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Positive topological entropy for monotone recurrence relations

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Abstract. We associate the topological entropy of monotone recurrence relations with the Aubry–Mather theory. If there exists an interval $[\rho_0, \rho_1]$ such that, for each $\omega \in (\rho_0, \rho_1)$, all Birkhoff minimizers with rotation number ω do not form a foliation, then the diffeomorphism on the high-dimensional cylinder defined via the monotone recurrence relation has positive topological entropy.

1. *Introduction*

In this paper, we study a criterion of positive topological entropy for a class of dynamical systems which generalizes the class of monotone twist maps on the two-dimensional cylinder. In fact, we discuss diffeomorphisms defined on the high-dimensional cylinder via monotone recurrence relations; see [**1**, §10].

Let $1 \le r \in \mathbb{N}$ be a natural number indicating the range of interactions between particles. Let $h \in C^2(\mathbb{R}^{r+1}, \mathbb{R})$ satisfy some hypotheses specified in §2. For each configuration $\mathbf{x} = (x_i) \in \mathbb{R}^{\mathbb{Z}}$, define the formal Lagrangian $W(\mathbf{x}) = \sum_{i\in\mathbb{Z}} h_i(\mathbf{x})$, where $h_i(\mathbf{x}) = h(x_i, \ldots, x_{i+r})$, $i \in \mathbb{Z}$.

A monotone recurrence relation is defined by finding a stationary point of W:

$$\partial_j W(\mathbf{x}) = \sum_{i=j-r}^{j} \partial_j h_i(\mathbf{x}) = 0, \qquad j \in \mathbb{Z}. \tag{1.1}$$

We call h the generating function of the monotone recurrence relation (1.1). Indeed, for $r = 1$, h is the generating function of a monotone twist map; see [**6**] or [**1**, §2]. From the point of view of physical applications, the generating function h describes the interaction of a particle with its neighborhood and the Lagrangian W denotes the energy of a system of particles.

By the twist condition of h we can define (see §2 for details) a diffeomorphism φ_h: $\mathbb{R}^{2r} \to \mathbb{R}^{2r}$ from monotone recurrence relation (1.1). Furthermore, by the periodicity condition of h we define on the $2r$-dimensional cylinder $S^1 \times \mathbb{R}^{2r-1}$ a diffeomorphism $\bar{\varphi}_h$, which is a generalization of monotone twist maps.

The main aim of this paper is to discuss a criterion of positive topological entropy for $\bar{\varphi}_h$.

It is well known that if a diffeomorphism possesses a transversal homoclinic point, then it has a horseshoe and hence positive topological entropy. If homoclinic points are not transversal, further criteria are provided in [**8**] along with analogous results for heteroclinic points; see also [**16**], where topologically crossing homoclinic points are considered.

Another way of obtaining positive topological entropy (see [**24**, Theorem 7.2]) is to construct an invariant set restricted to which the system is semi-conjugate to the full shift of N-symbols; see also [**14**].

A practical approach to create positive topological entropy is to use the concept of anti-integrability [**3**]. For example, it was shown [**11**] by applying the implicit function theorem that a billiard system generically admits a set of non-degenerate anti-integrable orbits corresponding bijectively to a topological Markov chain of arbitrarily large topological entropy.

We remark that the anti-integrable limit method is applicable to those which are far from integrable systems. However, the diffeomorphisms we are considering may be close to integrable systems. For example, consider the standard map on the cylinder

$$(x, y) \mapsto (x + y + k \sin 2\pi x,\ y + k \sin 2\pi x).$$

The case with large k corresponds to an anti-integrable limit [**3**], while small k corresponds to a nearly integrable system. What we are concerned with is, roughly speaking, under what conditions the generalized standard map with k not necessarily large defined on the high-dimensional cylinder carries positive topological entropy.

To guarantee positive topological entropy for monotone recurrence relations, Angenent [**1**] provided a criterion by showing the existence of two solutions of (1.1) exchanging rotation numbers.

Assume $\mathbf{x}^1 = (x_n^1)$ and $\mathbf{x}^2 = (x_n^2)$ are two solutions of (1.1). It is said they exchange rotation numbers if

$$\lim_{n\to+\infty} \frac{x_n^1}{n} \geq \omega_1, \quad \lim_{n\to-\infty} \frac{x_n^2}{n} \geq \omega_1, \quad \lim_{n\to+\infty} \frac{x_n^2}{n} \leq \omega_0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{x_n^1}{n} \leq \omega_0 \tag{1.2}$$

hold for some $\omega_0 < \omega_1$. For a more general definition, see [**1**, §6].

A criterion presented in [**1**] shows that if there exist two solutions of (1.1) exchanging rotation numbers, then the diffeomorphism $\bar{\varphi}_h$ defined by (1.1) has positive topological entropy; see [**1**, Theorem 7.1].

For exact area-preserving monotone twist maps Angenent [**2**] has obtained the following beautiful theorem.

PROPOSITION. *Let A be the annulus $S^1 \times [0, 1]$, and $f : A \to A$ be an area-preserving twist homeomorphism of A. Let $\rho_0 < \rho_1$ be the rotation numbers of f restricted to the boundaries. If the topological entropy $h_{\text{top}}(f)$ of f vanishes, then f must have an invariant circle of rotation number ω, for any $\omega \in (\rho_0, \rho_1)$.*

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In fact, what Angenent proved in [**2**] is that if one of the invariant circles of f is missing, which is equivalent to the existence of a Birkhoff region of instability, then the topological entropy is positive.

As mentioned in [**2**], although there is an alternative way of proving the above proposition by using a result of [**10**] on the existence of non-Birkhoff periodic orbits, Angenent gave an 'elementary' proof by combining the criterion in [**1**] and a theorem of Birkhoff providing finite orbit segments which stay close to the boundaries of a Birkhoff region of instability.

In fact, for monotone twist maps, Mather's connecting theorem [**17**] provides a general way of constructing two orbits exchanging rotation numbers in a Birkhoff region of instability.

A compact region on the cylinder is called a Birkhoff region of instability of an exact area-preserving twist map f if it is f-invariant and its boundaries consist of two invariant circles and no other invariant circles in between. Let ρ_0 and ρ_1 denote the rotation numbers of f restricted to the lower and upper boundaries, respectively.

According to the Aubry–Mather theory, for each $\omega \in (\rho_0, \rho_1)$, there is an invariant set Σ_ω, called the Aubry–Mather set, such that every orbit in Σ_ω has rotation number ω. If ω is irrational, then Σ_ω is a Cantor set. Mather's connecting theorem [**17**] says that given any two Aubry–Mather sets Σ_{ω_0} and Σ_{ω_1} inside a Birkhoff region of instability, there is a trajectory α-asymptotic to Σ_{ω_0} and ω-asymptotic to Σ_{ω_1}. This immediately leads to the existence of two orbits exchanging rotation numbers and hence positive topological entropy by Angenent's criterion [**1**].

In the present paper, we shall investigate an analogous problem of positive topological entropy for diffeomorphisms defined on the high-dimensional cylinder via monotone recurrence relation (1.1).

We remark that for diffeomorphisms on the high-dimensional cylinder there is no analogue of the Birkhoff region of instability. However, we know that for monotone twist maps the existence of a Birkhoff region of instability implies the non-existence of invariant circles with rotation number $\omega \in (\rho_0, \rho_1)$, which is equivalent to $p_0(\mathscr{M}_\omega) \neq \mathbb{R}$ (see the penultimate paragraph of [**6**, §4]) for each $\omega \in (\rho_0, \rho_1)$, where $\mathscr{M}_\omega$ denotes the set of minimal energy configurations (also called global minimizers or simply minimizers) with rotation number ω and p_0 denotes the projection $p_0(\mathbf{x}) = x_0$.

Now if the generating function h satisfies conditions (H1)–(H4) in §2, then we have the corresponding Aubry–Mather theory for monotone recurrence relation (1.1) [**9**, **12**, **13**, **15**, **19**]: for each $\omega \in \mathbb{R}$, there exists a Birkhoff minimizer with rotation number ω. Let $\mathscr{M}_\omega$ denote the set of all Birkhoff minimizers with rotation number ω of monotone recurrence relation (1.1). Then the generalization of Angenent's theorem in [**2**] to monotone recurrence relation (1.1), which is the main conclusion of this paper, can be stated as follows.

THEOREM A. *If there exist $\rho_0 < \rho_1$ such that, for each $\omega \in (\rho_0, \rho_1)$, $p_0(\mathscr{M}_\omega) \neq \mathbb{R}$, then the diffeomorphism $\bar{\varphi}_h$ defined by (1.1) has positive topological entropy.*

We should emphasize that there are at least two essential differences between monotone twist maps and monotone recurrence relations with long range of interactions (or the corresponding high-dimensional cylinder diffeomorphisms). First, if a monotone twist map has an invariant circle, then the cylinder is divided into two connected components, each of which is invariant. There is no such property for high-dimensional cylinder diffeomorphisms. Second, Aubry's lemma [**4**] implies that each minimal energy configuration for twist maps is Birkhoff, but we do not have such a property for general monotone recurrence relations. In fact, it was observed in [**9**] that there are minimizers that are not Birkhoff.

The idea of constructing two solutions of (1.1) exchanging rotation numbers is borrowed from [**23**], in which the gradient flow is investigated in configuration space with bounded action. We should mention that the method in [**23**] depends heavily on Aubry's lemma that two minimizers cross at most once, which we do not have for general monotone recurrence relations.

In order to make the approach of [**23**] work for monotone recurrence relations, we shall first establish the following result: for monotone recurrence relation (1.1), each minimizer with bounded action is Birkhoff (Theorem 3.15). We remark that by proving such a conclusion we in fact give as a byproduct an affirmative answer to a question of Blank in the case of monotone recurrence relations; see the last paragraph of [**9**, §1]. Note that a similar question for elliptic partial differential equations on the torus has also been posed by Bangert; see [**7**, §8].

Based on this result, we construct in configuration space with bounded action two solutions of (1.1) exchanging rotation numbers using the method of Slijepčević in [**23**] and hence arrive at the conclusion by Angenent's criterion in [**1**].

2. *Preliminaries*

We say that $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ is a stationary configuration if $\mathbf{x}$ is a solution of (1.1). We denote by $\mathcal{S}$ the set of all stationary configurations, $\mathcal{S}^+$ the set of all configurations $\mathbf{x}$ such that $\partial_i W(\mathbf{x}) \geq 0$ for each $i \in \mathbb{Z}$, and $\mathcal{S}^-$ the set of all configurations $\mathbf{x}$ satisfying $\partial_i W(\mathbf{x}) \leq 0$ for each $i \in \mathbb{Z}$.

A partial order '$\leq$' in $\mathbb{R}^{\mathbb{Z}}$ is defined as follows. For $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n) \in \mathbb{R}^{\mathbb{Z}}$, we say that $\mathbf{x} \leq \mathbf{y}$ if $x_n \leq y_n$ for all $n \in \mathbb{Z}$. We say that $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. We say that $\mathbf{x} \ll \mathbf{y}$ if $x_n < y_n$ for all $n \in \mathbb{Z}$. Two configurations $\mathbf{x} \neq \mathbf{y}$ are said to be strictly ordered if $\mathbf{x} \ll \mathbf{y}$ or $\mathbf{y} \ll \mathbf{x}$.

