Compositio Mathematica **129:** 301–323, 2001.
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Derived-tame Tree Algebras

Dedicated to Idun Reiten on the occasion of her 60th birthday

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(Received: 16 September 1999; accepted in final form: 29 September 2000)

Abstract. In this note we classify the derived-tame tree algebras up to derived equivalence. A tree algebra is a basic algebra $A = kQ/I$ whose quiver Q is a tree. The algebra A is said to be derived-tame when the repetitive category $\hat{A}$ of A is tame. We show that the tree algebra A is derived-tame precisely when its Euler form χ_A is non-negative. Moreover, in this case, the derived equivalence class of A is determined by the following discrete invariants: The number of vertices, the corank and the Dynkin type of χ_A. Representatives of these derived equivalence classes of algebras are given by the following algebras: the hereditary algebras of finite or tame type, the tubular algebras and a certain class of poset algebras, the so-called semichain-algebras which we introduce below.

Mathematics Subject Classifications (2000). 16G20, 16G60, 18E30.

Key words. tame derived category, Euler form, tree algebras.

1. Introduction

Let A be a finite-dimensional algebra (associative, with 1) over an algebraically closed field k. We denote by mod A the category of finite-dimensional, left A-modules and by $D^b(A)$ the derived category of bounded complexes over mod A. If the algebra A has finite global dimension, then $D^b(A)$ is triangle equivalent to the stable category $\underline{\text{mod}}\,\hat{A}$, where $\hat{A}$ denotes the repetitive category of A (see [16]). By Drozd's Tame–Wild dichotomy ([12]; see also [10] and [13]), precisely one of the following two cases occurs: The first possibility is that the repetitive category $\hat{A}$ is tame, i.e. for each dimension $d \in \mathbb{N}$ and each object X of $\hat{A}$, almost all indecomposable modules $M: \hat{A} \to \text{mod}\, k$ with $\sum_{Y \in \hat{A}} \dim_k M(Y) = d$ and $M(X) \neq 0$ occur in a finite number of one-parameter families. In the other case, the category $\hat{A}$ is wild, i.e. the conjugacy classes of pairs of square matrices (whose classification is a well-known unsolved problem) occur as isomorphism classes of certain $\hat{A}$-modules. Following [11], we say that A is *derived-tame* if the category $\hat{A}$ is tame. We are concerned with the question which algebras A are derived-tame.

The category $D^b(A)$ is best understood when A is derived equivalent to a hereditary algebra of finite or tame type or to a tubular algebra ([16, 17]). These algebras are also characterized by the fact that the derived category $D^b(A)$ is cycle-finite ([2])

https://doi.org/10.1023/A:1012591326777 Published online by Cambridge University Press

or that the push-down functor mod $\hat{A} \to$ mod $T(A)$ is dense ([5]), where $T(A)$ denotes the trivial extension of A. In all these cases, the Euler form of A is non-negative. Recall that the *Euler form* χ_A of A is the quadratic form which is associated to the homological bilinear form $\langle , \rangle$ on the Grothendieck group of A given by $\langle [X], [Y] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}^i_A(X, Y)$ for A-modules X and Y.

The knowledge of $D^b(A)$ is rather limited when the category $D^b(A)$ is not cycle-finite. Instances of derived-tame algebras whose derived category is not cycle-finite can be found among gentle algebras ([1]; see also [18, 23, 24]), and another class is provided by pg-critical algebras (NS). Motivated by his study of selfinjective algebras, A. Skowroński raised around 1990 the question whether the repetitive algebra of each simply connected algebra is tame if and only if the Euler form is non-negative.

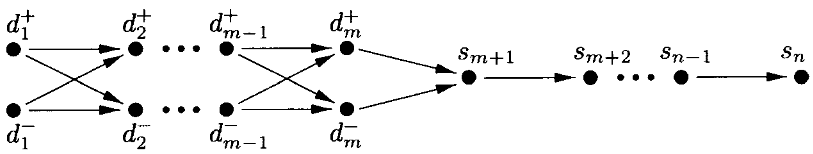
Clearly, this statement can hold only for classes of algebras A which are uniquely determined by the (discrete) datum of their Euler form χ_A. One instance is the class of *tree algebras:* Those basic algebras $A = kQ/I$ whose quiver Q is a tree. In this case, it has been conjectured explicitely in [11] that a tree algebra is derived-tame precisely when its Euler form is non-negative, and the conjecture could be verified in [7] for tree algebras containing a convex subalgebra which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p (p = 6, 7, 8)$ or to a tubular algebra.

We obtain in this paper the complete answer:

THEOREM 1.1. *A tree algebra is derived-tame precisely when its Euler form is non-negative.*

Moreover, we classify the derived-tame tree algebras up to derived equivalence. Using the results of [7], it is sufficient to concentrate on those derived-tame tree algebras which do *not* contain a convex subalgebra which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p (p = 6, 7, 8)$ or to a tubular algebra. We show that these algebras are derived equivalent to a so-called semichain $S(n, m)$:

For $m \leqslant n \in \mathbb{N}$ we define $S(n, m)$ to be the poset algebra which has the following quiver



and satisfies, as a poset algebra, that all squares are commutative. For $m = 0$ or $m = 1$, the algebras $S(n, m)$ are representation-finite hereditary with Dynkin quiver of type $\mathbb{A}_n$ for $m = 0$ and of type $\mathbb{D}_{n+1}$ for $n \geqslant 3$ and $m = 1$. The algebra $S(2, 2)$ is tame hereditary of type $\widetilde{\mathbb{A}}_3$, and the algebras $S(n, 2)$ are derived equivalent to tame hereditary algebras of type $\widetilde{\mathbb{D}}_{n+1}$. More generally, all the algebras $S(n, m)$ with

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$n > m \geqslant 1$ have a non-negative Euler form of corank $m-1$ and Dynkin type $\mathbb{D}_{n+1}$ (see [9] for the definition of the Dynkin type of a non-negative unit form).

The derived equivalence classes of derived-tame tree algebras are then described as follows:

THEOREM 1.2. *Let A be a connected derived-tame tree algebra. Then A is derived equivalent to a hereditary algebra of type* $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ *or to a tubular alge bra or to precisely one of the algebras* $S(n, m)$. *In particular, the tree algebra A is up to derived equivalence determined by the number of vertices, the corank and the Dynkin type of its Euler form.*

From the complete classification of derived tame tree algebras up to derived equivalence, we get some consequences: In [8] it was conjectured for a strongly simply connected algebra A that if the Euler form χ_A is non-negative of corank $\geqslant 3$, then the Dynkin type of χ_A has to be $\mathbb{D}_n$. We can confirm this conjecture in the case of tree algebras, and give the complete picture in Table I. There we list in the rows the possible coranks m of Euler forms χ of tree algebras, and the columns indicate the possible Dynkin types. The entries of the diagram show algebras whose Euler form is non-negative and has the required properties. By the sign '–' we indicate that no tree algebra exists whose Euler form has the corresponding properties.

Moreover, we obtain the following corollary

COROLLARY 1.3. *A derived-tame tree algebra contains a subcategory derived equivalent to* $\mathbb{E}_6$ *precisely when it contains a convex subcategory which is derived equivalent to hereditary algebra of type* $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p, p = 6, 7, 8$ *or to a tubular algebra.*

The tree algebras derived equivalent to $\mathbb{E}_6$ are classified in [7], thus one can use this handy list to determine whether a given derived-tame tree algebra has Dynkin type $\mathbb{D}_n$ or $\mathbb{E}_p$.

This note is organized as follows. In Section 2 we recall several important techniques, in particular the reflections at a sink or source of the algebra A. We further introduce the class of semi-tree algebras, this is the class of algebras where it is possible to work out the proof of the Theorems inductively. We finally divide

Table I.

	Type $\mathbb{A}_n$	Type $\mathbb{D}_n$	Type $\mathbb{E}_p$
$m = 0$	$k\mathbb{A}_n$	$k\mathbb{D}_n$	$k\mathbb{E}_p$
$m = 1$	–	$k\widetilde{\mathbb{D}}_n$	$k\widetilde{\mathbb{E}}_p$
$m = 2$	–	$S(n\text{-}1, 3)$	tubular
$m \geqslant 3$	–	$S(n\text{-}1, m\text{+}1)$	–

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the proof of the main theorems in two parts, formulated in Proposition 2.4 and 2.5. The remaining two sections are then devoted to the proofs of these two propositions.

When preparing this manuscript, we learned that Ch. Geiss obtained the same results (Theorem 1 and 2) by using quite different methods [15].

