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POWERS OF p-VALENT FUNCTIONS

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Abstract

If f is areally mean p-valent in the unit disc, if $\lambda > 0$, and if f^λ is defined as a single-valued analytic function on the unit disc with finitely many arcs removed, several results in the recent literature suggest that f^λ might be areally mean $p\lambda$-valent. The purpose of this note is to determine the valence of f^λ when f is areally mean p-valent, and also to characterize those functions for which f^λ is $p\lambda$-valent for all $\lambda > 0$. Analogous results are obtained for functions which are either s-dimensionally mean p-valent or logarithmically mean p-valent.

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1. Introduction and statement of results

If f is regular in $\gamma = \{z: |z| < 1\}$, set $n(r, w, f)$ equal to the number of roots in $\gamma_r = \{z: |z| < r\}$ of $f(z) = w$, and put $p(r, R, f) = (1/2\pi)\int_0^{2\pi} n(r, Re^{i\theta}, f)\,d\theta$. If $p(1, R, f) \leqslant p$ for all $R > 0$, f is called *circumferentially mean p-valent*, and we write $f \in C(p)$.

Denote the area, according to multiplicity, of $f(\gamma_r) \cap \{w: |w| < R\}$ by $A^*(r, R, f)$. It is easily verified that $A^*(r, R, f) = \int_0^{2\pi}\int_0^R n(r, te^{i\theta}, f)\,t\,dt\,d\theta$. If $A^*(1, R, f) \leqslant p\pi R^2$ for all $R > 0$, f is called areally mean p-valent, and we write $f \in S(p)$.

Two additional classes of p-valent functions appearing in the literature are the class of s-dimensionally mean p-valent functions (Spencer, 1940) and the class of logarithmically mean p-valent functions (Jenkins and Oikawa, 1971). We denote these classes by $S_s(p)$ and $L(p)$, respectively. We say that $f \in S_s(p)$ if

$$\int_0^R p(1, R, f)\,d(R^s) \leqslant pR^s \quad (R > 0),$$

while $f \in L(p)$ if

$$\int_{R_1}^{R_2} \frac{p(1, R, f)}{R}\,dR \leqslant p\left(\log\frac{R_2}{R_1} + \frac{1}{2}\right) \quad (0 < R_1 < R_2).$$

Many results in the recent literature have been concerned with the determination of growth rates of various quantities associated with p-valent functions. For

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example, if $f \in S(p)$ and if $f(z)^\lambda = z^\mu \sum_{-\infty}^{+\infty} a_n(\lambda) z^n$ in an annulus $\{z: 0 \leq r_1 < |z| < 1\}$, then $a_n(1) = O(1) n^{2p-1}$ if $p > \frac{1}{4}$, and more generally $a_n(\lambda) = O(1) n^{2p\lambda-1}$ if $p\lambda > \frac{1}{4}$ (Hayman, 1967, p. 104).

Results of this nature suggest that if f is p-valent and $\lambda > 0$, then f^λ is $p\lambda$-valent. Some positive results are known. If $f \in C(p)$ and $\lambda > 0$, then $f^\lambda \in C(p\lambda)$ (Hayman, 1967, p. 95). Also, if $f \in S(p)$ and $0 < \lambda \leq 1$, then $f^\lambda \in S(p\lambda)$ (Eke, 1967, p. 189). The purpose of this note is to determine the valence of f^λ when f is p-valent, and also to characterize those functions for which f^λ is $p\lambda$-valent for all $\lambda > 0$.

Before proceeding further, we must specify the meaning of expressions such as f^λ. In general, if f has zeros in γ, f^λ will not be single-valued. However, we shall be dealing exclusively with functions f which are p-valent in one of the above senses, and so, as is well known (Hayman, 1967, p. 103), f can vanish at most finitely many times in γ. If we now connect the zeros of f by a simple smooth arc α, one of whose end points lies on the circumference $|z| = 1$, a single-valued analytic branch of f^λ can be defined on the simply connected domain $\gamma_1 = \gamma \setminus \alpha$. With this understanding, expressions such as $f^\lambda \in S(p\lambda)$ shall mean that this analytic branch of f^λ is areally mean $p\lambda$-valent on the domain γ_1. Alternatively, we could have defined an analytic branch of f^λ in a suitable annulus $\{z: 1-\varepsilon < |z| < 1\}$, cut if necessary by a radius. All results in this paper are valid with either understanding of f^λ.

We first determine the valence of f^λ when $f \in S(p)$.

