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CENTRALIZERS IN HOUGHTON'S GROUPS

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Abstract We calculate the centralizers of elements, finite subgroups and virtually cyclic subgroups of Houghton's group H_n. We discuss various Bredon (co)homological finiteness conditions satisfied by H_n including the Bredon (co)homological dimension and $\underline{\mathrm{FP}}_n$ conditions, which are analogues of the ordinary cohomological dimension and FP_n conditions, respectively.

Keywords: Houghton's group; centralizers; finiteness conditions

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1. Introduction

Houghton's group H_n was introduced in [**11**] as an example of a group acting on a set S with $H^1(H_n, A \otimes \mathbb{Z}[S]) = A^{n-1}$ for any abelian group A.

In [**5**], Brown used an important new technique to show that the Thompson–Higman groups $F_{n,r}$, $T_{n,r}$ and $V_{n,r}$ were FP_∞. In the same paper he showed that Houghton's group H_n is interesting from the viewpoint of cohomological finiteness conditions; namely, he showed that H_n is FP_{n-1} but not FP_n. Thompson's group F was previously shown by different methods to be FP_∞ [**6**], thus providing the first known example of a torsion-free FP_∞ group with infinite cohomological dimension.

There has been recent interest in the structure of the centralizers of Thompson's groups. In [**22**] the centralizers of finite subgroups of generalizations of Thompson's groups T and V were calculated and this data was used to give information about Bredon (co)homological finiteness conditions satisfied by these groups. The results obtained in [**22**, Theorems 4.4 and 4.8] have some similarity with those obtained here. In [**2**] a description of centralizers of elements in the Thompson–Higman group V_n were given.

In § 2 we give the necessary background on Bredon (co)homological finiteness conditions. An analysis of the centralizers of finite subgroups in Houghton's group is contained in § 3. As Corollary 3.7, we obtain that centralizers of finite subgroups are FP_{n-1} but not FP_n. This should be compared with [**13**], where examples are given of soluble groups of type FP_n with centralizers of finite subgroups that are not FP_n, and also with [**21**], where it is shown that in virtually soluble groups of type FP_∞ the centralizers of all finite subgroups are of type FP_∞.

In § 4 our analysis is extended to arbitrary elements and virtually cyclic subgroups. Using this information, elements in H_n are constructed whose centralizers are FP_i for any $0 \leqslant i \leqslant n-3$. In § 5 the space that Brown constructed in [**5**] in order to prove that H_n is FP_{n-1} but not FP_n is shown to be a model for $\underline{\mathrm{E}}\, H_n$, the classifying space for proper actions of H_n. Finally, § 6 contains a discussion of Bredon (co)homological finiteness conditions that are satisfied by Houghton's group. Namely, we show in Proposition 6.1 that H_n is not quasi-$\underline{\mathrm{FP}}_0$ and in Proposition 6.3 that the Bredon cohomological dimension and the Bredon geometric dimension with respect to the family of finite subgroups are both equal to n. See § 2 for the definitions of quasi-$\underline{\mathrm{FP}}_n$ and of Bredon cohomological and geometric dimension.

Fixing a natural number $n > 1$, define *Houghton's group* H_n to be the group of permutations of $S = \mathbb{N} \times \{1, \dots, n\}$ that are 'eventually translations', i.e. for any given permutation $h \in H_n$ there are collections $\{z_1, \dots, z_n\} \in \mathbb{N}^n$ and $\{m_1, \dots, m_n\} \in \mathbb{Z}^n$ with

$$h(i, x) = (i + m_x, x) \quad \text{for all } x \in \{1, \dots, n\} \text{ and all } i \geqslant z_x. \tag{1.1}$$

Define a map ϕ as follows:

$$\phi\colon H_n \to \Big\{(m_1, \dots, m_n) \in \mathbb{Z}^n\colon \sum m_i = 0\Big\} \cong \mathbb{Z}^{n-1}, \tag{1.2}$$

$$\phi\colon h \mapsto (m_1, \dots, m_n). \tag{1.3}$$

Its kernel is exactly the permutations that are 'eventually zero' on S, i.e. the infinite symmetric group Sym_∞ (the finite support permutations of a countable set).

2. A review of Bredon (co)homological finiteness conditions

Throughout this section G is a discrete group and $\mathcal{F}$ is a family of subgroups of G that is closed under taking subgroups and conjugation. The *orbit category*, denoted by $\mathcal{O}_{\mathcal{F}}G$, is the small category whose objects are the transitive G-sets G/H for $H \in \mathcal{F}$ and whose arrows are all G-maps between them. Any G-map $G/H \to G/K$ is determined entirely by the image of the coset H in G/K and $H \mapsto xK$ determines a G-map if and only if $x^{-1}Hx \leqslant K$.

An $\mathcal{O}_{\mathcal{F}}G$-module, or *Bredon module*, is a contravariant functor from $\mathcal{O}_{\mathcal{F}}G$ to the category of abelian groups. As such, the category $\mathcal{O}_{\mathcal{F}}G$-*Mod* of $\mathcal{O}_{\mathcal{F}}G$-modules is abelian and exactness is defined pointwise: a short exact sequence

$$M' \to M \to M''$$

is exact if and only if

$$M'(G/H) \to M(G/H) \to M''(G/H)$$

is exact for all $H \in \mathcal{F}$. The category of $\mathcal{O}_{\mathcal{F}}G$-modules can be shown to have enough projectives. If Ω_1 and Ω_2 are G-sets, then we denote by $\mathbb{Z}[\Omega_1, \Omega_2]$ the free abelian group on the set of all G-maps $\Omega_1 \to \Omega_2$. If $H \in \mathcal{F}$, the $\mathcal{O}_{\mathcal{F}}G$-module $\mathbb{Z}[-, G/K]$ defined by

$$\mathbb{Z}[-, G/K](G/H) = \mathbb{Z}[G/H, G/K]$$

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is free and such modules serve as the building blocks of free $\mathcal{O}_{\mathcal{F}}G$-modules. More precisely, any free $\mathcal{O}_{\mathcal{F}}G$-module is a direct sum of such modules.

An $\mathcal{O}_{\mathcal{F}}G$-module M is said to be finitely generated if it admits an epimorphism from some free $\mathcal{O}_{\mathcal{F}}G$-module

$$\bigoplus_{i\in I}\mathbb{Z}[-,G/H_i]\twoheadrightarrow M$$

with I a finite set.

