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ON A CONJECTURE ON SHIFTED PRIMES WITH LARGE PRIME FACTORS, II

YUCHEN DING

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Abstract

Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding x. Let $P^+(n)$ be the largest prime factor of n, with the convention $P^+(1) = 1$, and $T_c(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p-1) \ge p^c\}$. Motivated by a conjecture of Chen and Chen ['On the largest prime factor of shifted primes', *Acta Math. Sin. (Engl. Ser.)* **33** (2017), 377–382], we show that for any c with $8/9 \le c < 1$,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \le 8(1/c - 1),$$

which clearly means that

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \to 0 \quad \text{as } c \to 1.$$

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1. Introduction

The investigation of shifted primes with large prime factors began in a brilliant article of Goldfeld [12]. Historically, this topic had aroused great interest because of its unexpected connection with the first case of Fermat's last theorem, thanks to the theorems of Fouvry [11] and Adleman and Heath-Brown [1].

For any positive integer n, let $P^+(n)$ be the largest prime factor of n with the convention $P^+(1) = 1$. Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding x. For $0 < c < 1$, let $T_c(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p-1) \ge p^c\}$. As early as 1969, Goldfeld [12] proved

$$\liminf_{x\to\infty} T_{1/2}(x)/\pi(x) \ge 1/2.$$

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Goldfeld further remarked that his argument also leads to

$$\liminf_{x\to\infty} T_c(x)/\pi(x) > 0, \tag{1.1}$$

provided that $c < 7/12$. It turns out that exploring large c which satisfy (1.1) is rather difficult and important. For improvements on the values of c, see Motohashi [21], Hooley [15, 16], Deshouillers and Iwaniec [7], and Fouvry [11]. Up to now, the best numerical value of c satisfying (1.1), with a cost of replacing $\pi(x)$ with $\pi(x)/\log x$, is 0.677, obtained by Baker and Harman [3].

In an earlier note [8] on this topic, I showed that

$$\limsup_{x\to\infty} T_c(x)/\pi(x) < 1/2 \tag{1.2}$$

holds for some absolute constant $c < 1$. As a corollary, I disproved a 2017 conjecture of Chen and Chen [6] that

$$\liminf_{x\to\infty} T_c(x)/\pi(x) \geq 1/2$$

for any c with $1/2 \leq c < 1$. The proof in my earlier note is based on the following deep result which is a corollary of the Brun–Titchmarsh inequality.

Proposition 1.1 [24, Lemma 2.2]. *There exist two functions $K_2(\theta) > K_1(\theta) > 0$, defined on the interval $(0, 17/32)$ such that for each fixed real $A > 0$ and all sufficiently large $Q = x^\theta$, the inequalities*

$$K_1(\theta)\frac{\pi(x)}{\varphi(m)} \leq \pi(x; m, 1) \leq K_2(\theta)\frac{\pi(x)}{\varphi(m)}$$

hold for all integers $m \in (Q, 2Q]$ with at most $O(Q(\log Q)^{-A})$ exceptions, where the implied constant depends only on A and θ. Moreover, for any fixed $\varepsilon > 0$, these functions can be chosen to satisfy the following properties:

- *$K_1(\theta)$ is monotonic decreasing and $K_2(\theta)$ is monotonic increasing;*
- *$K_1(1/2) = 1 - \varepsilon$ and $K_2(1/2) = 1 + \varepsilon$.*

The constant c in (1.2) is not specified because of the indeterminate nature of $K_1(\theta)$ in Proposition 1.1. In fact, $K_1(\theta)$ (and hence c) can be explicitly given if one checks carefully the articles of Baker and Harman [2] for $1/2 \leq \theta \leq 13/25$, and Mikawa [19] for $13/25 \leq \theta \leq 17/32$. This gives $K_1(\theta) \geq 0.16$ for $1/2 \leq \theta \leq 13/25$ [2, Theorem 1] and $K_1(\theta) \geq 1/100$ for Mikawa's range [19, (4)]. However, it seems that the constant c in (1.2) obtained in this way will be very close to 1 (see the proofs in [8]).