We say that $\mathbf{x} <_\alpha \mathbf{y}$ ($\mathbf{x} <_\omega \mathbf{y}$, $\mathbf{x} >_\alpha \mathbf{y}$, $\mathbf{x} >_\omega \mathbf{y}$), if there exists n_0 such that $x_n \leq y_n$ for each $n < n_0$ ($x_n \leq y_n$ for each $n > n_0$, etc.).

Let $\{\sigma_{k,l} \mid k, l \in \mathbb{Z}\}$ denote the translational group on $\mathbb{R}^{\mathbb{Z}}$ defined by

$$(\sigma_{k,l}\mathbf{x})_i = x_{i-k} + l.$$

A configuration $\mathbf{x} = (x_n)$ is said to be (p, q)-periodic if $\sigma_{q,p}\mathbf{x} = \mathbf{x}$, where $q, p \in \mathbb{Z}$.

Definition 2.1. A configuration $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ is said to be Birkhoff if for all $k, l \in \mathbb{Z}$, either $\sigma_{k,l}\mathbf{x} \gg \mathbf{x}$ or $\sigma_{k,l}\mathbf{x} = \mathbf{x}$ or $\sigma_{k,l}\mathbf{x} \ll \mathbf{x}$.

Each Birkhoff configuration $\mathbf{x} = (x_n)$ has a rotation number $\omega = \lim_{n\to\infty} x_n/n$. Furthermore, it follows that

$$|x_n - x_k - (n-k)\omega| \leq 1 \quad \text{for all } n, k \in \mathbb{Z}. \tag{2.1}$$

We assume that the generating function h satisfies the following hypotheses.

(H1) $h(\xi_1 + 1, \ldots, \xi_{r+1} + 1) = h(\xi_1, \ldots, \xi_{r+1})$.
(H2) $\max_{1 \le i \le j \le r+1} \|\partial_{i,j} h\|_{\sup} \le K$.
(H3) h is bounded from below and $h(\xi_1, \ldots, \xi_{r+1}) \to \infty$ if $|\xi_2 - \xi_1| \to \infty$.
(H4) Twist condition:

$$\partial_{1,k} h(\xi_1, \ldots, \xi_{r+1}) \le -\lambda < 0 \quad \text{for all } 2 \le k \le r+1,$$

$$\partial_{i,k} h(\xi_1, \ldots, \xi_{r+1}) \le 0 \quad \text{for } k \ne i.$$

If we denote $h_j(\mathbf{x}) = h(x_j, \ldots, x_{j+r})$, then the twist condition implies that

$$\partial_{i,k} h_j \le 0 \text{ for } k \ne i \quad \text{and} \quad \partial_{i,k} h_i \le -\lambda < 0 \text{ for } k = i+1, \ldots, i+r, \tag{2.2}$$

which will be frequently used in this paper. Let

$$\Lambda(x_{j-r}, \ldots, x_j, \ldots, x_{j+r}) = -\partial_j W(\mathbf{x}) = -\sum_{i=j-r}^{j} \partial_j h_i(\mathbf{x}), \quad j \in \mathbb{Z}.$$

Then (1.1) is equivalent to

$$\Lambda(x_{j-r}, \ldots, x_j, \ldots, x_{j+r}) = 0, \quad j \in \mathbb{Z}.$$

The twist condition (2.2) ensures that the function Λ is strictly increasing with respect to all its variables except possibly x_j. So if $(x_{j-r}, \ldots, x_j, \ldots, x_{j+r-1})$ is given, we can solve (1.1) for x_{j+r}. In this way we define a continuous map φ_h from $\mathbb{R}^{2r}$ to $\mathbb{R}^{2r}$ by

$$\varphi_h(x_{j-r}, \ldots, x_{j+r-1}) = (x_{j-r+1}, \ldots, x_{j+r}). \tag{2.3}$$

Similarly we can solve (1.1) for x_{j-r} if the other variables are given. Thus φ_h is a diffeomorphism of $\mathbb{R}^{2r}$ onto itself. Taking into account the periodicity condition (H1) of h, we consider $\mathbb{R}^{2r}/\mathbb{Z}$ which is homeomorphic to the high-dimensional cylinder $S^1 \times \mathbb{R}^{2r-1}$. Furthermore, we have

$$\varphi_h(x_{i-r} + 1, \ldots, x_{i+r-1} + 1) = \varphi_h(x_{i-r}, \ldots, x_{i+r-1}) + 1,$$

from which we define a map $\bar{\varphi}_h : S^1 \times \mathbb{R}^{2r-1} \to S^1 \times \mathbb{R}^{2r-1}$, a generalization of monotone twist maps on the cylinder.

We denote by $B = [i_0 - r, i_1]$ an arbitrary finite connected component of $\mathbb{Z}$ with $i_1 \ge i_0$, $\mathrm{int}(B) = [i_0, i_1]$ the interior of B, $\bar{B} = [i_0 - r, i_1 + r]$ its closure, $\partial B = \bar{B} \setminus \mathrm{int}(B)$ the boundary of B, $\partial B^- = [i_0 - r, i_0 - 1]$, $\partial B^+ = [i_1 + 1, i_1 + r]$, and $|B|$ the number of elements in B. Let

$$W_B(\mathbf{x}) = \sum_{i \in B} h_i(\mathbf{x}),$$

which is a function of coordinates of $\mathbf{x}$ with indices in $\bar{B}$. Denote by $\mathrm{supp}(\mathbf{v})$ the support of $\mathbf{v} = (v_n)$, i.e., $\mathrm{supp}(\mathbf{v}) = \{n \mid v_n \ne 0\}$.

Definition 2.2. A configuration $\mathbf{x}$ is called a minimizer if $W_B(\mathbf{x}) \le W_B(\mathbf{x} + \mathbf{v})$ for all finite connected component $B \subset \mathbb{Z}$ and all $\mathbf{v}$ with $\mathrm{supp}(\mathbf{v}) \subset \mathrm{int}(B)$.

We remark that if $\mathbf{x}$ is a minimizer, so is $\sigma_{k,l}\mathbf{x}$ for all $(k, l) \in \mathbb{Z}^2$. Meanwhile, the set of all minimizers is closed with respect to the product topology. Each minimizer is a stationary point of W. We denote by $\mathscr{M}_{p,q}$ the set of (p, q)-periodic Birkhoff minimizers, and by $\mathscr{M}_\omega$ the set of Birkhoff minimizers with rotation number ω. Then from the Aubry–Mather theory for monotone recurrence relations [**9**, **12**, **13**, **15**, **19**], we know that under conditions (H1)–(H4), the following conclusions hold.

LEMMA 2.3. *$\mathscr{M}_{p,q} \neq \emptyset$ for $p, q \in \mathbb{Z}$ ($q \neq 0$) and $\mathscr{M}_\omega \neq \emptyset$ for each $\omega \in \mathbb{R}$.*

Definition 2.4. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}}$, we call a connected component $D \subset \mathbb{Z}$ the crossing domain of $\mathbf{x}$ and $\mathbf{y}$ if D is the minimal connected component such that $\mathbf{x} < \mathbf{y}$ or $\mathbf{x} > \mathbf{y}$ always holds on each component of $\mathbb{Z}\backslash D$.

We say that $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ cross on $B \subset \mathbb{Z}$ if there exist $i, j \in B$ such that $x_i > y_i$ and $x_j < y_j$. They do not cross on B if $x_n - y_n \geq 0$ or $x_n - y_n \leq 0$ for all $n \in B$. So the crossing domain D is the minimal connected component of $\mathbb{Z}$ such that $\mathbf{x}$ and $\mathbf{y}$ do not cross on each component of $\mathbb{Z}\backslash D$.

We need to study the following gradient dynamics:

$$\dot{x}_i = -\partial_i W(\mathbf{x}) = -\sum_{j=i-r}^{i} \partial_i h_j(\mathbf{x}), \quad i \in \mathbb{Z}, \tag{2.4}$$

with initial conditions in Banach space

$$\mathscr{X} = \left\{ \mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \;\middle|\; \|\mathbf{x}\| = \sum_{n\in\mathbb{Z}} \frac{|x_n|}{2^{|n|}} < \infty \right\}.$$

Condition (H2) ensures the existence of a unique solution $\mathbf{x}(t)$ of (2.4) with $\mathbf{x}(0) \in \mathscr{X}$ for all $t \in \mathbb{R}$ so that we can define a flow $\{\phi^t\}_{t\in\mathbb{R}}$ on $\mathscr{X}$ from (2.4). The periodic condition (H1) makes it possible to consider $\{\phi^t\}$ in $\mathscr{X}/\langle\mathbf{1}\rangle$, where $\mathbf{1}$ denotes an element in $\mathscr{X}$ with all its components equal to 1. Moreover, ϕ^t commutes with $\sigma_{k,l}$:

$$\phi^t(\sigma_{k,l}\mathbf{x}) = \sigma_{k,l}(\phi^t\mathbf{x}) \quad \text{for all } t \in \mathbb{R}, \text{ and } k, l \in \mathbb{Z}. \tag{2.5}$$

The twist condition (H4) guarantees the monotonicity of ϕ^t for $t > 0$, as described in the following lemmas.

LEMMA 2.5. *Assume that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two solutions of (2.4) with $\mathbf{x}(0) \leq \mathbf{y}(0)$. Then $\mathbf{x}(t) \leq \mathbf{y}(t)$ for all $t > 0$. Furthermore, if $\mathbf{x}(0) < \mathbf{y}(0)$, then $\mathbf{x}(t) \ll \mathbf{y}(t)$ for all $t > 0$.*

LEMMA 2.6. *Assume that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two solutions of (2.4) with $\mathbf{x}(0) = \mathbf{x} = (x_i)$ and $\mathbf{y}(0) = \mathbf{y} = (y_i)$ satisfying $x_i \leq y_i$ for $i \geq i_0$ (or $i \leq i_0$) and that $x_{i_0-k}(t) \leq y_{i_0-k}(t)$ (or $x_{i_0+k}(t) \leq y_{i_0+k}(t)$) for all $k \in \{1, 2, \ldots, r\}$ and $t \in [0, T]$. Then $x_i(t) \leq y_i(t)$ for $i \geq i_0$ (or $i \leq i_0$) and $t \in [0, T]$.*

LEMMA 2.7. *The solution $\mathbf{x}(t)$ of (2.4) with $\mathbf{x}(0) \in \mathcal{S}^-$ (or $\mathbf{x}(0) \in \mathcal{S}^+$) is increasing (or decreasing) for $t \geq 0$.*

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We remark that the conclusions of Lemma 2.5 are standard results for gradient systems like (2.4) with twist condition (H4); see [**15**, Lemma 4.3], [**19**, Theorem 6.2], or [**12**, Lemmas 1 and 2]. The conclusions of Lemma 2.6 describe the monotonicity of rays, the proof of which is exactly the same as that of Lemma 2.5; see [**22**] for the proof of nearest neighbor coupling case. The proof of Lemma 2.7 is also the same as Lemma 2.5 and hence omitted here.

Definition 2.8. We say that a configuration $\mathbf{x} = (x_i) \in \mathbb{R}^{\mathbb{Z}}$ has bounded action if there exists $C > 0$, such that $|x_i - x_{i-1}| \le C$ for all $i \in \mathbb{Z}$.

