^{-1}}{2\alpha_{\rho i}}\right)$$

$$+\sum_{i=1}^{2}\left(\left(2\rho_i+\frac{\zeta_{xi}}{f_i}\right)|x_{iu1}|\,|x_{iu2}|+\frac{\zeta_{xi}}{\sqrt{f_i}}\|x_{ic}\|\,|x_{iu1}|+\frac{2\left\|P_{ic}\tilde{B}_{ic}\right\|+\zeta_{xi}}{\sqrt{f_i}}\|x_{ic}\|\,|x_{iu2}|\right),$$

where the constants $\alpha_{\rho i}$ $(i=1,2)$ are chosen such that $0<\alpha_{\rho i}<1-\rho_i$ $(i=1,2)$. Then defining $\bar{x}_i=[\,\|x_{ic}\|\;|x_{iu1}|\;|x_{iu2}|\,]^T$ $(i=1,2)$, one has

$$\dot{V}\leq-\tau V-\sum_{i=1}^{2}\left(\bar{x}_i^T R_i\bar{x}_i-\frac{\zeta_{ci}^2}{2\alpha_{\rho i}f_i}\right), \tag{18}$$

where τ is a positive constant and satisfies that $\tau\,\|P_{ic}\|<1$ $(i=1,2)$ and

$$R_i=\begin{bmatrix}1-\tau\,\|P_{ic}\| & -\frac{\zeta_{xi}}{2\sqrt{f_i}} & -\frac{2\|P_{ic}\tilde{B}_{ic}\|+\zeta_{xi}}{2\sqrt{f_i}}\\ -\frac{\zeta_{xi}}{2\sqrt{f_i}} & \frac{2p_i^{-1}-\tau}{2f_i}-1-\rho_i-\alpha_{\rho i} & -\rho_i-\frac{\zeta_{xi}}{2f_i}\\ -\frac{2\|P_{ic}\tilde{B}_{ic}\|+\zeta_{xi}}{2\sqrt{f_i}} & -\rho_i-\frac{\zeta_{xi}}{2f_i} & 1-\rho_i-\alpha_{\rho i}-\frac{2\zeta_{xi}+\tau}{2f_i}\end{bmatrix}.$$

If p_i^{-1} and f_i $(i=1,2)$ are parameters with sufficiently large values satisfying $p_i^{-1}\gg f_i>0$, one can see that $R_i>0$ $(i=1,2)$. Then it follows that

$$V(t)\leq V(t_0)e^{-\tau(t-t_0)}+\sum_{i=1,2}\frac{\zeta_{ci}^2}{2\tau\alpha_{\rho i}f_i}. \tag{19}$$

In addition, from (15), one has

$$|x_{i2}|\leq|x_{ic2}|+|x_{iu2}|/\sqrt{f_i},\quad |\tilde{x}_{i2}|\leq|x_{ic3}|+|x_{iu2}|/\sqrt{f_i}. \tag{20}$$

Hence, from (19) and (20), one can see that for the given bounded initial conditions, and the given positive constant ε, if f_i $(i=1,2)$ are sufficiently large, then the states $\tilde{x}_i(t)$, $x_{iu1}(t)$, and $x_{iu2}(t)$ $(i=1,2)$ are bounded and there exists a positive constant T such that $|y_i(t)|<\varepsilon$ $(i=1,2)$, $\forall t\geq T$. □

Remark 3. *One can see that only the bounds of the uncertainties of the 3-DOF helicopter system are necessary for the robust controller design. However, in practical applications, the bounds of the disturbances may not be known. In this case, the robust compensator parameters can be determined by an on-line tuning way. One can set f_i and p_i^{-1} $(i=1,2)$ to some certain initial values and satisfy $p_i^{-1}\gg f_i>0$ and run the helicopter system. If the tracking performances are not satisfactory, then one can tune f_i and p_i^{-1} $(i=1,2)$ to large values satisfying $p_i^{-1}\gg f_i>0$ until the desired tracking performances are achieved.*

5. Experimental Results and Discussions

The attitude angles are measured by two encoders with effective position resolution of 0.0879°.[21] The encoder data are transmitted to a dSPACE System, where the robust control scheme is implemented. The control signals are outputted to the front and back motors. The desired elevation and pitch angles are obtained by $r_i(s)=\varpi_i(s)/(\phi_i s+1)$ $(i=1,2)$, where ϖ_i $(i=1,2)$ are the reference input

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Table I. Nominal parameters of a 3-DOF helicopter.