We denote by $\mathbb{Z}_{\mathcal{F}}$ the $\mathcal{O}_{\mathcal{F}}G$-module taking all objects to $\mathbb{Z}$ and all arrows to the identity map. Analogously to ordinary group cohomology, we define the Bredon *cohomological dimension* of a $\mathcal{O}_{\mathcal{F}}G$-module M to be the shortest length of a projective resolution of M by $\mathcal{O}_{\mathcal{F}}G$-modules, and the cohomological dimension of a group G to be the shortest length of a projective resolution of the $\mathcal{O}_{\mathcal{F}}G$-module $\mathbb{Z}_{\mathcal{F}}$. These two integers are denoted $\mathrm{pd}_{\mathcal{F}}\,M$ and $\mathrm{cd}_{\mathcal{F}}\,G$. If $\mathcal{F}=\mathscr{F}in$ (the family of finite subgroups), then the notation $\underline{\mathrm{cd}}\,G$ is used and if $\mathcal{F}=\mathscr{V}\mathscr{C}yc$ (the family of virtually cyclic subgroups), then the notation $\underline{\underline{\mathrm{cd}}}\,G$ is used.

The Bredon *geometric dimension* of a group G, denoted by $\mathrm{gd}_{\mathcal{F}}\,G$, is defined to be the minimal dimension of a model for $\mathrm{E}_{\mathcal{F}}\,G$. Recall that a G-CW-complex is a CW-complex with a cellular rigid G-action, where a rigid action is one where the pointwise and setwise stabilizers of all cells coincide. A model for $\mathrm{E}_{\mathcal{F}}\,G$ is defined to be a G-CW-complex X such that

$$X^H\simeq\begin{cases}\mathrm{pt} & \text{if } H\in\mathcal{F},\\ \emptyset & \text{if } H\notin\mathcal{F}.\end{cases}$$

By an application of the equivariant Whitehead theorem [**16**, Theorem 2.4], this is unique up to G-homotopy equivalence. In the case where $\mathcal{F}=\mathscr{T}riv$, the family consisting of only the trivial subgroup, a model for $\mathrm{E}_{\mathscr{T}riv}\,G$ is the universal cover $\mathrm{E}\,G$ of an Eilenberg–Mac Lane space $\mathrm{K}(G,1)$. An n-dimensional model for $\mathrm{E}_{\mathcal{F}}\,G$ gives rise to a free resolution of $\mathcal{O}_{\mathcal{F}}G$-modules C_* by setting $C_n(G/H)=K_n(X^H)$, where K_n denotes the cellular chain complex of a CW-complex. Immediately we deduce that $\mathrm{cd}_{\mathcal{F}}\,G\leqslant\mathrm{gd}_{\mathcal{F}}\,G$.

A theorem of Lück and Meintrup gives an inequality in the other direction.

Theorem (Lück and Meintrup [19, Theorem 0.1]). $\mathrm{gd}_{\mathcal{F}}\,G\leqslant\max\{\mathrm{cd}_{\mathcal{F}}\,G,3\}$.

If $\mathcal{F}=\mathscr{F}in$, we denote the geometric dimension by $\underline{\mathrm{gd}}\,G$ and if $\mathcal{F}=\mathscr{V}\mathscr{C}yc$, we denote it by $\underline{\underline{\mathrm{gd}}}\,G$. Dunwoody has shown that $\underline{\mathrm{cd}}\,G=1$ implies that $\underline{\mathrm{gd}}\,G=1$ [**8**], and hence $\underline{\mathrm{cd}}\,G=\underline{\mathrm{gd}}\,G$ unless $\underline{\mathrm{cd}}\,G=2$ and $\underline{\mathrm{gd}}\,G=3$. Brady *et al*. showed in [**3**] that this can indeed happen.

There are many groups for which good models for $\underline{\mathrm{E}}\,G$ are known; [**18**] is a good reference for these.

The $\underline{\mathrm{FP}}_n$-conditions are natural generalizations of the FP_n conditions of ordinary group cohomology. An $\mathcal{O}_{\mathcal{F}}G$-module M is $\underline{\mathrm{FP}}_n$ if it admits a projective resolution by $\mathcal{O}_{\mathscr{F}in}G$-modules that is finitely generated in all dimensions less than or equal to n. A group G is $\underline{\mathrm{FP}}_n$ if $\mathbb{Z}_{\mathscr{F}in}$ is $\underline{\mathrm{FP}}_n$.

The following lemma details an alternative algebraic description of the condition $\underline{\mathrm{FP}}_n$ that is easier to calculate.

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Proposition (Kropholler *et al.* [14, Lemmas 3.1 and 3.2]).

(1) *G is $\underline{\mathrm{FP}}_0$ if and only if G has finitely many conjugacy classes of finite subgroups.*

(2) *An $\mathcal{O}_{\mathcal{F}}G$-module M is $\underline{\mathrm{FP}}_n$ ($n \geqslant 1$) if and only if G is $\underline{\mathrm{FP}}_0$ and $M(G/K)$ is of type FP_n over the Weyl group $WK = N_G K/K$ for all finite subgroups $K \leqslant G$.*

Corollary. *The following are equivalent for a group G:*

(1) *G is $\underline{\mathrm{FP}}_n$;*

(2) *G is $\underline{\mathrm{FP}}_0$ and the Weyl groups $WK = N_G K/K$ are FP_n for all finite subgroups K;*

(3) *G is $\underline{\mathrm{FP}}_0$ and the centralizers $C_G K$ are FP_n for all finite subgroups K.*

Proof. By the previous proposition, (1) and (2) are equivalent. To see the equivalence of (2) and (3), consider the short exact sequence

$$0 \to K \to N_G K \to WK \to 0.$$

K is finite and hence FP_∞, so WK is FP_n if and only if $N_G K$ is FP_n [**1**, Proposition 2.7]. K is finite, so $C_G K$ is of finite index in $N_G K$ [**23**, 1.6.13] and, as such, $C_G K$ is FP_n if and only if $N_G K$ is FP_n. Hence, WK is FP_n if and only if $C_G K$ is FP_n. □

The condition that a group G has only finitely many conjugacy classes of finite subgroups is extremely strong. In [**22**] the weaker condition quasi-$\underline{\mathrm{FP}}_n$ is introduced.

Definition 2.1.

(1) G is quasi-$\underline{\mathrm{FP}}_0$ if and only if there are finitely many conjugacy classes of finite subgroups isomorphic to a given finite subgroup.

(2) G is quasi-$\underline{\mathrm{FP}}_n$ if and only if G is quasi-$\underline{\mathrm{FP}}_0$ and WK is FP_n for every finite $K \leqslant G$.

Many results about finiteness in ordinary group cohomology carry over into the Bredon case. For example, in [**22**, § 5] versions of the Bieri–Eckmann criterion for both $\underline{\mathrm{FP}}_n$ and quasi-$\underline{\mathrm{FP}}_n$ $\mathcal{O}_{\mathcal{F}}G$-modules were shown to hold (see [**1**, § 1.3] for the classical case).

3. Centralizers of finite subgroups in H_n

First we recall some properties of group actions on sets before narrowing our focus to Houghton's group.