THEOREM 1. *Let* $f \in S(p)$, *and let* $\lambda > 0$ *be given. Set* $\Lambda = \max(\lambda, \lambda^2)$. *Then* $f^\lambda \in S(p\Lambda)$, *and the valence* $p\Lambda$ *is best possible.*

Note that for $\lambda > 1$, f^λ need not belong to $S(p\lambda)$. We now characterize those functions for which $f^\lambda \in S(p\lambda)$ for all $\lambda > 0$. The characterization seems somewhat interesting, since the areal behavior of f is characterized in terms of the circumferential behavior.

THEOREM 2. *Let* $p > 0$. *Then* $f^\lambda \in S(p\lambda)$ *for all* $\lambda > 0$ *if and only if* $f \in C(p)$.

Both Theorem 1 and Theorem 2 continue to hold for the classes $S_s(p)$ and $L(p)$. We have

THEOREM 3. *Let* $\lambda > 0$, $\Lambda = \max(\lambda, \lambda^2)$. *If* $f \in S_s(p)\,(L(p))$, *then* $f^\lambda \in S_s(p\Lambda)(L(p\Lambda))$, *and in each case the valence is best possible. Also,* $f \in C(p)$ *if and only if* $f^\lambda \in S_s(p\lambda)\,(L(p\lambda))$ *for all* $\lambda > 0$.

2. Proofs of positive results

We begin by noting that if f is regular in γ and if $\lambda > 0$, then

$$A^*(r, R, f^\lambda) = 2\pi\lambda^2 \int_0^{R^{1/\lambda}} t^{2\lambda-1} p(r, t, f)\, dt. \tag{1}$$

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In order to prove this, we recall (Hayman, 1967, p. 96) that

$$p(r, t^{\lambda}, f^{\lambda}) = \lambda p(r, t, f). \tag{2}$$

After using (2) and changing variables in the formula defining $A^*(r, R, f^{\lambda})$, we arrive at (1).

If $0 < \lambda \leqslant 1$, then $\Lambda = \lambda$, and the fact that $f^{\lambda} \in S(p\lambda)$ when $f \in S(p)$ is known (see Eke, 1967, p. 189 or Hayman, 1967, p. 45). We also note that this fact follows easily from (1) upon integrating by parts.

We now assume $\lambda > 1$, so that $\Lambda = \lambda^2$. If $f \in S(p)$, it follows from (1) that

$$\begin{aligned} A^*(1, R, f^{\lambda}) &= 2\pi\lambda^2 \int_0^{R^{1/\lambda}} t^{2\lambda-2}\, t p(1, t, f)\, dt \\ &\leqslant \lambda^2 R^{2-2/\lambda} A^*(1, R^{1/\lambda}, f) \\ &\leqslant \pi p \lambda^2 R^2. \end{aligned}$$

Hence $f^{\lambda} \in S(p\Lambda)$. An example showing that the valence $p\Lambda$ is best possible will be presented in Section 3.

We now prove Theorem 2. If $f \in C(p)$, it is well known (Hayman, 1967, p. 95) that $f^{\lambda} \in C(p\lambda)$, and hence $f^{\lambda} \in S(p\lambda)$ for all $\lambda > 0$.

If $f^{\lambda} \in S(p\lambda)$ for all $\lambda > 0$, then $A^*(1, R, f^{\lambda}) \leqslant \lambda p \pi R^2$ for all $\lambda > 0$, $R > 0$. Upon changing variables, we see that this is equivalent to

$$\int_0^T p(1, t, f)\, d(t^{2\lambda}) \leqslant p T^{2\lambda}$$

for all $\lambda > 0$, $T > 0$. It now follows from a theorem of Spencer (Spencer, 1940, p. 421) that $p(1, R, f) \leqslant p$ for all $R > 0$, and hence $f \in C(p)$.

The proof of Theorem 3 in the case of $S_s(p)$ is essentially the same as the proof in the case of $S(p) = S_2(p)$, and hence it will be omitted.

If $f \in L(p)$, then

$$\begin{aligned} \int_{R^1}^{R_2} \frac{p(1, R, f^{\lambda})}{R}\, dR &= \lambda^2 \int_{R_1^{1/\lambda}}^{R_2^{1/\lambda}} \frac{p(1, s, f)}{s}\, ds \\ &\leqslant p\Lambda\left(\log\frac{R_2}{R_1} + \frac{1}{2}\right), \end{aligned}$$

and so $f^{\lambda} \in L(p\Lambda)$. A simple modification of an example due to Jenkins and Oikawa (Jenkins and Oikawa, 1971, pp. 402–403) shows that the valence $p\Lambda$ is best possible.