2. APR-Tilts, Reflections and Blowing-Up

2.1. NOTATION

Let $A = kQ/I$ be a tree algebra, i.e. Q is a finite quiver whose underlying graph is a tree and I is an admissible ideal of the path algebra kQ. We will always suppose that our algebras are connected. When talking of a vertex of A we mean a vertex of the quiver Q. Since A is a tree algebra, there is a minimal set of paths generating the ideal I. We refer to these generating monomials $\rho = a_1 \to \cdots \to a_t$ in kQ as *relations* of A and occasionally call the set $\{a_1, \ldots, a_t\}$ the *support of the* relation ρ.

We also identify A with a k-category whose objects are the vertices of A and whose morphism space $A(x, y)$ from x to y is $e_y A e_x$, where e_x denotes the idempotent element of A associated with the vertex x, see [14]. We say that B is a convex subcategory of A if $B = kQ'/I'$ where Q' is a path-closed subquiver of Q and $I' = I \cap kQ'$. When viewing the algebra A as a category, we interpret the A-modules as k-linear functors $M: A \to \operatorname{mod} k$. Thus, we can speak of the derived category $D^b(A)$ of a k-category A. Moreover, the k-category B is *derived-equivalent* to a k-category C, if there is a triangle equivalence $F: D^b(B) \to D^b(C)$. We recall that the triangle equivalence F induces an isomorphism of the corresponding Grothendieck groups together with their homological bilinear forms, hence the Euler form χ_B of B is non-negative if and only if χ_C is, and, in this case, corank χ_B = corank χ_C. Also the derived-tameness of a k-category is preserved under triangle equivalences, see [11].

2.2. We recall from [7] the following result.

THEOREM 2.1. *Let A be a tree algebra containing a convex subcategory which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p (p = 6, 7, 8)$ or to a tubular algebra. Then A is derived-tame if and only if χ_A is non-negative. Moreover, in this case, the algebra A itself is derived equivalent to some hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p$ or to a tubular algebra.*

To prove our main theorems, we therefore can restrict to those tree algebras A that do *not* admit a subcategory which is derived equivalent to a hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p (p = 6, 7, 8)$ or to a tubular algebra. We show in Proposition 2.4 and 2.5 below that, under these conditions, the algebra A is derived equivalent to one of the algebras $S(n, m)$ provided its Euler form is non-negative or A is

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derived-tame. This, together with Theorem 2.1, shows the first part of Theorem 1.2: every derived-tame tree algebra is derived-equivalent to a hereditary algebra of type $\mathbb{E}_p, \widetilde{\mathbb{E}}_p (p = 6, 7, 8)$ or to a tubular algebra or to one of the algebras $S(n, m)$.

As we mentioned in the introduction, the tubular and the hereditary algebras of finite or tame type are derived-tame and have non-negative Euler form. The same holds for the semichain $S(n, m)$, see [11]. Moreover, these algebras can be distinguished by the number of vertices, the corank and the Dynkin type of its Euler form, and thus we obtain the second part of Theorem 1.2.

Propositions 2.4 and 2.5 also imply Theorem 1.1: Both assumptions on the tree algebra A, being derived-tame or having a non-negative Euler form, lead (up to derived equivalence) to the same class of algebras. More over, both properties are stable under derived equivalences (see [11]), thus the class of derived tame tree algebras coincides with the class of tree algebras whose Euler form is non-negative. In fact, the derived equivalences in Propositions 2.4 and 2.5 are realized by reflections and APR-tilts (which clearly preserve the derived-tameness), so we do not even need the abstract result from [11] that derived-tameness is preserved under derived equivalences.

2.3. APR-TILTS, REFLECTIONS

Let $A = kQ/I$. For a vertex x of A, we denote by $P(x)$ the projective cover of the simple A-module which is concentrated in x, and by $I(x)$ the corresponding injective hull. The Auslander-Reiten translations of mod A are denoted by τ_A and τ_A^-, cf. [22] or [6].

If x is a sink of A, then the module $T^{(x)} = \tau_A^- P(x) \oplus (\oplus_{y \neq x} P(y))$ is called an *APR-tilting module*, and the algebra $B = \operatorname{End} T^{(x)}$ is an *APR-tilt of A*. Of course, the algebras A and B are derived equivalent. It is easy to see that if no relation of A has the sink x as its terminal point, then the algebra B is obtained from A by reversing each arrow $y \to x$ to an arrow $y \leftarrow x^*$, cf. [1]. The notion of an APR-cotiliting module is defined dually for a source x of A.

We recall that the *one-point extension $A[M]$ of A by the A-module M* contains A as a convex subcategory and has one extra vertex x that becomes a source in $A[M]$ by $A[M](x, y) = M(y)$ for $y \neq x$ and $A[M](x, x) = k$. Conversely, given a source x in A, we can write $A = A_0[M]$ where $A_0 = A \backslash x$ and $M = \operatorname{rad} P(x)$, cf. [22]. Dually, co-extensions $[M]A$ are defined by adding sinks to the category A.

We recall from [19] the concept of reflections: Let x be a sink in Q. Then we write $A = [M]A_0$ as a one-point co-extension and define the *reflection of A at the sink x* to be the algebra $S_x^+ A = A_0[M]$. Thus, the algebra A_0 is a convex subcategory of both A and $S_x^+ A$, and the sink x of A is replaced by a source x^* in $S_x^+ A$, with $A(y, x) \cong S_x^+ A(x^*, y)$ for all $y \in A_0$. Dually, we consider the reflection $S_x^- A$ of A at a source x of A.

It is easy to see that the repetitive categories of A and $S_x A$ are isomorphic. Moreover, $S_x^+ A$ is tilting-cotilting equivalent to A, see [25].

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2.4. BLOWING-UP

When proving Proposition 2.5, we proceed by induction on the number of zero-relations. Now the algebra $S(n,m)$ we are aiming at is not a tree algebra, so we have to set up our induction within a slightly larger class of algebras which we prepare to introduce now.

Let $A = kQ/I$ be a basic algebra, and let $v \in Q_0$ be a vertex of A and F a finite set. Then we define the *blowing-up of A at the vertex v by F* to be the algebra $\tilde{A} = k\tilde{Q}/\tilde{I}$ given by the following quiver $\tilde{Q}$ and ideal $\tilde{I}$: The quiver $\tilde{Q}$ is obtained from Q by replacing the vertex v by the vertices v^f with $f \in F$ and each arrow $\alpha: x \to v$ by arrows $\alpha^f: x \to v^f, f \in F$ and dually for each arrow $\beta: v \to y$.

There is an obvious quiver epimorphism $\tilde{Q} \to Q$ which extends uniquely to an epimorphism of k-algebras $\pi: k\tilde{Q} \to kQ$. We define the ideal $\tilde{I}$ of $k\tilde{Q}$ as the inverse image of I under π. So the algebra A is a subcategory of $\tilde{A}$ (only convex if v is a sink or source), and we have $A(x, v) \cong \tilde{A}(x, v^f)$ and $A(v, x) \cong \tilde{A}(v^f, x)$ for all $x \neq v$ and $f \in F$.

LEMMA 2.2. *Let A be an algebra with a sink v. Let $\tilde{A}$ be the blowing-up of A at the vertex v by the set $F = \{1, \ldots, r\}$. Then the iterated reflection $S^+_{v^1} \ldots S^+_{v^r}\tilde{A}$ at the sinks v^i, $i \in F$ coincides with the blowing-up of $S^+_v A$ at the vertex v^* by the set F.*

Proof. As v is a sink in A, we can write $A = [M]A_0$ as a one-point co-extension by some A_0-module M with extension-vertex v. Then the algebra $\tilde{A}$ is just the r–fold one-point co-extension by the same module M and extension vertices $v^1, \ldots, v^r$. By definition, the algebra obtained by iterated reflections at the sinks v^i is the r–fold one-point extension by the module M, so $S^+_{v^1} \ldots S^+_{v^r}\tilde{A} = (\ldots(A_0[M])\ldots[M])$. On the other hand, first reflecting in v turns $A = [M]A_0$ into $S^+_v A = A_0[M]$ with source v^*, and blowing-up in v^* results in the same algebra $(\ldots(A_0[M])\ldots[M])$ as before. □

LEMMA 2.3. *Let A be an algebra with a sink v such that no relation of A has v as its terminal point. Let $\tilde{A}$ be the blowing-up of A at some vertex w by the set F. If w is not a neighbour of v, then the iterated APR-tilts of $\tilde{A}$ at the sinks $w^f, f \in F$ yield the same algebra as the blowing-up of $B = \operatorname{End} T^{(v)}$ at the vertex v^* by the set F.*

This follows easily from the explicit construction of the algebra B as explained in Section 2.3. Only the case when w is a neighbour of v has to be excluded, since the blowing-up in w could then possibly produce a commutativity relation with terminal vertex v.