Proposition 3.1. *If G is a group acting on a countable set X and H is any subgroup of G, then the following hold.*

(1) *If x and y are in the same G-orbit, then their isotropy subgroups G_x and G_y are G-conjugate.*

(2) *If $g \in C_G(H)$, then $H_{gx} = H_x$ for all $x \in X$.*

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(3) *Partition X into $\{X_a\}_{a=1}^t$, where $t \in \mathbb{N} \cup \{\infty\}$, via the equivalence relation $\sim$, where $x \sim y$ if and only if H_x is H-conjugate to H_y. Any two points in the same H-orbit will lie in the same partition and any $c \in C_G(H)$ maps X_a onto X_a for all a.*

(4) *Let G act faithfully on X with the property that, for all $g \in G$ and $X_a \subseteq X$ as in the previous section, there exists a group element $g_a \in G$ that fixes $X \setminus X_a$ and acts as g does on X_a. Then $C_G(H) = C_1 \times \cdots \times C_t$, where C_a is the subgroup of $C_G(H)$ acting trivially on $X \setminus X_a$.*

Proof. (1) and (2) are standard results.

(3) This follows immediately from (1) and (2).

(4) This follows from (3) and our new assumption on G. Let $c \in C_G(H)$ and c_a be the element given by the assumption. Since the action of G on X is faithful, c_a is necessarily unique. That the action is faithful also implies that $c = c_1 \cdots c_t$ and that any two c_a and c_b commute in G because they act non-trivially only on distinct X_a. Thus we have the necessary isomorphism $C_G(H) \to C_1 \times \cdots \times C_t$. □

Let $Q \leqslant H_n$ be a finite subgroup of Houghton's group H_n and let $S_Q = S \setminus S^Q$ be the set of points of S that are *not fixed* by Q. The finiteness of Q implies that $\phi(Q) = 0$ since any element q with $\phi(q) \neq 0$ necessarily has infinite order. For every $q \in Q$ there exists $\{z_1, \dots, z_n\} \in \mathbb{N}^n$ such that

$$q(i, x) = (i, x) \quad \text{if } i \geqslant z_x.$$

Taking z'_x to be the maximum of these z_x over all elements in Q, it follows that Q must fix the set $\{(i, x) \colon i \geqslant z'_x\}$ and, in particular, $S_Q \subseteq \{(i, x) \colon i < z'_x\}$ is finite.

We need to see that the subgroup $Q \leqslant H_n$ acting on the set S satisfies the conditions of Proposition 3.1 (4). We give the following lemma in more generality than is needed here as it will be useful later on. That the action is faithful is automatic since an element $h \in H_n$ is uniquely determined by its action on the set S.

Lemma 3.2. *Let $Q \leqslant H_n$ be a subgroup that is either finite or of the form $F \rtimes \mathbb{Z}$ for F a finite subgroup of H_n. Partition S with respect to Q into sets $\{S_a\}_{a=1}^t$ as in Proposition 3.1 (3) applied to the action of H_n on S and the subgroup Q of H_n. Then the conditions of Proposition 3.1 (4) are satisfied.*

Proof. Fix $a \in \{1, \dots, t\}$ and let h_a denote the permutation of S that fixes $S \setminus S_a$ and acts as h does on S_a. We wish to show that h_a is an element of H_n.

There are only finitely many elements in Q with finite order so, as in the argument just before this lemma, we may choose integers z_x for $x \in \{1, \dots, n\}$ such that if q is a finite-order element of Q, then $q(i, x) = (i, x)$ whenever $i \geqslant z_x$. If Q is a finite group, then one of the following statements holds.

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- S_a is fixed by Q, in which case

$$\{(i,x)\colon i \geqslant z_x,\ x \in \{1,\dots,n\}\} \subseteq S_a$$

so $h_a(i,x) = h(i,x)$ for all $i \geqslant z_x$. In particular, for large enough i, h_a acts as a translation on (i,x) and is hence an element of H_n.

- S_a is not fixed by Q, in which case

$$S_a \subseteq \{(i,x)\colon i < z_x,\ x \in \{1,\dots,n\}\}.$$

In particular, S_a is finite and $h_a(i,x) = (i,x)$ for all $i \geqslant z_x$. Hence, h_a is an element of H_n.

It remains to treat the case in which $Q = F \rtimes \mathbb{Z}$. Write w for a generator of $\mathbb{Z}$ in $F \rtimes \mathbb{Z}$. By choosing a larger z_x if needed, we may assume that w acts either trivially or as a translation on (i,x) whenever $i \geqslant z_x$. Hence, for any $x \in \{1,\dots,n\}$, the isotropy group in Q of $\{(i,x)\colon i \geqslant z_x\}$ is either F or Q.

If S_a has isotropy group Q or F, then for some $x \in \{1,\dots,n\}$, one of the following statements holds.

- We have

$$S_a \cap \{(i,x)\colon i \geqslant z_x\} = \{(i,x)\colon i \geqslant z_x\},$$

in which case $h_a(i,x) = h(i,x)$ for $i \geqslant z_x$. In particular, for large enough i, h_a acts as a translation on (i,x) and hence is an element of H_n.

- We have

$$S_a \cap \{(i,x)\colon i \geqslant z_x\} = \emptyset,$$

in which case $h_a(i,x) = (i,x)$ for $i \geqslant z_x$. In particular, for large enough i, h_a fixes (i,x) and hence is an element of H_n.

If S_a is the set corresponding to an isotropy group not equal to F or Q, then

$$S_a \subseteq \{(i,x)\colon i \geqslant z_x,\ x \in \{1,\dots,n\}\}.$$

It follows that h_a fixes (i,x) for $i \geqslant z_x$, and hence h_a is an element of H_n. □

Partition S into disjoint sets according to the Q-conjugacy classes of the stabilizers, as in Proposition 3.1 (3). The set with isotropy in Q equal to Q is S^Q and since S_Q is finite the partition is finite, and thus

$$S = S^Q \cup S_1 \cup \cdots \cup S_t.$$

Proposition 3.1 (4) gives that

$$C_{H_n}(Q) = H_n|_{S^Q} \times C_1 \times \cdots \times C_t,$$

where each C_a acts only on S_a and leaves S^Q and S_b fixed for $a \neq b$ (where $a, b \in \{1, \dots, t\}$). The first element of the direct product decomposition is the subgroup of $C_{H_n}(Q)$ acting only on S^Q and leaving $S \setminus S^Q$ fixed. This is $H_n|_{S^Q}$ (H_n restricted to S^Q) because, as the action of Q on S^Q is trivial, any permutation of S^Q will centralize Q. Choose a bijection $S^Q \to S$ such that, for all x, $(i, x) \mapsto (i + m_x, x)$ for large enough i and some $m_x \in \mathbb{Z}$; this induces an isomorphism between $H_n|_{S^Q}$ and H_n.