In order to complete the proof of Theorem 3, we note that if $f \in C(p)$, then $f^{\lambda} \in C(p\lambda) \subset S(p\lambda) \subset L(p\lambda)$. Conversely, if $f^{\lambda} \in L(p\lambda)$ for all $\lambda > 0$, then

$$\begin{aligned} \int_{R_1}^{R_2} \frac{p(1, t, f^{\lambda})}{t}\, dt &= \lambda^2 \int_{R_1^{1/\lambda}}^{R_2^{1/\lambda}} \frac{p(1, s, f)}{s}\, ds \\ &\leqslant \lambda p\left(\log\frac{R_2}{R_1} + \frac{1}{2}\right) \end{aligned}$$

for all $0 < R_1 < R_2$. Thus, for any interval $I \subset (0, \infty)$, we have

$$\lambda^2 \int_I \frac{p(1,s,f)-p}{s}\, ds \leqslant \lambda p/2. \tag{3}$$

If there exists s_0 with $p(1,s_0,f) > p$, then the fact that $p(1,s,f)$ is a lower semi-continuous function implies the existence of an interval I on which $p(1,s,f) > p$. This in turn contradicts the fact that (3) holds for all intervals I and for all $\lambda > 0$. Therefore, $p(1,s,f) \leqslant p$ for all s, and so $f \in C(p)$.

3. Example

We now present an example to complete the proof of Theorem 1. Given $p > 0$, $\lambda > 1$ and $0 < \varepsilon < p\lambda^2$, we construct $f \in S(p)$ such that $f^\lambda \notin S(p\Lambda - \varepsilon)$. We begin by choosing $x \in (0,1)$ and setting $y = (1-x^2)^{-1}$. Put $A(x) = \{t e^{i\theta} : x^{1/yp} < t < 1, \theta \in (0, 2\pi)\}$. Let h map γ conformally onto the simply connected domain $A(x)$, and set $f = h^{yp}$. Elementary geometric arguments now show that with $m = [yp]$, we have

$$n(1, Re^{i\theta}, f) = \begin{cases} m+1, & \theta \in [0, 2\pi(yp-m)), \quad R \in (x,1), \\ m, & \theta \in [2\pi(yp-m), 2\pi), \quad R \in (x,1), \\ 0 & R \notin (x,1). \end{cases}$$

This in turn implies that

$$p(1,R,f) = \begin{cases} yp, & R \in (x,1), \\ 0, & R \notin (x,1). \end{cases}$$

We now claim that $f \in S(p)$. If $0 < R \leqslant x$, it is trivially true that

$$A^*(1,R,f) \leqslant p\pi R^2.$$

If $x < R < 1$, then

$$\begin{aligned} A^*(1,R,f) &= 2\pi \int_0^R t p(1,t,f)\, dt \\ &= 2\pi \int_x^R ypt\, dt \\ &= \pi p(1-x^2)^{-1}(R^2 - x^2). \end{aligned}$$

Straightforward computations now show that $A^*(1,R,f) \leqslant p\pi R^2$. If $R \geqslant 1$, then $A^*(1,R,f) = p\pi \leqslant p\pi R^2$. Hence $f \in S(p)$.

We now complete the construction by choosing x (and hence f) so that $f^\lambda \notin S(p\Lambda - \varepsilon)$. If $x^\lambda < R < 1$, it follows from (1) that

$$A^*(1,R,f^\lambda) = p\Lambda\pi(R^2 - x^{2\lambda})\, y/\lambda.$$

Therefore

$$\sup\left\{\frac{A^*(1,R,f^\lambda)}{\pi R^2}: x^\lambda < R < 1\right\} = p\Lambda \frac{1-x^{2\lambda}}{\lambda(1-x^2)}.$$

As $x \to 1$, $(1-x^{2\lambda})/\lambda(1-x^2) = x^{2(\lambda-1)} + o(1)$. Hence, given $\varepsilon > 0$, we choose $x < 1$ such that

$$\sup\left\{\frac{A^*(1,R,f^\lambda)}{\pi R^2}: x^\lambda < R < 1\right\} > p\Lambda - \varepsilon.$$

With such an x, we have $f \in S(p)$, yet $f^\lambda \notin S(p\Lambda - \varepsilon)$.

To construct a corresponding example for $S_s(p)$, we merely define y to be $y = (1-x^s)^{-1}$, and proceed as above.

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