2.5. THE ALGEBRA $A[D]$

We will deal in the following only with blowing-up by sets $F = \{+, -\}$ of cardinality 2. Moreover, if we blow up an algebra in several vertices, the resulting algebra is independent on the order of the vertices chosen to be blown up. Thus, given

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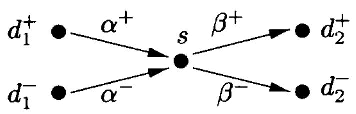
an algebra $A = kQ/I$, we just specify a set D of vertices and denote by $A[D] = kQ[D]/I[D]$ the blowing up of A in the vertices $d \in D$ by the set $\{+, -\}$. We often denote the vertices of $A[D]$ by d^+, d^- if $d \in D$ and by s if $s \in A \backslash D$. The algebra A will be a tree algebra later, thus the ideal I will be generated by zero-relations. Consequently, the ideal $I[D]$ is generated by all paths w in $kQ[D]$ with $\pi(w) \in I$ together with all commutativity relations $\beta^+\alpha^+ = \beta^-\alpha^-$ whenever there are arrows $x \xrightarrow{\alpha^+} d^+ \xrightarrow{\beta^+} y$ and $x \xrightarrow{\alpha^-} d^- \xrightarrow{\beta^-} y$ in $A[D]$ with $\pi(d^+) = \pi(d^-)$ and $\pi(\alpha^+) = \pi(\alpha^-)$ as well as $\pi(\beta^+) = \pi(\beta^-)$.

EXAMPLES. (1) If $A = kQ$ is a chain

$$d_1 \to d_2 \to \cdots \to d_m \to s_{m+1} \to \cdots \to s_n$$

with $D = \{d_1, \ldots, d_m\}$, then the algebra $A[D]$ coincides with the semichain $S(n, m)$ introduced above.

(2) If $A = kQ/I$ is given by the quiver $Q = d_1 \xrightarrow{\alpha} s \xrightarrow{\beta} d_2$ with ideal I generated by the relation $\beta\alpha = 0$ and $D = \{d_1, d_2\}$, then the algebra $A[D]$ is given by the following quiver with relations $\beta^+\alpha^+ = \beta^+\alpha^- = \beta^-\alpha^+ = \beta^-\alpha^- = 0$:



2.6. SEMI-TREE ALGEBRAS

We now introduce the class of algebras we are finally dealing with.

DEFINITION. Let $A = kQ/I$ be a tree algebra and D a set of vertices of A. We say that the algebra $A[D]$ is a *semi-tree algebra* provided the following conditions are satisfied:

(D1) At each vertex of D starts at most one arrow and at each vertex of D stops at most one arrow.

(D2) The ideal I is generated by relations of length 2 or 3.

(D3) If $\varepsilon = a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is one of the generators of the ideal I, then the middle vertex b does not belong to D. Moreover, all other generators of I that contain the arrow α stop in the vertex b, and all generators different from ε that contain the arrow β start in b.

(D4) The generators of I of length 3 have the form $\varepsilon = a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c'$ where c' is an end vertex of Q or dually $\varepsilon' = a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c$ with end vertex a' or they come as a pair ($\varepsilon = a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c', \varepsilon' = a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c$). In each case, the vertices a', b and c' do not be long to D, no other generator of the ideal I contains one of the arrows α or β, and in a' and c' do not start or stop any other arrows.

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(D5) Each convex hereditary subcategory H of A is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$. In case H is of type $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$, the end vertices of H do not belong to D.

PROPOSITION 2.4. *Let B be a tree algebra that does not contain a convex subcategory which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ or to a tubular algebra. If B is derived-tame or if the Euler form of B is non-negative, then the algebra B is derived equivalent to a semi-tree algebra $A[D]$.*

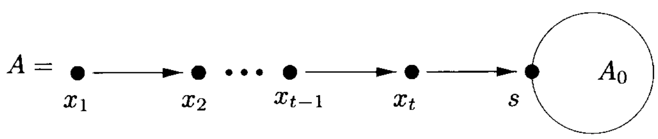
The second part of the proof of our main theorems is worked out in the class of semi-tree algebras:

PROPOSITION 2.5. *Let $A[D]$ be a semi-tree algebra that does not contain a convex subcategory which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ or to a tubular algebra. If $A[D]$ is derived-tame or if the Euler form of $A[D]$ is non-negative, then $A[D]$ is derived equivalent to some semichain $S(n, m)$.*

3. Proof of Proposition 2.5

We are given a tree algebra $A = kQ/I$ with a set of vertices D of A such that the conditions (D1) to (D5) hold. In order to show that the semi-tree algebra $A[D]$ is derived-equivalent to a semichain $S(n, m)$, we proceed by induction on the number of relations of A. We construct the derived equivalence using the operations considered in Lemmas 2.2 and 2.3 above. As these operations commute with the blowing-up procedure, it is sufficient to show that the tree algebra A can be transformed into the chain from Example (1) of Section 2.5; the blowing-up applied afterwards then yields the desired semichain.

LEMMA 3.1. *Let $A = kQ/I$ be an algebra and D a set of vertices of A. Suppose that A is the union of two convex sucategories A_0 and S with $A_0 \cap S = \{s\}$ where S is hereditary with quiver $x_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{t-1}} x_t \xrightarrow{\alpha_t} s$. Assume further that $s \notin D$ and no relation of A starts in one of the vertices x_j. Denote by $A^* = kQ^*/I^*$ the algebra whose quiver Q^* is obtained from Q by replacing the subcategory S by $S^* = x_t \xleftarrow{\alpha_t^*} \cdots \xleftarrow{\alpha_2^*} x_1 \xleftarrow{\alpha_1^*} s$ and whose ideal I^* is generated by the same relations as the ideal I of A. Then $A[D]$ is derived equivalent to $A^*[D]$.*



Proof. If $t = 1$, we apply the APR-tilt in the source x_1 to turn the arrow $x_1 \xrightarrow{\alpha_1} s$ into $x_1 \xleftarrow{\alpha_1^*} s$. This is possible since the neighbour s of x_1 does not belong to D. In case $t > 1$, we use induction on t: We first apply iterated reflections at the sources

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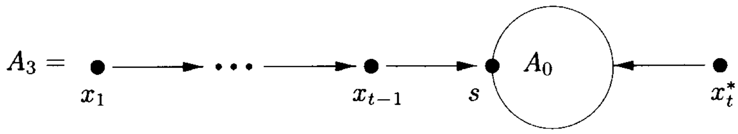
$x_1, \ldots, x_{t-1}$ and obtain $A_1 := S^-_{x_{t-1}} \ldots S^-_{x_1} A$. In A_1, the vertex x_t is a source, and no relation starts in x_t. Thus we can reverse the arrow $x_t \to s$ by an APR-tilt as in the case $t = 1$ and call the resulting algebra A_2. Next, reflecting in the sink x_t and in the iterated sinks $x^*_{t-1}, \ldots, x^*_1$ of A_2, we get $A_3 = S^+_{x^*_1} \ldots S^+_{x^*_{t-1}} S^+_{x_t} A_2$. The algebra A_3 contains A_0 as a convex subcategory with $A_3 \backslash A_0$ formed by the disjoint union of a source x^*_t together with a subquiver $S' = x_1 \overset{\alpha_1}{\to} \cdots \overset{\alpha_{t-2}}{\to} x_{t-1} \overset{\alpha_{t-1}}{\to} s$ that satisfies the conditions of the lemma and has one vertex less than S.



Thus, by induction, we can turn S' as described in the lemma and denote the resulting al gebra by A_4. Finally, we reflect the source x^*_t back and obtain $A^* = S^-_{x^*_t} A_4$ of the desired form. □

Remark. In the proofs of this section, we just specify the reflections or APR-tilts to be applied and describe the resulting algebras. Thus, when proving the lemma above, we leave it to the reader to write the algebra A in the form $A = A_0[M_1] \ldots [M_t]$ and check that, for instance, the algebra $S^-_{x_{t-1}} \ldots S^-_{x_1} A = [M_t] \ldots [M_2] A_0 [M_1]$ has the form claimed above.