Let

$$\mathscr{B}_C = \{\mathbf{x} = (x_i) \in \mathbb{R}^{\mathbb{Z}} \mid |x_i - x_{i-1}| \le C,\ i \in \mathbb{Z}\} \quad \text{and} \quad \mathscr{B} = \bigcup_{C \ge 0} \mathscr{B}_C.$$

LEMMA 2.9. *The product topology is equivalent to the topology induced by the norm $\|\cdot\|$ on $\mathscr{B}_C$ for each $C > 0$.*

Proof. It suffices to show that if $\mathbf{x}^n, \mathbf{x}^* \in \mathscr{B}_C$ and $\mathbf{x}^n \to \mathbf{x}^*$ pointwise, then $\|\mathbf{x}^n - \mathbf{x}^*\| \to 0$ as $n \to \infty$. Suppose that $\lim_{n\to\infty} x_i^n = x_i^*$ for all $i \in \mathbb{Z}$. We may assume that $|x_0^n - x_0^*| \le 1$. Since $\mathbf{x}^n, \mathbf{x}^* \in \mathscr{B}_C$, we have

$$|x_i^n - x_i^*| \le 2C\,|i\,| + 1 \quad \text{for all } i \in \mathbb{Z}.$$

The convergence of the series $\sum_{i\in\mathbb{Z}}(2C\,|i| + 1)/2^{|i|}$ implies that for every $\varepsilon > 0$ there is $i_0 > 0$ such that

$$\sum_{|i|>i_0} \frac{2C\,|i| + 1}{2^{|i|}} < \varepsilon/2.$$

Meanwhile, there exists $N \in \mathbb{N}$ such that for $n > N$,

$$|x_i^n - x_i^*\,| < \varepsilon/6 \quad \text{for all } |i\,| \le i_0.$$

It follows that for all $n > N$,

$$\|\mathbf{x}^n - \mathbf{x}^*\| = \sum_{|i\,|>i_0} \frac{|x_i^n - x_i^*\,|}{2^{\,|i\,|}} + \sum_{|i\,|\le i_0} \frac{|x_i^n - x_i^*\,|}{2^{\,|i\,|}} < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

LEMMA 2.10. *For $l \in \mathbb{N}$, $\mathscr{B}_l \cap \mathcal{S}^-$ (or $\mathscr{B}_l \cap \mathcal{S}^+$) is forward-invariant for ϕ^t $(t > 0)$.*

Proof. Assume that $\mathbf{x}(t)$ is a solution of (2.4) with $\mathbf{x}(0) = \mathbf{x} \in \mathscr{B}_l$. It suffices to show that $\mathbf{x}(t) \in \mathscr{B}_l$ for all $t > 0$ since $\mathcal{S}^-$ and $\mathcal{S}^+$ are forward-invariant for ϕ^t with $t \ge 0$ by Lemma 2.7. Indeed, $\mathbf{x} \in \mathscr{B}_l$ implies that $-l \le x_{i+1} - x_i \le l$ for all $i \in \mathbb{Z}$, i.e., $\sigma_{-1,-l}\mathbf{x} \le \mathbf{x} \le \sigma_{-1,l}\mathbf{x}$. From (2.5) and Lemma 2.5 we have

$$\phi^t(\sigma_{-1,-l}\mathbf{x}) \le \phi^t(\mathbf{x}) \le \phi^t(\sigma_{-1,l}\mathbf{x}) \;\Rightarrow\; -l \le x_{i+1}(t) - x_i(t) \le l \text{ for } t > 0,$$

implying that $\mathbf{x}(t) \in \mathscr{B}_l$ for all $t > 0$. $\square$

Therefore, we consider in §4 the dynamics of (2.4) in $\mathscr{B}_C$ with product topology. We know from Tychonov's theorem that each $E \subset \mathscr{B}_C$ is compact provided $p_0(E)$ is bounded and E is closed.

A criterion for positive topological entropy has been presented by Angenent; see [**1**, Theorem 7.1].

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PROPOSITION 2.11. *Let the monotone recurrence relation (1.1) have two solutions satisfying (1.2). Then the map $\bar{\varphi}_h$ defined by (1.1) has positive topological entropy.*

3. *Minimizers with bounded action*

In this section, we investigate some properties of minimizers, especially those for minimizers with bounded action.

We remark that the twist condition (H4) is weaker than that in [**20**] in that we require $\partial_{i,k}h \leq 0$ while in [**20**] it is assumed that $\partial_{i,k}h \equiv 0$, for $i \neq 1$, $k \neq 1$, and $i \neq k$. So we cannot simply apply directly the conclusions of [**20**], especially the dichotomy theorem for minimizers. Nevertheless, inspired by the ideas of Mramor and Rink in [**20**, **21**] and Bangert in [**5**, **6**] (we even borrow most of the notation from [**20**]) we obtain the conclusion we need that a minimizer with bounded action is Birkhoff.

LEMMA 3.1. *Assume that $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n) \in \mathbb{R}^{\mathbb{Z}}$ have bounded action. Then there exists a constant $d > 0$ such that for each finite connected component $B \subset \mathbb{Z}$, it follows that*

$$|W_B(\mathbf{x}) - W_B(\mathbf{y})| \leq d \sum_{k \in \bar{B}} |x_k - y_k|.$$

Moreover, if $\mathbf{x} = \mathbf{y}$ on $\mathrm{int}(B)$*, then*

$$|W_B(\mathbf{x}) - W_B(\mathbf{y})| \leq d \sum_{k \in \partial B} |x_k - y_k|. \tag{3.1}$$

Proof. Assume that $\mathbf{x}, \mathbf{y} \in \mathscr{B}_C$ for some $C > 0$. Then

$$|W_B(\mathbf{x}) - W_B(\mathbf{y})| \leq \sum_{i \in B} |h_i(\mathbf{x}) - h_i(\mathbf{y})| = \sum_{i \in B} \left| \int_0^1 \frac{d}{ds} h_i(s\mathbf{x} + (1-s)\mathbf{y})\, ds \right|$$
$$\leq \sum_{i \in B} \sum_{k=i}^{i+r} \int_0^1 |\partial_k h_i(s\mathbf{x} + (1-s)\mathbf{y})|\, ds \cdot |x_k - y_k|.$$

Taking into account the fact that $s\mathbf{x} + (1-s)\mathbf{y} \in \mathscr{B}_C$ and hypothesis (H1), we obtain a constant $\tilde{d} > 0$ depending on C and r such that $|\partial_k h_i(s\mathbf{x} + (1-s)\mathbf{y})| \leq \tilde{d}$, and hence

$$|W_B(\mathbf{x}) - W_B(\mathbf{y})| \leq d \sum_{k \in \bar{B}} |x_k - y_k|,$$

where $d = (r+1)\tilde{d}$. □

Definition 3.2. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}}$, define $M, m \in \mathbb{R}^{\mathbb{Z}}$ by $M_i = \max\{x_i, y_i\}$ and $m_i = \min\{x_i, y_i\}$. We call

$$W_B^c(\mathbf{x}, \mathbf{y}) = W_B(\mathbf{x}) + W_B(\mathbf{y}) - W_B(M) - W_B(m)$$

the crossing energy of $\mathbf{x}$ and $\mathbf{y}$ on B.

Let $\alpha = M - \mathbf{x}$, $\beta = m - \mathbf{x}$. Then $\alpha_i \geq 0$, $\beta_i \leq 0$ and $\alpha_i \beta_i = 0$ for all $i \in \mathbb{Z}$.

LEMMA 3.3. *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}}$ and an arbitrary finite connected component $B \subset \mathbb{Z}$, we have $W_B^c(\mathbf{x}, \mathbf{y}) \geq 0$. Moreover, if $\mathbf{x}$ and $\mathbf{y}$ cross on a connected component $B_1 \subset \mathbb{Z}$ such that $|B_1| \leq r + 1$, then $W_B^c(\mathbf{x}, \mathbf{y}) > 0$ for each $\bar{B} \supset B_1$.*

Proof. Note that $\mathbf{y} = \mathbf{x} + \alpha + \beta$. Then it follows from (2.2) that

$$\begin{aligned} W_B^c(\mathbf{x}, \mathbf{y}) &= W_B(\mathbf{x} + \alpha + \beta) - W_B(\mathbf{x} + \alpha) - W_B(\mathbf{x} + \beta) + W_B(\mathbf{x}) \\ &= \sum_{i \in B} \int_0^1 \left[\frac{d}{dt} h_i(\mathbf{x} + \alpha + t\beta) - \frac{d}{dt} h_i(\mathbf{x} + t\beta) \right] dt \\ &= \sum_{i \in B} \int_0^1 \int_0^1 \frac{d}{dt} \frac{d}{ds} h_i(\mathbf{x} + s\alpha + t\beta)\ ds\, dt \\ &= \sum_{i \in B} \sum_{j,k=i}^{i+r} \int_0^1 \int_0^1 \partial_{j,k} h_i(\mathbf{x} + s\alpha + t\beta)\, ds\, dt \cdot (\alpha_j \beta_k) \\ &\geq \sum_{i \in B} \sum_{j=i}^{i+r} \int_0^1 \int_0^1 \partial_{i,j} h_i(\mathbf{x} + s\alpha + t\beta)\, ds\, dt \cdot (\alpha_i \beta_j + \alpha_j \beta_i) \\ &\geq -\lambda \sum_{i \in B} \sum_{j=i}^{i+r} (\alpha_i \beta_j + \alpha_j \beta_i), \end{aligned}$$

implying that $W_B^c(\mathbf{x}, \mathbf{y}) \geq 0$.

From the above proof, it follows that $W_B^c(\mathbf{x}, \mathbf{y}) > 0$ whenever there exist $i \in B$ and $j \in \{i + 1, i + 2, \ldots, i + r\}$ such that $\alpha_i \beta_j < 0$ or $\alpha_j \beta_i < 0$, i.e., $\mathbf{x}$ and $\mathbf{y}$ cross on some $B_1 \subset \bar{B}$ with $|B_1| \leq r + 1$. □

LEMMA 3.4. *Let $B \subset \mathbb{Z}$ be a finite connected component and $\mathbf{x}$ and $\mathbf{y}$ be two solutions of (1.1). Then $\mathbf{x} \ll \mathbf{y}$ on int(B) if $\mathbf{x} < \mathbf{y}$ on $\bar{B}$.*

Proof. Assume that the conclusion is not true, i.e., $\alpha_i = 0$ for some $i \in \mathrm{int}(B)$. Then, by (2.2),

$$\begin{aligned} 0 = \sum_{j=i-r}^{i} \partial_i h_j(\mathbf{y}) - \partial_i h_j(\mathbf{x}) &= \sum_{j=i-r}^{i} \int_0^1 \frac{d}{dt} \partial_i h_j(t\mathbf{y} + (1-t)\mathbf{x})\, dt \\ &= \sum_{j=i-r}^{i} \sum_{k=j}^{j+r} \int_0^1 \partial_{i,k} h_j(t\mathbf{y} + (1-t)\mathbf{x})\, dt \cdot (y_k - x_k) \\ &\leq \sum_{k=i}^{i+r} \int_0^1 \partial_{i,k} h_i(t\mathbf{y} + (1-t)\mathbf{x})\, dt \cdot \alpha_k \\ &\quad + \sum_{j=i-r}^{i-1} \int_0^1 \partial_{j,i} h_j(t\mathbf{y} + (1-t)\mathbf{x})\, dt \cdot \alpha_j. \end{aligned}$$

From the assumption $\alpha_i = 0$ and again the twist condition (2.2) and $\alpha_k \geq 0$ for $k \neq i$, it follows that $\alpha_k = 0$ for all $k = i - r, i - r + 1, \ldots, i + r$. By induction, we derive that $\mathbf{x} = \mathbf{y}$ on $\bar{B}$, in contradiction to the assumption $\mathbf{x} < \mathbf{y}$ on $\bar{B}$. □

Lemma 3.4 immediately leads to the following conclusion.

