BEYOND ENDOSCOPY VIA THE TRACE FORMULA – III THE STANDARD REPRESENTATION

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Abstract   We finalize the analysis of the trace formula initiated in S. A. Altuğ [Beyond endoscopy via the trace formula-I: Poisson summation and isolation of special representations, *Compos. Math.* **151**(10) (2015), 1791–1820] and developed in S. A. Altuğ [Beyond endoscopy via the trace formula-II: asymptotic expansions of Fourier transforms and bounds toward the Ramanujan conjecture. Submitted, preprint, 2015, Available at: arXiv:1506.08911.pdf], and calculate the asymptotic expansion of the beyond endoscopic averages for the standard L-functions attached to weight $k \geqslant 3$ cusp forms on $GL(2)$ (cf. Theorem 1.1). This, in particular, constitutes the first example of beyond endoscopy executed via the Arthur–Selberg trace formula, as originally proposed in R. P. Langlands [Beyond endoscopy, in *Contributions to Automorphic Forms, Geometry, and Number Theory*, pp. 611–698 (The Johns Hopkins University Press, Baltimore, MD, 2004), chapter 22]. As an application we also give a new proof of the analytic continuation of the L-function attached to Ramanujan's Δ-function.

Keywords: beyond endoscopy; trace formula

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Contents

1. Introduction

Let us begin by briefly recalling beyond endoscopy. Since our aim here is to put the current paper in context rather than to give a comprehensive overview, rather than trying to make precise statements we simply introduce the main problem. For a more through overview we refer the reader to the original article [11] and the recent exposition in [3].

Let G be a reductive algebraic group defined over a global field F and let S be a finite set of places of F containing all the archimedean places. The main aim of beyond endoscopy is to isolate those automorphic representations of G (unramified outside of S) that are functorial transfers from other groups. The strategy proposed in [11] is to use the partial automorphic L-functions, $L^S(s, \pi, \rho)$, for various irreducible representations $\rho : {}^L G \to GL(V)$ to detect functorial transfers. It is based on the expectation that if $L^S(s, \pi, \rho)$ has a pole in $\Re(s) \geqslant 1$ then π should be a functorial transfer, and conversely if π is a functorial transfer then, by a theorem of Chevalley (cf. [4]), one can find a ρ such that $L^S(s, \pi, \rho)$ has a pole in $\Re(s) \geqslant 1$.

To analyze the poles of

$$L^S(s, \pi, \rho) = \sum_{n \geqslant 1}' \frac{a_{\pi,\rho}(n)}{n^s},$$

where the prime indicates that we are summing over n that are not divisible by any of the finite primes in S, in the region $\Re(s) \geqslant 1$ one can study the asymptotic behavior, in the variable X, of the partial averages

$$\sum_{n < X}' a_{\pi,\rho}(n),$$

up to terms of size $o(X)$. The strategy proposed in [11] is to use the trace formula to study the averages above. More precisely, let $\mathbb{A}_S = \prod_{v \in S} \mathbb{Q}_v$, $\mathbb{A}^S = \prod_{v \notin S}' \mathbb{Q}_v$, and $f_S \in C_c^\infty(G(\mathbb{A}_S))$. Then, for each $n \geqslant 1$ that is relatively prime to every prime in S there exist a function $f^{n,\rho} \in C_c^\infty(G(\mathbb{A}^S))$ such that

$$\sum_{\pi \in L^2_{\text{disc}}(G)} \text{tr}(\pi_S(f_S)) a_{\pi,\rho}(n) = \text{tr}(R_{\text{disc}}(f_S f^{n,\rho})),$$

where L^2_{disc} denotes the discrete part of the automorphic spectrum[1] of G. Substituting this expression into the partial averages of coefficients above, we can now state the problem

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[1] We are ignoring the issues about central characters, stability etc. since a thorough discussion of such would take us too far afield.

of beyond endoscopy as to study the asymptotic behavior of the averages,

$$\sideset{}{'}\sum_{n<X} \operatorname{tr}(R_{\mathrm{disc}}(f_S f^{n,\rho})), \tag{1}$$

using the Arthur–Selberg trace formula. In [1] the study of the averages in (1) was initiated for $G = GL(2)$, $S = \{\infty\}$, and $\rho = \mathrm{Sym}^r$, the rth symmetric power representation of $^L G = GL(2, \mathbb{C})$. The geometric side of the trace formula consists of a sum over rational conjugacy classes of (weighted) orbital integrals multiplied by certain arithmetic volume factors.

The critical part of these sums are the ones over elliptic conjugacy classes, those conjugacy classes whose characteristic polynomials are irreducible over $\mathbb{Q}$. In order to analyze the elliptic part, an appropriate approximate functional equation was introduced in [1] (cf. (4') of *loc. cit.*) and a Poisson summation was applied on the so-called Steinberg–Hitchin base (i.e., the space of characteristic polynomials in this case) to isolate the contribution of certain special representations (which, in general, give non-zero contribution to the asymptotic expansion of (1), see [11, (31) and (65)]) were isolated. In the subsequent paper [2] the remaining terms after Poisson summation were analyzed giving a firm control over the asymptotic behavior of various Fourier transforms that appear after Poisson summation (cf. [2, Theorems A.14 and A.15]). We remark that both of the papers cited above are concerned with a single trace formula, in other words they are not concerned with the averages in (1) but rather prove results for an individual n.

The current paper puts all of the previous work together and executes the asymptotic analysis of (1) for $G = GL(2)$ over $\mathbb{Q}$, $S = \{\infty\}$, and $\rho = \mathrm{Sym}^1$, the standard representation of $GL(2, \mathbb{C})$, where we specialize our test function to pick up holomorphic discrete series representation at the archimedean place. This puts us in the framework of classical holomorphic modular forms of weight k. Let us also recall the (normalized) Hecke operators in this setting

$$T_k(n)(f)(z) = n^{\frac{k-1}{2}} \sum_{\substack{ad=n \\ a,d>0 \\ b \bmod d}} d^{-k} f\left(\frac{az+b}{d}\right),$$

where f is any holomorphic modular form of weight k and full level. Note that with this normalization the Ramanujan conjecture reads as $\operatorname{tr}(T_k(n)) = O_{k,\epsilon}(n^\epsilon)$ for every $\epsilon > 0$. Our main theorem can be stated as follows:

Theorem 1.1. *Let $k \geqslant 3$ be an integer and for any $n \geqslant 1$ let $T_k(n)$ denote the nth Hecke operator (normalized as above) acting on the space S_k of holomorphic cusp forms of weight k. Then, for every $\epsilon > 0$*

$$\sum_{n<X} \operatorname{tr}(T_k(n)) = O_{k,\epsilon}(X^{\frac{31}{32}+\epsilon}).$$

We comment on the generality of the methods, the dependence on various assumptions, and possible improvements on the exponent $\frac{31}{32}$ below in the next section.

To end the introduction we also present an application of the above theorem and give a new proof of the analytic continuation of the L-function attached to the Ramanujan Δ-function. Recall that $\Delta(z)$ and $\tau(n)$ are defined by

$$\Delta(z) := e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24} = \sum_{n=1}^{\infty} \tau(n) e(nz).$$

It is well-known (cf. [15, p. 84]) that $\Delta(z)$ is a weight 12 cusp form of full level. Attached to Δ is the L-function

$$L(s, \Delta) := \sum_{n=1}^{\infty} \frac{\tilde{\tau}(n)}{n^s},$$

where

$$\tau(n) = \tilde{\tau}(n) n^{\frac{11}{2}}.$$

We remark that the above normalization of the L-function is so that it aligns with our normalization of the Hecke operators. It follows from the elementary Hecke bound that $L(s, \Delta)$ is absolutely convergent[2] for $\Re(s) > \frac{3}{2}$ (*loc. cit.* (39) on p. 94). Theorem 1.1 then has the following immediate corollary.

Corollary 1.2. $L(s, \Delta)$ *extends to a holomorphic function in* $\Re(s) > \frac{31}{32}$.

Let us also remark that the novelty of Corollary 1.2 is not the end result, which is well-known and dates back to the works of Hecke (cf. [8]), but rather the method of proof, which gives the analytic continuation of an automorphic L-function using solely the trace formula.

1.1. Several remarks and comments

- First, we would like to emphasize that the novelty of the paper is in the method rather than the result. The fact that the standard L-functions for $GL(2)$ have analytic continuation is well-known (it goes back to Hecke in the setting of the current paper and is known in much greater generality (cf. [7])). It is the method by which we prove the analytic continuation that is new and constitutes the first example of beyond endoscopy carried out via the Arthur–Selberg trace formula, following the original proposal in [11]. We should also remark that following the ideas of Beyond Endoscopy, in [12], has proved the analytic continuation of the standard L-functions using the Kuznetsov formula.

- Let us make a couple of remarks on the generality of the method and the various restrictions we have in Theorem 1.1. As already mentioned, the specific choice of test functions at the archimedean place puts us in the framework of the classical Eichler–Selberg trace formula. It also avoids the contribution of the continuous and the non-tempered spectrum (i.e., the trivial representation) with the price that it picks up only holomorphic cusp forms (no Maass forms interfere).

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[2]Indeed, Rankin–Selberg theory implies that $L(s, \Delta)$ is absolutely convergent for $\Re(s) > 1$. We will, however, not be needing this fact.

The reason we chose this approach is that it avoids peripheral issues and go directly to the heart of the matter which is the analysis of the averages in (1) over the elliptic part of the trace formula. Our primary aim in this paper is to analyze the beyond endoscopic averages in (1) rather than to give a proof of the analytic continuation of the standard L-functions for $GL(2)$. The analytic difficulties related to the elliptic part are already present in the Eichler–Selberg trace formula, and this is why we chose this approach rather than working in the generality of [1, 2], where the general the geometric side was analyzed for an arbitrary smooth test function of compact support at the archimedean place. We should also note the analysis carried out in these references is sufficient to carry out the analysis without any restriction in Theorem on the archimedean test function, and that we will be using the results of [1, 2] (in particular [1, Theorem 6.1] and Theorem 1.1 and the estimates in the appendices of [2]) throughout the paper.

The assumption about $S = \{\infty\}$ (i.e., we are restricting to representations unramified at every finite place, or classically, forms of full level), on the other hand, is harmless analytically and can be removed without any trouble. It brings in congruence conditions in the sums (depending on the allowed ramification) and does not effect the analysis in any serious way. It just brings extra work on the algebra and would complicate (the already complicated) notation.

- One can, without much effort, improve the exponent $\frac{31}{32}$ of Theorem 1.1. (Our estimates, especially in §4.2, are far from optimal.) Since our aim in this paper is to execute the beyond endoscopic averages in (1), which is equivalent to getting an $o(X)$ in Theorem 1.1, we did not, in any way, aim for optimality in the exponent.

- We would also like to briefly mention the connection of the paper with the ρ-trace formula of [3] (note that ρ is denoted by r in that article). Instead of repeating various definitions and constructions of Arthur we simply refer the reader to [3, pp. 8–14]. The result of Theorem 1.1, in particular, proves the ρ-trace formula for $G = GL(2)$ and ρ = standard representation. In terms of Arthur's notation (cf. (2.7) of *loc. cit.*) it can be stated as

$$S^{\rho}_{\mathrm{cusp}}(f) = 0,$$

where f denotes our specific choice of test function.

- The last remark is on a peculiar phenomenon about the averages in (1). To describe the issue let us first start with the expected result and work our way back. As in Theorem 1.1 let us fix an integer $k \geqslant 3$. If π is an cuspidal automorphic representation of $GL(2)$ attached to a cusp form of weight k of full level, then the standard L-function, $L(s, \pi)$, is holomorphic in $\Re(s) \geqslant 1$. Therefore, denoting the nth Hecke operator acting on the space of weight k cusp forms of full level, normalized as in Theorem 1.1, by $T_k(n)$, we expect to have

$$\sum_{\pi} \mathrm{Res}_{s=1} L(s, \pi) = \lim_{X \to \infty} \frac{1}{X} \sum_{n < X} \mathrm{tr}(T_k(n)) = 0. \tag{2}$$

This indeed is implied by the main result of the paper; however, one sees a surprising feature in the calculation of this limit by the trace formula. Using the trace formula

one gets[3] (cf. § 3)

$$\lim_{X \to \infty} \frac{1}{X} \sum_{n < X} \operatorname{tr}(T_k(n)) = \lim_{X \to \infty} \frac{1}{X} \sum_{n < X} \{(i_n) + (ii_n) + (iii_n)\},$$

where (i_n) is the contribution of the elliptic conjugacy classes, (ii_n) is the contribution of the hyperbolic and unipotent conjugacy classes (hyperbolic contribution is the sum over all $d \mid n$ that satisfies $d \neq \sqrt{n}$, and the unipotent contribution comes from $d = \sqrt{n}$, which exists only if n is a perfect square), and (iii_n) is the contribution of the identity element.