LEMMA 3.2. *Let $A = kQ$ be a hereditary algebra with quiver Q of the form $Q = 1 \to 2 \to \cdots \to n$ and D a set of vertices of A. Then $A[D]$ is derived equivalent to $S(n, m)$ with $m = |D|$.*

Proof. Reflections act on the set of vertices of Q as cyclic permutations, thus we may assume that the first t vertices of Q belong to D and the last r vertices do not belong to D. When we know that some vertex of Q belongs to D or to $Q \backslash D$, then we denote it by d_i or by s_j, respectively; otherwise, we use the letter x.

$$Q = d_1 \to \cdots d_t \to s_{t+1} \to \cdots \to x_{n-r-1} \to d_{n-r} \to s_{n-r+1} \to \cdots \to s_n.$$

Now, if $t = m$, then $t + r = n$ and we are done. Otherwise, we present in the following a sequence of reflections and APR-tilts that moves the vertex d_{n-r} to the first place. Thus, we increase the number of vertices at the beginning that belong to D by one. We repeat this movement until $t = m$.

In the first step we shift the vertex d_{n-r} to the beginning by setting $A_1 = S^+_{d_{n-r}} S^+_{s_{n-r+1}} \ldots S^+_{s_n} A$. Then we apply Lemma 3.1 to the subquiver $S = d_{n-r} \to s_{n-r+1} \to \cdots \to s_n$ in order to change the orientation of its arrows and denote the resulting algebra by A_2. It is hereditary with quiver

$$d_{n-r} \leftarrow s_{n-r+1} \leftarrow \cdots \leftarrow s_n \to d_1 \to \cdots d_t \to s_{t+1} \to \cdots \to x_{n-r-1}.$$

Then we set $A_3 = S^+_{d_{n-r}} A_2$ and reverse by Lemma 3.1 the orientation of the subquiver

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$s_{n-r+1} \leftarrow \cdots \leftarrow s_n$ of A_3; the resulting algebra is denoted by A_4. Putting $A_5 = S^+_{d_1} \ldots S^+_{d_t} S^+_{s_{t+1}} \ldots S^+_{x_{n-r-1}} A_4$, we get that A_5 is hereditary with quiver

$$d_1 \to \cdots \to d_t \to s_{t+1} \to \cdots \to x_{n-r-1} \to s_{n-1} \to \cdots \to s_{n-r+1} \to s_n \leftarrow d_{n-r}.$$

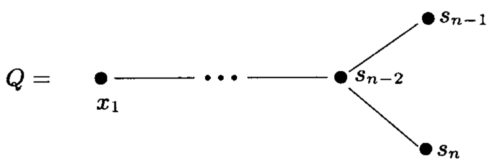
We now just reverse the arrow $s_n \leftarrow d_{n-r}$ of A_5 by an APR-tilt and, with A_6 being the resulting algebra, set $A_7 = S^+_{d_{n-r}} A_6$. This algebra is finally hereditary with quiver

$$d_{n-r} \to d_1 \to \cdots \to d_t \to s_{t+1} \to \cdots \to x_{n-r-1} \to s_{n-1} \to \cdots \to s_{n-r+1} \to s_n\,,$$

thus we increased the number of vertices in D at the beginning by one. $\square$

LEMMA 3.3. *Let $A = kQ$ be a hereditary algebra and D a set of ver tices of A such that $A[D]$ is a semi-tree algebra. Then $A[D]$ is derived equivalent to $S(n, m)$ with $m = |D|$.*

Proof. As $A[D]$ is a hereditary semi-tree algebra, we get by condition (D5) that the quiver of A is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$. Up to derived equivalence, we can even suppose that it is of type $\mathbb{A}_n$: Assume A has the quiver Q where



Then, by (D1) and (D5), the vertices with index $n-2$, $n-1$ and n do not belong to the set D. Moreover, up to an APR-tilt in s_n, we can suppose that the arrows $s_{n-2} - s_{n-1}$ and $s_{n-2} - s_n$ point in the same direction. But now we get the same algebra $A[D]$ if we replace the two vertices s_{n-1} and s_n by one vertex d_{n-1} that belongs to D. The same argument applied to the case $\widetilde{\mathbb{D}}_n$ shows that we can restrict to quivers of type $\mathbb{A}_n$.

By Lemma 3.2, it is sufficient to show that $A[D]$ is derived equivalent to an algebra $A'[D']$ where $A' = kQ'$ is hereditary with a linear oriented quiver Q' of type $\mathbb{A}$. If the quiver $Q = x_1 - x_2 - \cdots - x_l$ of A has no internal source or sink, then we are done. Otherwise, we denote by $s = x_j$ the vertex with minimal index $j > 1$ such that x_j is a sink or a source. Suppose (up to duality) that s is a sink. By condition (D1), the vertex s does not belong to D. Thus, by Lemma 3.1, we can turn the subquiver $x_1 \to \cdots \to x_{j-1} \to s$ of Q into $x_{j-1} \leftarrow \cdots \leftarrow x_1 \leftarrow s$ and thus reduce the number of sinks of Q. The statement follows by induction on the number of sinks or sources of Q. $\square$

Lemma 3.3 proves Proposition 2.5 in case the algebra A is hereditary. Now we start to deal with the case when the algebra A has some relations. The proof is by induction on the number of relations of A, hence it is sufficient to show how one can get rid of one relation. The 'right' choic e which relation should be eliminated first is a bit delicate, and we need to introduce some notation to explain that.

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By the condition (D3) and (D4) in the definition of a semi-tree algebra, each relation determines a trisection of the quiver of A as follows: Given two vertices x, y of the tree Q, we denote by $\triangleright(x, y)$ the subtree of Q with root y pointing towards x, i.e. the interval $[y, x]$ together with all vertices z of Q whose distance from z to x is smaller than the di stance from z to y. In the example below, for instance, the subtree $\triangleright(4, 3)$ with root 3 pointing towards 4 is supported by the vertices $\{3, 4, 5, 6, 7, 10, 11\}$.

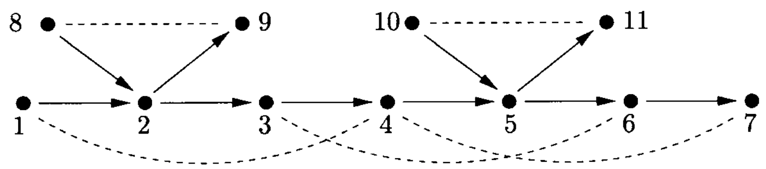
Now let $\varepsilon = a \to b \to c$ be a relation of A of length 2 as considered in (D3). Then we define a convex subcategory N_ε^- of A by $N_\varepsilon^- = \triangleright(a, b)$, and dually we set $N_\varepsilon^+ = \triangleright(c, b)$. Moreover, we define $N_\varepsilon^0 = \bigcup_x \triangleright(x, b)$, where the union runs over al l neighbours x of b except a and c. Then, by construction, the union $N_\varepsilon^- \cup N_\varepsilon^0 \cup N_\varepsilon^+$ covers the whole quiver Q and, by condition (D3), all relations of A different from ε have a support that is completely contained in one of the sets $N_\varepsilon^-, N_\varepsilon^0, N_\varepsilon^+$.

We define a trisection with the same properties for relations of length 3. By condition (D4), we either have one relation $\varepsilon = a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c'$ where c' is an end vertex of Q or dually $\varepsilon' = a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c$ with end vertex a', or there is a pair

$$(\varepsilon = a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c', \qquad \varepsilon' = a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c).$$

In each case, we associate to the datum ε or ε' or $(\varepsilon, \varepsilon')$ a trisection of Q by convex subcategories $N_\varepsilon^-, N_{lon}^0, N_\varepsilon^+$ as follows: We define $N_\varepsilon^- = \triangleright(a, a')$, $N_\varepsilon^+ = \triangleright(c, c')$ and $N_\varepsilon^0 = \bigcup_x \triangleright(x, b)$, where the union runs over all neighbours x of b except a' and c'. Note: N_ε^+ may be empty in case c' is an end vertex of Q, dually for N_ε^-. Then again by condition (D4), all the other relations of A have a support that is completely contained in one of the sets $N_\varepsilon^-, N_\varepsilon^0, N_\varepsilon^+$.

EXAMPLE. We illustrate the constructions above on the algebra $A = kQ/I$, given by the quiver Q below and the relations $\varepsilon(1): 1 \to 2 \to 3 \to 4$, $\varepsilon(2): 3 \to 4 \to 5 \to 6$, $\varepsilon(3): 4 \to 5 \to 6 \to 7$ and $\varepsilon(4): 8 \to 2 \to 9$, $\varepsilon(5): 10 \to 5 \to 11$. Then the subcategory $N_{\varepsilon(1)}^-$, for instance, is empty, whereas $N_{\varepsilon(1)}^0$ is supported by $\{2, 8, 9\}$ and $N_{\varepsilon(1)}^+$ is supported by $\{3, 4, 5, 6, 7, 10, 11\}$.



