Robert I. Soare, ***Turing Computability***, Theory and Applications of Computability, Springer-Verlag, Berlin, Heidelberg, 2016, xxxvi + 263 pp.

This book has big shoes to fill. The author's previous textbook (Springer-Verlag, 1987) is today a classic in the field of computability theory, with several generations of students of the subject having by now been "raised" on it around the world. The work under review is intended in part as a new edition of that work, but mostly as an update. That much is already clear from the updated title: from ***Recursively enumerable sets and degrees*** in the 1987 version, to ***Turing computability*** in the present one. Going beyond nomenclature, an update is not a small task. Computability theory, over the past three decades, has enjoyed some truly extensive developments, with major shifts in research focus and countless new results. The field is certainly much larger, and much more specialized. At the same time, the 1987 book arguably owes much of its success to not trying to be an all-encompassing reference, but focusing instead on bringing the reader up to speed—meaning, to being able to understand and think about research problems—as rapidly as possible. Perhaps this is a hallmark of any great introductory text, but it clearly becomes more challenging for a broader subject. Nonetheless, the book under review seems to find the right balance.

The principal theme of modern computability theory is that of *relative* computability, of a set X of natural numbers being computed from another such set Y, meaning that there is an algorithm that, given information about which numbers belong and do not belong to Y, can answer which numbers belong and do not belong to X. This notion, along with a precise formalization of the concept of an algorithm, was a seminal achievement of Alan Turing in the 1930s. But it was Email Post who really developed this notion, and coined the term *Turing reducibility* for the process of computing one set from another. And it was Post who introduced the systematic study of (*Turing*) *degrees*, or equivalence classes under this reducibility (which Post called *degrees of unsolvability*), as a way of calibrating how far a given set of natural numbers—or by extension, a mathematical object that can be coded as such a set—is from being computable.

Much of the work in this direction, which has come to be called *degree theory*, focused on the degrees of *computably enumerable* (*c.e.*) sets—sets which can be effectively listed, but not in any particular order, and which may consequently not be computable. Two natural examples of such sets are the empty set, which is of course computable, and the *halting set*—the set of all numbers 2^x3^y such that the algorithm with code x halts on input y—which, famously, is not. *Post's problem*, to find a c.e. set strictly in-between these two in Turing degree, spurred on the initial work in degree theory, and set the stage for much of the work that would take place in the coming decades. By the late 1980s, when Soare's first textbook came out, degree theory was a rich and widely investigated subject, responsible for the development of myriad new techniques for building c.e. sets with various combinatorial and computational properties. Many of those techniques are featured in the later chapters of the 1987 text, with a view towards open problems and future research. Yet, the next decade would see a gradual but steady waning of degree theory as the dominant focus of computability theory, yielding to areas like computable structure theory, algorithmic randomness, and reverse mathematics. While the powerful methods from the study of the c.e. degrees found application here, too, each of these areas necessitated its own, different techniques and ideas.

And so it makes sense that we find in the book under review none of the advanced topics from the 1987 version: there are no results about the lattice of c.e. sets under inclusion, no mention of $0'''$-priority arguments, in fact, nothing in-depth on infinite injury methods at all. Instead, the author largely sticks to themes that are common across all the different facets of computability being worked on today. Of course, the omitted topics can still be found in Soare's older book, as well as in newer texts by Odifreddi (***Classical Recursion Theory***, North-Holland, 1989), by Lerman (***A Framework for Priority Arguments***, Cambridge University Press, 2010), and others. With updated terminology and notation, it is now easy to segue from the new book to these and other specialized texts, especially those—like Ash and Knight (***Computable Structures and the Hyperarithmetical Hierarchy***, North-Holland, 2000), Downey and Hirschfeldt (***Algorithmic Randomness and Complexity***, Springer, 2010), and Simpson (***Subsystems of Second Order Arithmetic***, Springer-Verlag, 1999), to name just a few—that became standard references in their respective subfields after

Soare's 1987 textbook was first published. The new book provides a rigorous but accessible common foundation to them all.

Not much in this book is in any sense a re-edition of anything in the 1987 text. Only Part III (out of five parts in total), on minimal degrees, covers material that can be wholly found in the original. That goes even for Part I, which is the main introduction to the subject. This comprises the first seven chapters, which are roughly analogous to the first seven in the old book. We see the basic definitions of computable sets and functions, and of c.e. sets; standard first results like the *s-m-n* theorem and the recursion theorem; the definition of the jump operator and the arithmetical hierarchy, and Post's theorem relating the two; an exploration of the Δ^0_2 sets and the limit lemma; key notions like immunity and hyperimmunity, and their applications to Post's problem; and essential techniques like permitting, finite extensions and forcing, and finite injury arguments. But there are key differences here. For one, the author begins straight away with the definition of Turing machines, omitting any mention of the primitive recursive functions or the unbounded search operator, or any lengthy formal analysis of the equivalence between the partial recursive functions and Turing computations. Rather, the author postpones this material to the largely expository Part V, on the history of computability. This is a departure from most standard introductions to the subject, but it very much reflects the contemporary view.

Instead, there are a great number of new topics added here. Chapter 2, for example, covers Lachlan games, giving an especially elegant way to think of and organize various c.e. set constructions, and obtain c.e. sets satisfying desired requirements. There is much more on games in Part IV (Chapters 14–16), including on Banach-Mazur games and Gale-Stewart games, along with a brief discussion of determinacy. This material complements that in Part I very nicely, and could easily be added to it in any first graduate course in computability.