The remarkable point is that the limits of the averages of the individual contributions are *non-zero*. More precisely, the proof of Theorem 1.1 shows that

$$\lim_{X \to \infty} \frac{1}{X} \sum_{n < X} (i_n) = \frac{1}{k-1}, \quad \lim_{X \to \infty} \frac{1}{X} \sum_{n < X} (ii_n) = \frac{1}{1-k}, \quad \lim_{X \to \infty} \frac{1}{X} \sum_{n < X} (iii_n) = 0. \quad (3)$$

So when we sum up the contributions of the terms the result is 0; However, individually the elliptic and hyperbolic contributions[4] are both non-zero and cancel each other!

There is one more peculiarity we would like to highlight. If instead of the residue one is interested just in the pole of the L-function, taking logarithmic derivatives of the L-functions, one replaces (2) with a sum over prime powers and expects

$$\sum_{\pi} \operatorname{ord}_{s=1} L(s, \pi) = \lim_{X \to \infty} \frac{1}{X} \sum_{\substack{p^r < X \\ p:\text{ prime}}} \log(p) \operatorname{tr}(T_k(p^r)) = 0, \quad (4)$$

where we used 'ord' to denote the order of the pole. It is not hard to see that in the corresponding sums in (3), the individual averages of (i_{p^r}), (ii_{p^r}), and (iii_{p^r}) are all 0. i.e.,

$$\lim_{X \to \infty} \frac{1}{X} \sum_{\substack{p^r < X \\ p:\text{ prime}}} \log(p)(i_{p^r}) = \lim_{X \to \infty} \frac{1}{X} \sum_{\substack{p^r < X \\ p:\text{ prime}}} \log(p)(ii_{p^r})$$

$$= \lim_{X \to \infty} \frac{1}{X} \sum_{\substack{p^r < X \\ p:\text{ prime}}} \log(p)(iii_{p^r}) = 0. \quad (5)$$

We emphasize that this is an interesting phenomenon that was not previously observed. In the initial stages of the ideas of Beyond Endoscopy Langlands had suggested the use of the logarithmic derivative to capture the order of the pole (cf. [11]). This approach was later revisited in the letter of Sarnak [12], where he suggested the use of the

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[3]In what follows one should keep in mind that the contribution of the non-tempered spectrum (the trivial representation in this case), which was shown to contribute to (i_n) (cf. [1, Theorem 1]), is 0 in this setting, so we can ignore it. In general one needs to subtract it from (i_n) to make sense of the claims below.

[4]The proof of Proposition 3.1 shows that the non-zero contribution is indeed coming from the weighted orbital integrals over hyperbolic conjugacy classes rather than the unipotent ones.

L-function rather than the logarithmic derivative with the idea that sums over primes are harder to handle from an analytic perspective.

The above observation, on the other hand, suggests that the sums over integers, although may be friendlier from a purely analytic number theory perspective, may complicate matters from the viewpoint of the trace formula (as more terms in the trace formula contribute to the limit, and internal cancellations arise).

An interesting question, also raised by Arthur (cf. [3, Problem VI]), is to understand if there is a conceptual explanation for the averages over integers and primes to behave in the way described above. One can also make a further comment on these matters. It was shown in [1] that the trivial representation and the special term, denoted by $\operatorname{tr}(\xi_0(f_\infty))$, which comes from the continuous spectrum contribute to the term $\xi = 0$ after Poisson summation on the Steinberg–Hitchin basis. It turns out that the non-zero limit of the sums (over integers) of the weighted orbital integrals of the hyperbolic conjugacy classes (cf. the second limit in (3)) also gets cancelled with a part of this dominant term (cf. Corollary 4.7). Is there any connection? (This is, in a sense, combining [3, Problems V and VI].) Although the questions are intriguing so far we have no explanation for this besides the calculations below.

1.2. Outline of the paper

The paper is organized as follows: in §2, we prove Theorem 1.1 and Corollary 1.2 assuming various estimates on the averages of the geometric side of the trace formula. We hope that this motivates the various estimates to follow in the subsequent sections.

In §3, we introduce the Selberg trace formula and prove the estimates on the identity, hyperbolic, and unipotent contributions (§3.1).

Section 4 is the heart of the paper and constitutes the analysis of the elliptic part. Following [1], we first rewrite the elliptic part and then introduce an approximate functional equation. Then in §4.1, we apply Poisson summation on the Steinberg–Hitchin basis (Concretely, the conjugacy classes in $GL(2)$ are parametrized by their characteristic polynomials, and for fixed determinant the only variable is the trace. We apply Poisson summation on this variable, denoted by the variable m in §4) and analyze the term corresponding to $\xi = 0$. In Corollary 4.7, we show that the contribution of this term in the elliptic part exactly cancels (as it should!) the contribution of the hyperbolic part. Section 4.2 forms the main part of the analysis of the elliptic part. In this section, we analyze the rest of the terms after Poisson summation. Our strategy[5] is to bring in the sum over n and to use Poisson summation. We do this in two steps. In §4.2.1, we first strip off, for each fixed n, certain parts of the elliptic term that does not give any contribution to the asymptotic expansion. Then in §4.2.2, we finally bring the n-sum in and use Poisson summation.

In §5, we prove certain estimates on Fourier transforms and character sums that are used in the analysis of §4.2.2.

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[5]We remark that applying Poisson summation on the *n*-sum works well for the standard representation and the symmetric square, however stops being productive for higher symmetric powers. This point was observed by Sarnak and we refer to [12] for further discussion of this issue.

1.3. Notation and conventions

Notation

- $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ as usual will denote the sets of integers, real, and complex numbers respectively. $\mathbb{N} = \{0, 1, 2, \ldots\}$ will denote the set of natural numbers.

- The Mellin transform of a function Φ is defined as usual, $\tilde{\Phi}(u) = \int_0^\infty \Phi(x) x^{u-1} \, dx$.

- $\sum_{a \bmod \times N}$ will mean $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. For an integer l, we denote its radical (i.e., the square-free part of it) by $\mathrm{rad}(l) := \prod_{p | l} p$.

- $\mathcal{S}(\mathbb{R})$ will denote the Schwartz space, $\mathcal{S}(\mathbb{R}) = \{\Phi \in C^\infty(\mathbb{R}) \mid \sup |x^\alpha \Phi^{(\beta)}(x)| < \infty \, \forall \alpha, \beta \in \mathbb{N}\}$. We also remark that for functions Φ which are only defined on $\mathbb{R}^+$, by abuse of notation, we use $\Phi \in \mathcal{S}(\mathbb{R})$ to mean that $\Phi(x)$ and all of its derivatives decay faster than any polynomial for $x > 0$.

- For a domain $D \subset \mathbb{C}$, $f = O_h(g)$ means that there exists a constant K, depending only on h, such that $|f(x)| \leqslant K|g(x)|$ for every $x \in D$. Most of the times D will be clear from the context and will not be specified. $f \ll_h g$ means $f = O_h(g)$.

- $\sqrt{\cdot}$ denotes the branch of the square-root function that is positive on $\mathbb{R}^+$, $\left(\frac{D}{\cdot}\right)$ denotes the Kronecker symbol, and $e(x) := e^{2\pi i x}$.

- We also note a slight change of notation from [2] to the current paper. The function that we denote by θ_∞ in this paper was denoted by $\theta_{\infty,1}^{pos}$ in [2]. We hope that this simplifies the notation and does not cause any confusion.

Conventions

- Throughout the paper, unless otherwise explicitly stated (cf. Corollary 1.2), we always normalize the Hecke operators so that the Ramanujan conjecture is $a_\pi(n) = O(n^\epsilon)$ for every $\epsilon > 0$.

- There is an auxiliary function F that is introduced in the approximate functional equation (cf. (13)). All of the estimates in §4 depend on the choice of this function. Since this function is fixed once and for all, and does not depend on anything else we suppress this dependence and do not mention it in any of the estimates.

- Since the function θ_∞ depends only on the weight k (cf. (12)), instead of $O_{\theta_\infty}(\cdot)$ we simply write $O_k(\cdot)$.

2. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we give proofs of Theorem 1.1 and Corollary 1.2 using the results of the rest of the paper so that the reader can follow how each estimate is used.

Proof of Theorem 1.1. First note that if k is odd then the space of cusp forms of full level and weight k is empty so the theorem is trivially true. For the rest of the proof assume k is even. Then, by the Eichler–Selberg trace formula

$$\mathrm{tr}(T_k(n)) = (i_n) + (ii_n) + (iii_n).$$

(For an explanation of the terms involved, and what they actually are, see §3.)

Substituting this in the average gives

$$\sum_{n<X} \text{tr}(T_k(n)) = \sum_{n<X} \{(i_n) + (ii_n) + (iii_n)\}.$$

By (11),

$$\sum_{n<X} (iii_n) = O_k(\log(X)). \tag{6}$$

By Proposition 3.1,

$$\sum_{n<X} (ii_n) = \frac{X}{1-k} + O_k(\sqrt{X}). \tag{7}$$

To asymptotically evaluate the average of (i_n) we use Corollaries 4.7, 4.9, 4.11, 4.13, and 4.15. For any $\kappa, \alpha > 0$ such that $2\kappa + \alpha < \frac{1}{12}$, and every $N > 0$ these give

$$\sum_{n<X} (i_n) = \frac{X}{k-1} + O(X^{2-N\kappa} + X^{\frac{3+2\kappa}{4}} \log^2(X) + X^{1-\frac{3\alpha}{2}} \log(X) + X^{1-\kappa} \log^2(X)$$
$$+ X^{\frac{11}{12} + \kappa + \alpha + \frac{11\epsilon}{6}}),$$

where the implied constant above depends only on k, N, κ, α, and ϵ. Choosing $\kappa = \frac{1}{32} - \frac{\epsilon}{2}, \alpha = \frac{1}{48} - \frac{\epsilon}{3}$, and $N = \frac{2}{\kappa}$ then gives,

$$\sum_{n<X} (i_n) = \frac{X}{k-1} + O(X^{\frac{31}{32} + \epsilon}). \tag{8}$$

The theorem follows from (6), (7), and (8). □

Proof of Corollary 1.2. The proof is a straightforward application of partial summation. For any $N_1 > N_0$ we get

$$\sum_{n=N_0}^{N_1} \frac{\tilde{\tau}(n)}{n^s} = \frac{B(N_1)}{N_1^s} - \frac{B(N_0-1)}{N_0^s} + \sum_{n=N_0}^{N_1-1} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) B(n), \tag{9}$$

where $B(n) := \sum_{m=1}^{n} \tilde{\tau}(m)$. It is well-known that $\Delta(z)$ is a cuspidal Hecke eigenform of weight 12 (cf. [15, p. 84]), and that the space of cusp forms of weight 12 for the full modular group has dimension 1 (cf. p. 96 of *loc. cit.*). Keeping in mind the normalization we have in Theorem 1.1 we therefore have $\tilde{\tau}(n) = \text{tr}(T_{12}(n))$ for every $n \geqslant 1$. Hence, by Theorem 1.1 for every $\epsilon > 0$ we have

$$B(n) = O(n^{\frac{31}{32} + \epsilon}).$$

Substituting these bounds in (9), and keeping in mind that $\Re(s) > \frac{31}{32}$, then gives

$$\sum_{n=N_0}^{N_1} \frac{\tilde{\tau}(n)}{n^s} \ll \frac{1}{N_0^{s - \frac{31}{32} - \epsilon}}.$$

Therefore we conclude that the partial sums converge uniformly to $L(s, \Delta)$ and the corollary follows. □

3. The trace formula

Let $k > 2$ be an *even* integer. Recall (cf. [14] or [10] p. 370) that the Eichler–Selberg trace formula expresses the trace of the nth Hecke operator acting on cusp forms of weight k and full level as the sum

$$\operatorname{tr}(T_k(n)) = -\frac{1}{2n^{\frac{k-1}{2}}} \sum_{m < \sqrt{n}} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \sum_{\substack{f^2 \mid (4n - m^2) \\ \frac{m^2 - 4n}{f^2} \equiv 0,1 \bmod 4}} h_w\left(\frac{m^2 - 4n}{f^2}\right) \qquad (i_n)$$

$$-\frac{1}{2}\sum_{d \mid n} \min\left(\frac{d}{\sqrt{n}}, \frac{\sqrt{n}}{d}\right)^{k-1} \qquad\qquad (ii_n)$$

$$+\frac{k-1}{12\sqrt{n}}\delta_\square(n) \qquad\qquad\qquad\qquad\qquad (iii_n)$$

where

$$\rho = \frac{m + \sqrt{m^2 - 4n}}{2}, \quad \bar{\rho} = \frac{m - \sqrt{m^2 - 4n}}{2}, \qquad (10)$$

and

$$\delta_\square(n) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

The function $h_w(\alpha)$ is defined as the class number of the order of discriminant α weighted by $1/2$ or $1/3$ if $\alpha = -4$ or $\alpha = -3$ respectively.