LEMMA 3.5. *Assume that $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and $\mathbf{x} < \mathbf{y}$. Then $\mathbf{x} \ll \mathbf{y}$.*

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Definition 3.6. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}}$ and $B \subset \mathbb{Z}$. Define

$$M^B(\mathbf{x}) = M \text{ on int}\,(B) \text{ and } M^B(\mathbf{x}) = \mathbf{x} \text{ on } \mathbb{Z} \setminus \text{int}\,(B),$$

$$m^B(\mathbf{x}) = m \text{ on int}\,(B) \text{ and } m^B(\mathbf{x}) = \mathbf{x} \text{ on } \mathbb{Z} \setminus \text{int}\,(B),$$

$$M^B(\mathbf{y}) = M \text{ on int}\,(B) \text{ and } M^B(\mathbf{y}) = \mathbf{y} \text{ on } \mathbb{Z} \setminus \text{int}\,(B),$$

$$m^B(\mathbf{y}) = m \text{ on int}\,(B) \text{ and } m^B(\mathbf{y}) = \mathbf{y} \text{ on } \mathbb{Z} \setminus \text{int}\,(B).$$

Note that $M = \max\{\mathbf{x}, \mathbf{y}\}$ and $m = \min\{\mathbf{x}, \mathbf{y}\}$. Then it is obvious that

$$M = \max\{M^B(\mathbf{x}), m^B(\mathbf{y})\} \quad \text{and} \quad m = \min\{M^B(\mathbf{x}), m^B(\mathbf{y})\}. \tag{3.2}$$

LEMMA 3.7. *Assume that* $\mathbf{x}$ *and* $\mathbf{y}$ *are minimizers. Then for each finite connected component* $B \subset \mathbb{Z}$*, it follows that*

$$W_B^c(\mathbf{x}, \mathbf{y}) \le W_B^c(M^B(\mathbf{x}), m^B(\mathbf{y})) \quad \textit{and} \quad W_B^c(\mathbf{x}, \mathbf{y}) \le W_B^c(M^B(\mathbf{y}), m^B(\mathbf{x})). \tag{3.3}$$

Proof. Because both $\mathbf{x}$ and $\mathbf{y}$ are minimizers, it follows that for each finite connected component $B \subset \mathbb{Z}$,

$$W_B(\mathbf{x}) + W_B(\mathbf{y}) \le W_B(M^B(\mathbf{x})) + W_B(m^B(\mathbf{y}))$$

and

$$W_B(\mathbf{x}) + W_B(\mathbf{y}) \le W_B(M^B(\mathbf{y})) + W_B(m^B(\mathbf{x})).$$

Subtracting $W_B(M) + W_B(m)$ on both sides of the above two inequalities, we obtain (3.3) by (3.2). □

LEMMA 3.8. *Assume that* $\mathbf{x}$ *and* $\mathbf{y}$ *are minimizers. Let* $k_0 \le k_1$ *be integers such that* $x_i \le y_i$ *for all* $i \in [k_0 - r, k_0 - 1] \cup [k_1 + 1, k_1 + r]$*. Then either* $\mathbf{x} \ll \mathbf{y}$ *or* $\mathbf{x} = \mathbf{y}$ *on* $[k_0, k_1]$.

Proof. Let $B = [k_0 - r, k_1]$. Then $\text{int}(B) = [k_0, k_1]$ and $\bar{B} = [k_0 - r, k_1 + r]$. The assumption implies that $M^B(\mathbf{y}) = M$ and $m^B(\mathbf{x}) = m$ on $\bar{B}$. From Lemma 3.3 it follows that

$$W_B(\mathbf{x}) + W_B(\mathbf{y}) \ge W_B(m) + W_B(M) = W_B(m^B(\mathbf{x})) + W_B(M^B(\mathbf{y})).$$

On the other hand, we have $W_B(\mathbf{x}) \le W_B(m^B(\mathbf{x}))$ and $W_B(\mathbf{y}) \le W_B(M^B(\mathbf{y}))$ since both $\mathbf{x}$ and $\mathbf{y}$ are minimizers. Thus

$$W_B(\mathbf{x}) = W_B(m^B(\mathbf{x})) \quad \text{and} \quad W_B(\mathbf{y}) = W_B(M^B(\mathbf{y})),$$

implying that both $m^B(\mathbf{x})$ and $M^B(\mathbf{y})$ are minimizers because

$$W_{B'}(\mathbf{x}) = W_{B'}(m^B(\mathbf{x})) \quad \text{and} \quad W_{B'}(\mathbf{y}) = W_{B'}(M^B(\mathbf{y})),$$

for each finite connected set $B' \supset B$. Consequently, it follows from Lemma 3.5 that $\mathbf{x} = m^B(\mathbf{x})$ and $M^B(\mathbf{y}) = \mathbf{y}$ and hence $\mathbf{x} \le \mathbf{y}$ on $\bar{B}$, implying by Lemma 3.4 that $\mathbf{x} \ll \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$ on $\text{int}(B) = [k_0, k_1]$. □

Lemma 3.8 immediately leads to the following conclusions which we state without proof.

LEMMA 3.9. *If the crossing domain of two minimizers* $\mathbf{x}$ *and* $\mathbf{y}$ *is bounded and non-empty, then* $\mathbf{x} <_\alpha \mathbf{y}$ *implies* $\mathbf{x} >_\omega \mathbf{y}$*, or* $\mathbf{x} >_\alpha \mathbf{y}$ *implies* $\mathbf{x} <_\omega \mathbf{y}$.

LEMMA 3.10. *If the crossing domain of two minimizers* $\mathbf{x}$ *and* $\mathbf{y}$ *is unbounded, then there exist at most two disjoint connected components* B_1 *and* B_2 *with* $|B_1| \geq r$ *and* $|B_2| \geq r$ *such that* $\mathbf{x}$ *and* $\mathbf{y}$ *do not cross on* B_1 *and on* B_2.

We say that $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ are ω-asymptotic (α-asymptotic) if $|x_n - y_n| \to 0$ as $n \to +\infty$ $(-\infty)$. For $n \in \mathbb{Z}$, define $B(n, r) = \{i + n \mid i \in \mathbb{Z} \text{ and } -r \leq i \leq r\}$.

Definition 3.11. Two configurations $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ are called almost ω-asymptotic (α-asymptotic) if there exists a subsequence $\{n_k\} \subset \mathbb{Z}^+$ $(\mathbb{Z}^-)$, such that

$$\lim_{k\to\infty} |x_{n_k+l} - y_{n_k+l}| = 0, \quad l \in B(0, r).$$

$\mathbf{x}$ and $\mathbf{y}$ are said to be almost asymptotic if they are both almost α-asymptotic and almost ω-asymptotic.

LEMMA 3.12. *Assume that* $\mathbf{x} = (x_n)$ *and* $\mathbf{y} = (y_n)$ *are minimizers with bounded action, and that one of the following three conditions is satisfied:*

(1) $\mathbf{x}$ *and* $\mathbf{y}$ *are almost asymptotic;*

(2) $\mathbf{x} <_\alpha$ *(or* $>_\alpha$*)* $\mathbf{y}$ *and* $\mathbf{x}$ *and* $\mathbf{y}$ *are almost* ω*-asymptotic;*

(3) $\mathbf{x} <_\omega$ *(or* $>_\omega$*)* $\mathbf{y}$ *and* $\mathbf{x}$ *and* $\mathbf{y}$ *are almost* α*-asymptotic.*

Then $\mathbf{x}$ *and* $\mathbf{y}$ *are strictly ordered.*

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ are almost asymptotic. Then there exist $\{n_k\} \subset \mathbb{Z}^+$ and $\{\bar{n}_k\} \subset \mathbb{Z}^-$ such that

$$\lim_{k\to+\infty} |x_{n_k+l} - y_{n_k+l}| = 0 \quad \text{and} \quad \lim_{k\to+\infty} |x_{\bar{n}_k+l} - y_{\bar{n}_k+l}| = 0, \quad l \in B(0, r). \tag{3.4}$$

We claim that both $M = \max\{\mathbf{x}, \mathbf{y}\}$ and $m = \min\{\mathbf{x}, \mathbf{y}\}$ are minimizers.

Indeed, if M is not a minimizer (the proof for the case that m is not a minimizer is similar), then there exist a finite connected set $A \subset \mathbb{Z}$ and $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ with $\operatorname{supp}(\mathbf{v}) \subset \operatorname{int}(A)$ such that for all $B \supset A$,

$$W_B(M + \mathbf{v}) = W_B(M) - \delta, \tag{3.5}$$

for some $\delta > 0$. Let $B_k = [\bar{n}_k - r, n_k] \cap \mathbb{Z}$. Then $A \subset B_k$ for k large enough. From Lemma 3.3 and (3.5) it follows that

$$W_{B_k}(M + \mathbf{v}) + W_{B_k}(m) + \delta \leq W_{B_k}(\mathbf{x}) + W_{B_k}(\mathbf{y}).$$

Meanwhile, it is easy to check by (3.1) and (3.4) that

$$\lim_{k\to+\infty} W_{B_k}(M) - W_{B_k}(M^{B_k}(\mathbf{x})) = \lim_{k\to+\infty} W_{B_k}(m) - W_{B_k}(m^{B_k}(\mathbf{y})) = 0, \tag{3.6}$$

implying the existence of $N \in \mathbb{N}$ such that for all $k > N$,

$$|W_{B_k}(M + \mathbf{v}) - W_{B_k}(M^{B_k}(\mathbf{x}) + \mathbf{v})| < \delta/3 \quad \text{and} \quad |W_{B_k}(m) - W_{B_k}(m^{B_k}(\mathbf{y}))| < \delta/3.$$

Consequently, taking large k, we have

$$W_{B_k}(M^{B_k}(\mathbf{x}) + \mathbf{v}) + W_{B_k}(m^{B_k}(\mathbf{y})) < W_{B_k}(\mathbf{x}) + W_{B_k}(\mathbf{y}),$$

in contradiction to the assumption that both $\mathbf{x}$ and $\mathbf{y}$ are minimizers.

Therefore, both M and m are minimizers and hence either $\mathbf{x} = M$ or $\mathbf{y} = M$, implying that $\mathbf{x}$ and $\mathbf{y}$ are strictly ordered.