Parameter	Value	Parameter	Value	Parameter	Value
a_1	0	a_2	0	a_3	–2.53
b_1	0.0858	b_2	0.581		

Table II. Controller parameters.

Parameter	Value	Parameter	Value	Parameter	Value
σ_1	1000	f_1	5	p_1	0.0278
θ_1	0	ϕ_1	1	l_{x1}	10
σ_2	1000	f_2	3.33	p_2	0.0417
θ_2	0.67	ϕ_2	0.7	l_{x2}	10

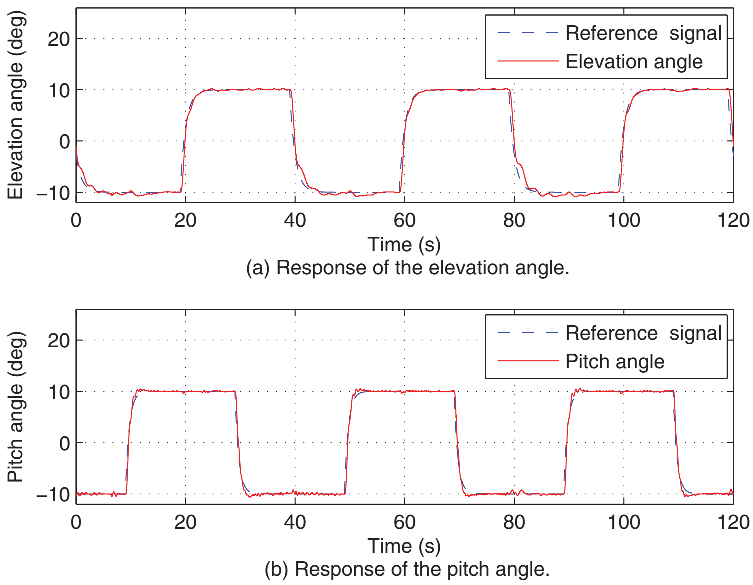

Fig. 4. Responses of the two channels in case 1.

commands for the elevation and pitch channels respectively. Table I presents the nominal values of the helicopter parameters. In order to obtain the desired output tracking performances, the controller parameters are selected as in Table II. Actually, there exist control input constraints on the front and back motors. The control inputs $u_i(t)$ $(i = f, b)$ are required to satisfy $u_i(t) > 0$ to avoid rotations of the two rotors in an opposite direction. Furthermore, there exist upper bounds on control inputs as $u_i(t) < 22$ $(i = f, b)$.

Case 1: Aggressive missions are taken in serious coupling condition for the laboratory helicopter. The tracking responses of two interacting channels are presented in Fig. 4. Note that in ref. [14], the motion control for one channel was considered while other angles were required to be stabilized at 0^o . Only the pitch angle motion control was discussed in ref. [13]. One can see that by the proposed robust optimal control approach, the closed-loop system has good attitude tacking properties under aggressive maneuvers with strong coupling. The control inputs in the mission are shown in Fig. 5, and one can observe that $u_f(t)$ and $u_b(t)$ do not reach the upper or lower bounds of control inputs.

Case 2: In this experiment, the step response performances are evaluated with comparison. The reference input commands ϖ_1 and ϖ_2 are step signals in this case, and the responses are shown in Figs. 6 and 7 respectively. Steady-state errors, 5%-zone setting time, and overshoots are about 0.1^o and 0.2^o, 2.82 s and 1.65 s, and 0.44% and 2.64% for the elevation and pitch channels respectively.