Note that the classical Eichler–Selberg trace formula, as stated in [14], is $n^{\frac{k-1}{2}}$ times the one given above. This is due to the normalization of the eigenvalues of the Hecke operators. We normalize them so that the Ramanujan conjecture reads as $|a_\pi(p)| \leqslant 2$.

We also remark that in the above expression, (iii_n) corresponds to the contribution of the identity element, (i_n) corresponds to the contribution of conjugacy classes of elliptic elements, and (ii_n) combines the contribution of the hyperbolic conjugacy classes (terms for which $n \neq d^2$) and unipotent conjugacy classes (appears only if $\delta_\square(n) = 1$, and is the term corresponding to $d^2 = n$).

3.1. Identity, hyperbolic, and unipotent contributions

The contribution of the identity element is easily seen to be

$$\frac{k-1}{12}\sum_{n < X}\frac{\delta_\square(n)}{\sqrt{n}} = O_k\left(\log(X)\right). \qquad (11)$$

As explained in §1.1, quite interestingly, the hyperbolic conjugacy classes give a non-zero contribution to the limit (we remark that this contribution will be cancelled by the contribution of the elliptic term to the limit, cf. Corollary 4.7).

Proposition 3.1.

$$-\frac{1}{2}\sum_{n < X}\sum_{d \mid n}\min\left(\frac{d}{\sqrt{n}}, \frac{\sqrt{n}}{d}\right)^{k-1} = \frac{X}{1-k} + O_k(\sqrt{X}).$$

Proof. First note that for any d dividing n such that $d \neq \sqrt{n}$, d and $\frac{n}{d}$ give the same contribution to the inner sum. Therefore,

$$\sum_{d|n} \min\left(\frac{d}{\sqrt{n}}, \frac{\sqrt{n}}{d}\right)^{k-1} = 2 \sum_{\substack{d|n \\ d<\sqrt{n}}} \left(\frac{d}{\sqrt{n}}\right)^{k-1} + \delta_\square(n).$$

Next, we trivially have

$$\sum_{n<X} \delta_\square(n) = O(\sqrt{X}). \tag{$*$}$$

For the rest of the sum, first note that

$$\sum_{n<X} \sum_{\substack{d|n \\ d<\sqrt{n}}} \left(\frac{d}{\sqrt{n}}\right)^{k-1} = \sum_{\substack{ab<X \\ a<b}} \left(\frac{a}{b}\right)^{\frac{k-1}{2}} = \sum_{\substack{b\leqslant\sqrt{X} \\ a<b}} \left(\frac{a}{b}\right)^{\frac{k-1}{2}} + \sum_{\substack{\sqrt{X}<b<X \\ a<\frac{X}{b}}} \left(\frac{a}{b}\right)^{\frac{k-1}{2}}. \tag{$**$}$$

Then, by Euler–Maclaurin formula

$$\sum_{a<b} a^{\frac{k-1}{2}} = \frac{2}{k+1} b^{\frac{k+1}{2}} + O_k(b^{\frac{k-1}{2}}) \quad \text{and} \quad \sum_{a<\frac{X}{b}} a^{\frac{k-1}{2}} = \frac{2}{k+1}\left(\frac{X}{b}\right)^{\frac{k+1}{2}} + O_k\left(\left(\frac{X}{b}\right)^{\frac{k-1}{2}}\right).$$

Therefore,

$$\sum_{\substack{b\leqslant\sqrt{X} \\ a<b}} \left(\frac{a}{b}\right)^{\frac{k-1}{2}} = \frac{X}{k+1} + O_k(\sqrt{X}) \quad \text{and} \quad \sum_{\substack{\sqrt{X}<b<X \\ a<\frac{X}{b}}} \left(\frac{a}{b}\right)^{\frac{k-1}{2}} = \frac{2X}{(k+1)(k-1)} + O_k(\sqrt{X}).$$

Substituting the above estimates in $(**)$ and combining the result with $(*)$ finishes the proof. $\qquad\square$

4. Contribution of the elliptic conjugacy classes

Following [1] we begin the analysis by rewriting the elliptic part so that it takes a more suitable form. Let,[6]

$$\theta_\infty(x) := \begin{cases} \dfrac{i}{2\pi}\left\{\left(x+\sqrt{x^2-1}\right)^{k-1} - \left(x-\sqrt{x^2-1}\right)^{k-1}\right\} & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{Z}$, $n \in \mathbb{N}$ let $\rho, \bar{\rho}$ be the two roots of $X^2 - mX + n = 0$ as defined in (10). Then,

$$-\frac{1}{2n^{\frac{k-1}{2}}} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} = \frac{\pi i}{\sqrt{m^2-4n}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right). \tag{12}$$

---

[6]Let us remark that the function $\theta_\infty(x)$ is, up to a constant, the orbital integral of a matrix coefficient of a weight k discrete series representation of $GL_2(\mathbb{R})$, which explains the notation, cf. [10, Proposition 14.1].

Lemma 4.1. *Let $m, n \in \mathbb{N}$ be such that $m^2 - 4n < 0$ and h_w be defined as in §2. Then,*

$$\sum_{\substack{f^2 | (4n - m^2) \\ \frac{m^2 - 4n}{f^2} \equiv 0, 1 \bmod 4}} h_w\left(\frac{m^2 - 4n}{f^2}\right) = \frac{\sqrt{4n - m^2}}{\pi} L(1, m^2 - 4n),$$

where $L(s, m^2 - 4n)$ is the following weighted sum of quadratic Dirichlet L-functions

$$L(s, m^2 - 4n) := \sum_{\substack{f^2 | m^2 - 4n \\ \frac{m^2 - 4n}{f^2} \equiv 0, 1 \bmod 4}} \frac{1}{f^{2s-1}} L\left(s, \left(\frac{(m^2 - 4n)/f^2}{\cdot}\right)\right).$$

Proof. Recall that for a fundamental discriminant $D < 0$ the class number formula states

$$L\left(1, \left(\frac{D}{\cdot}\right)\right) = \frac{2\pi h(D)}{w_D \sqrt{|D|}},$$

where w_D is the number of roots of unity in $\mathbb{Q}(\sqrt{D})$ and $h(D)$ is the class number of the same field.

Let $m^2 - 4n = D(m, n)s(m, n)^2$, where $D(m, n)$ is a fundamental discriminant (i.e., the discriminant of the field $\mathbb{Q}(\sqrt{m^2 - 4n})$). Then for any $f \mid s(m, n)$, by [5, Theorem 7.24] we have

$$h_w\left(\frac{m^2 - 4n}{f^2}\right) = \frac{s(m, n)}{f} \frac{2h(D(m, n))}{w_{D(m, n)}} \prod_{q | \frac{s(m, n)}{f}} \left(1 - \left(\frac{D(m, n)}{q}\right)\frac{1}{q}\right).$$

Moreover the condition that $f^2 \mid m^2 - 4n$ such that $\frac{m^2 - 4n}{f^2} \equiv 0, 1 \bmod 4$ is equivalent to $f \mid s(m, n)$. Therefore,

$$\sum_{\substack{f^2 | m^2 - 4n \\ \frac{m^2 - 4n}{f^2} \equiv 0, 1 \bmod 4}} h_w\left(\frac{m^2 - 4n}{f^2}\right) = \frac{2s(m, n)h(D(m, n))}{w_{D(m, n)}} \sum_{f | s(m, n)} \frac{1}{f} \prod_{q | \frac{s(m, n)}{f}} \left(1 - \left(\frac{D(m, n)}{q}\right)\frac{1}{q}\right)$$

$$= \frac{\sqrt{4n - m^2}}{\pi} L(1, m^2 - 4n). \qquad \square$$

Corollary 4.2. *The elliptic part of the trace formula can be expressed as*

$$\sum_{m \in \mathbb{Z}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right) L(1, m^2 - 4n),$$

where $\theta_\infty(x)$ is defined in (12) and $L(1, m^2 - 4n)$ is as in Lemma 4.1.

Proof. Follows from Lemma 4.1. $\qquad \square$

4.1. Approximate functional equation, Poisson summation, and the contribution of the dominant term

Next, let us recall the approximate functional equation introduced in [1]. Let[7] $F(x)$ be

$$F(x) = \frac{1}{2K_0(2)} \int_x^\infty e^{-y-\frac{1}{y}} \frac{dy}{y}, \tag{13}$$

where $K_0(2)$ just normalizes the integral to be 1 and is given by $\sum_{k=1}^\infty \Gamma'(k)\Gamma^{-3}(k)$. The only property of the above function that we use (cf. [9, pp. 257–258]) is the following:

The Mellin transform, $\tilde{F}(u)$, is holomorphic except for a simple pole at $u = 0$ with residue 1.

Let $m < 2\sqrt{n}$. Then, by [1, Corollary 3.5], where we have taken $A = 4n - m^2$ and $\iota_\delta = 1$, we have

$$L(1, m^2 - 4n) = \sum_{\substack{f^2|m^2-4n \\ \frac{m^2-4n}{f^2} \equiv 0,1 \bmod 4}} \frac{1}{f} \sum_{l=1}^\infty \frac{1}{l}\left(\frac{(m^2-4n)/f^2}{l}\right)$$

$$\times \left[F\left(\frac{lf^2}{4n-m^2}\right) + \frac{lf^2}{\sqrt{4n-m^2}} H\left(\frac{lf^2}{4n-m^2}\right) \right],$$

where

$$H(y) := \frac{\sqrt{\pi}}{2\pi i} \int_{(1)} \frac{\Gamma\left(\frac{1+u}{2}\right)}{\Gamma\left(\frac{2-u}{2}\right)} (\pi y)^{-u} \tilde{F}(u)\, du. \tag{14}$$

4.1.1. Poisson summation.

We begin with introducing a shorthand notation for the Fourier transform that will frequently appear throughout the text.

$$I_{l,f}(\xi, n) := \int_{-1}^1 \theta_\infty(x) \left\{ F\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) + \frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}} H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) \right\}$$

$$\times e\left(\frac{-x\xi\sqrt{n}}{2lf^2}\right) dx.$$

Lemma 4.3.

$$\sum_{m\in\mathbb{Z}} \theta_\infty\left(\frac{m}{2\sqrt{n}}\right) L(1, m^2 - 4n) = \frac{\sqrt{n}}{2} \sum_{f,l=1}^\infty \frac{1}{(lf^2)^{\frac{3}{2}}} \sum_{\xi\in\mathbb{Z}} \frac{Kl_{l,f}(\xi, n)}{\sqrt{l}} I_{l,f}(\xi, n), \tag{15}$$

where

$$Kl_{l,f}(\xi, \mp p^k) := \sum_{\substack{a \bmod 4lf^2 \\ a^2 \pm 4p^k \equiv 0 \bmod f^2 \\ \frac{a^2 \pm 4p^k}{f^2} \equiv 0,1 \bmod 4}} \left(\frac{(a^2 \pm 4p^k)/f^2}{l}\right) e\left(\frac{a\xi}{4lf^2}\right).$$

---

[7]We remark that the specific choice of this function is irrelevant to the rest of the argument. Only the pole of its Mellin transform is important.