Assume that condition (2) is satisfied. Assume also that $\mathbf{x} <_\alpha \mathbf{y}$ and $\mathbf{x}$ and $\mathbf{y}$ are almost ω-asymptotic. Then there exists $\{n_k\} \subset \mathbb{Z}^+$ such that

$$\lim_{k\to+\infty} |x_{n_k+l} - y_{n_k+l}| = 0, \quad l \in B(0, r).$$

Let $B_k = [-n_k, n_k] \cap \mathbb{Z}$. Then we have by (3.1) that

$$\lim_{k\to+\infty} W_{B_k}(M) - W_{B_k}(M^{B_k}(\mathbf{y})) = \lim_{k\to+\infty} W_{B_k}(m) - W_{B_k}(m^{B_k}(\mathbf{x})) = 0.$$

The rest of the proof is similar to that of case (1). The proof for condition (3) is similar to condition (2) and is omitted. □

LEMMA 3.13. *If the crossing domain D of two minimizers $\mathbf{x}$ and $\mathbf{y}$ with bounded action is unbounded, then there exists $\varepsilon_0 > 0$ such that $W^c_{B(n,4r)}(\mathbf{x}, \mathbf{y}) \geq \varepsilon_0$ for each $n \in D$.*

Proof. If this is not true, then there exists a sequence $\{n_k\} \subset D$ such that

$$\lim_{k\to+\infty} W^c_{B(n_k,4r)}(\mathbf{x}, \mathbf{y}) = 0.$$

Without loss of generality we may assume that $n_k \to +\infty$ as $k \to +\infty$. Choosing $l_k \in \mathbb{Z}$ such that $(\sigma_{-n_k,l_k}\mathbf{x})_0 = x_{n_k} + l_k \in [0, 1]$, we deduce from Tychonov's theorem that there are convergent (in the sense of product topology) subsequences of $\{\sigma_{-n_k,l_k}\mathbf{x}\}$ and $\{\sigma_{-n_k,l_k}\mathbf{y}\}$, not relabeled, such that

$$\lim_{k\to+\infty} \sigma_{-n_k,l_k}\mathbf{x} = \bar{\mathbf{x}} \quad \text{and} \quad \lim_{k\to+\infty} \sigma_{-n_k,l_k}\mathbf{y} = \bar{\mathbf{y}}.$$

Note that $\bar{\mathbf{x}} = (\bar{x}_n)$ and $\bar{\mathbf{y}} = (\bar{y}_n)$ are minimizers and that

$$W^c_{B(n_k,4r)}(\mathbf{x}, \mathbf{y}) = W^c_{B(0,4r)}(\sigma_{-n_k,l_k}\mathbf{x}, \sigma_{-n_k,l_k}\mathbf{y}).$$

It then follows that $W^c_{B(0,4r)}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 0$.

If $\bar{\mathbf{x}} \neq \bar{\mathbf{y}}$, then $\bar{x}_n - \bar{y}_n$ changes sign at most once for $n \in B(0, 4r)$. Indeed, from Lemma 3.3 we know that for each connected component $B \subset B(0, 4r)$ with $|B| = r$, $\bar{x}_n - \bar{y}_n$ does not change sign for all $n \in B$. As a consequence of Lemma 3.8, we conclude that $\bar{x}_n - \bar{y}_n$ changes sign at most once for $n \in B(0, 4r)$. This implies the existence of a connected component $B_1 \subset B(0, 4r)$ such that $|\bar{B}_1| \geq 3r$ and $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ do not cross on $\bar{B}_1$, i.e., $\bar{\mathbf{x}} \leq \bar{\mathbf{y}}$ or $\bar{\mathbf{y}} \leq \bar{\mathbf{x}}$ on $\bar{B}_1$. From Lemma 3.4 we derive that $\bar{\mathbf{x}} \ll \bar{\mathbf{y}}$ or $\bar{\mathbf{x}} \gg \bar{\mathbf{y}}$ on $\mathrm{int}(B_1)$ since $\bar{\mathbf{x}} \neq \bar{\mathbf{y}}$ on $\bar{B}_1$. Consequently, for k sufficiently large, we have $\sigma_{-n_k,l_k}\mathbf{x} \ll \sigma_{-n_k,l_k}\mathbf{y}$ or $\sigma_{-n_k,l_k}\mathbf{x} \gg \sigma_{-n_k,l_k}\mathbf{y}$ on $\mathrm{int}(B_1)$ and hence $\mathbf{x} \ll \mathbf{y}$ or $\mathbf{x} \gg \mathbf{y}$ on infinitely many intervals with length no less than r, in contradiction to Lemma 3.10.

If $\bar{\mathbf{x}} = \bar{\mathbf{y}}$, then $x_{n_k+l} - y_{n_k+l} \to 0$ as $k \to +\infty$ for each $l \in \mathbb{Z}$, implying that $\mathbf{x}$ and $\mathbf{y}$ are almost ω-asymptotic. If $\mathbf{x} <_\alpha \mathbf{y}$ or $\mathbf{x} >_\alpha \mathbf{y}$, then Lemma 3.12 leads to the conclusion that $\mathbf{x}$ and $\mathbf{y}$ are strictly ordered. Otherwise we may take a sequence $\{n'_k\} \subset D$ with $n'_k \to -\infty$ as $k \to +\infty$ and $l'_k \in \mathbb{Z}$ such that

$$\lim_{k\to+\infty} W^c_{B(n'_k,4r)}(\mathbf{x}, \mathbf{y}) = 0, \quad \lim_{k\to+\infty} \sigma_{-n'_k,l'_k}\mathbf{x} = \mathbf{x}', \quad \text{and} \quad \lim_{k\to+\infty} \sigma_{-n'_k,l'_k}\mathbf{y} = \mathbf{y}'.$$

If $\mathbf{x}' \neq \mathbf{y}'$, then we obtain a contradiction as above. If $\mathbf{x}' = \mathbf{y}'$, we derive that $\mathbf{x}$ and $\mathbf{y}$ are almost α-asymptotic, and hence, by the fact that $\mathbf{x}$ and $\mathbf{y}$ are almost ω-asymptotic shown above and by Lemma 3.12, that $\mathbf{x}$ and $\mathbf{y}$ are strictly ordered, which contradicts the assumption. □

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LEMMA 3.14. *The crossing domain of two minimizers* $\mathbf{x}$ *and* $\mathbf{y}$ *with bounded action is bounded.*

Proof. Assume that the crossing domain D of $\mathbf{x}$ and $\mathbf{y}$ is unbounded. For each bounded connected component $B \subset \mathbb{Z}$, we have by Lemma 3.7 that

$$W_B^c(\mathbf{x}, \mathbf{y}) \le W_B^c(M^B(\mathbf{x}), m^B(\mathbf{y})). \tag{3.7}$$

Note that both $\mathbf{x}$ and $\mathbf{y}$ have bounded action, i.e., there is a constant $C > 0$ such that $|x_n - x_{n-1}| \le C$ and $|y_n - y_{n-1}| \le C$ for $n \in \mathbb{Z}$. Then $|x_n - y_n| \le 2rC$ for $n \in \partial B$ since $\mathbf{x}$ and $\mathbf{y}$ cross on ∂B^- and on ∂B^+ by Lemma 3.10. From (3.1) we have

$$\begin{aligned} W_B^c(M^B(\mathbf{x}), m^B(\mathbf{y})) &= W_B(M^B(\mathbf{x})) - W_B(M) + W_B(m^B(\mathbf{y})) - W_B(m) \\ &\le 2d \sum_{n \in \partial B} |x_n - y_n| \le 2d \cdot 2r \cdot 2rC, \end{aligned}$$

and hence the right-hand side of (3.7) is bounded independently of the choice of B. On the other hand, we may choose B large enough so that the left-hand side of (3.7) is large enough because of Lemma 3.13. This is a contradiction. □

THEOREM 3.15. *If* $\mathbf{x}$ *is a minimizer with bounded action, then* $\mathbf{x}$ *is Birkhoff.*

Proof. Assuming that $k, l \in \mathbb{Z}$, we compare $\mathbf{x}$ with $\sigma_{k,l}\mathbf{x}$. According to Lemmas 3.14 and 3.9, we may assume that $\sigma_{k,l}\mathbf{x} <_\alpha \mathbf{x}$ and $\sigma_{k,l}\mathbf{x} >_\omega \mathbf{x}$ (the cases $\sigma_{k,l}\mathbf{x} >_\alpha \mathbf{x}$ and $\sigma_{k,l}\mathbf{x} <_\omega \mathbf{x}$ can be treated similarly). Without loss of generality we may assume that $k > 0$ ($k = 0$ is trivial and $k < 0$ is analogous).

There exist $i_0, i_1 \in \mathbb{Z}$ such that

$$(\sigma_{k,l}\mathbf{x})_i \le x_i \text{ for } i \le i_0 \quad \text{and} \quad (\sigma_{k,l}\mathbf{x})_i \ge x_i \text{ for } i \ge i_1,$$

i.e.,

$$x_{i-k} + l \le x_i \text{ for } i \le i_0 \quad \text{and} \quad x_i \ge x_{i+k} - l \text{ for } i \ge i_1 - k,$$

implying

$$n \in \mathbb{N} \mapsto (\sigma_{k,l}^n \mathbf{x})_i = x_{i-nk} + nl \text{ is non-increasing for } i \le i_0$$

and

$$n \in \mathbb{N} \mapsto (\sigma_{-k,-l}^n \mathbf{x})_i = x_{i+nk} - nl \text{ is also non-increasing for } i \ge i_1 - k.$$

From Lemma 2.3 we obtain a periodic Birkhoff minimizer $\bar{\mathbf{x}} \in \mathscr{M}_{l,k}$ such that $\bar{x}_{i-k} + l = \bar{x}_i$ for $i \in \mathbb{Z}$.

According to Lemma 3.9 we have two cases, either $\bar{\mathbf{x}} <_\alpha \mathbf{x}$ or $\bar{\mathbf{x}} <_\omega \mathbf{x}$ ($\bar{\mathbf{x}} = \mathbf{x}$ is trivial). We assume that $\bar{\mathbf{x}} <_\alpha \mathbf{x}$ (the other case can be treated similarly), i.e., $\bar{x}_i \le x_i$ for $i \le i_{-1} \le i_0$. Consequently, the sequence $\{x_{i-nk} + nl\}$ is non-increasing and bounded below by $\bar{x}_{i-nk} + nl = \bar{x}_i$. Thus

$$\tilde{x}_i = \lim_{n \to +\infty} x_{i-nk} + nl$$

exists and $\tilde{x}_{i-k} + l = \tilde{x}_i$ for $i \le i_{-1}$. From the periodicity of $(\tilde{x}_i)_{i \le i_{-1}}$ we conclude that both $\mathbf{x}$ and $\sigma_{k,l}\mathbf{x}$ are α-asymptotic to $(\tilde{x}_i)_{i \le i_{-1}}$ and hence $\mathbf{x}$ and $\sigma_{k,l}\mathbf{x}$ are α-asymptotic. It then follows from Lemma 3.12 that $\mathbf{x}$ is Birkhoff. □

4. *Foliation of Birkhoff minimizers*

Following Moser [**18**], we say that $\mathscr{M}_\omega$ forms a foliation if $p_0(\mathscr{M}_\omega) = \mathbb{R}$, otherwise we say that it constitutes a lamination. In this section, we shall show that if there is an unbounded solution of (2.4) in $\mathscr{B} \cap \mathcal{S}^-$ (or $\mathscr{B} \cap \mathcal{S}^+$), then we can construct a foliation $\mathscr{M}_\omega$ for some $\omega \in \mathbb{R}$. We should remark that the idea is borrowed from Slijepčević [**23**]. The proof of Theorem 4.3 in [**23**] can be adapted, almost word for word, to the case of monotone recurrence relations. So the proof of Theorem 4.5 is given only for completeness.