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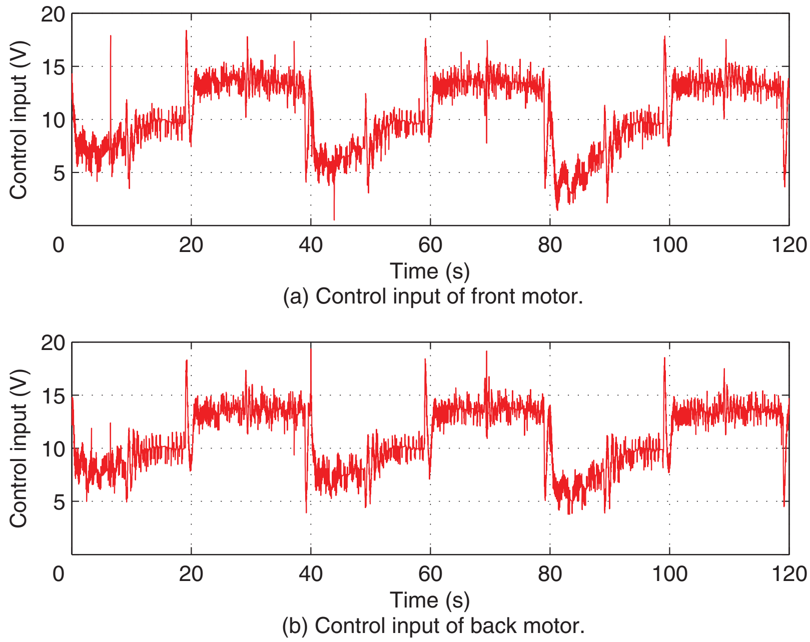

Fig. 5. Control inputs in case 1.

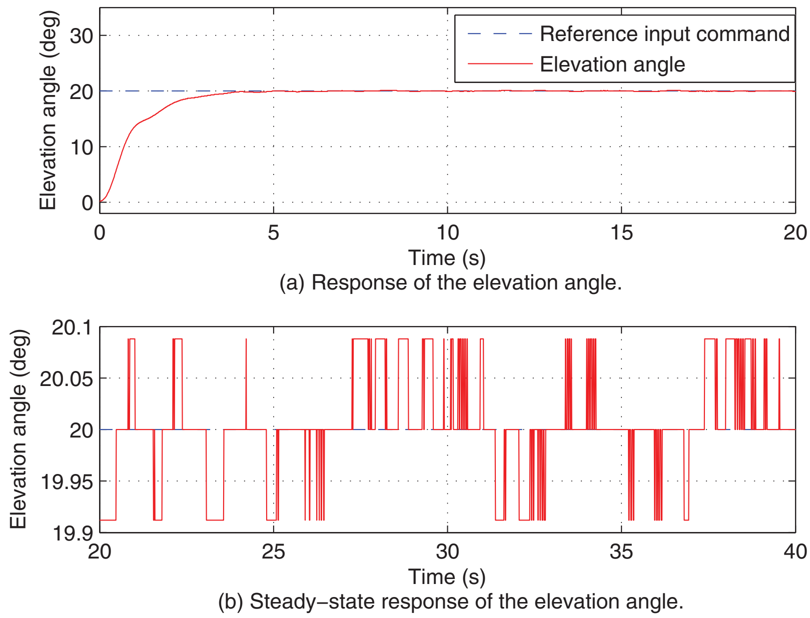

Fig. 6. Step response of the elevation channel.

In contrast, the tracking performances of 10 s settling time with 10% overshoot and 1^{o} state error with 10% overshoot were achieved for the elevation angle in refs. [14] and [21] respectively. In ref. [17], 5%-zone setting time and steady-state errors were 4.6 s and 3.8 s, and 0.2^{o} and $0.4^{o}$for the elevation channel and pitch channel respectively. One can see that both dynamical and steady-state performances are improved by our proposed control method.