We also remark that $Kl_{l,f}(\xi, n)$ is periodic *in ξ and n mod $4lf^2$, which will be used throughout the paper without reference.*

Proof. This is just a restatement of [1, Theorem 4.2], where we take $\alpha = \frac{1}{2}$. □

4.1.2. Analysis of the term $\xi = 0$.

Lemma 4.4. *Let $\beta, z \in \mathbb{C}$ be such that $z > 1$ and $\beta - \frac{z}{2} > 1$. Then,*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\beta - \frac{z}{2}}} \sum_{f,l=1}^{\infty} \frac{1}{(lf^2)^{z+\frac{1}{2}}} \frac{Kl_{l,f}(0,n)}{\sqrt{l}} = 4 \frac{\zeta(2z)\zeta\left(\beta + \frac{z}{2}\right)\zeta\left(\beta - \frac{z}{2}\right)}{\zeta(z+1)}.$$

Proof. By [2, Corollary B.8] $Kl_{l,f}(0,n) = O(\log(lf^2)l \gcd(n, f^2))$, hence the left hand side is absolutely convergent. By [1, Corollary 5.4] we have

$$\sum_{f,l=1}^{\infty} \frac{1}{(lf^2)^{z+\frac{1}{2}}} \frac{Kl_{l,f}(0,n)}{\sqrt{l}} = 4 \frac{\zeta(2z)}{\zeta(z+1)} \prod_{p|n} \frac{1 - p^{-z(v_p(n)+1)}}{1 - p^{-z}}.$$

Summing this over n gives

$$4 \frac{\zeta(2z)}{\zeta(z+1)} \sum_{n=1}^{\infty} \frac{1}{n^{\beta - \frac{z}{2}}} \prod_{p|n} \frac{1 - p^{-z(v_p(n)+1)}}{1 - p^{-z}} = 4 \frac{\zeta(2z)\zeta(z)}{\zeta(z+1)} \prod_{p} \left(\sum_{m=0}^{\infty} \frac{1 - p^{-z(m+1)}}{p^{m(\beta - \frac{z}{2})}} \right)$$

$$= 4 \frac{\zeta(2z)\zeta\left(\beta + \frac{z}{2}\right)\zeta\left(\beta - \frac{z}{2}\right)}{\zeta(z+1)}. \qquad \square$$

Proposition 4.5. *For $k \geqslant 3$ we have,*

$$\frac{1}{2} \sum_{n<X} \sqrt{n} \sum_{f,l=1}^{\infty} \frac{1}{(lf^2)^{\frac{3}{2}}} \frac{Kl_{l,f}(0,n)}{\sqrt{l}} \int_{-1}^{1} \theta_{\infty}(x) F\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) dx$$

$$= 2X^{\frac{3}{2}} \zeta(2) \int \theta_{\infty}(x) dx + 2^{\frac{-1}{2}} X^{\frac{5}{4}} \tilde{F}\left(\frac{-1}{2}\right) \int \frac{\theta_{\infty}(x)}{(1-x^2)^{\frac{1}{4}}} dx + O_k(1).$$

The first term on the right hand side is the contribution of the trivial representation, cf. [1, Lemma 6.2] (which, in fact, is 0 in our case. cf. Corollary 4.7).

Proof. The proof is a simple application of Mellin inversion and contour shifting. By Perron's formula the left hand side is equal to

$$\frac{1}{4\pi i} \int_{(4)} \frac{X^u}{u} \left[\sum_{n=1}^{\infty} \frac{1}{n^{u-\frac{1}{2}}} \sum_{f,l=1}^{\infty} \frac{1}{(lf^2)^{\frac{3}{2}}} \frac{Kl_{l,f}(0,n)}{\sqrt{l}} \int_{-1}^{1} \theta_{\infty}(x) F\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) dx \right] du. \quad (*)$$

We note that the absolute convergence of the triple sum for $\Re(u) > 2$ is guaranteed by the estimate $\sum_{l,f} \cdots = O_{\theta_{\infty}, F}(\sqrt{n})$, which follows from [1, Theorem 6.1]. Next, we use

Mellin inversion on F which gives

$$(*) = \frac{1}{2(2\pi i)^2} \int_{(4)} \int_{(2)} \frac{X^u \tilde{F}(w) 2^w}{u}$$

$$\times \left[\sum_{n=1}^{\infty} \frac{1}{n^{u-\frac{1+w}{2}}} \sum_{f,l=1}^{\infty} \frac{1}{(lf^2)^{w+\frac{3}{2}}} \frac{Kl_{l,f}(0,n)}{\sqrt{l}} \int_{-1}^{1} (1-x^2)^{\frac{w}{2}} \theta_{\infty}(x)\, dx \right] dw\, du.$$

We note that the same bound as above guarantees the absolute convergence of the triple sum since $\Re(u - \frac{1+w}{2}) = \frac{5}{2} > \frac{3}{2}$. Using Lemma 4.4, with $\beta = u$ and $z = w+1$, in the inner sums gives

$$(*) = \frac{2}{(2\pi i)^2} \int_{(4)} \int_{(2)} \frac{X^u \tilde{F}(w) 2^w}{u}$$

$$\times \left[\frac{\zeta(2(w+1))\zeta\left(u+\frac{w+1}{2}\right)\zeta\left(u-\frac{w+1}{2}\right)}{\zeta(w+2)} \int_{-1}^{1} (1-x^2)^{\frac{w}{2}} \theta_{\infty}(x)\, dx \right] dw\, du.$$

The rest of the proof is contour shifting. We shift the w-contour to right and the u-contour to left. To shift the contours first note that the only pole of

$$\frac{\zeta(2(w+1))\zeta\left(u+\frac{w+1}{2}\right)\zeta\left(u-\frac{w+1}{2}\right)}{\zeta(w+2)}$$

in the region $\Re(w) \geqslant 2$ is simple and is at $w = 2u - 3$ with residue $-2\zeta(4u - 4)$ and all the other terms depending on w are holomorphic in $\Re(w) \geqslant 2$. Therefore, moving the w-contour to $\Re(w) = 6$ gives[8]

$$(*) = \frac{4}{2\pi i} \int_{(4)} \frac{X^u \tilde{F}(2u-3) 2^{2u-3}}{u} \zeta(4u-4) \left[\int_{-1}^{1} (1-x^2)^{u-\frac{3}{2}} \theta_{\infty}(x)\, dx \right] du$$

$$+ \frac{2}{(2\pi i)^2} \int_{(4)} \int_{(6)} \frac{X^u \tilde{F}(w) 2^w}{u}$$

$$\times \left[\frac{\zeta(2(w+1))\zeta\left(u+\frac{w+1}{2}\right)\zeta\left(u-\frac{w+1}{2}\right)}{\zeta(w+2)} \int_{-1}^{1} (1-x^2)^{\frac{w}{2}} \theta_{\infty}(x)\, dx \right] dw\, du.$$

We first handle the second integral above. Since $F \in C_c^{\infty}(\mathbb{R}_+)$, $\tilde{F}(w)$ is rapidly decreasing on the line $\Re(w) = 6$. Moreover the ratio of ζ-functions is rapidly decreasing in vertical strips and hence we can interchange the u and w integrals. Moreover the u-integrand is holomorphic in $4 \geqslant \Re(u) \geqslant -1$, except with a simple pole at $u = 0$. Hence by interchanging the u and w integrals and moving the u-contour to $\Re(u) = -\frac{1}{2}$ (which picks up the residue at $u = 0$) we get that the second integral is $O_k(1)$.

---

[8]We remark that since we are moving the contour from left to right the residue formula has an extra '−' sign.

For the first integral, we again move the u-contour to $\Re(u) = -\frac{1}{2}$ (Note that the x-integral still converges since we are assuming that the weight, k, of the forms we are working with satisfies $k \geqslant 3$ which implies $\theta_\infty(x)$ vanishes to order at least 2 at $x = \pm 1$, hence $(1-x^2)^{-3/2}\theta_\infty(x)$ is integrable around $x = \pm 1$.) This picks up the pole of $\zeta(4u-4)$ at $u = \frac{5}{4}$ with residue $\frac{1}{4}$, the pole of $\tilde{F}(2u-3)$ at $u = \frac{2}{3}$ with residue $\frac{1}{2}$, and the pole of $\frac{1}{u}$ at $u = 0$ with residue 1. Therefore,

$$(*) = 2X^{\frac{3}{2}}\zeta(2)\int \theta_\infty(x)\,dx + 2^{\frac{-1}{2}}X^{\frac{5}{4}}\tilde{F}\left(\frac{-1}{2}\right)\int \frac{\theta_\infty(x)}{(1-x^2)^{\frac{1}{4}}}\,dx + O_k(1).$$

Combining the estimates above for the two integrals finishes the proof. $\qquad\square$

Proposition 4.6. *For $k \geqslant 3$ we have,*

$$\frac{1}{4}\sum_{n<X}\sum_{f,l=1}^{\infty}\frac{1}{\sqrt{lf^2}}\frac{Kl_{l,f}(0,n)}{\sqrt{l}}\int_{-1}^{1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right)dx$$

$$= 2^{\frac{-1}{2}}X^{\frac{5}{4}}\tilde{F}\left(\frac{1}{2}\right)\int\frac{\theta_\infty(x)}{(1-x^2)^{\frac{1}{4}}}\,dx + \pi X\zeta(0)\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\,dx + O_k(1).$$

Proof. The proof follows exactly the same argument as in the proof of Proposition 4.5. We give the details for completeness. By Perron's formula the left hand side is equal to

$$\frac{1}{8\pi i}\int_{(4)}\frac{X^u}{u}\left[\sum_{n=1}^{\infty}\frac{1}{n^u}\sum_{f,l=1}^{\infty}\frac{1}{\sqrt{lf^2}}\frac{Kl_{l,f}(0,n)}{\sqrt{l}}\int_{-1}^{1}\frac{\theta_\infty(x)}{\sqrt{1-x^2}}H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right)dx\right]du. \quad (*)$$

Once again we note that the absolute convergence of the triple sum and the interchange of the integrals is guaranteed by the estimate $\sum_{l,f}\cdots = O_{\theta_\infty,F}(\sqrt{n})$, which follows from [1, Theorem 6.1]. Next, substituting (14) for the function H we get that $(*)$ is

$$\frac{\sqrt{\pi}}{4(2\pi i)^2}\int_{(4)}\int_{(1)}\frac{X^u\tilde{F}(w)}{u}\frac{2^w\Gamma\left(\frac{1+w}{2}\right)}{\pi^w\Gamma\left(\frac{2-w}{2}\right)}$$

$$\times\left[\sum_{n=1}^{\infty}\frac{1}{n^{u-\frac{w}{2}}}\sum_{f,l=1}^{\infty}\frac{1}{(lf^2)^{w+\frac{1}{2}}}\frac{Kl_{l,f}(0,n)}{\sqrt{l}}\int_{-1}^{1}(1-x^2)^{\frac{w-1}{2}}\theta_\infty(x)\,dx\right]dw\,du,$$

where the interchange of the integrals with the triple sum is justified by the same bound above using $\Re(u-\frac{w}{2}) = 3 > \frac{3}{2}$. By Lemma 4.4 with $\beta = u$ and $z = w$ we have

$$(*) = \frac{\sqrt{\pi}}{(2\pi i)^2}\int_{(4)}\int_{(1)}\frac{X^u\tilde{F}(w)}{u}\frac{2^w\Gamma\left(\frac{1+w}{2}\right)}{\pi^w\Gamma\left(\frac{2-w}{2}\right)}$$

$$\times\left[\frac{\zeta(2w)\zeta\left(u+\frac{w}{2}\right)\zeta\left(u-\frac{w}{2}\right)}{\zeta(w+1)}\int_{-1}^{1}(1-x^2)^{\frac{w-1}{2}}\theta_\infty(x)\,dx\right]dw\,du.$$