We should mention that the definition of strictly ordered curves in this paper is different from that of rotational invariant circles of [**23**] in that we do not require translation invariance. The second difference is that in the setting of twist maps [**23**], each stationary solution intersects a circle configuration at most once (see [**23**, Lemma 2.1]), a property like Aubry's lemma which we do not have for monotone recurrence relations. The third difference is that for high-dimensional cylinder maps induced from monotone recurrence relations there is no concept analogous to Birkhoff's region of instability for twist maps. The difficulties resulting from these differences are overcome by using Theorem 3.15, the main result of §3.

We always assume the product topology on $\mathscr{B}$ by Lemma 2.9.

Definition 4.1. The image ℓ of a continuous function $\gamma\colon \mathbb{R} \to \mathscr{B}_C \cap \mathcal{S}$ ($C > 0$) is said to be a strictly ordered curve if γ is strictly increasing, i.e., $\gamma(s_1) \ll \gamma(s_2)$ for $s_1 < s_2$, and $\{\gamma_0(s) \mid s \in \mathbb{R}\} = \mathbb{R}$.

LEMMA 4.2. *Each element in a strictly ordered curve is a minimizer.*

Proof. Assume that $\ell \subset \mathscr{B}_C \cap \mathcal{S}$ is a strictly ordered curve, γ its parametrization with γ_j the jth projection, $\mathbf{x} = (x_i) \in \ell$, and $B \subset \mathbb{Z}$ a finite connected component.

Assume that $\mathbf{x}^* = (x_i^*)_{i \in \bar{B}}$ is a minimal segment such that $x_i^* = x_i$ for $i \in \partial B$ and $W_B(\mathbf{x}^* + \mathbf{v}) \geq W_B(\mathbf{x}^*)$ for all $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$ with $\operatorname{supp}(\mathbf{v}) \subset \operatorname{int}(B)$. The existence of $\mathbf{x}^*$ is guaranteed by hypothesis (H3).

Let $\bar{\mathbf{x}} = \gamma(\bar{s}) \in \ell$, where $\bar{s} = \max\{s \in \mathbb{R} \mid \gamma_j(s) \leq x_j^*,\ j \in \bar{B}\}$. The existence of $\bar{s}$ is ensured by the facts that γ_0 is surjective, $\gamma(s) \in \mathscr{B}_C$ for $s \in \mathbb{R}$, and γ is continuous. Note that $\bar{B}$ is finite and $\bar{\mathbf{x}} \in \ell$. Then there exists $i \in \bar{B}$ such that $\bar{x}_i = x_i^*$. We distinguish two cases.

(i) If $i \in \partial B$, then $\bar{x}_i = x_i^* = x_i$. Since γ is strictly increasing, then $\mathbf{x} = \bar{\mathbf{x}}$, and hence $x_n = \bar{x}_n \leq x_n^*$ for $n \in \bar{B}$.

(ii) If $i \in \operatorname{int}(B)$, then $\partial_i W(\mathbf{x}^*) = 0$ because $\mathbf{x}^* = (x_i^*)_{i \in \bar{B}}$ is a minimal segment. Consequently, for each $i \in \operatorname{int}(B)$,

$$
\begin{aligned}
0 = \partial_i W(\mathbf{x}^*) - \partial_i W(\bar{\mathbf{x}}) &= \sum_{j=i-r}^{i} \partial_i h_j(\mathbf{x}^*) - \partial_i h_j(\bar{\mathbf{x}}) \\
&= \sum_{j=i-r}^{i} \int_0^1 \frac{d}{d\tau} \partial_i h_j(\tau \mathbf{x}^* + (1-\tau)\bar{\mathbf{x}})\ d\tau \\
&= \sum_{j=i-r}^{i} \sum_{k=j}^{j+r} \int_0^1 \partial_{i,k} h_j(\tau \mathbf{x}^* + (1-\tau)\bar{\mathbf{x}})\ d\tau \cdot (x_k^* - \bar{x}_k).
\end{aligned}
$$

Noting that $x_i^* = \bar{x}_i$, $\partial_{i,k} h_j \le 0$ for $k \ne i$, and $\partial_{i,k} h_i < 0$ for $k = i+1$, we claim that $x_{i+1}^* = \bar{x}_{i+1}$. Then by induction we derive that there exists $j \in \partial B$ such that $x_j^* = \bar{x}_j$ and hence, by case (i), $x_n \le x_n^*$ for $n \in \bar{B}$.

Analogously we have $x_n \ge x_n^*$ for $n \in \bar{B}$ and hence $x_n = x_n^*$ for $n \in \bar{B}$, implying that $\mathbf{x}$ is a minimizer since B is arbitrary. □

Remark. If there exists a strictly ordered curve ℓ, then each $\mathbf{x} \in \ell$ is a minimizer with bounded action, and hence it is Birkhoff by Theorem 3.15 and has a rotation number. Since all elements in ℓ have the same rotation number ω (otherwise they will intersect), it follows that $\ell \subset \mathscr{M}_\omega$, and $p_0(\mathscr{M}_\omega) = \mathbb{R}$ since $p_0(\ell) = \mathbb{R}$.

In fact, in order to define a strictly ordered curve, it is enough to assume that $p_0(\ell) = \mathbb{R}$ and that γ is strictly increasing.

LEMMA 4.3. *Assume that $\gamma\colon \mathbb{R} \to \mathscr{B}_C \cap \mathcal{S}$ is strictly increasing and $\{\gamma_0(s) \mid s \in \mathbb{R}\} = \mathbb{R}$. Then $\ell = \{\gamma(s) \mid s \in \mathbb{R}\}$ is a strictly ordered curve.*

Proof. It suffices to verify the continuity of γ. Choose an increasing sequence $\{s_i\} \subset \mathbb{R}$ such that $s_i \to s$ as $i \to \infty$. Then $\gamma(s_i)$ is an increasing sequence bounded above by $\gamma(s)$ and hence $\gamma(s_i) \to \mathbf{x} \in \mathcal{S}$. Since $\mathbf{x} = (x_n) \le \gamma(s)$, we have $x_0 = \gamma_0(s)$ due to the assumption that γ_0 is surjective. It follows from Lemma 3.5 that $\mathbf{x} = \gamma(s)$. □

Assume that $\mathbf{u}(t)$ $(t > 0)$ is a solution of (2.4) with $\mathbf{u}(0) \in \mathscr{B}_C \cap \mathcal{S}^-$ for some $C \in \mathbb{N}$. Then from Lemma 2.7 we know that $\mathbf{u}(t)$ is increasing for $t \ge 0$. Moreover, from Lemma 2.10 we deduce that $\mathbf{u}(t)$ has bounded action for all $t \ge 0$. If $\{\mathbf{u}(t)\}_{t \ge 0}$ is bounded, then $\mathbf{y} = \lim_{t\to\infty} \mathbf{u}(t)$ is an equilibrium state of (2.4), i.e., a solution of (1.1). If it is unbounded, we can nevertheless obtain an equilibrium state of (2.4). In fact, we can furthermore construct a strictly ordered curve from the unbounded solution $\mathbf{u}(t)$. To this end, we need a lemma which has been proved in [**23**, §4].

Let $\mathscr{D} = \mathscr{B}_C \cap \mathcal{S}^-$ or $\mathscr{D} = \mathscr{B}_C \cap \mathcal{S}^+$ for some $C \in \mathbb{N}$. We denote by $\mathscr{F}(I, \mathscr{D})$, where I is an interval in $\mathbb{R}$, the topological space of all increasing functions $f : I \to \mathscr{D}$ with the induced product topology. Let $\triangle$ denote a family of subsets $\mathscr{D}_s \subset \mathscr{D}$, $s \in I$. Define

$$\mathscr{F}(\triangle) = \{f \mid f : I \to \mathscr{D},\ f(s) \in \mathscr{D}_s,\ s \in I, \text{ and } f \text{ is increasing}\} \subset \mathscr{F}(I, \mathscr{D}).$$

LEMMA 4.4. *If $\triangle$ is a family of compact sets, then $\mathscr{F}(\triangle)$ is sequentially compact.*

THEOREM 4.5. *Assume that $\mathbf{u}(0) \in \mathscr{B}_C \cap \mathcal{S}^-$ for some $C \in \mathbb{N}$ and that $\mathbf{u}(t)$ $(t \ge 0)$ is a unbounded solution of (2.4). Then there exist a sequence of times t_k, and $n_k, l_k \in \mathbb{Z}$, such that $\lim_{k\to\infty} \sigma_{n_k,l_k} \mathbf{u}(t_k) = \mathbf{z}$. Moreover, there exists a strictly ordered curve $\ell \subset \mathscr{B}_C \cap \mathcal{S}$ such that $\mathbf{z} \in \ell$.*

Proof. It follows from Lemma 2.10 that $\mathbf{u}(t) \in \mathscr{B}_C$, i.e., $|u_{i+1}(t) - u_i(t)| \le C$ for all $i \in \mathbb{Z}$ and $t \ge 0$. Taking into account the periodicity of h, we deduce the existence of $b > 0$ such that

$$|\partial_i h_j(\mathbf{u}(t))| \le b, \quad |\dot{u}_i(t)| \le b, \quad \text{and} \quad |\ddot{u}_i(t)| \le b \quad \text{for all } i, j \in \mathbb{Z} \text{ and } t \ge 0. \tag{4.1}$$

Choose a sequence of times $\{t_k\}$ and $\{l_k\} \subset \mathbb{Z}$ such that

$$\mathbf{u}(t_k) + l_k \cdot \mathbf{1} \triangleq \hat{\mathbf{u}}(t_k) \to \mathbf{y} = (y_i) \quad \text{as } k \to \infty. \tag{4.2}$$

Furthermore, we take a subsequence of $\{t_k\}$ if necessary, not relabeled, such that $|t_{k+1} - t_k| \geq 2$ for $k \in \mathbb{N}$, and

$$|\hat{u}_i(t_{k+1}) - \hat{u}_i(t_k)| \leq \delta_k \leq 1/k \quad \text{for } i \in [-k^2 - r, k^2 + r], \tag{4.3}$$

where $\delta_k > 0$ is chosen such that for $j \in [-k^2 - r, k^2]$,

$$|h_j(\mathbf{u}(t_{k+1})) - h_j(\mathbf{u}(t_k))| = |h_j(\hat{\mathbf{u}}(t_{k+1})) - h_j(\hat{\mathbf{u}}(t_k))| \leq \frac{b}{2k^2 + r + 1}. \tag{4.4}$$

By simple calculations we obtain

$$\begin{aligned}
\sum_{j=-k^2-r}^{k^2} h_j(\mathbf{u}(t_{k+1})) - h_j(\mathbf{u}(t_k)) &= \sum_{j=-k^2-r}^{k^2} \sum_{i=j}^{j+r} \int_{t_k}^{t_{k+1}} \partial_i h_j(\mathbf{u}(\tau))\, \dot{u}_i(\tau)\, d\tau \\
= \sum_{j=-k^2-r}^{-k^2-1} \sum_{i=j}^{-k^2-1} \int_{t_k}^{t_{k+1}} \partial_i h_j(\mathbf{u}(\tau))\, \dot{u}_i(\tau)\, d\tau &- \sum_{j=-k^2}^{k^2} \int_{t_k}^{t_{k+1}} |\dot{u}_j(\tau)|^2\, d\tau \\
+ \sum_{j=k^2-r+1}^{k^2} \sum_{i=k^2+1}^{j+r} \int_{t_k}^{t_{k+1}} \partial_i h_j(\mathbf{u}(\tau))\, \dot{u}_i(\tau)\, d\tau.
\end{aligned}$$

From (4.1), (4.4), and the above calculations we deduce that

$$\begin{aligned}
\sum_{j=-k^2}^{k^2} \int_{t_k}^{t_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2\, d\tau &= \sum_{j=-k^2}^{k^2} \int_{t_k}^{t_{k+1}} |\dot{u}_j(\tau)|^2\, d\tau \\
\leq \frac{(2k^2 + r + 1) \cdot b}{2k^2 + r + 1} + br \sum_{i=-k^2-r}^{-k^2-1} [u_i(t_{k+1}) - u_i(t_k)] &+ br \sum_{i=k^2+1}^{k^2+r} [u_i(t_{k+1}) - u_i(t_k)].
\end{aligned} \tag{4.5}$$

Now we have two cases.