Case 3: In order to check the dynamical tracking performances of the closed-loop system, a 3-DOF helicopter is required to track non-stationary sinusoidal signals for elevation and pitch angles simultaneously as shown in ref. [17]. The corresponding control input commands can be expressed

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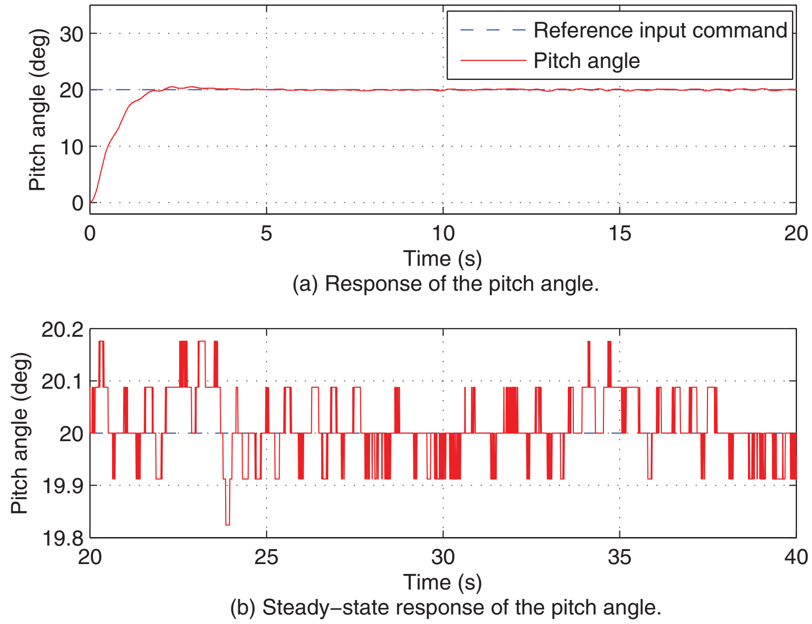

Fig. 7. Step response of the pitch channel.

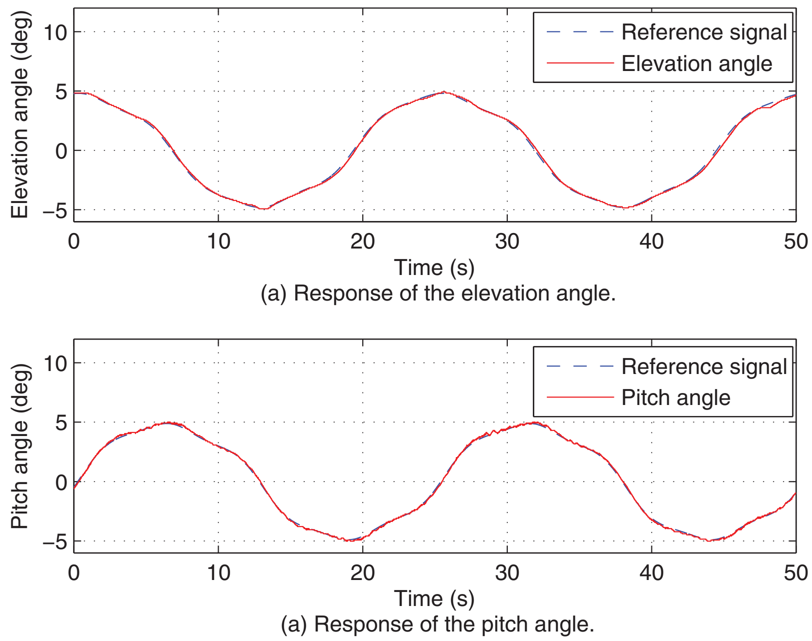

Fig. 8. Responses of two channels in case 3.

as follows:

$$\dot{\varpi}_1(t) = -(0.25 + 0.15 \sin t)\varpi_2(t),$$
$$\dot{\varpi}_2(t) = (0.25 + 0.15 \sin t)\varpi_1(t),$$

where $\varpi_1(0) = -5$ and $\varpi_2(0) = 0$. Figure 8 presents the responses of the two channels with the designed robust optimal controllers. The tracking errors and control inputs in this case are depicted in Figs. 9 and 10. One can see that the tracking errors are guaranteed to be less than 0.4° and 0.3°for the