The end of Chapter 3 covers trees and the low basis theorem, fixing something that, while not a serious omission in the old book when it came out in 1987, has certainly turned into one since then. Indeed, it is a bit of a historical irony that one of Soare's most famous theorems—the low basis theorem, proved jointly with Carl Jockusch (Π^0_1 classes and degrees of theories, ***Journal of Symbolic Logic***, vol. 173 (1972), pp. 33–56)—appears merely as an exercise in the 1987 text, with no broader discussion. That theorem is today among the most widely and ubiquitously used results in the entire field, and Chapter 3 gives it a nice introduction. Trees, Π^0_1 classes, and various other basis results are further explored in Part II of the book (Chapters 8–11), along with some much more recent related topics, like the *anti-basis* theorems of Kent and Lewis (On the degree spectrum of a Π^0_1 class, ***Transactions of the American Mathematical Society***, vol. 362 (2010), pp. 5283–5319). Theorem 10.3.3 in this section gives a nice equivalence between different characterizations of a set having *PA degree*, i.e., computing a complete consistent extension of Peano arithmetic. This, too, is a central topic, yet this equivalence is not commonly found in book form. Chapter 11 gives a brief introduction to randomness, mostly as a means of further illustrating some of the utility of Π^0_1 classes, but with a number of important results included.

The aforementioned Part V at the end of the book covers (some of) the history of computability theory in the 20th century, starting with Hilbert's issuance of his ICM problems in 1900, and subsequently his *Entscheidungsproblem* in 1928. We are told about some of the key figures in this history, and the author makes the compelling and convincing case for why, morally, "Turing got it right" with his model of computation, over, say, Kleene's μ-recursive functions, or the more syntactic Herbrand-Gödel functions, even though formally these are all equivalent. The subsequent discussion of the development of relative computability is fascinating. To be sure, there is much that happened later in the subject that is not covered here, but that is for other books to talk about. The author focuses on a narrow but important early time for the subject, and the inclusion of this discussion feels appropriate even for an otherwise highly technical text, making the case (all too often disregarded in mathematics) for *why* we are studying what we are studying, and also for *how* we are studying it. It is a pleasure to read for both expert and non-expert alike.

At a time when computability theory is enjoying remarkable activity and fruitfulness, and benefiting from having a large number of students and young researchers, there is no question that the subject is ready for a new standard introductory text. The present book shares all the features that helped its predecessor become such a standard thirty years ago, and at the same time, it is modern, and it is relevant to today's state of the field. The subject will be well-served by it.

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Picture yourself a country with trails, grasslands, etc., inhabited by beautiful unicorns, in quite a number. In fact, most of the trails avoid those unicorns, but a few of them have the astonishing particularity of hosting herds of unicorns. Similarly, the unicorns are quite scarce on many grasslands, while on others they do appear but only along the trails, and on still other grasslands they appear in surprisingly large numbers throughout. For a long time, geographers could not really understand what was so special about the geography of those trails and grasslands, that were fully inhabited by unicorns, despite an attractive and convincing suggestion by unicorn ecologists. Remarkably, this inspired nereid ecologists to wonder whether a similar suggestion could explain the population of nereids in some rare rivers and ponds of a neighboring country.

This small book aims at unveiling a similar mathematical mystery.

Our mathematical countries, no less fantastic but absolutely real, are the Abelian varieties and the Shimura varieties, named after the mathematicians Niels Abel (1802–1829) and Goro Shimura (1930–). There is less poetry, however, in the name given to our magical beasts, respectively torsion points or special points.

By *varieties*, we mean here algebraic varieties, that is, loci defined by polynomial equations, say, with complex coefficients. Our trails are curves, our grasslands, surfaces, etc.

The most elementary examples of *Abelian varieties* are given by elliptic curves, each of them being the set of solutions in the projective plane of some cubic equation with nonzero discriminant. By Weierstrass's theory of bi-periodic functions, elliptic curves can also be described as the quotient of the complex plane $\mathbf{C}$ by a lattice $\mathbf{Z} + \mathbf{Z}\tau$, where τ, a complex number of positive imaginary part, is an element of Poincaré's upper half plane $\mathfrak{h}$.

More generally, Abelian varieties are those irreducible varieties which are endowed with a group law, defined by polynomials as well, and are, moreover, "compact" or, more precisely, projective; they can also be understood from the point of view of complex function theory, where they appear as (particular) complex tori, quotients of a complex affine space $\mathbf{C}^g$ by a lattice Λ. Torsion points are then defined as in group theory. A basic property is that an Abelian variety of dimension g contains n^{2g} points a such that $n \cdot a = 0$, for every integer $n \geq 1$; these are the images modulo Λ of the points of $n^{-1}\Lambda$.

Around 1960, Yuri Manin and David Mumford had conjectured that irreducible subvarieties of an Abelian variety which contain a *dense* set of torsion points must be Abelian subvarieties, or the image of such a subvariety under that translation by a torsion point. By "dense", we mean that those points are not contained in a subvariety of a smaller dimension—on remarkable grasslands, unicorns are not soleley populated along a few trails. This conjecture has been proved by Michel Raynaud in 1983 and many new beautiful proofs have been given since.

The simplest example of a *Shimura variety* is the *modular curve*, which parameterizes elliptic curves. Namely, it is just the quotient of the upper half plane by identifying two elements τ and τ' for which the lattices $\mathbf{Z} + \mathbf{Z}\tau$ and $\mathbf{Z} + \mathbf{Z}\tau'$ give rise to the same elliptic curve. It comes out that this corresponds to quotienting the upper half-plane $\mathfrak{h}$ by the group $\mathrm{SL}(2, \mathbf{Z})$ acting by homographies. In this case, the Jacobi j-function identifies this quotient with the complex plane $\mathbf{C}$. More generally, Shimura varieties are relatively easily defined from the point of view of complex function theory, where they appear as quotients of "symmetric

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