To give an explicit definition of the group C_a we need three lemmas.

Lemma 3.3. *C_a is isomorphic to the group T of Q-set automorphisms of S_a.*

Proof. An element $c \in C_a$ determines a Q-set automorphism of S_a, giving a map $C_a \to T$. Since the action of C_a on S_a is faithful, this map is injective. Any Q-set automorphism α of S_a may be extended to a Q-set automorphism of S, where α acts trivially on $S \setminus S_a$. Since S_a is a finite set, α acts trivially on (i, x) for large enough i and any $x \in \{1, \dots, n\}$, and hence α is an element of H_n. Finally, since α is a Q-set automorphism, $q\alpha s = \alpha q s$ (equivalently $\alpha^{-1} q \alpha s = s$ for all $s \in S$ and $q \in Q$) showing that $\alpha \in C_a$ and so the map $C_a \to T$ is surjective. □

Lemma 3.4. *S_a is Q-set isomorphic to the disjoint union of r copies of Q/Q_a, where Q_a is an isotropy group of S_a and $r = |S_a|/|Q : Q_a|$.*

Proof. S_a is finite and so splits as a disjoint union of finitely many Q-orbits. Choose orbit representatives $\{s_1, \dots, s_r\} \subset S_a$ for these orbits. These s_k may be chosen to have the same Q-stabilizers: if $Q_{s_1} \neq Q_{s_2}$, then there is some $q \in Q$ such that $Q_{qs_2} = qQ_{s_2}q^{-1} = Q_{s_1}$ (the partitions S_a were chosen to have this property by Proposition 3.1). Iterating this procedure, we get a set of representatives who all have isotropy group Q_{s_1}. Now set $Q_a = Q_{s_1}$ and note that there are $|Q : Q_a|$ elements in each of the Q-orbits, so $r|Q : Q_a| = |S_a|$. □

Recall that if G is any group and $r \geqslant 1$ is some natural number, then the wreath product $G \wr \mathrm{Sym}_r$ is the semi-direct product

$$G \wr \mathrm{Sym}_r = \prod_{k=1}^{r} G \rtimes \mathrm{Sym}_r,$$

where the symmetric group Sym_r acts by permuting the factors in the direct product.

Recall also that for any subgroup H of a group G, the Weyl group $W_G H$ is defined to be $W_G H = N_G H / H$.

Lemma 3.5. *The group C_a is isomorphic to the wreath product $W_Q Q_a \wr \mathrm{Sym}_r$, where Q_a is some isotropy group of S_a and $r = |S_a|/|Q : Q_a|$.*

Proof. Using Lemmas 3.3 and 3.4, C_a is isomorphic to the group of Q-set automorphisms of the disjoint union of r copies of Q/Q_a.

To begin, we show that the group of automorphisms of the Q-set Q/Q_a is isomorphic to $W_Q Q_a$. An automorphism $\alpha\colon Q/Q_a \to Q/Q_a$ is determined by the image $\alpha(Q_a) = qQ_a$ of the identity coset and such an element determines an automorphism if and only if

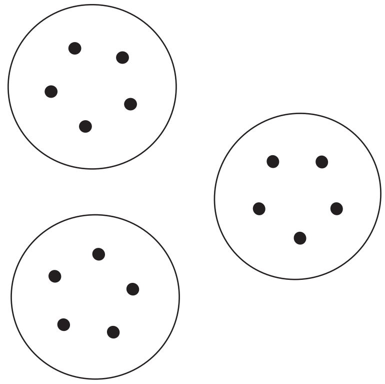

Figure 1. A representation of S_a. The large circles are the sets $\{Qs_1, \ldots, Qs_r\}$ (in this figure $r = 3$). Elements of Sym_r permute only the large circles, while elements of $\prod_{k=1}^{r} W_Q Q_a$ leave the large circles fixed and permute only elements inside them.

$q^{-1}Q_a q \leqslant Q_a$, or equivalently $q \in N_Q Q_a$. Since two elements $q_1, q_2 \in Q$ will determine the same automorphism if and only if $q_1 Q_a = q_2 Q_a$, the group of Q-set automorphisms of Q/Q_a is the Weyl group $W_Q Q_a$.

For the general case, note that if $c \in C_a$, then c permutes the Q-orbits $\{Qs_1, \ldots, Qs_r\}$, so there is a map $C_a \to \mathrm{Sym}_r$. Assume that the representatives $\{s_1, \ldots, s_r\}$ have been chosen, as in the proof of Lemma 3.4, to have the same Q-stabilizers. The map π is split by the map

$$\begin{aligned} \iota\colon \mathrm{Sym}_r &\to C_a, \\ \sigma &\mapsto (\iota(\sigma)\colon qs_k \mapsto qs_{\sigma(k)} \text{ for all } q \in Q). \end{aligned}$$

Each $\iota(\sigma)$ is a well-defined element of H_n since

$$qs_k = \tilde{q}s_k \iff \tilde{q}^{-1}q \in Q_{s_k} = Q_{s_{\sigma(k)}} \iff qs_{\sigma(k)} = \tilde{q}s_{\sigma(k)}.$$

The kernel of the map π is exactly the elements of C_a that fix each Q-orbit but may permute the elements inside the Q-orbits; by the previous part, this is exactly $\prod_{k=1}^{r} W_Q Q_a$. For any $\sigma \in \mathrm{Sym}_r$, the element $\iota(\sigma)$ acts on $\prod_{k=1}^{r} W_Q Q_a$ by permuting the factors, so the group C_a is indeed isomorphic to the wreath product. □

The centralizer $C_{H_n} Q$ can now be completely described.

Proposition 3.6. *The centralizer $C_{H_n}(Q)$ of any finite subgroup $Q \leqslant H_n$ splits as a direct product*

$$C_{H_n}(Q) \cong H_n|_{S^Q} \times C_1 \times \cdots \times C_t,$$

where $H_n|_{S^Q} \cong H_n$ is Houghton's group restricted to S^Q and, for all $a \in \{1, \ldots, t\}$,

$$C_a \cong W_Q Q_a \wr \mathrm{Sym}_r$$

for Q_a an isotropy group of S_a and $r = |S_a|/|Q : Q_a|$. In particular, H_n has finite index in $C_{H_n}(Q)$.

Proof. We have already proven that

$$C_{H_n}(Q) \cong H_n|_{S^Q} \times C_1 \times \cdots \times C_t$$

and Lemma 3.5 gives the required description of C_a. □

Corollary 3.7. *If Q is a finite subgroup of H_n, then the centralizer $C_{H_n}(Q)$ is* FP_{n-1} *but not* FP_n.