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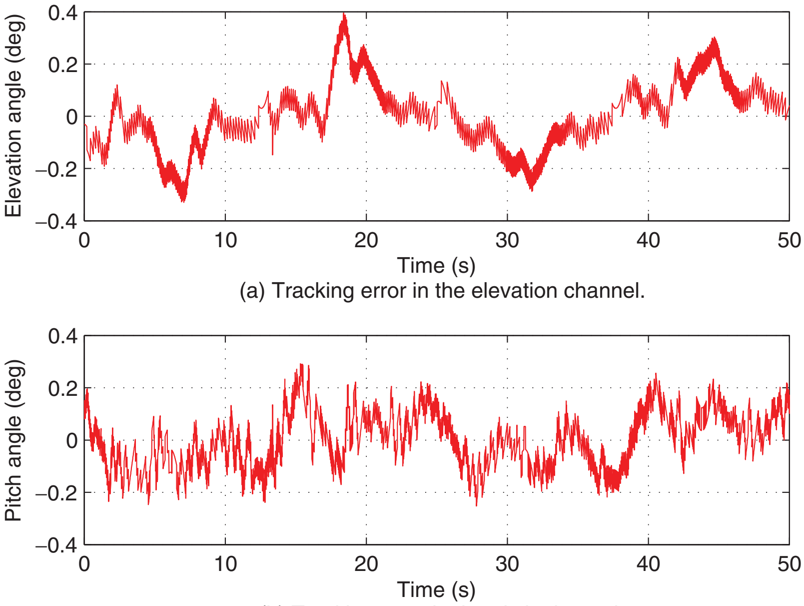
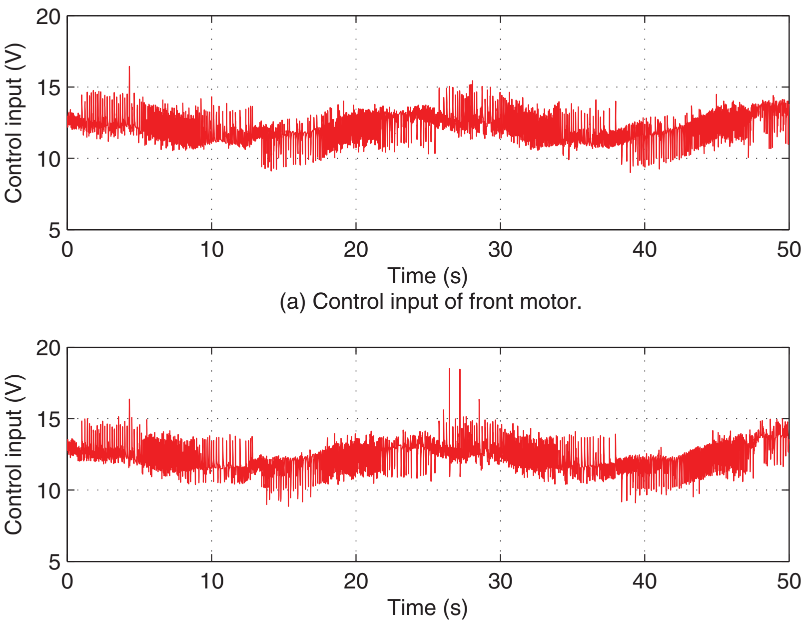


(a) Tracking error in the elevation channel.

(b) Tracking error in the pitch channel.

Fig. 9. Tracking errors in case 3.



(a) Control input of front motor.

(b) Control input of back motor.

Fig. 10. Control inputs in case 3.

elevation and pitch channels respectively. Compared with the experimental results in ref. [17], which achieved tracking errors of 0.5° and 1.1° for the two angles, this output control method improves the dynamical tracking performances of the closed-loop system.

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6. Conclusions

A robust output control approach combining the optimal method with the robust compensation technique was proposed to address the attitude control problem for a 3-DOF laboratory helicopter without angular velocity measurements. This robust control scheme results in a linear time-invariant controller consisting of an observer, a nominal optimal controller, and a robust output compensator which is easy to be implemented in practical applications. The asymptotical tracking properties and robust stability of the closed-loop system were proved. Experimental results on the laboratory helicopter showed good dynamical and steady-state tracking performances for elevation and pitch channels.

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