The following lemma describes which relations can be used in the inductive step.

LEMMA 3.4. *One can choose a relation ε or a pair $(\varepsilon, \varepsilon')$ of relations in A in such a way that at most one of the convex subcategories $N_\varepsilon^-, N_\varepsilon^0, N_\varepsilon^+$ defined above contains some relations.*

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Proof. Let $\varepsilon(1)$ be some relation or $(\varepsilon(1), \varepsilon(1)')$ some pair. If at least two of the categories $N^-_{\varepsilon(1)}$, $N^0_{\varepsilon(1)}$, $N^+_{\varepsilon(1)}$ contain the support of some relation of A, then we choose one containing a relation $\varepsilon(2)$. Now, one of $N^-_{\varepsilon(2)}$, $N^0_{\varepsilon(2)}$, $N^+_{\varepsilon(2)}$ contains the relation $\varepsilon(1)$. If the two others do not contain any relation, we are done. Otherwise, we proceed in the same way. But, since we work with a finite tree Q, this procedure has to stop with some relation $\varepsilon = \varepsilon(n)$ that has the required properties. □

EXAMPLE. If we start in the example above with the relation $\varepsilon(1)$, both sets $N^0_{\varepsilon(1)}$ and $N^+_{\varepsilon(1)}$ contain further relations. Choose one of them, say $N^0_{\varepsilon(1)}$ supporting the relation $\varepsilon(4)$. Now $N^0_{\varepsilon(4)}$ contains relations, but $N^-_{\varepsilon(4)}$ and $N^+_{\varepsilon(4)}$ don't. Thus the relation $\varepsilon(4)$ is a good candidate to start the induction.

We suppose from now on that $A[D]$ is a semi-tree algebra satisfying the assumptions from Proposition 2.5. We start with the inductive step of the proof of proposition 2.5. Let $\varepsilon = a \to b \to c$ be a relation of A of length 2 as considered in (D3) and suppose that N^-_ε and N^0_ε contain no relation.

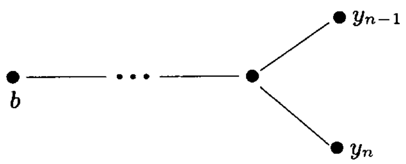
LEMMA 3.5. *Up to derived equivalence, the convex subcategories N^-_ε and N^0_ε chosen above have the form*

$$N^-_\varepsilon = x_1 \to \cdots \to x_t \to a \to b$$

and

$$N^0_\varepsilon = b \to y_1 \to \cdots \to y_s .$$

Proof. Consider first the convex subcategory N^0_ε. It is hereditary, contains the vertex b and by (D5) it is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$. Now, if N^0_ε contains a subquiver of the form $x - b - y_1 - y_2$ (with arbitrary orientation of the arrows), then the full subcategory of $A[D]$ with vertex set $\{a, b, c, x, y_1, y_2\}$ is derived equivalent to $\mathbb{E}_6$, in contradiction to the assumptions of Proposition 2.5. On the other hand, if the vertex b has three direct neighbours $\{x, y, z\}$ in N^0_ε, then the full subcategory of $A[D]$ with vertex set $\{a, b, c, x, y, z\}$ is not derived tame and has no non-negative Euler form. Thus, the quiver of N^0_ε has the form



and by applying the same considerations as in Lemma 3.3 we can suppose that N^0_ε has a quiver $b - y_1 - \cdots - y_s$ of type $\mathbb{A}_{s+1}$. If this quiver is linearly oriented, we are done (change, if necessary, the orientation by Lemma 3.1). Otherwise, let $j < s$ be the maximal index such that the vertex y_j is a source or sink. Then the subcategory

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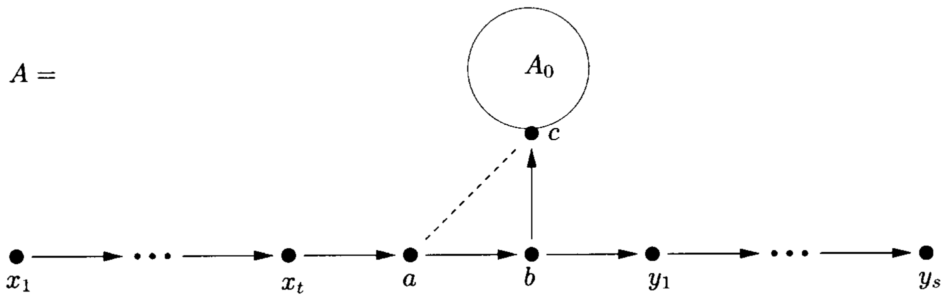
$y_j - \cdots - y_s$ is linearly oriented, and the vertex y_j does not belong to D by (D1). Thus we can apply Lemma 3.1 and reduce the number of sinks or sources until we reach the linear oriented case. For N_ε^- the arguments are the same, except that the vertex a may be in D. But then it is no source by (D1), and we apply the considerations above to the source in N_ε^- with smallest distance to a. □

Thus we can suppose that A has the following form with a convex subcategory A_0 and relation $\varepsilon = a \to b \to c$:

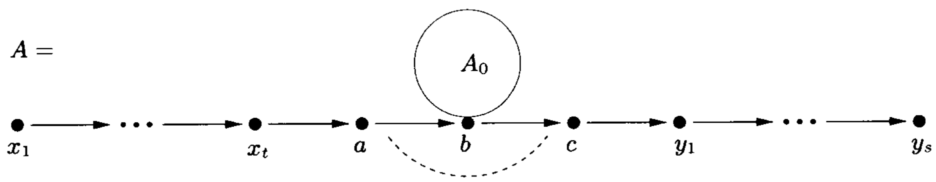


Then the algebra $A_2 := S_a^- S_{x_t}^- \ldots S_{x_1}^- A$ has the form

$$a^* \leftarrow x_t^* \leftarrow \cdots \leftarrow x_1^* \leftarrow y_s \leftarrow \cdots \leftarrow y_1 \leftarrow b \to c - A_0 .$$

It has one relation less than A and is again a semi-tree algebra, hence we can continue by induction.

Now consider the case when $\varepsilon = a \to b \to c$ is a relation of A of length 2 as considered in (D3) and N_ε^- and N_ε^+ contain no relation. Then, after a suitable change of the subcategories N_ε^- and N_ε^+ as in Lemma 3.5, the algebra A has the form



with some convex subcategory A_0. Now set $A_2 := S_a^- S_{x_t}^- \cdots S_{x_1}^- A$. In the algebra A_2, the subcategory $S = b \to c \to y_1 \to \cdots \to y_s$ satisfies the conditions of Lemma 3.1, thus we can change the orientation in S and denote the resulting algebra by A_3. Then the algebra $A_4 := S_{x_1^*}^+ \ldots S_{x_t^*}^+ S_{a^*}^+ A$ has the form

$$x_1 \to \cdots x_t \to a \to c \to y_1 \to \cdots \to y_s \to b - A_0 .$$

It has one relation less than A and is again a semi-tree algebra, hence we can continue by induction.