We now finish the proof by shifting contours. We shift the w-contour to right and the u-contour to left. To shift the contours first note that the only pole of

$$\frac{\Gamma\left(\frac{1+w}{2}\right)}{\Gamma\left(\frac{2-w}{2}\right)}\frac{\zeta(2w)\zeta\left(u+\frac{w}{2}\right)\zeta\left(u-\frac{w}{2}\right)}{\zeta(w+1)}$$

in the region $\Re(w) \geqslant 1$ is at $w = 2u - 2$ with residue $\frac{-2\Gamma\left(u-\frac{1}{2}\right)\zeta(4u-4)}{\Gamma(2-u)}$ and all the other terms depending on w are holomorphic in $\Re(w) \geqslant 1$. Therefore, moving the w-contour to $\Re(w) = 6$ gives[9]

$$(*) = \frac{2\sqrt{\pi}}{2\pi i}\int_{(4)}\frac{X^u\tilde{F}(2u-2)}{u}\frac{2^{2u-2}\Gamma\left(u-\frac{1}{2}\right)}{\pi^{2u-2}\Gamma(2-u)}\zeta(4u-4)\left[\int_{-1}^{1}(1-x^2)^{u-\frac{3}{2}}\theta_\infty(x)\,dx\right]du$$

$$+ \frac{2}{(2\pi i)^2}\int_{(4)}\int_{(6)}\frac{X^u\tilde{F}(w)}{u}\frac{2^w\Gamma\left(\frac{1+w}{2}\right)}{\pi^w\Gamma\left(\frac{2-w}{2}\right)}$$

$$\times\left[\frac{\zeta(2(w+1))\zeta\left(u+\frac{w+1}{2}\right)\zeta\left(u-\frac{w+1}{2}\right)}{\zeta(w+2)}\int_{-1}^{1}(1-x^2)^{\frac{w}{2}}\theta_\infty(x)\,dx\right]dw\,du.$$

As in Proposition 4.5 we can move the u-contour in the second integral to $\Re(u) = -\frac{1}{2}$ picking only the residue of $\frac{1}{u}$, which implies that the second integral is $O_k(1)$.

For the first line, we again move the u-contour to $\Re(u) = -\frac{1}{2}$ (Note, once again, that the x-integral still converges since we are assuming $k \geqslant 3$ which implies $\theta_\infty(x)$ vanishes to order at least 2 at $x = \pm 1$, hence $(1-x^2)^{-3/2}\theta_\infty(x)$ is integrable around $x = \pm 1$.). This picks up the pole of $\zeta(4u-4)$ at $u = \frac{5}{4}$ with residue $\frac{1}{4}$, the pole of $\tilde{F}(2u-2)$ at $u = 1$ with residue $\frac{1}{2}$, and the pole of $\frac{1}{u}$ at $s = 0$ with residue 1. Therefore,

$$(*) = 2^{\frac{-1}{2}}X^{\frac{5}{4}}\tilde{F}\left(\frac{1}{2}\right)\int\frac{\theta_\infty(x)}{(1-x^2)^{\frac{1}{4}}}\,dx + \pi X\zeta(0)\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\,dx + O_k(1).$$

The lemma follows from substituting $\zeta(0) = -1/2$. $\qquad\square$

Corollary 4.7. *For $k \geqslant 3$ we have*

$$\sum_{n<X}\sqrt{n}\sum_{f,l=1}^{\infty}\frac{1}{(lf^2)^{\frac{3}{2}}}\frac{Kl_{l,f}(0,n)}{\sqrt{l}}I_{l,f}(\xi,n) = \frac{X}{k-1} + O_k(1).$$

Proof. By Propositions 4.5 and 4.6 the LHS is

$$2X^{\frac{3}{2}}\zeta(2)\int\theta_\infty(x)\,dx + 2^{\frac{-1}{2}}X^{\frac{5}{4}}\tilde{F}\left(\frac{-1}{2}\right)\int\frac{\theta_\infty(x)}{(1-x^2)^{\frac{1}{4}}}\,dx$$

$$+ 2^{\frac{-1}{2}}X^{\frac{5}{4}}\tilde{F}\left(\frac{1}{2}\right)\int\frac{\theta_\infty(x)}{(1-x^2)^{\frac{1}{4}}}\,dx + \pi\zeta(0)X\int\frac{\theta_\infty(x)}{\sqrt{1-x^2}}\,dx + O_k(1).$$

---

[9] As in Proposition 4.5 since we are moving the contour from left to right the residue formula has an extra '−' sign.

By [1, Lemma 3.3] $\tilde{F}$ is an odd function hence the second and third terms cancel each other (We remark that this is not a consequence of our particular choice of the function F. If, instead of the current choice, we had chosen an F so that $\tilde{F}$ is not odd, the function H that appear in the approximate functional equation would change and the third term above would have $-\tilde{F}\left(\frac{-1}{2}\right)$ instead of $\tilde{F}\left(\frac{1}{2}\right)$.).

As we already remarked in Proposition 4.5 the first term is the contribution of the trivial representation. In general we would have to remove this term since we are only interested in the cuspidal part of the spectrum. In the current case, the operator used in the Eichler–Selberg trace formula is a projection on a subset of the cuspidal spectrum and since the trivial representation is orthogonal to the cuspidal spectrum this contribution is 0. We show this fact directly by showing that the integral vanishes.

By the definition of the function $\theta_\infty(x)$ given in (12)

$$\int \theta_\infty(x)\,dx = \frac{i}{2\pi} \int_{-1}^{1} \left\{ (x + \sqrt{x^2-1})^{k-1} - (x - \sqrt{x^2-1})^{k-1} \right\} dx$$
$$= \frac{i^k}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{-i(k-1)\alpha} - (-1)^{k-1}e^{i(k-1)\alpha}) \cos(\alpha)\,d\alpha.$$

Since $k \equiv 0 \bmod 2$, $(-1)^{k-1} = -1$ and we have

$$\frac{i^k}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{-i(k-1)\alpha} - (-1)^{k-1}e^{i(k-1)\alpha}) \cos(\alpha)\,d\alpha = \frac{i^k}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos((k-1)\alpha)\cos(\alpha)\,d\alpha$$
$$= \frac{i^k}{4\pi} \left(\frac{\sin(k\alpha)}{k} + \frac{\sin((k-2)\alpha)}{k-2} \right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= 0,$$

where in the last line we used $k \equiv 0 \bmod 2$ again. Finally we calculate the integral that appears in the fourth term. Proceeding as above,

$$\int \frac{\theta_\infty(x)}{\sqrt{1-x^2}}\,dx = \frac{i}{2\pi} \int_{-1}^{1} \frac{(x+\sqrt{x^2-1})^{k-1} - (x - \sqrt{x^2-1})^{k-1}}{\sqrt{1-x^2}}\,dx$$
$$= \frac{i^k}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{-i(k-1)\alpha} - (-1)^{k-1}e^{i(k-1)\alpha})\,d\alpha$$
$$= \frac{i^k}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos((k-1)\alpha)\,d\alpha$$
$$= \frac{2}{\pi(1-k)},$$

where in the last line we used $k \equiv 0 \bmod 2$. The corollary now follows from the fact that $\zeta(0) = -1/2$. $\qquad\square$

4.2. Analysis of the terms $\xi \neq 0$

The analysis for the terms $\xi \neq 0$ is more delicate than the previous calculations. To direct the reader in what is to come, we remark that we will be dealing with sums over the

variables l, f, ξ, and n. The analysis will be carried over in two steps: we first break the region of summation into two sets according to (disregarding epsilons etc.)

$$n^{1/4} \ll lf^2 \ll n^{1/2} \tag{16}$$

and its complement. The complement will be treated in §4.2.1, where the summation range is further divided into sub-ranges and each range is bounded by 'pointwise' estimates for each n. The critical range (16) is taken in §4.2.2. We remark that the estimates therein make use of the cancellation among the n-sum.

To be a little bit more precise, in the complement of the range (16) we bound the 'Kloosterman-like' sums $Kl_{l,f}(\xi, n)$ pointwise, ignoring the oscillation in the frequencies; whereas in the range (16) we keep track of the oscillation and make use of the further cancellation caused by this oscillation.

4.2.1. Preliminary estimates. In this section we give estimates on (15) for $\frac{lf^2\xi}{\sqrt{n}}$ running on certain ranges. We emphasize, once again, that these estimates are pointwise valid for every n. In other words, we are not yet bringing the n-sum of (1) in.

For any integer $n \geqslant 1$ and $\kappa, \alpha > 0$ let,

$$S_0(n, \kappa) := \sum_{lf^2\xi \gg n^{\frac{1}{2}+\kappa}} \frac{Kl_{l,f}(\xi, n)}{\sqrt{l}} \frac{I_{l,f}(\xi, n)}{(lf^2)^{\frac{3}{2}}}, \tag{17}$$

$$S_1(n, \kappa, \alpha) := \sum_{\substack{lf^2\xi \ll n^{\frac{1}{2}+\kappa} \\ \xi \gg n^{\frac{1}{6}+\kappa+\alpha}}} \frac{Kl_{l,f}(\xi, n)}{\sqrt{l}} \frac{I_{l,f}(\xi, n)}{(lf^2)^{\frac{3}{2}}}, \tag{18}$$

$$S_2(n, \kappa, \alpha) := \sum_{\substack{lf^2 \ll n^{\frac{1}{4}-\kappa} \\ 0 < \xi \ll n^{\frac{1}{6}+\kappa+\alpha}}} \frac{Kl_{l,f}(\xi, n)}{\sqrt{l}} \frac{I_{l,f}(\xi, n)}{(lf^2)^{\frac{3}{2}}}. \tag{19}$$

A heuristic discussion. To make the proofs easier to follow let us begin by giving a heuristic discussion of Theorems 4.8, 4.10, and 4.12. There are certain structural points about these sums that we first would like to discuss, with the hope that these clarify the estimates a bit.

(1) First of all, note that in all of the estimates below the parameters l and f appear as the product lf^2 rather than individually. So it is the range over which lf^2 runs over that matters.

(2) Next, by[10] [2, Corollary B.8], for a square-free l, and ξ and n such that $\gcd(f, n) = 1$ and $\gcd(l, \xi) = 1$, we have $Kl_{l,f}(\xi, n) \ll \sqrt{l}$. Therefore the quotient $\frac{Kl_{l,f}(\xi,n)}{\sqrt{l}}$, at least when the above conditions are satisfied, is a complex number of absolute value at most 1, and for the heuristics we may as well assume that this is the case disregarding the conditions.

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[10] There is an extra $\log(lf^2)$ factor in [2, Corollary B.8]; however, for square-free l one can remove that factor. This, anyways, is not the main point of the heuristic discussion.

(3) Finally, by [2, Proposition 4.7] (where their $\theta^{pos}_{\infty,1}$ is our θ_∞) we have $I_{l,f}(\xi, n) \ll \left(\frac{\sqrt{n}}{lf^2\xi}\right)^N \frac{1}{\xi^2}$ (cf. (21) below) for every $N \geqslant 0$ and every $l, f, n, \xi \neq 0$, and by Proposition 4.9 of *loc. cit.* we have $I_{l,f}(\xi, n) \ll \left(\frac{\xi\sqrt{n}}{lf^2}\right)^{-\frac{3}{2}}$ for $\frac{lf^2\xi}{\sqrt{n}} \ll 1$ (cf. (23) below).

We can now discuss the contents of the theorems below. By the third remark above, $I_{l,f}(\xi, n)$ decays very rapidly when $lf^2\xi \gg \sqrt{n}$. Therefore we expect S_0 to be very small and this is the content of Theorem 4.8.