Proof. H_n is finite index in the centralizer $C_{H_n}(Q)$, by Proposition 3.6. Appealing to Brown's result [**5**, Theorem 5.1] that H_n is FP_{n-1} but not FP_n, and that a group is FP_n if and only if a finite index subgroup is FP_n [**4**, Proposition VIII.5.5.1], we deduce that $C_{H_n}(Q)$ is FP_{n-1} but not FP_n. □

4. Centralizers of elements in H_n

If $q \in H_n$ is an element of finite order, then the subgroup $Q = \langle q \rangle$ is a finite subgroup and the previous section may be used to describe the centralizer $C_{H_n}(q) = C_{H_n}(Q)$. Thus, for an element q of finite order, $C_{H_n}(q) \cong C \times H_n$ for some finite group C.

If $q \in H_n$ is an element of infinite order and $Q = \langle q \rangle$, then we may apply Proposition 3.1 (3) to split up S into a disjoint collection $\{S_a : a \in A \subseteq \mathbb{N}\} \cup S^Q$ (S^Q is the element of the collection associated with the isotropy group Q). Assume that S_0 is the set associated with the trivial isotropy group. Since q is a translation on $(i, x) \in S = \mathbb{N} \times \{1, \ldots, n\}$ for large enough i, and points acted on by such a translation have trivial isotropy, there are only finitely many elements of S whose isotropy group is neither the trivial group nor Q. Hence, S_a is finite for $a \neq 0$ and the set A is finite. From now on let $A = \{0, \ldots, t\}$. We now use Lemma 3.2 and Proposition 3.1 (4) as in the previous section: $C_{H_n}(Q)$ splits as

$$C_{H_n}(Q) \cong C_0 \times C_1 \times \cdots \times C_t \times H_n|_{S^Q},$$

where C_a acts only on S_a and $H_n|_{S^Q}$ is Houghton's group restricted to S^Q. Unlike in the last section, $H_n|_{S^Q}$ may not be isomorphic to H_n. Let $J \subseteq \{1, \ldots, n\}$ satisfy

$$x \in J \text{ if and only if } (i, x) \in S^Q \text{ for all } i \geqslant z_x, \text{ some } z_x \in \mathbb{N}.$$

If $x \notin J$, then, for large enough i, q must act as a non-trivial translation on (i, x) and the set $(\mathbb{N} \times \{x\}) \cap S^Q$ is finite. Clearly, $|J| \leqslant n-2$, but different elements q may give values $0 \leqslant |J| \leqslant n-2$. In the case $|J| = 0$, S^Q is necessarily finite and so $H_n|_{S^Q}$ is isomorphic to a finite symmetric group on S^Q. It is also possible that $S^Q = \emptyset$, in which case $H_n|_{S^Q}$ is just the trivial group. If $|J| \neq 0$, then the argument proceeds as in the previous section by choosing a bijection

$$S^Q \to \mathbb{N} \times J$$

such that $(i, x) \mapsto (i + m_x, x)$ for some $m_x \in \mathbb{Z}$ whenever i is large enough and $x \in J$. This set map induces a group isomorphism between $H_n|_{S^Q}$ and $H_{|J|}$ (Houghton's group on the set $J \times \mathbb{N}$).

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Lemma 3.5 describes the groups C_a for $a \neq 0$, so it remains only to treat the case $a = 0$. We cannot use the arguments used for $a \neq 0$ here as the set S_0 is not finite. In particular, Lemma 3.3 does not apply: every Q-set isomorphism of S_0 is realized by an element of the infinite support permutation group on S_0, but there are Q-set isomorphisms of S_0 that are not realized by an element of H_n.

The next three lemmas are needed to describe C_0 and this description will use the graph Γ, which we now describe. The vertices of Γ are those $x \in \{1, \dots, n\}$ for which q acts non-trivially on infinitely many elements of $\mathbb{N} \times \{x\}$. Equivalently, the vertices are the elements of $\{1, \dots, n\} \setminus J$. There is an edge from x to y in Γ if there exists $s \in S_0$ and $N \in \mathbb{N}$ such that, for all $m \geqslant N$, we have $q^{-m}s \in \mathbb{N} \times \{x\}$ and $q^m s \in \mathbb{N} \times \{y\}$. Let $\pi_0\Gamma$ denote the path components of Γ and for any vertex x of Γ denote by $[x]$ the element of $\pi_0\Gamma$ corresponding to that vertex.

Let $z \in \mathbb{N}$ be some integer such that, for all $i \geqslant z$, q acts trivially or as a translation on (i, x) for all $x \in \{1, \dots, n\}$. Fix z for the remainder of this section.

For each path component $[x]$ in $\pi_0\Gamma$, let $S_0^{[x]}$ denote the smallest Q-subset of S_0 containing the set $\{(i, y)\colon i \geqslant z,\ y \in [x]\}$. Note that $(i, y) \notin S_0^{[x]}$ for any $y \notin [x]$ and $i \geqslant z$ since if (i, x) and (j, y) are two elements of S_0 in the same Q-orbit with $i \geqslant z$ and $j \geqslant z$, then there is an edge between x and y in Γ: if $(i, x) = q^k(j, y)$ and q acts as a positive translation on the element (i, x), then let $N = k$ and $s = (i, x)$, and similarly for when q acts as a negative translation. This gives a Q-set decomposition of S_0 as

$$S_0 = \coprod_{[x] \in \pi_0\Gamma} S_0^{[x]},$$

where $\coprod$ denotes disjoint union.

Lemma 4.1. *Let $[x] \in \pi_0\Gamma$. If $C_0^{[x]}$ denotes the subgroup of C_0 that acts non-trivially only on $S_0^{[x]}$, then there is an isomorphism*

$$C_0 \cong C_0^{[x_1]} \times \cdots \times C_0^{[x_r]},$$

where $[x_1], [x_2], \dots, [x_r]$ are all elements of $\pi_0\Gamma$.

Proof. If $c \in C_0$ and $[x] \in \pi_0\Gamma$, then let $c_{[x]}$ denote the permutation of S such that $c_{[x]}$ acts as c does on $S_0^{[x]}$, and acts trivially on $S \setminus S_0^{[x]}$. We will show that $c_{[x]}$ is an element of C_0. Since the action of C_0 on S_0 is faithful, it follows that the elements $c_{[x]}$ and $c_{[y]}$ commute and

$$c = c_{[x_1]} c_{[x_2]} \cdots c_{[x_r]},$$

which suffices to prove the lemma.