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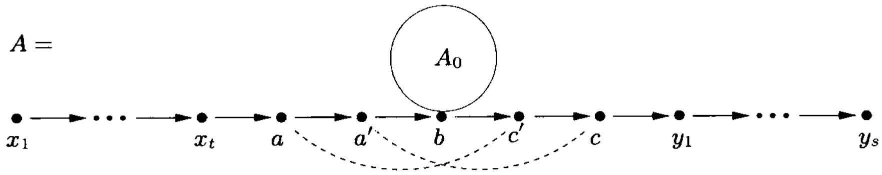
The case of a relation of length 3 is slightly more complicated. We just deal here with a pair $(\varepsilon, \varepsilon')$ of relations of length 3 on the subquiver $a \to a' \to b \to c' \to c$ of A, keeping in mind that one of the vertices a or c (and hence one of the relations $\varepsilon, \varepsilon'$) may not exist. First consider the case when both convex subcategories N_ε^- and N_ε^+ are hereditary. Then we may assume as in Lemma that A has the following form with a convex subcategory A_0:

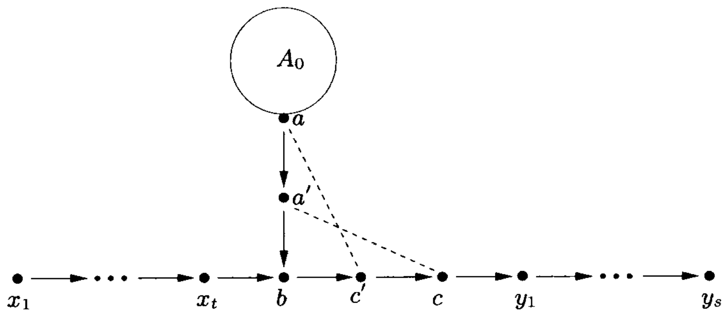


Now set $A_2 := S_c^+ S_{y_1}^+ \dots S_{y_s}^+ S_a^- S_{x_t}^- \dots S_{x_1}^- A$. The algebra A_2 contains A_0 as a convex subcategory, and one component of $A_2 \backslash A_0$ is of the form $a' \to b \to c'$. By condition (D4), the vertices a' and c' do not belong to D and are not involved in any relation of A_2. Thus we can turn the arrow $a'ob$ by an APR-tilt in a' and obtain $a' \leftarrow b \to c'$. But then we obtain the same blowing-up when we replace the two arrows $a' \leftarrow b \to c'$ by one arrow $b \to b'$ with $b' \in D$. Let us call this algebra A_3. Now set $A_4 := S_{y_s^*}^- \cdots S_{y_1^*}^- S_{c^*}^- A_3$, then A_4 has a component of $A_4 \backslash A_0$ of the form $b \to b' \to c \to y_1 \to \cdots \to y_s$, which we can turn by Lemma 3.1 into $b \leftarrow y_s \leftarrow \cdots \leftarrow y_1 \leftarrow c \leftarrow b'$. The resulting algebra being A_5, we finally set $A_6 := S_{x_1^*}^+ \dots S_{x_t^*}^+ S_{a^*}^+ A_3$. This algebra has the form

$$A_0 - b \leftarrow y_s \leftarrow \cdots \leftarrow y_1 \leftarrow c \leftarrow b' \leftarrow a \leftarrow x_t \leftarrow \cdots \leftarrow x_1 \,,$$

which was what we wanted.

The last case we have to deal with is when there is a pair $(\varepsilon, \varepsilon')$ of relations of length 3 on the subquiver $a \to a' \to b \to c' \to c$ of A and the convex subcategories N_ε^0 and N_ε^+ are hereditary. Then we may assume as in Lemma 3.4 that A has the following form with a convex subcategory A_0:



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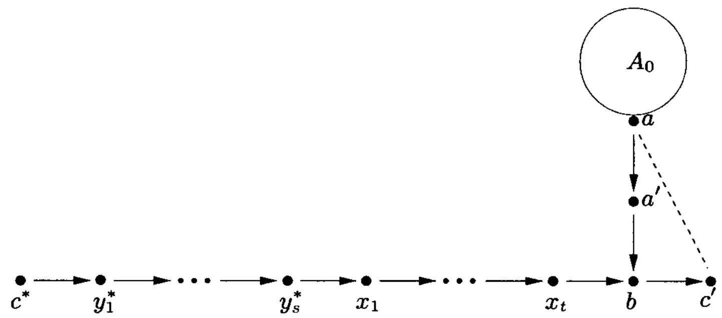
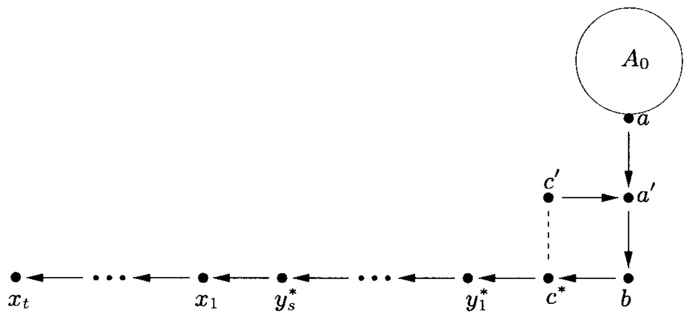
Now set $A_2 := S^+_c S^+_{y_1} \cdots S^+_{y_s} A$. This algebra has the following form, where all relations except $\varepsilon: a \to a' \to b \to c'$ are contained in A_0:



We turn the subalgebra $c^* \to \cdots \to b$ by Lemma 3.1 and call this algebra A_3. Then the algebra $A_4 := S^+_{c'} A_3$ has the following form, where all relations except $\eta: c' \to a' \to b \to c^*$ are contained in A_0:



We consider $A_5 := S^+_{c^*} S^+_{y_1^*} \cdots S^+_{y_s^*} S^+_{x_1} \cdots S^+_{x_t} A_4$. There, one component S of $A_5 \backslash A_0$ is formed by the vertices $\{a, a', b, c'\}$. As we have done before, we turn the direction of the arrow $c' \to a'$ and view the two arrows starting in a' as one arrow, thus we replace S by $S' = a \to a' \to b'$ with $b' \in D$ and denote this algebra by A_6. The algebra $S^-_{x_t^*} \ldots S^-_{x_1^*} S^-_{y_s} \ldots S^-_{y_1} S^-_c A$ then finally has the desired form

$$A_0 - a \to a' \to b' \to c^* \to y_1^* \to \cdots y_s^* \to x_1 \to \cdots \to x_t .$$

This finishes the last case in the proof of Proposition 2.5. □

4. Proof of Proposition 2.4

Let B be a tree algebra that does not contain a convex subcategory which is derived equivalent to some hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ or to a tubular algebra. Suppose further that B is derived-tame or the Euler form of B is non-negative. Then we have to show that B is derived equivalent to a semi-tree algebra $A[D]$.

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Note that if the Euler form of B is non-negative of corank 0 or 1, then B is derived equivalent to a hereditary algebra of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$, respectively ([4]). These algebras are described in [3, 20] and [26]. Comparing with these lists, we see that not all these algebras are semi-tree algebras: There do occur relations of length greater than 3. Thus our first aim is to eliminate these long relations. Let therefore $\rho: a_0 \to a_1 \dots a_{n-1} \to a_n$ be some relation of B. We say that ρ *is of type* (I) if

(R1) Both a_0 and a_n are end vertices of B and
(R2) The projective B-module $P(a_0)$ has support $\{a_0, \dots, a_{n-1}\}$ and the injective B-module $I(a_n)$ has support $\{a_1, \dots, a_n\}$.

Otherwise, we say that ρ is a relation of type (II).

LEMMA 4.1. *Let R be an arbitrary tree algebra which has some relation of type (I). Then there is a tree algebra S which is derived equivalent to R and has strictly less relations.*

Proof. Let $\rho: a_0 \to a_1 \dots a_{n-1} \to a_n$ be a relation of R of type (I). As a_0 and a_n are end vertices, the reflections $S^-_{a_o}R$ and $S^+_{a_n}R$ in these vertices are defined. Since ρ is of type (I), the algebras $S^-_{a_o}R$ and $S^+_{a_n}R$ are tree algebras of the following form: Reflecting in a_0 produces an arrow $a_{n-1} \to a_0^*$ and a relation $\sigma_x^*: x \to a_i \to a_{i+1} \dots a_{n-1} \to a_0^*$ for every x in R with a relation $\sigma_x: x \to a_i \to a_{i+1} \dots a_{n-1} \to a_n$. Dually, reflecting in a_n leads to an arrow $a_n^* \to a_1$ and to a relation $\tau_y^*: a_n^* o a_1 \dots a_{j-1} \to a_j \to y$ for every y in R with a relation $\tau_y: a_0 \to a_1 \dots a_{j-1} \to a_j \to y$. If we denote by ξ the number of relations σ_x and by η the number of relations τ_y, then reflecting in a_0 changes the number of relations by $\xi - \eta - 1$ whereas reflecting in a_n changes the corresponding number by $\eta - \xi - 1$. Since at least one of these numbers is strictly smaller than 0, the lemma holds. □

By iterated use of the lemma above we can suppose from now on (up to derived equivalence) that B has no relations of type (I). For later reference, we collect the conditions on B and say that B *satisfies condition* (E) if

(E1) all relations of B are of type (II),
(E2) B is derived-tame or the Euler form of B is non-negative and
(E3) B does not contain a convex subalgebra which is derived equivalent to some $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ or to a tubular algebra.