In [8], I also pointed out that Chen and Chen's conjecture is already in contradiction with the Elliott–Halberstam conjecture (from Pomerance [22], Granville [13], Wang [23] and Wu [24]). In fact,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) = \lim_{x\to\infty} T_c(x)/\pi(x) = \left(1 - \rho\left(\frac{1}{c}\right)\right) \to 0 \quad \text{as } c \to 1, \tag{1.3}$$

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under the assumption of the Elliott–Halberstam conjecture, where $\rho(u)$ is the Dickman function, defined as the unique continuous solution of the differential-difference equation

$$\begin{cases} \rho(u) = 1 & \text{for } 0 \le u \le 1, \\ u\rho'(u) = -\rho(u-1) & \text{for } u > 1. \end{cases}$$

However, there are earlier results related to the conjecture of Chen and Chen, and my earlier result (1.2). In fact, as indicated by the proof of a result of Erdős [9, Lemma 4], as early as 1935, one could already conclude from Erdős' proof combined with Lemma 2.2 of Wu (see below) that (1.3) is true in part.

THEOREM 1.2 (Erdős). *Unconditionally,*

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \to 0 \quad \text{as } c \to 1.$$

Essentially, Theorem 1.2 can be deduced from Erdős' proof by adding Wu's lemma (see Erdős' argument in [9, from page 212, line 6 to page 213, line 4]). Since Erdős' conclusion is not clearly formulated, it is meaningful to restate it explicitly as Theorem 1.2. It is also of interest to pursue Erdős' theorem a little further to reach the following quantitative form.

THEOREM 1.3. *For* $8/9 \le c < 1$,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \le 8(1/c - 1).$$

We note that the restriction on $c \ge 8/9$ in our theorem is natural since otherwise, the upper bound would exceed 1 which is certainly meaningless. Theorem 1.3 can also be compared with the results of Goldfeld [12], Luca *et al.* [18], and Chen and Chen [6] which state that

$$\liminf_{x\to\infty} T_c(x)/\pi(x) \ge 1 - c$$

for $0 < c \le 1/2$. These bounds were recently improved in part by Feng and Wu [10], and Liu, Wu and Xi [17]. From Theorem 1.3, we clearly have two corollaries, one of which is Erdős' theorem (Theorem 1.2) while the other revisits the main result (1.2) of my earlier note in a quantitative form.

COROLLARY 1.4. *For* $c > 16/17$,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) < 1/2.$$

2. Proofs

From now on, p will always be a prime. The proof of Theorem 1.3 is based on the following lemma deduced from the sieve method (see, for example, [14, Theorem 5.7, page 172]).

LEMMA 2.1. *Let g be a natural number and let a_i, b_i $(i = 1, 2, \ldots, g)$ be integers satisfying*

$$E := \prod_{i=1}^{g} a_i \prod_{1\le r<s\le g} (a_r b_s - a_s b_r) \neq 0.$$

Let $\rho(p)$ denote the number of solutions in n modulo p of

$$\prod_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod{p},$$

and suppose that

$$\rho(p) < p \quad \text{for all } p.$$

If the real numbers y and z satisfy $1 < y \le z$, then

$$|\{n : z - y < n \le z, a_i n + b_i \text{ prime for } i = 1, 2, \ldots, g\}|$$
$$\le 2^g g! \prod_p \Big(1 - \frac{\rho(p) - 1}{p - 1}\Big)\Big(1 - \frac{1}{p}\Big)^{-g+1} \frac{y}{\log^g y}\Big(1 + O\Big(\frac{\log\log 3y + \log\log 3|E|}{\log y}\Big)\Big),$$

where the constant implied by the O-symbol depends at most on g.

We also need the following important relation established by Wu [24, Theorem 2].