By the same remark, when $lf^2\xi \ll \sqrt{n}$ we have $\frac{I_{l,f}(\xi,n)}{(lf^2)^{\frac{3}{2}}} \ll (\sqrt{n}\xi)^{-\frac{3}{2}}$. Therefore, S_1 is going to be comparable to the double integral

$$n^{-\frac{3}{4}} \int_{n^{\frac{1}{6}}}^{n^{\frac{1}{2}}} \xi^{-\frac{3}{2}} \int_1^{\frac{n^{\frac{1}{2}}}{\xi}} 1 = O(n^{-\frac{1}{2}}).$$

This is the content of Theorem 4.10.

When $lf^2 \ll n^{\frac{1}{4}}$ and $\xi \ll n^{\frac{1}{6}}$ we necessarily have $lf^2\xi \ll n^{\frac{5}{12}} < \sqrt{n}$. Hence, again by the same remark, in this region $\frac{I_{l,f}(\xi,n)}{(lf^2)^{\frac{3}{2}}} \ll (\sqrt{n}\xi)^{-\frac{3}{2}}$. Therefore, we expect S_2 to be $O(n^{-\frac{1}{2}})$ and this is the content of Theorem 4.12.

Theorem 4.8. *Let $n \in \mathbb{Z}_{>0}$. Then for every $N > 0$ and $\kappa > 0$,*

$$S_0(n, \kappa) \ll n^{\frac{1}{2} - N\kappa},$$

where the implied constant depends only on k, N, and κ.

Proof. We recall two estimates from [2] that will be used throughout the proof. First, [2, Corollary B.8] states,

$$Kl_{l,f}(\xi, n) \ll \delta(n; f^2) \log(lf^2) \sqrt{l \gcd(n, f^2)} \sqrt{\gcd\left(\frac{\xi}{\sqrt{\gcd(n, f^2)}}, l\right)}, \qquad (20)$$

where $\delta(n; f^2)$ is 1 if n is a square modulo f^2 and 0 otherwise. The second estimate, [2, Corollary 4.8], is on the Fourier transform that appear in the sum. It states that for every $N > 0$

$$I_{l,f}(\xi, n) \ll \left(\frac{\sqrt{n}}{lf^2\xi}\right)^N \frac{1}{\xi^2}, \qquad (21)$$

where the implied constant depends only on k and N. Using these we can now prove the proposition. First, we show that the triple sum of the proposition converges absolutely. Combining this bound and (21) we get that for every $N > 0$,

$$S_0(n, \kappa) \ll n^{-N\kappa} \sum_{f,l=1}^\infty \frac{1}{(lf^2)^{\frac{3}{2}}} \sum_{lf^2\xi \gg n^{\frac{1}{2}+\kappa}} \frac{|Kl_{l,f}(\xi, n)|}{\sqrt{l}} \frac{1}{\xi^2}$$

$$\ll n^{-N\kappa} \sum_{f,l=1}^\infty \frac{1}{(lf^2)^{\frac{3}{2}}} \sum_{\xi \in \mathbb{Z}\setminus\{0\}} \frac{|Kl_{l,f}(\xi, n)|}{\sqrt{l}} \frac{1}{\xi^2}, \qquad (\circ)$$

where the implied constant depends only on k and N. Next, we trivially have

$$\gcd(n, f^2) \leqslant n \quad \text{and} \quad \sqrt{\gcd\left(\frac{\xi}{\sqrt{\gcd(n, f^2)}}, l\right)} \leqslant \sqrt{\xi}.$$

Substituting these two bounds in (20) gives

$$Kl_{l,f}(\xi, n) \ll \log(lf^2)\sqrt{nl\xi}. \tag{22}$$

Using (22) in (∘) shows that

$$S_0(n, \kappa) \ll n^{\frac{1}{2}-N\kappa} \sum_{\xi, f, l=1}^{\infty} \frac{\log(lf^2)}{(lf^2\xi)^{\frac{3}{2}}} \ll n^{\frac{1}{2}-N\kappa}. \qquad \square$$

Corollary 4.9. *For every $N, \kappa > 0$,*

$$\sum_{n < X} \sqrt{n}\, S_0(n, \kappa) = O(X^{2-N\kappa}),$$

where the implied constant depends only on k, κ, and N.

Proof. Follows from Theorem 4.8. $\qquad \square$

Theorem 4.10. *Let $n \in \mathbb{Z}_{>0}$. Then for every $\kappa, \alpha > 0$,*

$$S_1(n, \kappa, \alpha) \ll \frac{\log(n)}{n^{\frac{3-2\kappa}{4}}} \left(\sum_{\substack{d_0^2 | n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{\log(n)}{d_0^{1/2}} + \frac{n^{\frac{1-2(\kappa+3\alpha)}{4}}}{d_0^{3/2}} \right),$$

where the implied constant depends only on k, κ, and α.

Proof. To prove the theorem we bound

$$S_1(n, \kappa, \delta) := \sum_{\substack{lf^2\xi \ll n^{\frac{1}{2}+\kappa} \\ \xi \gg n^{\delta}}} \frac{Kl_{l,f}(\xi, n)}{l^2 f^3} I_{l,f}(\xi, n),$$

for any $\delta > \kappa$ (not to be confused with the function $\delta(n; f^2)$ of (24)) and then specialize to $\delta = \frac{1}{6} + \kappa + \alpha$. This on one hand avoids notational burden, and on the other hand proves that the exponent $\frac{1}{6}$ is the best possible via this argument. Let us first estimate the integral. We claim that

$$\int_{-1}^{1} \theta_\infty(x) F\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{n}}{2lf^2}\right) dx \ll \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{3}{2}} \frac{1}{\xi^{3/2}}, \qquad (\star)$$

$$\int_{-1}^{1} \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{n}}{2lf^2}\right) dx \ll \frac{lf^2}{\sqrt{n}\xi}, \qquad (\star\star)$$

where the implied constants depend only on k. These bounds, in fact, are the same bounds in [2, Proposition 4.9, lines (i) and (iii)]. The only difference is that in *loc. cit.* the bounds are proved for the range $\frac{lf^2\xi}{\sqrt{n}} \ll 1$; however, our range is $\frac{lf^2\xi}{\sqrt{n}} \ll n^\kappa$ and $\xi \gg n^{\frac{1}{6}+2\kappa}$. We claim that the bounds still hold for this range. This indeed follows from the proof of Proposition 4.9 of *loc. cit.* In order to give the details we first need to remind the reader how the assumption $\frac{lf^2\xi}{\sqrt{n}} \ll 1$ was used in the proof of Proposition 4.9. First, the proof itself depends on the asymptotic expansion of Theorem A.14 of *loc. cit.* The assumption $\frac{lf^2\xi}{\sqrt{n}} \ll 1$ implies that $\frac{\xi\sqrt{n}}{lf^2} = \frac{\sqrt{n}}{lf^2\xi}\xi^2 \gg 1$, and substituting this for the parameter D in the asymptotic expansion of Theorem A.14 implies that the first term dominates the asymptotic expansion, which is the result of Proposition 4.9. In other words, the essential ingredient is the bound $\frac{\sqrt{n}\xi}{lf^2} \gg 1$.

Coming back to our proof we have $\xi \gg n^\delta$, which implies that $\frac{\xi\sqrt{n}}{lf^2} \gg \frac{n^{\frac{1}{2}+\delta}}{lf^2} \gg n^{\delta-\kappa} \gg 1$, hence the conclusion of [2, Proposition 4.9] still holds. Note also that

$$lf^2\xi \ll n^{\frac{1}{2}+\kappa} \quad \Rightarrow \quad \frac{lf^2}{\sqrt{n}} \ll \frac{n^\kappa}{\xi}. \qquad (\star\star\star_{S_1})$$

Substituting $(\star\star\star_{S_1})$ in $(\star\star)$ then gives,

$$\int_{-1}^{1} \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{n}}{2lf^2}\right) dx \ll \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{1}{2}} \frac{n^{\frac{\kappa}{2}}}{\xi^{3/2}}. \qquad (\star\star_{S_1})$$

Using $(\star)$ and $(\star\star_{S_1})$ we then get

$$I_{l,f}(\xi,n) \ll \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{3}{2}} \frac{n^{\frac{\kappa}{2}}}{\xi^{3/2}}, \qquad (23)$$

where the implied constants depend only k and κ. Next, we bound the character sums, $Kl_{l,f}(\xi,n)$. By [2, Corollary B.8] we have,

$$Kl_{l,f}(\xi,n) \ll \begin{cases} \delta(n;f^2)\log(lf^2)\sqrt{l\gcd(n,f^2)}\sqrt{\gcd\left(\frac{\xi}{\sqrt{\gcd(n,f^2)}},l\right)} \dfrac{l\sqrt{\gcd(n,f^2)}}{\mathrm{rad}(l)} & \Big| \xi \\ \\ 0 & \text{otherwise,} \end{cases} \qquad (24)$$

where $\delta(n;f^2)$ is 1 if n is a square modulo f^2 and 0 otherwise, and $\mathrm{rad}(l) = \prod_{p|l} p$, where p denotes a prime. This, in particular, implies that if $\gcd(n,f^2)$ is not a perfect square then $Kl_{l,f}(\xi,n)$ vanishes. For l,f,ξ,n such that $Kl_{l,f}(\xi,n) \neq 0$ let,

$$d_0^2 := \gcd(n,f^2), \quad n = d_0^2 n_0, \quad f = d_0 f_0, \quad \xi = d_0 \xi_0,$$
$$d_1 := \gcd(l,\xi_0), \quad l = d_1 l_0, \quad \xi_0 = d_1 \xi_1.$$

Note also that l_0 is square-free. Substituting these in (24) gives

$$Kl_{l,f}(\xi,n) \ll \log(d_0^2 d_1 l_0 f_0^2) d_0 d_1 \sqrt{l_0}. \qquad (25)$$

Combining (23) and (25) implies that,

$$S_1(n, \kappa, \delta) \ll \frac{1}{n^{\frac{3-2\kappa}{4}}} \sum_{d_0^2 | n} \sum_{\substack{d_0^3 d_1^2 l_0 f_0^2 \xi_1 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \xi_1 \gg n^\delta}} \frac{\log(d_0^2 d_1 l_0 f_0^2)}{\sqrt{d_0} d_1 \xi_1^{3/2}}.$$

Note that $d_0^3 d_1^2 l_0 f_0^2 \xi_1 \ll n^{\frac{1}{2}+\kappa} \Rightarrow d_0 \ll n^{\frac{1+2\kappa}{6}}$. Therefore, in the summation range for $S_1(n, \kappa)$, d_0 runs through square divisors of n which are $\ll n^{\frac{1+2\kappa}{6}}$. Therefore,

$$S_1(n, \kappa, \delta) \ll \frac{1}{n^{\frac{3-2\kappa}{4}}} \left(\sum_{\substack{d_0^2 | n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{T(d_0, n, \kappa, \delta)}{\sqrt{d_0}} + \sum_{\substack{d_0^2 | n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{U(d_0, n, \kappa, \delta)}{\sqrt{d_0}} \right), \qquad (26)$$

where

$$T(d_0, n, \kappa, \delta) := \sum_{\substack{d_0^3 d_1^2 l_0 f_0^2 \xi_1 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \gg n^\delta}} \frac{\log(d_0^2 d_1 l_0 f_0^2)}{d_1 \xi_1^{3/2}}, \quad U(d_0, n, \kappa, \delta)$$

$$:= \sum_{\substack{d_0^3 d_1^2 l_0 f_0^2 \xi_1 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \ll n^\delta \\ d_0 d_1 \xi_1 \gg n^\delta}} \frac{\log(d_0^2 d_1 l_0 f_0^2)}{d_1 \xi_1^{3/2}}.$$

We now bound $T(d_0, n, \kappa, \delta)$ and $U(d_0, n, \kappa, \delta)$.

