# An Approximation connected with $e^{-x}$

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§ 1. *Introduction.*

One of the many interesting problems discussed by Ramanujan[1] is concerned with the effect of truncating at its maximum term $n^n/n!$ the exponential series for $e^n$, where $n$ is a positive integer. When $n$ is large, the sum of the first $n$ terms is, roughly speaking, half the sum of the whole series. More precisely, Ramanujan's conjecture was that, if

$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \ldots . + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!}\theta_n = \tfrac{1}{2} e^n,$$

then $\theta_n$ lies between $\frac{1}{2}$ and $\frac{1}{3}$, and that, when $n$ is large, $\theta_n$ may be represented asymptotically by the formula

$$\theta_n \sim \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \ldots .$$

The first rigorous proofs of these results and of the fact that $\theta_n$ decreases steadily as $n$ increases, were published independently by Szegö[2] and Watson.[3]

Dr A. C. Aitken recently informed me that he possessed strong numerical evidence for the existence of similar results in connexion with $e^{-n}$. He had defined the number $\phi_n$ by the equation

$$1 - \frac{n}{1!} + \frac{n^2}{2!} - \ldots . + (-1)^n \frac{n^n}{n!}\phi_n = e^{-n},$$

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[1] *Collected Papers of Srinivasa Ramanujan* (1927), xxvi.

[2] *Journal London Math. Soc.*, 3 (1928), 225-232.

[3] *Proc. London Math. Soc.*, (2), 29 (1928), 293-308.

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and had calculated the values of $\phi_n$ given in the table below.

| $n$ | $\phi_n$ |
|---|---|
| 0 | 1·0.... |
| 1 | ·63212055.... |
| 2 | ·56766764.... |
| 8 | ·51609845.... |
| 9 | ·51425871.... |
| 10 | ·51280208.... |
| 11 | ·51161414.... |
| 12 | ·51062785.... |
| 100 | ·50125310.... |

The present note is concerned with proving three theorems which Dr Aitken had conjectured on this numerical evidence.

§ 2. *Theorem* 1. *If* $\phi_n$ *be defined by*

$$1 - \frac{n}{1!} + \frac{n^2}{2!} - \dots + (-1)^n \frac{n^n}{n!} \phi_n = e^{-n},$$

*then* $\phi_n \to \frac{1}{2}$ as $n \to \infty$.

Let us consider the sum of the exponential series for $e^{-n}$, truncated at its maximum term; it is

$$\begin{aligned} 1 - \frac{n}{1!} + \frac{n^2}{2!} - \dots + (-1)^n \frac{n^n}{n!} &= (-1)^n \frac{n^n}{n!} \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!} n^{-r} \\ &= (-1)^n \frac{n^n}{n!} \int_0^\infty e^{-u} \left(1 - \frac{u}{n}\right)^n du \\ &= (-1)^n \frac{n^n}{n!} I, \text{ say.} \end{aligned}$$

We split up the integral $I$ into two parts, *viz.*

$$\begin{aligned} I &= \int_0^n e^{-u} \left(1 - \frac{u}{n}\right)^n du + \int_n^\infty e^{-u} \left(1 - \frac{u}{n}\right)^n du \\ &= I_1 + I_2, \end{aligned}$$

and consider the behaviour of $I_1$ and $I_2$ for large values of the integer $n$.

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Now, since

$$\left(1 - \frac{u}{n}\right)^n \to e^{-u}$$

as $n \to \infty$, we have, by a formal limiting process,

$$\lim_{n \to 0} I_1 = \int_0^\infty e^{-2u}\, du = \tfrac{1}{2}.$$

That this process is valid follows from the use of Tannery's Theorem for integrals.[1] On the other hand, if we make the substitution $u = n + v$ in $I_2$, we obtain

$$I_2 = (-1)^n \frac{e^{-n}}{n^n} \int_0^\infty e^{-v} v^n\, dv = (-1)^n \frac{e^{-n}}{n^n} n!\,.$$

We have thus shewn that

$$1 - \frac{n}{1!} + \frac{n^2}{2!} - \ldots . + (-1)^n \frac{n^n}{n!} = (-1)^n \frac{n^n}{n!} I_1 + e^{-n},$$

which is the result stated in Theorem 1.

§ 3. *Theorem* 2. *$\phi_n$ decreases steadily from* 1 *to* $\frac{1}{2}$ *as the integer* $n$ *increases from* 0 *to* $\infty$.

It follows from § 2 that

$$\phi_n = 1 - \int_0^n e^{-u}\left(1 - \frac{u}{n}\right)^n du$$

$$= 1 - n\int_0^1 [e^{-v}(1 - v)]^n\, dv.$$

Now, as $v$ increases from 0 to 1, $e^{-t} = e^{-v}(1 - v)$ decreases from 1 to 0, so that $t$ increases from 0 to $\infty$. Thus[2] we have

$$\phi_n = 1 - n\int_0^\infty e^{-nt} \frac{dv}{dt}\, dt.$$

A second formula for $\phi_n$ may be obtained by integration by parts, which gives

$$\phi_n = 1 + \left[e^{-nt} \frac{dv}{dt}\right]_0^\infty - \int_0^\infty e^{-nt} \frac{d^2 v}{dt^2}\, dt.$$

But since

$$t = v - \log(1 - v),$$

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[1] Bromwich, *Infinite Series* (1926), 485.