Beforehand, we show that some particular algebras cannot occur under condition (E2) and (E3):

LEMMA 4.2. *Let E be the algebra with quiver $x \leftarrow a_0 \to a_1 \dots a_{n-1} \to a_n$ and relation $a_0 \to \dots \to a_n$ or the algebra with quiver $x \to a_0 \to a_1 \dots a_{n-1} \to a_n$ and relation $a_0 \to \dots \to a_n$ and possibly one relation $x \to \dots \to a_i$ for some*

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$1 \leqslant i \leqslant n-1$. *Then E is derived equivalent to* $\mathbb{E}_{n+2}$ *in case* $4 \leqslant n \leqslant 6$, *to* $\widetilde{\mathbb{E}}_8$ *in case* $n = 7$ *and to a wild hereditary algebra in case* $n \geqslant 8$.

Proof. Consider the reflection $S^+_{a_n}E$ of E in the sink a_n. In case there is an arrow $x \leftarrow a_0$ or there is no relation $x \to \cdots \to a_i$, the algebra $S^+_{a_n}E$ is of the form $\mathbb{E}_{n+2}$ in case $4 \leqslant n \leqslant 6$, or $\widetilde{\mathbb{E}}_8$ in case $n = 7$ and it is wild in case $n \geqslant 8$. If there is a relation $x \to \cdots \to a_i$, then it is easy to see that the algebra $S^-_x S^+_{a_n}E$ is tilted of the type stated above. □

To prove Proposition 2.4, it is sufficient to show the following claim: *Let B be a tree algebra satisfying conditions (E1) to (E3). Then B is of the form* $A[D]$ *for some tree algebra A with some set of vertices D satisfying (D1) to (D5).*

Suppose it is possible to write the algebra B as a blowing-up $A[D]$. Since B is a tree algebra, no internal vertex of A belongs to the set D, hence condition (D1) holds automatically. The same is true for condition (D5): If H is some convex hereditary subcategory of B, then H is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$ due to condition (E3). So we concentrate in the following on the relations which can occur in B.

LEMMA 4.3. *Let B be a tree algebra satisfying condition (E). Then all relations of B have length 2 or 3.*

Proof. Let $\rho: a_0 \to a_1 \dots a_{n-1} \to a_n$ be a relation of B and suppose $n \geqslant 4$. If a_0 is no end vertex and connected with some x, then the convex subalgebra of B with support $\{x, a_0, a_1, \dots, a_n\}$ is by Lemma 4.2 derived equivalent to some $\mathbb{E}_p$ or to $\widetilde{\mathbb{E}}_8$ or to a wild hereditary algebra, in contradiction to conditions (E2) or (E3). Now suppose that both a_0 and a_n are end vertices. Since ρ is of type (II) by condition (E1), there is (up to duality) an arrow $a_i \to y$ with $y \neq a_{i+1}$ for some $i \in \{2, \dots, n-1\}$ such that the path $a_0 \to \cdots \to y$ does not vanish in B. Now consider the convex subalgebra E of B with support $a_0, \dots, a_n, y$. Reflecting in y leads to an algebra $S^+_y E$ that has the form considered in Lemma 4.2, in contradiction to (E2) or (E3). Therefore, the only possible values for n are 2 and 3. □

We now start to investigate the possible combinations of relations.

LEMMA 4.4. *Let B be a tree algebra satisfying condition (E). Let ρ and σ be relations of B and suppose that ρ has length 3. If ρ and σ share precisely one arrow α of B, then α is the start arrow of ρ and the end arrow of σ or vice versa.*

Proof. Suppose first that σ has length 2. Then we have, up to duality, to exclude the cases when $\sigma = a_0 \stackrel{\alpha}{\to} a_1 \to a_2$ and $\rho = a_0 \stackrel{\alpha}{\to} a_1 \to b_1 \to b_2$ or $\rho = b_1 \to a_0 \stackrel{\alpha}{\to} a_1 \to b_2$. In both cases, the assumption that ρ is of type (II) yields the existence of an additional vertex y connected with one of the vertices of ρ. Using the list [7] of tree algebras which are derived equivalent to $\mathbb{E}_6$, it is easy to see that in each case the convex subalgebra of B formed by the vertices $\{a_0, a_1, a_2, b_1, b_2, y\}$ is derived equivalent to $\mathbb{E}_6$, in contradiction to condition (E3).

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Now suppose that σ and ρ both have length 3. Up to duality, we have to exclude the following cases: Either α is the first arrow of $\sigma = a_0 \stackrel{\alpha}{\to} a_1 \to a_2 o a_3$ and $\rho = a_0 \stackrel{\alpha}{\to} a_1 \to b_1 \to b_2$ or $\rho = b_1 \to a_0 \stackrel{\alpha}{\to} a_1 \to b_2$, or else α is the middle arrow of σ, hence $\sigma = a_0 \to a_1 {}^{\alpha \to a_2 \to a_3}$ and $\rho = b_1 \to a_1 \stackrel{\alpha}{\to} a_2 \to b_2$. In each case, we find the convex subalgebra of B formed by the vertices $\{a_0, a_1, a_2, a_3, b_1, b_2\}$ in the list [7] of tree algebras which are derived equivalent to $\mathbb{E}_6$, in contradiction to condition (E3). □

LEMMA 4.5. *Let B be a tree algebra satisfying condition (E). Suppose that $\rho: a_1 \to b_1 \to b_2 \to b_3$ is a relation of length 3. Then, up to duality, either b_3 is an end vertex of B or there is a relation $\sigma: b_1 \to b_2 \to b_3 \to c_1$ and no relation $\tau: a_0 \to a_1 \to b_1 \to b_2$.*

Proof. If ρ contains an end vertex, we may assume up to duality that this is the vertex b_3. Otherwise, if a_1 is connected with an additional vertex a_0 and b_3 with some vertex c_1, then we consider the convex subalgebra of B formed by the vertices $\{a_0, a_1, b_1, b_2, b_3, c_1\}$. From the list [7] it follows easily that there must be precisely one relation of length 3 sharing two arrows with ρ, up to duality this is the relation σ from the lemma. □

In Figure 1 we show all configurations of relations that may occur in B. Here, the ringed vertices, denoted by $\circ$, are supposed to have no other neighbours than those shown in the figure, whereas the starred vertices, denoted by $\star$, are possibly connected with vertices of B that are not shown in Figure 1. The first three configurations describe all possibilities of the blowing-up of a relation $\varepsilon = a \to b \to c$ in A to a tree algebra $A[D]$. The remaining configurations then descri be all possible relations of length 3 in a semi-tree algebra of the form $A[D]$.

Thus, if this are all possible relations in the algebra B, then B is clearly of the form $A[D]$ where A is a tree whose internal vertices coincide with the internal vertices of B, and whose end vertices belong to D if there is in B a corresponding pair of end vertices (a_1, a_2) or (c_1, c_2) of Figure 1. Thus, the Proposition 4.6 below shows that B is of the form $A[D]$ where A and D satisfy conditions (D1) to (D5).

PROPOSITION 4.6. *Let B be a tree algebra satisfying condition (E), and let ρ be a relation of B. Then ρ is given by one of the diagrams sho wn in Figure 1 (or their duals). Moreover, if σ is another relation of B that contains some of the arrows shown in this diagram, then σ is either one of the relations shown in the diagram, or it stops or starts in the vertex b in case ρ has length 2, and it stops in the vertex b_1 or starts in the vertex b_3 in case ρ has length 3.*

Proof. Let ρ be a relation of B. By Lemma we know that ρ has length 2 or 3. Suppos e first that $\rho = a_1 \stackrel{\alpha_1}{\to} b \stackrel{\beta_1}{\to} c_1$ has length 2. Then we know by Lemma 4.4 that each relation of B that contains one of the arrows of ρ and does *not* start or stop in the vertex b is of the form $\sigma = a_1 \stackrel{\alpha_1}{\to} b \stackrel{\beta_2}{\to} c_2$ with $c_2 \neq c_1$ or $\tau = a_2 \stackrel{\alpha_2}{\to} b \stackrel{\beta_1}{\to} c_1$ with $a_2 \neq a_1$.