Let $y \in \{1, \dots, n\}$. The element $c_{[x]}$ acts trivially on (i, y) for $i \geqslant z$ if $y \notin [x]$, and acts as c does on (i, y) for $i \geqslant z$ if $y \in [x]$. Thus, $c_{[x]}$ is an element of H_n. Since $c_{[x]}$ is also a Q-set automorphism of S, $c_{[x]}$ is a member of C_0. □

Lemma 4.2. *Let $[x] \in \pi_0\Gamma$, $c \in C_0$ and let $z' \in \mathbb{N}$ such that c acts either trivially or as a translation on (i, x) for all $x \in \{1, \dots, n\}$ and $i \geqslant z'$. The action of c on some element $(i, x) \in S$ for $i \geqslant z'$ then completely determines the action of c on $S_0^{[x]}$.*

Proof. Firstly, note that knowing the action of c on some element (i, x) for $i \geqslant z'$ determines the action of c on the set $\{(i, x)\colon i \geqslant z'\}$ since we chose z' in order to have this property.

Let $y \in [x]$ such that there is an edge from x to y and so there is a natural number N and element $s \in S_0^{[x]}$ such that $q^N s = (i, x)$ and $q^{-N} s = (j, y)$ for some natural numbers i and j. By choosing N larger if necessary, we can take $i, j \geqslant z'$. The action of c on (j, y) is now completely determined by the action on (i, x) since

$$c(j, y) = cq^{-2N}(i, x) = q^{-2N}c(i, x).$$

For any $y \in [x]$ there is a path from x to y in Γ, so we have determined the action of c on the set $X = \{(j, y)\colon j \geqslant z',\ y \in [x]\}$. If $s \in S_0^{[x]} \setminus X$, then, since $S_0^{[x]} \setminus X$ is finite, there is some integer m with $q^m s = x \in X$. So $cs = cq^{-m}x = q^{-m}cx$, which completely determines the action of c on s. □

Lemma 4.3. *For any $[x] \in \pi_0\Gamma$ there is an isomorphism*

$$C_0^{[x]} \cong \mathbb{Z}.$$

Proof. By Lemma 4.2 the action is completely determined by the action on some element (i, x) for large enough i, and the action on this element is necessarily by translation by some element $m_x(c)$. This defines an injective homomorphism $C_0^{[x]} \to \mathbb{Z}$, sending $c \mapsto m_x(c)$. Let $q_{[x]}$ be the element of $C_0^{[x]}$ described in the proof of Lemma 4.1: $q_{[x]}$ is a non-trivial element of $C_0^{[x]}$, so $C_0^{[x]}$ is mapped isomorphically onto a non-trivial subgroup of $\mathbb{Z}$. □

Combining Lemmas 4.1 and 4.3 shows that $C_0 \cong \mathbb{Z}^r$, where $r = |\pi_0\Gamma|$.

Recall that the vertices of Γ are indexed by the set $\{1, \dots, n\} \setminus J$. Since there are no isolated vertices in Γ, $|\pi_0\Gamma| \leqslant \lfloor (n - |J|)/2 \rfloor$ (where $\lfloor\cdot\rfloor$ denotes the integer floor function). Recalling that $0 \leqslant |J| \leqslant n - 2$, the set $\{1, \dots, n\} \setminus J$ is necessarily non-empty, so $1 \leqslant |\pi_0\Gamma|$. Combining these gives

$$1 \leqslant |\pi_0\Gamma| \leqslant \lfloor (n - |J|)/2 \rfloor.$$

We can now completely describe the centralizer $C_{H_n}(q)$.

Theorem 4.4.

(1) *If $q \in H_n$ is an element of finite order, then*

$$C_{H_n}(q) \cong H_n|_{S^Q} \times C_1 \times \cdots \times C_t,$$

where $H_n|_{S^Q} \cong H_n$ is Houghton's group restricted to S^Q and, for all $a \in \{1, \dots, t\}$,

$$C_a \cong W_Q Q_a \wr \mathrm{Sym}_r$$

for Q_a an isotropy group of S_a and $r = |S_a|/|Q : Q_a|$. In particular, H_n is finite index in $C_{H_n}Q$.

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(2) *If $q \in H_n$ is an element of infinite order, then either*

$$C_{H_n}(q) \cong H_k \times \mathbb{Z}^r \times C_1 \times \cdots \times C_t$$

or

$$C_{H_n}(q) \cong F \times \mathbb{Z}^r \times C_1 \times \cdots \times C_t,$$

where F is some finite symmetric group, H_k is Houghton's group with $0 \leqslant k \leqslant n-2$, and the groups C_a are as in the previous part. In the first case $1 \leqslant r \leqslant \lfloor (n-k)/2 \rfloor$ and in the second case $1 \leqslant r \leqslant \lfloor n/2 \rfloor$.

In Corollary 3.7 it was proved that for an element q of finite order, $C_{H_n}(q)$ is FP_{n-1} but not FP_n. The situation is much worse for elements q of infinite order, in which case the centralizer may not even be finitely generated. For example, when n is odd and q is the element acting on $S = \mathbb{N} \times \{1, \ldots, n\}$ as

$$q\colon \begin{cases} (i,x) \mapsto (i+1,x) & \text{if } x \leqslant (n-1)/2, \\ (i,x) \mapsto (i-1,x) & \text{if } (n+1)/2 \leqslant x \leqslant n-1 \text{ and } i \neq 0, \\ (0,x) \mapsto (0, x-((n-1)/2)) & \text{if } (n+1)/2 \leqslant x \leqslant n-1, \\ (i,n) \mapsto (i,n), \end{cases}$$

the only fixed points are on the ray $\mathbb{N} \times \{n\}$. The argument leading up to Theorem 4.4 shows that the centralizer is a direct product of groups, one of which is Houghton's group H_1, which is isomorphic to the infinite symmetric group and hence is not finitely generated. In particular, for this q, the centralizer $C_{H_n}(q)$ is not even FP_1. A similar example can easily be constructed for the case in which n is even.

All the groups in the direct product decomposition from Theorem 4.4 except H_k are FP_∞, being built by extensions from finite groups and free abelian groups. By choosing various infinite-order elements q, for example by modifying the example of the previous paragraph, the centralizers can be chosen to be FP_k for $0 \leqslant k \leqslant n-3$. The upper bound of $n-3$ arises because any infinite-order element q must necessarily be 'eventually a translation' (in the sense of (1.1)) on $\mathbb{N} \times \{x\}$ for *at least* two x. As such, the copy of Houghton's group in the centralizer can act on at most $n-2$ rays and is thus at largest H_{n-2}, which is FP_{n-3}.

Corollary 4.5. *If Q is an infinite virtually cyclic subgroup of H_n, then either*

$$C_{H_n}(Q) \cong H_k \times \mathbb{Z}^r \times C_1 \times \cdots \times C_t$$

or

$$C_{H_n}(Q) \cong F \times \mathbb{Z}^r \times C_1 \times \cdots \times C_t,$$

where the elements in the decomposition are all as in Theorem 4.4.

This corollary can be proved by reducing to the case of Theorem 4.4, but before that we require the following lemma.