[2] *Cf.* Watson, *loc. cit.*

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we have

$$\frac{dv}{dt} = \frac{1-v}{2-v}, \qquad \frac{d^2 v}{dt^2} = -\frac{(1-v)}{(2-v)^3};$$

hence

$$\phi_n = \tfrac{1}{2} + \int_0^\infty e^{-nt} G(t)\, dt$$

where $G(t) = (1-v)/(2-v)^3$ is positive. From this formula the monotonic character of $\phi_n$ at once follows. To complete the theorem, we note that $\phi_0$ is obviously 1, and that we have already shewn that $\phi_n \to \frac{1}{2}$.

§ 4. *Theorem* 3. *$\phi_n$ possesses the asymptotic expansion*

$$\phi_n \sim \frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{13}{256n^4} + \ldots.$$

We obtain the asymptotic expansion of

$$\phi_n = 1 - n \int_0^\infty e^{-nt} \frac{dv}{dt}\, dt$$

by applying to the integral

$$J = \int_0^\infty e^{-nt} \frac{dv}{dt}\, dt$$

the following lemma, due to Watson.[1]

*Lemma. Let $F(t)$ be analytic when $|t| \leqq a + \delta$, where $a > 0$, $\delta > 0$, save possibly for a branch point at the origin; and let*

$$F(t) = \sum_{m=1}^\infty a_m t^{(m/r)-1}$$

*when $|t| \leqq a$, $r$ being positive; also let $|F(t)| < Ke^{bt}$, where $K$ and $b$ are positive numbers independent of $t$, when $t$ is positive and $t \geqq a$. Then the asymptotic expansion*

$$\int_0^\infty e^{-\nu t} F(t)\, dt \sim \sum_{m=1}^\infty a_m \Gamma(m/r)\, \nu^{-m/r}$$

*holds when $|\nu|$ is sufficiently large and $|\arg \nu| \leqq \frac{1}{2}\pi - \Delta$, where $\Delta$ is an arbitrary positive number.*

The function $v(t)$ which occurs in the integrand of $J$ was defined to be the solution of the equation in $w$,

$$t = w - \log(1 - w),$$

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[1] *Proc. London Math. Soc.* (2), 17 (1918), 133. It also occurs in his treatise on *Bessel Functions* (1922), 236.

which vanishes at $t = 0$. By reversion of series, it can be found, with little difficulty, that

$$v = \frac{t}{2} - \frac{t^2}{16} - \frac{t^3}{192} + \frac{t^4}{3072} + \frac{13t^5}{30720} + \dots$$

is the formal power series expansion of $v$.

To determine the radius of convergence of this power series, we make use of the theory of functions of a complex variable. When $t$ is complex, $v$ is one branch of the many-valued function $w$ defined by the equation

$$t = w - \log(1 - w).$$

The singular points of $w$ are the points where $dw/dt$ is infinite or zero. But since

$$\frac{dw}{dt} = \frac{1 - w}{2 - w},$$

the singularities are the points where $w$ takes the values 1 and 2, and so are the point at infinity and the points $t = 2 \pm (2p + 1)\pi i$, where $p = 0, 1, 2, \dots$.

It follows, then, that $v(t)$ is regular when $|t| < |2 + \pi i|$, and that the power series converges there. But this implies that $dv/dt$ is also regular there, and possesses the power series expansion

$$\frac{dv}{dt} = \frac{1}{2} = \frac{t}{8} - \frac{t^2}{64} + \frac{t^3}{768} + \frac{13t^4}{6144} + \dots$$

Lastly, we observe that, when $t \geqq 0$,

$$0 \leqq \frac{dv}{dt} = \frac{1 - v}{2 - v} \leqq \frac{1}{2},$$

so that all the conditions of Watson's lemma are satisfied.

Applying the lemma, we have

$$J \sim \frac{1}{2n} - \frac{1}{8n^2} - \frac{1}{32n^3} + \frac{1}{128n^4} + \frac{13}{256n^5} + \dots,$$

and hence, since $\phi_n = 1 - nJ$,

$$\phi_n \sim \frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{13}{256n^4} + \dots$$

It is, perhaps, of interest to point out that Dr Aitken conjectured correctly the first two coefficients in this asymptotic expansion and suggested that the third was either 1/32 or 7/225.

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§ 5. Dr Aitken also observes that "Ramanujan's result, enunciated in § 1, has an application to the theory of rare frequency. It implies that, as the mean $m$ of Poisson's function $y = e^{-m} m^x / \Gamma(x+1)$ increases, the median (which is the abscissa of the ordinate which bisects the area under the curve) tends, quite rapidly, to $m - \frac{1}{6}$; so that, since the mode (the abscissa of maximum ordinate) can readily be proved to tend to $m - \frac{1}{2}$, we have here an instance of the property, observed empirically in many skew curves of frequency, that the distances of the mode, median and mean are approximately in the ratio 2 : 1."