(i) Assume that $\{u_0(t_{k+1}) - u_0(t_k) \mid k \in \mathbb{N}\}$ is bounded, i.e., $\{l_k - l_{k+1}\}$ is bounded. Then there exist infinitely many k such that $l_k - l_{k+1} = l > 0$, and thanks to (4.3), for each $i \in [-k^2 - r, k^2 + r]$,

$$|u_i(t_{k+1}) - u_i(t_k) - l| \leq \delta_k \leq 1/k. \tag{4.6}$$

Now (4.5) and (4.6) imply that

$$\sum_{j=-k^2}^{k^2} \int_{t_k}^{t_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2\, d\tau \leq b + br^2(2l + 2) \triangleq b_1.$$

As a consequence, we can choose $n_k \in [-k^2 - r + k, k^2 + r - k]$ such that

$$\sum_{j=-n_k-k}^{-n_k+k-1} \int_{t_k}^{t_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2\, d\tau \leq b_1/k. \tag{4.7}$$

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It is easy to check by simple calculations that there is a constant $b_2 > 0$ such that for each $j \in [-n_k - k, -n_k + k - 1]$ and for each $t \in [t_k, t_{k+1}]$,

$$|\partial_j W(\mathbf{u}(t))| \leq b_2\, k^{-1/4}, \tag{4.8}$$

which will be verified in the appendix.

Taking $l'_k \in \mathbb{Z}$ to replace l_k, not relabeled for simplicity, such that

$$\hat{u}_{-n_k}(t_k) = u_{-n_k}(t_k) + l_k \in [0, 1], \tag{4.9}$$

and $\mathbf{z}^k = \sigma_{n_k, l_k} \mathbf{u}(t_k)$ has a convergent subsequence, not relabeled, such that $\mathbf{z}^k \to \mathbf{z}$ as $k \to \infty$, it is easy to check by (4.8) that $\mathbf{z}$ is an equilibrium of (2.4).

To construct a strictly ordered curve, we need to verify the conditions of Lemma 4.4. Note that u_{-n_k} is a strictly increasing function from $[0, +\infty)$ to $\mathbb{R}$. Let ν_k denote its inverse function, i.e., $u_{-n_k}(\nu_k(s)) = s$ for $s \in [u_{-n_k}(0), +\infty)$. Define γ^k by

$$\gamma^k(s) = \sigma_{n_k, l_k} \mathbf{u}(\nu_k(u_{-n_k}(t_k) + s)) \in \mathscr{D},$$

where

$$s \in [0, u_{-n_k}(t_{k+1}) - u_{-n_k}(t_k)] \supset [0, l - 1/k]$$

by (4.3). Note that by (4.9),

$$\gamma_0^k(s) = u_{-n_k}(t_k) + s + l_k \in [\hat{u}_{-n_k}(t_k), \hat{u}_{-n_k}(t_{k+1}) - \hat{u}_{-n_k}(t_k) + l] \subset [0, l + 1]. \tag{4.10}$$

Then $\gamma^k(s)$ belongs to a compact set of $\mathscr{D}$ by Tychononov's theorem for each k and $s \in [0, l)$. Applying Lemma 4.4, we obtain a convergent subsequence, not relabeled, such that $\gamma^k \to \xi$ and ξ: $[0, l) \to \mathscr{D}$ is increasing. We derive by (4.8) that $\xi(s) \in \mathscr{B}_C \cap \mathcal{S}$ for each $s \in [0, l)$. Since $\xi_0(s) = y_0 + s$ is strictly increasing by (4.2) and (4.10), Lemma 3.5 implies that ξ is strictly increasing.

Defining $\xi(l) = \lim_{s \to l} \xi(s)$, we derive $\xi(l) = \xi(0) + l \cdot \mathbf{1}$. So we can extend ξ to $\mathbb{R} \to \mathscr{B}_C \cap \mathcal{S}$ such that ξ is strictly increasing and ξ_0 is surjective by (4.10). As a consequence of Lemma 4.3, $\ell = \{\xi(s) \mid s \in \mathbb{R}\}$ is a strictly ordered curve and $\mathbf{z} = \xi(0) \in \ell$.

(ii) Assume that $\{u_0(t_{k+1}) - u_0(t_k) \mid k \in \mathbb{N}\}$ is unbounded. Choose a subsequence of $\{t_k\}$ if necessary, not relabeled, such that $l_k - l_{k+1} \geq k^2 + 2$, where l_k is the same as (4.2). Take $\delta_k < 1/k$ such that (4.4), with k^2 replaced by k^3, holds true for $j \in [-k^3 - r, k^3]$. Then we have an inequality similar to (4.5) with k^2 replaced by k^3, and hence

$$\sum_{j=-k^3}^{k^3} \int_{t_k}^{t_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2\, d\tau \leq b + br \cdot 2r d_k \leq b_3 d_k,$$

where $b_3 > 0$ is a constant and

$$d_k \triangleq \max_{i \in [-k^3 - r, k^3 + r]} |u_i(t_{k+1}) - u_i(t_k)| \leq l_k - l_{k+1} + 1.$$

Note that

$$\min_{i \in [-k^3 - r, k^3 + r]} |u_i(t_{k+1}) - u_i(t_k)| \geq l_k - l_{k+1} - 1 \geq k^2 + 1 \geq 2k.$$

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Then for each $n \in [-k^3, k^3]$, we can choose $[t'_k, t'_{k+1}] \subset [t_k, t_{k+1}]$ such that $u_n(t'_{k+1}) - u_n(t'_k) = 2k$ and

$$\sum_{j=-k^3}^{k^3} \int_{t'_k}^{t'_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2 \, d\tau \le 4b_3 \, k, \tag{4.11}$$

which will be verified in the appendix.

Therefore, we find $n_k \in [-k^3 - r + k, k^3 + r - k]$ and the corresponding t'_k and t'_{k+1}, such that $u_{-n_k}(t'_{k+1}) - u_{-n_k}(t'_k) = 2k$ and

$$\sum_{j=-n_k-k}^{-n_k+k-1} \int_{t'_k}^{t'_{k+1}} |\partial_j W(\mathbf{u}(\tau))|^2 \, d\tau \le \frac{4b_3}{k},$$

which is similar to (4.7) and implies that

$$|\partial_j W(\mathbf{u}(t))| \le b_4 \, k^{-1/4}, \tag{4.12}$$

for $j \in [-n_k - k, -n_k + k - 1]$ and $t \in [t'_k, t'_{k+1}]$, where $b_4 > 0$ is a constant. Choose $\tilde{t} \in [t'_k, t'_{k+1}]$ such that $u_{-n_k}(\tilde{t}) - u_{-n_k}(t'_k) = k$ and define $\gamma^k \colon [-k, k] \to \mathscr{D}$ by

$$\gamma^k(s) = \sigma_{n_k, l_k} \, \mathbf{u}(\nu_k(u_{-n_k}(\tilde{t}) + s)),$$

where $l_k \in \mathbb{Z}$ such that $\gamma_0^k(0) \in [0, 1]$ and ν_k is the inverse function of u_{-n_k}. By Lemma 4.4 and the fact $\gamma_0^k(s) \in [s, s+1]$, the sequence γ^k has a convergent subsequence, not relabeled, such that $\gamma^k \to \xi$, where $\xi \colon \mathbb{R} \to \mathscr{D}$, and $\xi(s) \in \mathcal{S}$ for $s \in \mathbb{R}$ by (4.12). Consequently, $\ell = \{\xi(s) \mid s \in \mathbb{R}\}$ is a strictly ordered curve, which can be verified analogously to case (i). □

5. *Proof of Theorem A*

In this section we construct under the assumption of Theorem A two solutions of (1.1) exchanging their rotation numbers.

LEMMA 5.1. *If* $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, *then* $\mathbf{z} = \max\{\mathbf{x}, \mathbf{y}\} \in \mathcal{S}^-$.

Proof. Choose $i \in \mathbb{Z}$ and assume that $z_i = x_i$. Thanks to $\partial_i W(\mathbf{x}) = 0$ and the twist condition $\partial_{i,k} h_j \le 0$ for $i \ne k$, we obtain

$$\begin{aligned}\partial_i W(\mathbf{z}) &= \sum_{j=i-r}^{i} \partial_i h_j(\mathbf{z}) - \partial_i h_j(\mathbf{x}) = \sum_{j=i-r}^{i} \int_0^1 \frac{d}{dt} \partial_i h_j(t\mathbf{z} + (1-t)\mathbf{x}) \, dt \\ &= \sum_{j=i-r}^{i} \sum_{k=j}^{j+r} \int_0^1 \partial_{i,k} h_j(t\mathbf{z} + (1-t)\mathbf{x}) \, dt \cdot (z_k - x_k) \le 0,\end{aligned}$$

implying $\mathbf{z} \in S^-$. □

LEMMA 5.2. *Let* $\omega_0 < \omega_1$. *Assume that* $\mathbf{u}$ *and* $\mathbf{v}$ *are Birkhoff minimizers with rotation numbers* ω_0 *and* ω_1, *respectively, and that* $p_0(\mathscr{M}_\omega) \ne \mathbb{R}$ *for each* $\omega \in [\omega_0, \omega_1]$. *Then the solution* $\mathbf{z}(t)$ *of (2.4) with* $\mathbf{z}(0) = \mathbf{z} = \max\{\mathbf{u}, \mathbf{v}\}$ *is bounded.*

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Proof. We assume that $\mathbf{z}(t)$ $(t \geq 0)$ is unbounded. Applying Theorem 4.5, we choose sequences t_k, n_k, and l_k, such that

$$\mathbf{x}^k = \sigma_{n_k,l_k}\mathbf{z}(t_k) \to \mathbf{y} \quad \text{as } k \to \infty,$$

where $\mathbf{y} \in \ell$, a strictly ordered curve. Since $\mathbf{u} \in \mathscr{M}_{\omega_0}$ and $\mathbf{v} \in \mathscr{M}_{\omega_1}$, we obtain by (2.1) that there exist $i_{-1}, i_0 \in \mathbb{Z}$, such that $i_{-1} < i_0$ and $z_i = u_i$ for $i \leq i_{-1}$ and $z_i = v_i > u_i$ for all $i \geq i_0$. Choose $l'_k \in \mathbb{Z}$ such that $\mathbf{v}^k = \sigma_{n_k,l'_k}\mathbf{v}$ satisfies

$$v^k_{i_0-j} \geq x^k_{i_0-j},\ j \in \{1, \ldots, r\} \quad \text{and} \quad v^k_{i_0} \in [x^k_{i_0} + N, x^k_{i_0} + N + 1] \text{ for some } N \geq 2. \tag{5.1}$$