$$T(d_0, n, \kappa, \delta) = \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \gg n^\delta}} \frac{1}{d_1} \sum_{f_0^2 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2} \rfloor} \sum_{l_0 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 f_0^2} \rfloor} \log(d_0^2 d_1 l_0 f_0^2) \sum_{\xi_1 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 f_0^2 l_0} \rfloor} \frac{1}{\xi_1^{3/2}}$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \gg n^\delta}} \frac{1}{d_1} \sum_{f_0^2 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2} \rfloor} \sum_{l_0 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 f_0^2} \rfloor} \log(d_0^2 d_1 l_0 f_0^2)$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \gg n^\delta}} \frac{1}{d_1} \sum_{f_0^2 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2} \rfloor} \left(\log(d_0^2 d_1 f_0^2) + \left(1 + \frac{n^{1/2+\kappa}}{d_0^3 d_1^2 f_0^2} \right) \log\left(\frac{n^{1/2+\kappa}}{d_0 d_1} \right) \right)$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \gg n^\delta}} \left(\frac{\log(d_0^2 d_1)}{d_1} + \frac{n^{1/2+\kappa}}{d_0^3 d_1^3} \log\left(\frac{n^{1/2+\kappa}}{d_0 d_1} \right) \right)$$

$$\ll \log^2(n) + \frac{n^{1/2+\kappa-2\delta}}{d_0} \log(n). \qquad (\circ)$$

We remark that all of the implied constants depend only on κ and δ. Moving on to $U(d_0, n, \kappa, \delta)$ we have:

$$U(d_0, n, \kappa, \delta) = \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \ll n^\delta}} \frac{1}{d_1} \sum_{\lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2} \rfloor \gg \xi_1 \gg \lfloor \frac{n^\delta}{d_0 d_1} \rfloor} \frac{1}{\xi_1^{3/2}} \sum_{f_0^2 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 \xi_1} \rfloor} \sum_{l_0 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 f_0^2 \xi_1} \rfloor} \log(d_0^2 d_1 l_0 f_0^2)$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \ll n^\delta}} \frac{1}{d_1} \sum_{\lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2} \rfloor \gg \xi_1 \gg \lfloor \frac{n^\delta}{d_0 d_1} \rfloor} \frac{1}{\xi_1^{3/2}}$$

$$\times \sum_{f_0^2 \ll \lfloor \frac{n^{\frac{1}{2}+\kappa}}{d_0^3 d_1^2 \xi_1} \rfloor} \left(\log\left(d_0^2 d_1 f_0^2\right) + \left(1 + \frac{n^{1/2+\kappa}}{d_0^3 d_1^2 f_0^2 \xi_1}\right) \log\left(\frac{n^{1/2+\kappa}}{d_0 d_1 \xi_1}\right) \right)$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \ll n^\delta}} \frac{1}{d_1} \sum_{\xi_1 \gg \lfloor \frac{n^\delta}{d_0 d_1} \rfloor} \left(\frac{\log(d_0^2 d_1)}{\xi^{3/2}} + \left(\frac{n^{1/2+\kappa}}{d_0^3 d_1^2 \xi_1^{5/2}} + \frac{1}{\xi^{3/2}}\right) \log\left(\frac{n^{1/2+\kappa}}{d_0 d_1 \xi_1}\right) \right)$$

$$\ll \sum_{\substack{d_0^3 d_1^2 \ll n^{\frac{1}{2}+\kappa} \\ d_0 d_1 \ll n^\delta}} \left(\frac{\sqrt{d_0} \log(d_0^2 d_1)}{n^{\delta/2} \sqrt{d_1}} + \frac{n^{(1+2\kappa-3\delta)/2} \log(n)}{\sqrt{d_0^3 d_1^3}} \right)$$

$$\ll \left(1 + \frac{\sqrt{d_0}}{n^{\delta/2}} + \frac{n^{(1+2\kappa-3\delta)/2}}{d_0^{3/2}} \right) \log(n). \tag{$\circ\circ$}$$

Once again, all of the implied constants depend only on κ and δ.

Finally, substituting ($\circ$) and ($\circ\circ$) into (26) gives,

$$S_1(n, \kappa, \delta) \ll \frac{\log(n)}{n^{\frac{3-2\kappa}{4}}} \left(\sum_{\substack{d_0^2 | n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{\log(n)}{d_0^{1/2}} + \frac{n^{(1+2\kappa-3\delta)/2}}{d_0^{3/2}} + \frac{1}{n^{\delta/2}} \right),$$

where the implied constant depends only on $k, \kappa,$ and δ. The theorem then follows from substituting $\delta = \frac{1}{6} + \kappa + \alpha$. $\qquad\square$

Corollary 4.11. *For every $\kappa, \alpha > 0$,*

$$\sum_{n < X} \sqrt{n}\, S_1(n, \kappa, \alpha) \ll \left(\log^2(X) X^{\frac{3+2\kappa}{4}} + \log(X) X^{1-\frac{3\alpha}{2}} \right).$$

where the implied constant depends only on $k, \kappa,$ and α.

Proof. By Theorem 4.10 we get

$$\sum_{n<X} \sqrt{n}\, S_1(n,\kappa,\alpha) \ll \sum_{n<X} \frac{\log(n)}{n^{\frac{1-2\kappa}{4}}} \left(\sum_{\substack{d_0^2|n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{\log(n)}{d_0^{1/2}} + \frac{n^{\frac{1-2(\kappa+3\alpha)}{4}}}{d_0^{3/2}} \right)$$

$$\ll \sum_{n<X} \frac{\log(n)}{n^{\frac{1-2\kappa}{4}}} \left(\sum_{\substack{d_0^2|n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{\log(n)}{d_0^{1/2}} + \frac{n^{\frac{1-2(\kappa+3\alpha)}{4}}}{d_0^{3/2}} \right).$$

Bounding each term in the double sum separately we get,

$$\sum_{n<X} \frac{\log^2(n)}{n^{\frac{1-2\kappa}{4}}} \sum_{\substack{d_0^2|n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{1}{d_0^{1/2}} = \sum_{d_0 \ll X^{\frac{1+2\kappa}{6}}} \frac{1}{d_0^{1-\kappa}} \sum_{n \ll \lfloor \frac{X}{d_0^2} \rfloor} \frac{\log^2(nd_0^2)}{n^{\frac{1-2\kappa}{4}}}$$

$$\ll \sum_{d_0 \ll X^{\frac{1+2\kappa}{6}}} \frac{\log^2(X)}{d_0^{1-\kappa}} \left(\left(\frac{X}{d_0^2}\right)^{\frac{3+2\kappa}{4}} + \left(\frac{d_0^2}{X}\right)^{\frac{1-2\kappa}{4}} + 1 \right)$$

$$\ll \log^2(X) X^{\frac{3+2\kappa}{4}},$$

$$\sum_{n<X} \frac{\log(n)}{n^{3\alpha/2}} \sum_{\substack{d_0^2|n \\ d_0 \ll n^{\frac{1+2\kappa}{6}}}} \frac{1}{d_0^{3/2}} = \sum_{d_0 \ll X^{\frac{1+2\kappa}{6}}} \frac{1}{d_0^{(3+6\alpha)/2}} \sum_{n \ll \lfloor \frac{X}{d_0^2} \rfloor} \frac{\log(nd_0^2)}{n^{3\alpha/2}}$$

$$\ll \sum_{d_0 \ll X^{\frac{1+2\kappa}{6}}} \frac{\log(X)}{d_0^{(3+6\alpha)/2}} \left(\left(\frac{X}{d_0^2}\right)^{1-\frac{3\alpha}{2}} + \left(\frac{d_0^2}{X}\right)^{\frac{3\alpha}{2}} + 1 \right)$$

$$\ll \log(X) X^{1-\frac{3\alpha}{2}}.$$

The corollary follows. □

Theorem 4.12. *Let $n \in \mathbb{Z}_{>0}$. Then for every $\kappa > 0$ and $\frac{1}{12} \geqslant \alpha > 0$,*

$$S_2(n,\kappa,\alpha) \ll \log^2(n) n^{-\frac{1}{2}-\kappa},$$

where the implied constant depends only on k, κ, and α.

Proof. The proof follows the same argument as in the proof of Theorem 4.10. First, note that since $lf^2 \ll n^{\frac{1}{4}-\kappa}$ and $\xi \neq 0$ we necessarily have $\frac{\xi\sqrt{n}}{lf^2} \gg 1$. Therefore the first paragraph of the proof of Theorem 4.10 goes through verbatim and implies the same

bounds as in ($\star$) and ($\star\star$) of that theorem. Moreover, since $S_2(n, \kappa, \alpha)$ has the summation ranges $lf^2 \ll n^{\frac{1}{4}-\kappa}$ and $\xi \ll n^{\frac{1}{6}+\kappa+\alpha}$ we have $lf^2\xi \ll n^{\frac{5}{12}+\alpha}$. Therefore,

$$\frac{lf^2}{\sqrt{n}} \ll \frac{1}{n^{1/12-\alpha}\xi}. \qquad (\star\star\star_{S_2})$$

Substituting ($\star\star\star_{S_2}$) into ($\star\star$) and using $\alpha \leqslant \frac{1}{12}$ gives

$$\int_{-1}^{1} \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H\left(\frac{lf^2(4n)^{-1/2}}{\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{n}}{2lf^2}\right) dx \ll \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{1}{2}} \frac{1}{\xi^{3/2}}. \qquad (\star\star_{S_2})$$

Finally, substituting ($\star$) and ($\star\star_{S_2}$) in $I_{l,f}(\xi, n)$ we get

$$I_{l,f}(\xi, n) \ll \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{3}{2}} \frac{1}{\xi^{3/2}}. \qquad (27)$$

We also remark that the implied constant is independent of l, f, ξ and n. Moving on to the character sum, $Kl_{l,f}(\xi, n)$, we once again have the bound in (24). Note that because of the presence of $\delta(n; f^2)$ implies that the $\gcd(n, f^2)$ has to be a perfect square otherwise the sum vanishes. Moreover, when $\gcd(n, f^2)$ is a square, the sum still vanishes unless $\sqrt{\gcd(n, f^2)} \mid \xi$. This implies that whenever $Kl_{l,f}(\xi, n) \neq 0$ we have to have $\xi = \sqrt{\gcd(n, f^2)}\xi_1$ for some ξ_1, and we have the bound

$$Kl_{l,f}(\xi, n) = Kl_{l,f}(\gcd(n, f^2)\xi_1, n) \ll \log(lf^2)\sqrt{l\gcd(n, f^2)\xi_1} = log(lf^2)\sqrt{l\xi}. \qquad (28)$$

Substituting the bounds in (27) and (28) into $S_2(n, \kappa)$ gives,

$$S_2(n, \kappa, \alpha) \ll \sum_{lf^2 \ll n^{\frac{1}{4}-\kappa}} \frac{1}{(lf^2)^{\frac{3}{2}}} \sum_{\xi \ll n^{\frac{1}{6}+\kappa+\alpha}} \frac{\log(lf^2)\sqrt{l\xi}}{\sqrt{l}} \left(\frac{lf^2}{\sqrt{n}}\right)^{\frac{3}{2}} \frac{1}{\xi^{\frac{3}{2}}} \ll \log^2(n)n^{-\frac{1}{2}-\kappa}. \qquad \square$$

Corollary 4.13. *For every $\kappa > 0$ and $\frac{1}{12} \geqslant \alpha > 0$,*

$$\sum_{n<X} \sqrt{n}\, S_2(n, \kappa, \alpha) \ll \log^2(X)X^{1-\kappa},$$

where the implied constant depends only on k, κ and α.