LEMMA 2.2. *For $0 < c < 1$, let*

$$T_c'(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p-1) \ge x^c\}.$$

Then for sufficiently large x,

$$T_c(x) = T_c'(x) + O\Big(\frac{x \log\log x}{(\log x)^2}\Big).$$

We now turn to the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Let x be a sufficiently large number throughout the proof. Instead of investigating $T_c(x)$, we first deal with $T_c'(x)$. For $1/2 \le c < 1$, it is easy to see that

$$T_c'(x) = \sum_{\substack{x^c \le q < x \\ q \in \mathcal{P}}} \sum_{\substack{p \le x \\ q | p-1}} 1. \tag{2.1}$$

On putting $p - 1 = qh$ in the sum (2.1) and then exchanging the order of summation,

$$T_c'(x) = \sum_{\substack{x^c \le q < x \\ q \in \mathcal{P}}} \sum_{\substack{h < x/q \\ qh+1 \in \mathcal{P}}} 1 \le \sum_{\substack{h < x^{1-c} \\ 2 | h}} \sum_{\substack{2 < q < x/h \\ q, qh+1 \in \mathcal{P}}} 1. \tag{2.2}$$

For any h with $2 \mid h$ and $h < x^{1-c}$, let $\rho(p)$ denote the number of solutions of

$$n(hn + 1) \equiv 0 \pmod{p}.$$

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Then

$$\rho(p) = \begin{cases} 1 & \text{if } p \mid h, \\ 2 & \text{otherwise.} \end{cases}$$

Now, by Lemma 2.1 with $g = 2, a_1 = 1, b_1 = 0, a_2 = h, b_2 = 1$ and $z = y = x/h$,

$$3|E| = 3h \ll x, \quad 3y = 3x/h \ll x \quad \text{and} \quad y = x/h \geq \sqrt{x},$$

from which it follows that

$$\sum_{\substack{2<q<x/h \\ q,qh+1\in\mathcal{P}}} 1 \leq 16\mathfrak{S} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \frac{x/h}{\log^2(x/h)} \Big(1 + O\Big(\frac{\log\log x}{\log x}\Big)\Big), \tag{2.3}$$

where an empty product for $\prod_{p|h,p>2}$ above denotes 1 as usual and

$$\mathfrak{S} = \prod_{p>2} \Big(1 - \frac{1}{p-1}\Big)\Big(1 - \frac{1}{p}\Big)^{-1} = \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

Inserting (2.3) into (2.2) gives

$$T_c'(x) \leq (1 + o(1))16\mathfrak{S} \sum_{\substack{h<x^{1-c} \\ 2|h}} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \frac{x/h}{\log^2(x/h)}. \tag{2.4}$$

Note that

$$\prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \leq 2 \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p}\Big) \tag{2.5}$$

since the gaps between odd primes are at least 2, from which we can already give a nontrivial upper bound of $T_c'(x)$ via partial summations. To make our bound more explicit than (2.5), we employ a nice result of Banks and Shparlinski [4, Lemma 2.3] (on taking $a = 1$ therein), which states that for $z \geq 2$,

$$S(z) := \sum_{\substack{h<z \\ 2|h}} \frac{1}{h} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) = \frac{1 + o(1)}{2\mathfrak{S}} \log z. \tag{2.6}$$

For $1 \leq z < 2$, we set $S(z) = 0$. By partial summation,

$$\sum_{\substack{h<x^{1-c} \\ 2|h}} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p}\Big) \frac{1/h}{\log^2(x/h)} = \frac{S(x^{1-c})}{(\log x^c)^2} - \int_1^{x^{1-c}} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2}. \tag{2.7}$$

Note also that for $z \geq 2$,

$$S(z) \leq \sum_{h<z} \frac{1}{h} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \leq \sum_{h<z} \frac{1}{h} \prod_{p|h} \Big(1 + \frac{3}{p}\Big) = \sum_{h<z} \frac{1}{h} \sum_{d|h} \frac{3^{\omega(d)}\mu^2(d)}{d},$$