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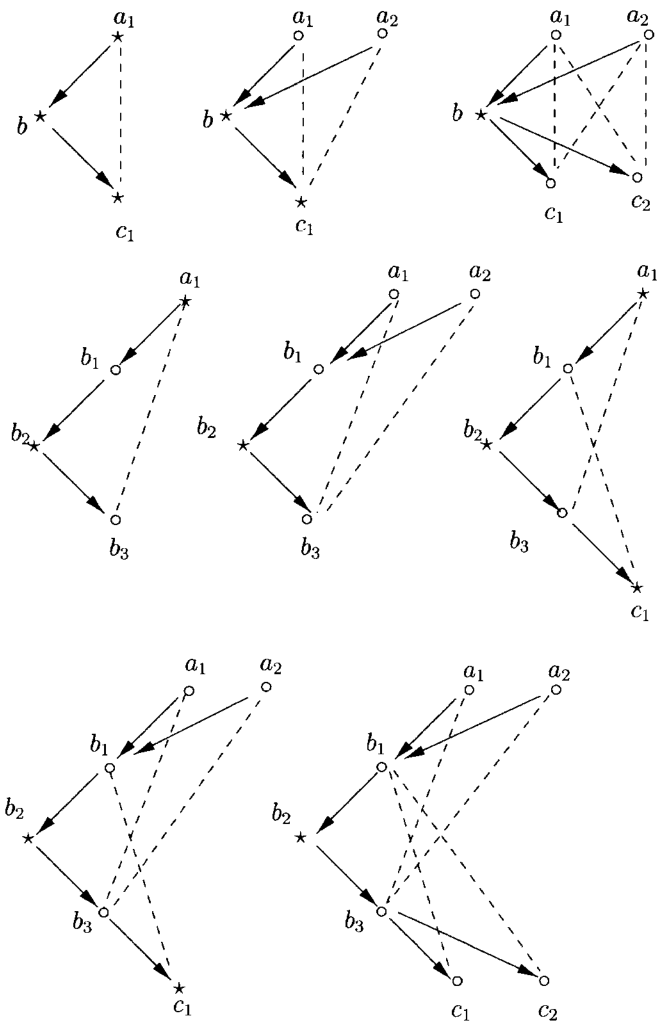

Figure 1. The possible relations of B.

Case 1: If there are no such relations σ and τ, then ρ is given by the first diagram shown in Figure 1, and the assertions of the proposition hold.

Case 2: Now we suppose that there is one relation τ, but no relation σ and show that we arrive at the second diagram from Figure 1: As before, by Lemma 4.4 each relation of length 3 that contains one of the arrows α_1, α_2 or β_1 either stops or starts in b. Since we assume that we have no relation of the form σ and on ly one relation of the form τ, there is only one possibility how a relation can contain one of the arrows α_1, α_2 or β_1, but not stop or start in b: This is a relation $\sigma' = a_2 \xrightarrow{\alpha_2} b \xrightarrow{\beta_3} c_3$ such that the product $\beta_3\alpha_1$ does not vanish in B. In this case, we make

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use of the fact that the relation τ is of type (II), which yields the existence of an additional vertex y. Using the list [7], it is then easy to see that in each case the convex subalgebra of B formed by the vertices $\{a_1, a_2, b, c_1, c_3, y\}$ is derived equivalent to $\mathbb{E}_6$, in contradiction to condition (E3). Hence, we are in the situation of the second diagram from Figure 1, and the proposition holds if we can also show that both a_1 and a_2 are end vertices. Suppose therefore that a_1 is connected with an additional arrow y_1. Then we use the fact that the relation τ is of type (II), hence ther e must exist an additional vertex y_2. Using the list [7], it is then easy to see that in each case the convex subalgebra of B formed by the vertices $\{a_1, a_2, b, c, y_1, y_2\}$ is derived equivalent to $\mathbb{E}_6$, in contradiction to condition (E3).

Case 3: If both σ and τ occur, we obtain the third diagram in Figure 1: By the considerations from the previous case it follows that there must be a relation $\rho': a_2 \xrightarrow{\alpha_2} b \xrightarrow{\beta_2} c_2$, and it only remains to show that a_1, a_2, c_1 and c_2 are end vertices of B. Suppose there is an additional vertex y connected to a_1, and consider the convex subalgebra C of B formed by $\{a_1, a_2, b, c_1, c_2, y\}$. Obviously, the algebra $S^+_{c_2} S^+_{c_1} C$ is tilted of a wild hereditary algebra, in contradiction to (E2).

The last thing we have to show when ρ has length 2 is that there cannot occur more than one relation of type u (or dually, of type σ). So suppose there are relations $\tau = a_2 \xrightarrow{\alpha_2} b \xrightarrow{\beta} c_1$ and $\tau' = a_3 \xrightarrow{\alpha_3} b \xrightarrow{\beta} c_1$. Then we deduce a contradiction from the fact that ρ, τ and τ' are of type (II): In case c_1 is no end vertex, hence connected with some y, consider the convex subalgebra C of B formed by $\{a_1, a_2, a_3, b, c_1, y\}$. Then the algebra $S^-_{a_1} S^-_{a_2} S^-_{a_3} C$ is tilted of a wil d hereditary algebra, in contradiction to condition (E2). If one of the following cases occurs: either a_1 is no end vertex, or there is an arrow $\gamma: y \to b$ such that the product $\beta\gamma$ does not vanish, or there is an arrow $\gamma: b \to y$ such that the products $\gamma\alpha_1, \gamma\alpha_2$ and $\gamma\alpha_3$ do not vanish, then we define a convex subalgebra C in the same way as above and see that $S^+_{c_1} C$ is tilted of a wild hereditary algebra. If there are finally se veral arrows $\gamma_i: b \to y_i$ such that for each of the vertices a_j some path to y_i does not vanish in B, then we find a convex subalgebra of B which is derived equivalent to $\mathbb{E}_6$.

This finishes the discussion when ρ has length 2, and we now turn to the case of a relation $\rho: a_1 \to b_1 \to b_2 \to b_3$. We know from Lemma 4.4 and Lemma 4.5 that, up to duality, each relation of B that shares common arrows with ρ and th at does not stop in b_1 must be a length 3 relation sharing two arrows with ρ. Moreover, if there is a relation sharing the last two arrows with ρ, then there is no relation sharing the first two arrows with ρ. Thus, we only have to consider the following additional relations:

$$\rho': a_2 \to b_1 \to b_2 \to b_3, \qquad \sigma: b_1 \to b_2 \to b_3 \to c_1 \quad \text{and}$$
$$\sigma': b_1 \to b_2 \to b_3 \to c_2.$$

As in the first half of the proof, we distinguish now several cases according to the

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existence of these relations and show that we arrive at the corresponding diagram from Figure 1.

Case 4: If none of the relations ρ', σ and σ' occurs, then we obtain diagram 4 from Figure 1, the vertex b_3 is an end vertex by our choices and all the conditions of the proposition concerning additional relations hold by Lemma 4.4 and 4.5. Thus it only remains to show that b_1 has only the two direct neighbours a_1 and b_2 in B: Either a_1 has a neighbour a_0, then an additional neighbour y of b_1 would lead to a convex subalgebra $\{a_0, a_1, b_1, b_2, b_3, y\}$ which is derived equivalent to $\mathbb{E}_6$. Otherwise, if a_1 is an end vertex, we can exchange b_1 and b_2 up to duality. Hence, the case to be considered is when *both* b_1 and b_2 have an additional neighbour, say y_1 and y_2 in B. Then the convex subalgebra $\{a_1, b_1, b_2, b_3, y_1, y_2\}$ is derived equivalent to $\mathbb{E}_6$.

Case 5: If ρ' occurs, but σ and σ' not, then we are in the situation of diagram 5. Again, by our choice of orientation, b_3 is an end vertex and there are no other relations possible involving one of the given arrows. If b_1 or a_1 would have an additional neighbour y in B, we denote by C the convex subalgebra of B formed by $\{a_1, a_2, b_1, b_2, b_3, y\}$. Then the reflection $S^+_{b_3}B$ is tilted of a wild hereditary algebra.

Case 6: If σ occurs, but ρ' and σ' not, then we are in the situation of diagram 6. As in case 4, everything works for the relations, and the vertex b_1 has no additional neighbour y in B, since otherwise the convex subalgebra formed by $\{a_1, b_1, b_2, b_3, c_1, y\}$ is derived equivalent to $\mathbb{E}_6$.

Case 7: If σ and ρ' occur, but σ' not, we are in the situation of diagram 7, and Case 8: If σ,σ' and ρ' occur, we are in the situation of diagram 8. Both these cases behave analogous to case 5.

We finally have to show that there are not three relations sharing their two end arrows. So suppose we have the relations ρ, ρ' and a third one $\rho'': a_3 \to b_1 \to b_2 \to b_3$. If C denotes the convex subalgebra C of B formed by $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, then the reflection $S^+_{b_3}B$ is a wild hereditary algebra. □

Acknowledgement

We thank P. Dräxler and C. M. Ringel for helpful discussions concerning this note.

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