We shall prove that $v^k_i \geq x^k_i$ for $i > i_0$ and $k \in \mathbb{N}$ by applying Lemma 2.6. To this end, we need to verify the following two statements:

(i) $v^k_{i_0-j} \geq (\sigma_{n_k,l_k}\mathbf{z}(t))_{i_0-j}$, for $j \in \{1, 2, \ldots, r\}$ and for all $t \in [0, t_k]$. Indeed, by Lemmas 2.7 and 5.1, we know that $\mathbf{z}(t)$ is increasing, i.e., $\mathbf{z}(t_1) \leq \mathbf{z}(t_2)$ for $0 \leq t_1 < t_2$. Thus we deduce from (5.1) that

$$v^k_{i_0-j} \geq x^k_{i_0-j} \geq (\sigma_{n_k,l_k}\mathbf{z}(t))_{i_0-j} \quad \text{for all } t \in [0, t_k].$$

(ii) $v^k_i \geq (\sigma_{n_k,l_k}\mathbf{z})_i$ for $i \geq i_0$. Indeed, since $\mathbf{z}(t) \gg \mathbf{v}$ for $t > 0$ by Lemma 2.5, it follows from (5.1) that

$$v^k_{i_0} = (\sigma_{n_k,l'_k}\mathbf{v})_{i_0} \geq x^k_{i_0} = (\sigma_{n_k,l_k}\mathbf{z}(t_k))_{i_0} > (\sigma_{n_k,l_k}\mathbf{v})_{i_0}.$$

Since $\mathbf{v}$ is Birkhoff,

$$v^k_i \geq (\sigma_{n_k,l_k}\mathbf{v})_i \quad \text{for all } i \in \mathbb{Z}. \tag{5.2}$$

If $n_k \leq 0$, then from (5.2) we have for $i \geq i_0$,

$$v^k_i \geq (\sigma_{n_k,l_k}\mathbf{v})_i = v_{i-n_k} + l_k \geq u_{i-n_k} + l_k \geq (\sigma_{n_k,l_k}\mathbf{u})_i.$$

If $n_k > 0$ (without loss of generality we may assume that $n_k > i_0 - i_{-1}$), then

$$v^k_{i_0} \geq x^k_{i_0} + N \geq (\sigma_{n_k,l_k}\mathbf{z})_{i_0} + N = z_{i_0-n_k} + l_k + N = u_{i_0-n_k} + l_k + N.$$

Combining

$$-1 \leq v^k_i - v^k_{i_0} - (i - i_0)\omega_1 \leq 1 \quad \text{and} \quad -1 \leq u_{i-n_k} - u_{i_0-n_k} - (i - i_0)\omega_0 \leq 1,$$

which follow from (2.1) since $\mathbf{v}^k$ and $\mathbf{u}$ are Birkhoff, we derive

$$\begin{aligned} v^k_i &\geq v^k_{i_0} + (i - i_0)\omega_1 - 1 \geq u_{i_0-n_k} + l_k + N + (i - i_0)\omega_1 - 1 \\ &\geq u_{i-n_k} - 1 - (i - i_0)\omega_0 + l_k + N + (i - i_0)\omega_1 - 1 \\ &= (\sigma_{n_k,l_k}\mathbf{u})_i + (i - i_0)(\omega_1 - \omega_0) + N - 2 \\ &\geq (\sigma_{n_k,l_k}\mathbf{u})_i \quad \text{for all } i \geq i_0. \end{aligned}$$

So in either case we have

$$v^k_i \geq (\sigma_{n_k,l_k}\mathbf{u})_i \quad \text{for } i \geq i_0,$$

and hence, by (5.2),

$$v^k_i \geq (\sigma_{n_k,l_k}\mathbf{z})_i \quad \text{for } i \geq i_0.$$

Now we can apply Lemma 2.6 to obtain

$$v_i^k \geq (\sigma_{n_k,l_k}\mathbf{z}(t_k))_i = x_i^k \quad \text{for } i \geq i_0,$$

implying $\mathbf{v}^k >_\omega \mathbf{x}^k$.

Due to (5.1), we can choose a convergent subsequence of $\{\mathbf{v}^k\}$, not relabeled, such that $\mathbf{v}^k \to \mathbf{v}'$ as $k \to \infty$, and hence $\mathbf{v}' >_\omega \mathbf{y}$.

Analogously, we can find translates $\mathbf{u}^k$ of $\mathbf{u}$, such that $\mathbf{u}^k >_\alpha \mathbf{x}^k$ and $\mathbf{u}^k \to \mathbf{u}'$ with $\mathbf{u}' >_\alpha \mathbf{y}$.

Note that $\mathbf{u}'$ and $\mathbf{v}'$ are Birkhoff minimizers with rotation number ω_0 and ω_1, respectively. We have that $\mathbf{y}$ is Birkhoff by Theorem 3.15 and hence has rotation number ω with $\omega_0 \leq \omega \leq \omega_1$. This implies from Theorem 3.15 that the strictly ordered curve ℓ consists of Birkhoff minimizers with rotation number ω, which contradicts the assumption that $p_0(\mathscr{M}_\omega) \neq \mathbb{R}$. □

Proof of Theorem A. Assume that $\mathbf{u}$ and $\mathbf{v}$ are Birkhoff minimizers with rotation number ω_0 and ω_1, respectively, with $\rho_0 < \omega_0 < \omega_1 < \rho_1$. Due to Lemma 5.2, the solution $\mathbf{z}(t)$ of (2.4) with the initial condition $\mathbf{z} = \max\{\mathbf{u}, \mathbf{v}\}$ is bounded. Let $\mathbf{y} = \lim_{t\to+\infty} \mathbf{z}(t)$ and assume that $z_i = v_i$ for $i \geq i_0$. Choose $N \in \mathbb{N}$ and let $\bar{\mathbf{v}} = \sigma_{0,N}\mathbf{v}$ such that $\bar{v}_{i_0-k} \geq y_{i_0-k}$ and hence $\bar{v}_{i_0-k} \geq z_{i_0-k}(t)$, for all $k \in \{1, 2, \ldots, r\}$ and for all $t \geq 0$. Since $\bar{v}_i \geq z_i = v_i$ for $i \geq i_0$, then from Lemma 2.6 we have $\bar{v}_i \geq z_i(t)$ for $i \geq i_0$ and $t \geq 0$, and hence $\bar{v}_i \geq y_i$ for $i \geq i_0$. Consequently, since $y_i \geq v_i$ for $i \geq i_0$, we derive

$$\omega_1 = \lim_{i\to+\infty} \frac{v_i}{i} \leq \lim_{i\to+\infty} \frac{y_i}{i} \leq \lim_{i\to+\infty} \frac{\bar{v}_i}{i} = \omega_1.$$

Analogously we have $\lim_{i\to-\infty} y_i/i = \omega_0$.

If we assume that $\mathbf{z}' = \min\{\mathbf{u}, \mathbf{v}\} \in \mathcal{S}^+$ and $\mathbf{z}'(t)$ is a solution of (2.4) with $\mathbf{z}'(0) = \mathbf{z}'$, then $\mathbf{z}'(t)$ is decreasing and bounded and we obtain a solution $\mathbf{y}' = \lim_{t\to\infty} \mathbf{z}'(t)$ of (1.1) with $\lim_{i\to+\infty} y_i'/i = \omega_0$ and $\lim_{i\to-\infty} y_i'/i = \omega_1$.

Therefore, we have constructed two solutions of (1.1). $\mathbf{y}$ and $\mathbf{y}'$. exchanging rotation numbers, i.e., satisfying (1.2), and hence we arrive at the conclusion by Proposition 2.11. □

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A. *Appendix*

In this appendix we verify (4.8) and (4.11). The notation is the same as in the proof of Theorem 4.5.

Proof of (4.8). If we denote $f(t) = |\partial_j W(\mathbf{u}(t))|^2 = |\dot{u}_j(t)|^2 \geq 0$ for $j \in [-n_k - k, -n_k + k - 1]$, then

$$\int_{t_k}^{t_{k+1}} f(\tau)\, d\tau \leq b_1/k. \tag{A.1}$$

Assume that f attains its maximum on $[t_k, t_{k+1}]$ at a. Then $a + 1/\sqrt{k} \in [t_k, t_{k+1}]$ ($a - 1/\sqrt{k} \in [t_k, t_{k+1}]$ is treated similarly) since $t_{k+1} - t_k \geq 2$. We deduce by (4.1) that

$$|f'(t)| = |2\dot{u}_j(t)\ddot{u}_j(t)| \leq b' \quad \text{for some } b' > 0 \text{ and } t \geq 0,$$

and hence

$$\min_{t\in[a,a+1/\sqrt{k}]} f(t) \geq f(a) - b'/\sqrt{k} \geq 0$$

(assume k large). Consequently, we have

$$\int_{t_k}^{t_{k+1}} f(\tau)\,d\tau \geq \int_a^{a+1/\sqrt{k}} f(\tau)\,d\tau \geq (f(a) - b'/\sqrt{k})/\sqrt{k} = f(a)/\sqrt{k} - b'/k,$$

which implies (4.8) by (A.1). □

Proof of (4.11). Let k_0 be the integer part of $d_k/2k$. We find $k_0 + 1$ points $t_k = \tau_0 < \tau_1 < \cdots < \tau_{k_0} \leq t_{k+1}$ such that $u_n(\tau_{i+1}) - u_n(\tau_i) = 2k$ for $i = 0, 1, \ldots, k_0 - 1$. Let

$$g(t) = \sum_{j=-k^3}^{k^3} |\partial_j W(\mathbf{u}(t))|^2, \quad A = \int_{t_k}^{t_{k+1}} g(\tau)\,d\tau, \quad G(t) = \int_t^{t'} g(\tau)\,d\tau,$$

where $t \in [\tau_0, \tau_{k_0-1}]$ and t' satisfies $u_n(t') - u_n(t) = 2k$. The function G is continuous on $[\tau_0, \tau_{k_0-1}]$. If we assume that $G(t) > A/k_0$ for each $t \in [\tau_0, \tau_{k_0-1}]$, then

$$A \geq \int_{\tau_0}^{\tau_{k_0}} g(\tau)\,d\tau = G(\tau_0) + G(\tau_1) + \cdots + G(\tau_{k_0-1}) > A,$$

a contradiction. Consequently, there exist $t'_k \in [\tau_0, \tau_{k_0-1}] \subset [t_k, t_{k+1}]$ and hence t'_{k+1} such that

$$G(t'_k) \leq A/k_0 \quad \text{and} \quad u_n(t'_{k+1}) - u_n(t'_k) = 2k,$$

that is,

$$\int_{t'_k}^{t'_{k+1}} g(\tau)\,d\tau \leq \frac{A}{k_0} \leq \frac{2A}{k_0+1} \leq \frac{2k}{d_k} 2A \leq 4b_3\,k,$$

which verifies (4.11). □

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