Lemma 4.6. *Every infinite virtually cyclic subgroup Q of H_n is finite-by-$\mathbb{Z}$.*

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Proof. By [**12**, Proposition 4], Q is either finite-by-$\mathbb{Z}$ or finite-by-D_∞, where D_∞ denotes the infinite dihedral group. We show that the latter cannot occur. Assume that there is a short exact sequence

$$0 \to F \hookrightarrow Q \xrightarrow{\pi} \mathrm{D}_\infty \to 0,$$

where F is regarded as a subgroup of Q. Let a and b generate D_∞ so that

$$\mathrm{D}_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

Let $p, q \in Q$ be lifts of a and b such that $\pi(p) = a$ and $\pi(q) = b$; then $p^2 \in F$. Since F is finite, p^2 has finite order and hence p has finite order. The same argument shows that q has finite order. $pq \in Q$ necessarily has infinite order as $\pi(pq)$ is of infinite order in D_∞.

However, since p and q are finite-order elements of H_n, by the argument at the beginning of § 3, they both permute only a finite subset of S. Thus, pq permutes a finite subset of S and is of finite order, but this contradicts the previous paragraph. □

Proof of Corollary 4.5. Using the previous lemma, write Q as $Q = F \rtimes \mathbb{Z}$, where F is a finite group. As F is finite, the set S_F of points not fixed by F is finite (see the argument at the beginning of § 3). Let $z \in \mathbb{N}$ be such that, for $i \geqslant z$, F acts trivially on (i, x) for all x and $\mathbb{Z}$ acts on (i, x) either trivially or as a translation. Applying Lemma 3.2 and Proposition 3.1, S splits as a disjoint union

$$S = S^Q \cup S_0 \cup S_1 \cup \cdots \cup S_t,$$

where S^Q is the fixed-point set, S_0 is the set with isotropy group F and the S_a for $1 \leqslant a \leqslant t$ are subsets of $\{(i, x) \colon i \leqslant z\}$, and hence are all finite. By Proposition 3.1, $C_{H_n}(Q)$ splits as a direct product

$$C = H_n|_{S^Q} \times C_0 \times C_1 \times \cdots \times C_t,$$

where $H_n|_{S^Q}$ denotes Houghton's group restricted to S^Q. The argument of Theorem 4.4 showing that $H_n|_{S^Q}$ is isomorphic to either a finite symmetric group or to H_k for some $0 \leqslant k \leqslant n-2$ applies without change, as does the proof of the structure of the groups C_a for $1 \leqslant a \leqslant t$. It remains to observe that because every element in S_0 is fixed by F, any element of H_n centralizing $\mathbb{Z}$ and fixing $S \setminus S_0$ necessarily also centralizes Q, and is thus a member of C_0. This reduces us again to the case of Theorem 4.4, showing that $C_0 \cong \mathbb{Z}^r$ for some natural number $1 \leqslant r \leqslant \lfloor (n-k)/2 \rfloor$, or $1 \leqslant r \leqslant \lfloor n/2 \rfloor$ if $H_n|_{S^Q}$ is a finite symmetric group. □

5. Brown's model for $\underline{\mathrm{E}}H_n$

The main result of this section will be Corollary 5.5, where the construction of Brown [**5**] that is used to prove that H_n is FP_{n-1} but not FP_n is shown to be a model for $\underline{\mathrm{E}}\, H_n$.

Remark 5.1. Since the main objects of study in this section are monoids, maps are written from left to right.

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Write $\mathcal{M}$ for the monoid of injective maps $S \to S$ with the property that every permutation is 'eventually a translation' (in the sense of (1.1)) and write T for the free monoid generated by $\{t_1, \dots, t_n\}$, where

$$(i,x)t_y = \begin{cases}(i+1,x) & \text{if } x = y,\\ (i,x) & \text{if } x \neq y.\end{cases}$$

The elements of T will be called *translations*. The map $\phi\colon H_n \to \mathbb{Z}^n$, defined in (1.2), extends naturally to a map $\phi\colon \mathcal{M} \to \mathbb{Z}^n$. Give $\mathcal{M}$ a poset structure by setting $\alpha \leqslant \beta$ if $\beta = t\alpha$ for some $t \in T$. The monoid $\mathcal{M}$ can be given the obvious action on the right by H_n, which in turn gives an action of H_n on the poset $(\mathcal{M}, \leqslant)$ since $\beta = t\alpha$ implies that $\beta h = t\alpha h$ for all $h \in H_n$. Let $|\mathcal{M}|$ be the geometric realization of this poset; namely, simplicies in $|\mathcal{M}|$ are finite ordered collections of elements in $\mathcal{M}$ with the obvious face maps. An element $h \in H_n$ fixes a vertex $\{\alpha\} \in |\mathcal{M}|$ if and only if $s\alpha h = s\alpha$, which is true if and only if h fixes $S\alpha$. Thus, the stabilizer $(H_n)_\alpha$ may only permute the finite set $S \setminus S\alpha$ and we may deduce the following proposition.

Proposition 5.2. *Stabilizers of simplicies in $|\mathcal{M}|$ are finite.*

We now build up to the proof that $|\mathcal{M}|$ is a model for $\underline{\mathrm{E}}\, H_n$ with a few lemmas.

Proposition 5.3. *If $Q \leqslant H_n$ is a finite group, then the fixed-point set $|\mathcal{M}|^Q$ is non-empty and contractible.*

Proof. For all $q \in Q$, choose $\{z_0(q), \dots, z_n(q)\}$ to be an n-tuple of natural numbers such that $(i,x)q = (i,x)$ whenever $i \geqslant z_x(q)$ for all i. Q then fixes all elements $(i,x) \in S$ with $i \geqslant \max_Q z_x(q)$. Define a translation $t = t_1^{\max_Q z_1(q)} \cdots t_n^{\max_Q z_n(q)}$, $t \in \mathcal{M}^Q$, so that $\{t\}$ is a vertex of $|\mathcal{M}|^Q$ and $|\mathcal{M}|^Q \neq \emptyset$.