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where $\mu(d)$ is the Möbius function and $\omega(d)$ is the number of distinct prime factors of d. Exchanging the order of summation,

$$S(z) \le \sum_{d<z} \frac{3^{\omega(d)}\mu^2(d)}{d} \sum_{\substack{h<z \\ d|h}} \frac{1}{h} \le 3 \sum_{d<z} \frac{3^{\omega(d)}\mu^2(d)}{d^2} \log z < 3K \log z, \tag{2.8}$$

where

$$K = 3 \sum_{d=1}^{\infty} \frac{3^{\omega(d)}\mu^2(d)}{d^2}.$$

From (2.8),

$$\int_1^{\log x} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2} \ll_K \frac{\log\log x}{(\log x)^2} = o((\log x)^{-1}). \tag{2.9}$$

Now, routine computations yield

$$\frac{S(x^{1-c})}{(\log x^c)^2} = \frac{1+o(1)}{\mathfrak{S}} \frac{(1-c)}{2c^2} (\log x)^{-1} \tag{2.10}$$

and

$$\begin{aligned}
\int_{\log x}^{x^{1-c}} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2} &= \frac{1+o(1)}{\mathfrak{S}} \int_{\log x}^{x^{1-c}} \frac{\log z}{z} \Big(\log \frac{x}{z}\Big)^{-3} dz \\
&= \frac{1+o(1)}{\mathfrak{S}} \int_{1}^{x^{1-c}} \frac{\log z}{z} \Big(\log \frac{x}{z}\Big)^{-3} dz + o((\log x)^{-1}) \\
&= \frac{1+o(1)}{\mathfrak{S}} \int_{x^c}^{x} \frac{\log x - \log u}{u} (\log u)^{-3}\, du + o((\log x)^{-1}) \\
&= \frac{1+o(1)}{\mathfrak{S}} \Big(\frac{1-c}{2c^2} + \frac{1}{2} - \frac{1}{2c}\Big)(\log x)^{-1},
\end{aligned} \tag{2.11}$$

thanks to the estimate (2.6). Combining (2.9), (2.10) and (2.11), one sees that the right-hand side of (2.7) equals

$$\frac{1+o(1)}{\mathfrak{S}} \Big(\frac{1}{2c} - \frac{1}{2}\Big)(\log x)^{-1}. \tag{2.12}$$

Taking (2.12) into (2.4), we immediately obtain

$$T_c'(x) \le (1+o(1))8\Big(\frac{1}{c} - 1\Big)\frac{x}{\log x}.$$

Therefore, by Lemma 2.2,

$$T_c(x) = T_c'(x) + O\Big(\frac{x \log\log x}{(\log x)^2}\Big) \le (1+o(1))8\Big(\frac{1}{c} - 1\Big)\frac{x}{\log x}.$$

Our theorem now follows from the prime number theorem. □

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3. Remarks

Under the assumption of the Elliott–Halberstam conjecture, it is reasonable to predict that the exact value of c in Corollary 1.4 should be $e^{-1/2} = 0.60653\ldots$ from (1.3) and the recursion formula (see, for example, [20, (7.6)]) for Dickman's function:

$$\rho(v) = u - \int_u^v \frac{\rho(t-1)}{t}\,dt \quad (1 \le u \le v).$$

It therefore seems to be of interest to improve, as far as possible, the numerical value of c in Corollary 1.4. We leave this as a challenge to readers.

Though we provided nontrivial upper bounds on $T_c(x)$ for $8/9 \le c < 1$ in Theorem 1.3, the extension of these bounds to $1/2 \le c < 1$ is an unsolved problem.

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YUCHEN DING, School of Mathematical Sciences,
Yangzhou University, Yangzhou 225002, PR China
e-mail: ycding@yzu.edu.cn