If $\{m\}, \{n\} \in |\mathcal{M}|^Q$, then let $a, b \in T$ be two translations such that

$$\phi(m) - \phi(n) = \phi(b) - \phi(a)$$

(recall that for a translation t, $\phi(t)$ must be an n-tuple of positive numbers). Thus, $\phi(am) = \phi(bn)$ and, since $am, bn \in \mathcal{M}$, there exist n-tuples $\{z_1, \dots, z_n\}$ and $\{z'_1, \dots, z'_n\}$ such that am acts as a translation for all $(i,x) \in S$ with $i \geqslant z_x$ and bn acts as a translation for all $(i,x) \in S$ with $i \geqslant z'_x$. Let

$$c = t_1^{\max\{z_1, z'_1\}} \cdots t_n^{\max\{z_n, z'_n\}}$$

so that $cam = cbn$. Further pre-composing c with a large translation (for example, that from the first section of this proof), we can assume that $cam = cbn \in \mathcal{M}^Q$ and $\{cam = cbn\} \in |\mathcal{M}|^Q$. This shows that the poset $\mathcal{M}^Q$ is directed and hence the simplicial realization $|\mathcal{M}^Q| = |\mathcal{M}|^Q$ is contractible. $\square$

Proposition 5.4. *If $Q \leqslant H_n$ is an infinite group, then $|\mathcal{M}|^Q = \emptyset$.*

Proof. Consider an infinite subgroup $Q \leqslant H_n$ with $|\mathcal{M}|^Q \neq \emptyset$ and choose some vertex $\{m\} \in |\mathcal{M}|^Q$. For any $q \in Q$, since $mq = m$, it must be that $\phi(m) + \phi(q) = \phi(m)$ and so $\phi(q) = 0$, and hence Q is a subgroup of $\mathrm{Sym}_\infty \leqslant H_n$. Furthermore, Q must permute an infinite subset of S (if it permuted just a finite set it would be a finite subgroup). That $mq = m$ implies that this infinite subset is a subset of $S \setminus Sm$, but this is finite by construction. So the fixed-point subset $|\mathcal{M}|^Q$ for any infinite subgroup Q is empty. □

Corollary 5.5. *$|\mathcal{M}|$ is a model for $\underline{\mathrm{E}}\, H_n$.*

Proof. Combine Propositions 5.2, 5.3 and 5.4. □

6. Finiteness conditions satisfied by H_n

Recall from § 2 that a group G is $\underline{\mathrm{FP}}_0$ if and only if it has finitely many conjugacy classes of finite subgroups. G satisfies the weaker quasi-$\underline{\mathrm{FP}}_0$ condition if and only if it has finitely many conjugacy classes of subgroups isomorphic to a given finite subgroup.

Proposition 6.1. *H_n is not quasi-$\underline{\mathrm{FP}}_0$.*

Before the above proposition is proved, we need a lemma. In the infinite symmetric group Sym_∞ acting on the set S, elements can be represented by products of disjoint cycles. We use the standard notation for a cycle: $(s_1, s_2, \ldots, s_m)$ represents the element of Sym_∞ sending $s_i \mapsto s_{i+1}$ for $i < n$ and $s_n \mapsto s_1$. Any element of finite order in H_n is contained in the infinite symmetric group Sym_∞ by the argument at the beginning of § 3. We say two elements of Sym_∞ have the same *cycle type* if they have the same number of cycles of length m for each $m \in \mathbb{N}$.

Lemma 6.2. *If q is a finite-order element of H_n and h is an arbitrary element of H_n, then hqh^{-1} is the permutation given in the disjoint cycle notation by applying h to each element in each disjoint cycle of q. In particular, if q is represented by the single cycle $(s_1, \ldots s_m)$, then hqh^{-1} is represented by $(hs_1, \ldots, hs_m)$.*

Furthermore, two finite-order elements of H_n are conjugate if and only if they have the same cycle type.

Proof. The proof of the first part is analogous to [**24**, Lemma 3.4]. Let q be an element of finite order and let h be an arbitrary element of H_n. If q fixes $s \in S$, then hqh^{-1} fixes hs. If $q(i) = j$, $h(i) = k$ and $h(j) = l$, for $i, j, k, l \in S$, then $hqh^{-1}(k) = l$, as required.

By the above, conjugate elements have the same cycle type. For the converse, notice that any two finite-order elements with the same cycle type necessarily lie in Sym_r for some $r \in \mathbb{N}$, so by [**24**, Theorem 3.5] they are conjugated by an element of Sym_r. □

Proof of Proposition 6.1. If q is any order 2 element of H_n, then $\{\mathrm{id}_{H_n}, q_1\}$ is a subgroup of H_n isomorphic to $\mathbb{Z}_2$. Choosing a collection of elements q_i for each $i \in \mathbb{N}_{\geqslant 1}$ so that q_i has i disjoint 2-cycles gives a collection of isomorphic subgroups that are all non-conjugate, by Lemma 6.2. □

Proposition 6.3. *$\underline{\mathrm{cd}}\, H_n = \underline{\mathrm{gd}}\, H_n = n$.*

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Proof. As described in the introduction, H_n can be written as

$$\mathrm{Sym}_\infty \hookrightarrow H_n \twoheadrightarrow \mathbb{Z}^{n-1}.$$

$\underline{\mathrm{gd}}\,\mathbb{Z}^{n-1} = n-1$ since a model for $\underline{\mathrm{E}}\,\mathbb{Z}^{n-1}$ is given by $\mathbb{R}^{n-1}$ with the obvious action. $\underline{\mathrm{gd}}\,\mathrm{Sym}_\infty = 1$ by [**20**, Theorem 4.3], as it is the colimit of its finite subgroups, each of which have geometric dimension 0, and the directed category over which the colimit is taken has homotopy dimension 1 [**20**, Lemma 4.2]. $\mathbb{Z}^{n-1}$ is torsion free and so has a bound of 1 on the orders of its finite subgroups and we deduce from [**17**, Theorem 3.1] that $\underline{\mathrm{gd}}\,H_n \leqslant n-1+1=n$.

To deduce the other bound, we use an argument due to Gandini [**9**]. Assume that $\underline{\mathrm{cd}}\,H_n \leqslant n-1$. By [**3**, Theorem 2],

$$\mathrm{cd}_{\mathbb{Q}} \leqslant \underline{\mathrm{cd}}\,H_n = n-1.$$

In [**5**, Theorem 5.1] it is proved that H_n is FP_{n-1} (but not FP_n). Combining this with [**15**, Proposition 1], we deduce that there is a bound on the orders of the finite subgroups of H_n, but this is obviously a contradiction. Thus,

$$n \leqslant \underline{\mathrm{cd}}\,H_n \leqslant \underline{\mathrm{gd}}\,H_n \leqslant n.$$

□

Remark 6.4. In [**7**, Theorem 1], it is proved that for every countable elementary amenable group G of finite Hirsch length h, $\underline{\underline{\mathrm{gd}}}\,G \leqslant h+2$ (see the beginning of [**10**] for a precise definition of Hirsch length for elementary amenable groups). From this we may deduce that since the Hirsch length of H_n is $h(H_n) = n-1$,

$$\underline{\underline{\mathrm{gd}}}\,H_n \leqslant n+1.$$

In [**20**, Corollary 5.4] it is proved that $\underline{\underline{\mathrm{gd}}}\,G \geqslant \underline{\mathrm{gd}}\,G - 1$ for any group G. Thus, we deduce that

$$n-1 \leqslant \underline{\underline{\mathrm{gd}}}\,H_n \leqslant n+1.$$

We finish with the following question.

Question 6.5. What is the exact geometric dimension of Houghton's group H_n with respect to the family of virtually cyclic subgroups?

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