Proof. Follows from Theorem 4.12. $\qquad \square$

4.2.2. Estimating the critical range. In this section, we estimate the critical range of summation where $X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}$. Let

$$S(X, \kappa, \alpha) := \sum_{n<X} \sqrt{n} \sum_{\substack{X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{l^2 f^3} \sum_{\substack{\xi \in \mathbb{Z}\setminus\{0\} \\ lf^2\xi \ll X^{\frac{1}{2}+\kappa} \\ \xi \ll X^{\frac{1}{6}+\kappa+\alpha}}} Kl_{l,f}(\xi, n) I_{l,f}(\xi, n). \qquad (29)$$

The basic strategy for estimating this sum is to apply Poisson summation to the n-sum. The heuristic reason is pretty simple:

Using a smooth dyadic partition we can assume that $n \sim X$. Moreover, in the region of summation $I_{l,f}(\xi, n)$ is roughly 1 (we have used all of its decay properties to get the estimates of the previous section). Therefore, bounding (29) reduces to bounding

$$\sqrt{X} \sum_{n \in \mathbb{Z}} G\left(\frac{n}{X}\right) \sum_{\substack{X^{\frac{1}{2}} \gg lf^2 \gg X^{\frac{1}{4}} \\ lf^2 \xi \ll X^{\frac{1}{2}} \\ \xi \ll X^{\frac{1}{6}}}} \frac{Kl_{lf}(\xi, n)}{l^2 f^3}.$$

Now notice that $Kl_{l,f}(\xi, n)$ is periodic in n modulo $4lf^2$. Therefore Poisson summation on the above sum gives

$$X^{\frac{3}{2}} \sum_{\substack{X^{\frac{1}{2}} \gg lf^2 \gg X^{\frac{1}{4}} \\ lf^2 \xi \ll X^{\frac{1}{2}} \\ \xi \ll X^{\frac{1}{6}}}} \frac{1}{l^3 f^5} \sum_{\nu \in \mathbb{Z}} \hat{G}\left(\frac{X\nu}{4lf^2}\right) \omega_{l,f}(\xi, n),$$

where $\omega_{l,f}(\xi, \nu)$ is as in (38), and it is roughly of size lf (cf. Corollary 5.9). Since $lf^2 \ll \sqrt{X}$ we deduce that as long as $\nu \neq 0$ the decay of $\hat{G}$ will guarantee that the sum is very small in terms of the variable X. The only remaining point is to analyze the term corresponding to $\nu = 0$. By a local analysis we can show that this term is 0 unless ν and ξ satisfy certain divisibility conditions and in our range of summation those conditions cannot be satisfied (see the proof of Theorem 4.14).

The only difficulty in executing this simple strategy is that the function G is not alone. It comes as the product of $G(\frac{n}{X}) I_{l,f}(\xi, n)$ so that one needs to get the decay properties of the Fourier transform of this product uniformly in all the variables, and this is done in Proposition 5.2 of §5.

We can now go on and execute the strategy described above. First of all, let $G \in C_c^\infty([\frac{1}{4}, \frac{5}{4}])$ be a smooth function. In Corollary 4.15, we specialize this mollifier to a smooth approximation to the characteristic function of the interval of $[\frac{1}{2}, 1)$ to get back from the estimates on smoothed sums to estimating (29). Set,

$$S_G(X, \kappa, \alpha) := \sum_{n \in \mathbb{Z}} G\left(\frac{n}{X}\right) \sqrt{n} \sum_{\substack{X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{l^2 f^3} \sum_{\substack{\xi \in \mathbb{Z} \setminus \{0\} \\ lf^2 \xi \ll X^{\frac{1}{2}+\kappa} \\ \xi \ll X^{\frac{1}{6}+\kappa+\alpha}}} Kl_{l,f}(\xi, n) I_{l,f}(\xi, n). \quad (30)$$

Theorem 4.14. *For every* $G \in C_c^\infty([\frac{1}{4}, \frac{5}{4}])$, $M \geq 2$, $\kappa, \alpha > 0$ *such that* $\frac{1}{12} > 2\kappa + \alpha$ *we have*

$$S_G(X, \kappa, \alpha) = O(\|G\|_{M,1} X^{\frac{17}{12} + 2\kappa + \alpha - M(\frac{1}{3} - \kappa - \alpha)} \log(X)),$$

where

$$\|G\|_{M,1} = \sum_{j=0}^{M} \|G^{(j)}\|_1,$$

$G^{(j)}$ *denoting the* jth *derivative of* G *(i.e., the Sobolev* $W^{M,1}$*-norm of* G*), and the implied constant depends only on* $k, \kappa, \alpha,$ *and* M.

Proof. Note that the character sum, $Kl_{l,f}(\xi, n)$, is periodic in n modulo $4lf^2$. Using this we interchange the n-sum with the rest of the terms and break it up into arithmetic progressions $\mod 4lf^2$. This gives,

$$S_G(X, \kappa, \alpha) = \sum_{\substack{X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{l^2 f^3} \sum_{\substack{\xi \in \mathbb{Z} \setminus \{0\} \\ lf^2 \xi \ll X^{\frac{1}{2}+\kappa} \\ \xi \ll X^{\frac{1}{6}+\kappa+\alpha}}} \sum_{b \bmod 4lf^2} Kl_{l,f}(\xi, b)$$

$$\times \sum_{\substack{n \in \mathbb{Z} \\ n \equiv b \bmod 4lf^2}} \sqrt{n}\, G\left(\frac{n}{X}\right) I_{l,f}(\xi, n).$$

We now apply Poisson summation to the n-sum and get

$$S_G(X, \kappa, \alpha) = \sum_{\substack{X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{l^2 f^3} \sum_{\substack{\xi \in \mathbb{Z} \setminus \{0\} \\ lf^2 \xi \ll X^{\frac{1}{2}+\kappa} \\ \xi \ll X^{\frac{1}{6}+\kappa+\alpha}}} \sum_{\nu \in \mathbb{Z}} \frac{\omega_{l,f}(\xi \nu)}{4lf^2} J_{l,f}(\xi, \nu, X), \qquad (31)$$

where $J_{l,f}(\xi, \nu, X)$ is the Fourier transform defined in (37) and $\omega_{l,f}(\xi, \nu)$ is the character sum defined in (38). In order to bound $J_{l,f}(\xi, \nu)$ and $\omega_{l,f}(\xi, \nu)$, we use Proposition 5.2 and Corollary 5.9 respectively. For any $M, N \geqslant 0$ and $\nu \neq 0$, Proposition 5.2 gives the following bound on $I_{l,f}(\xi, \nu)$,

$$J_{l,f}(\xi, \nu) \ll \|G\|_{M,1} \frac{X^{\frac{N-M+3}{2}}}{\nu^M (lf^2)^N} \left[\left(\frac{lf^2}{\sqrt{X}}\right)^M + \xi^M \right], \qquad (\bullet)$$

where the implied constant depends only on k, G, M, and N. For $\omega_{l,f}(\xi, \nu)$ Corollary 5.9 gives

$$\omega_{l,f}(\xi, \nu) \ll \begin{cases} \log(lf) lf \sqrt{\gcd(lf^2, \nu) \gcd(l, \nu)} & \dfrac{l}{\operatorname{rad}(l)} \mid \nu, \quad \gcd(lf^2, \nu) \mid \xi \\ 0 & \text{otherwise,} \end{cases} \qquad (\bullet\bullet)$$

where the implied constant is absolute. Going back to (31) we break the analysis of the ν sum into two according to $\nu = 0$ and $\nu \neq 0$.

- $\nu = 0$. In this case the character sum $\omega_{l,f}(\xi, 0)$ does not vanish only if $lf^2 \mid \xi$. But the ranges in $S_G(X, \kappa, \alpha)$ are $lf^2 \gg X^{\frac{1}{4}-\kappa}$ and $\xi \ll X^{\frac{1}{6}+\kappa+\alpha}$. Since $2\kappa + \alpha < \frac{1}{12}$ these ranges do not intersect, hence for all l, f, ξ in the range for $S_G(X, \kappa, \alpha)$ the term corresponding to $\nu = 0$ vanishes.

- $\nu \neq 0$. We will be using the bounds in ($\bullet$) and ($\bullet\bullet$). We first need to separate the gcd-factors from ($\bullet\bullet$). Let $\gcd(l, \nu) = d_0$. Then,

$$\omega_{l,f}(\xi, \nu) \neq 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} l = d_0 l_0 \\ \nu = d_0 \nu_0 \\ \xi = d_0 \xi_0 \end{array} \middle| \gcd(l_0, \nu_0) = 1 \right\}.$$

Then (●●) implies that,

$$\omega_{d_0 l_0, f}(d_0 \xi_0, d_0 \nu_0) \ll \log(d_0 l_0 f) d_0^2 l_0 f \sqrt{\gcd(f^2, \nu_0)} \leqslant \log(d_0 l_0 f) d_0^2 l_0 f \sqrt{\nu_0}. \qquad (\bullet\bullet')$$

Now, taking $N = 0$ in (●) and using (●●), we get the following bound valid for every $M \geqslant 2$,

$$S_G^{\nu \neq 0}(X, \kappa, \alpha) \ll \|G\|_{M,1} X^{\frac{3-M}{2}} \sum_{X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}} \frac{\log(d_0 l_0 f)}{d_0 (l_0 f^2)^2}$$

$$\times \sum_{\substack{\xi_0, \nu_0 \in \mathbb{Z} \setminus \{0\} \\ d_0^2 l_0 f^2 \xi_0 \ll X^{\frac{1}{2}+\kappa} \\ d_0 \xi_0 \ll X^{\frac{1}{6}+\kappa+\alpha}}} \frac{1}{\nu_0^{M-\frac{1}{2}}} \left[\left(\frac{l_0 f^2}{\sqrt{X}} \right)^M + \xi_0^M \right],$$

where the implied constant is independent of X and G. Since the ν_0-sum converges absolutely (recall that $M \geqslant 2$) and since $d_0 l_0 f^2 \ll X^{\frac{1}{2}+\kappa} \Rightarrow \log(d_0 l_0 f) \leqslant \log(X)$, we have

$$S_G^{\nu \neq 0}(X, \kappa, \alpha) \ll \|G\|_{M,1} X^{\frac{3-M}{2}} \log(X) \sum_{X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}} \frac{1}{d_0 (l_0 f^2)^2}$$

$$\times \sum_{\substack{\xi_0 \in \mathbb{Z} \setminus \{0\} \\ d_0^2 l_0 f^2 \xi_0 \ll X^{\frac{1}{2}+\kappa} \\ d_0 \xi_0 \ll X^{\frac{1}{6}+\kappa+\alpha}}} \left[\left(\frac{l_0 f^2}{\sqrt{X}} \right)^M + \xi_0^M \right]$$

$$= \|G\|_{M,1} X^{\frac{3-M}{2}} \log(X) \left(S_{\nu \neq 0}^{(1)}(X, \kappa, \alpha) + S_{\nu \neq 0}^{(2)}(X, \kappa, \alpha) \right), \qquad (\circ)$$

where

$$S_G^{\nu \neq 0, (1)}(X, \kappa, \alpha) := X^{-\frac{M}{2}} \sum_{X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}} \frac{(l_0 f^2)^{M-2}}{d_0} \sum_{\substack{\xi_0 \in \mathbb{Z} \setminus \{0\} \\ d_0^2 l_0 f^2 \xi_0 \ll X^{\frac{1}{2}+\kappa} \\ d_0 \xi_0 \ll X^{\frac{1}{6}+\kappa+\alpha}}} 1,$$

$$S_G^{\nu \neq 0, (2)}(X, \kappa, \alpha) := \sum_{X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}} \frac{1}{d_0 (l_0 f^2)^2} \sum_{\substack{\xi_0 \in \mathbb{Z} \setminus \{0\} \\ d_0^2 l_0 f^2 \xi_0 \ll X^{\frac{1}{2}+\kappa} \\ d_0 \xi_0 \ll X^{\frac{1}{6}+\kappa+\alpha}}} \xi_0^M.$$

We now bound $S_G^{\nu \neq 0, (1)}(X, \kappa, \alpha)$ and $S_G^{\nu \neq 0, (2)}(X, \kappa, \alpha)$.

$$S_G^{\nu \neq 0, (1)}(X, \kappa, \alpha) \ll X^{-\frac{M}{2}} \sum_{X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}} \frac{(l_0 f^2)^{M-2}}{d_0} \left(1 + \min \left\{ \frac{X^{\frac{1}{2}+\kappa}}{d_0^2 l_0 f^2}, \frac{X^{\frac{1}{6}+\kappa+\alpha}}{d_0} \right\} \right)$$

$$\ll X^{-\frac{1}{2}+\kappa(M-1)}\log(X) + X^{-\frac{M}{2}}$$

$$\times \sum_{l_0 f^2 \ll X^{\frac{1}{2}+\kappa}} (l_0 f^2)^{M-2} \min\left\{\frac{X^{\frac{1}{2}+\kappa}}{l_0 f^2}, X^{\frac{1}{6}+\kappa+\alpha}\right\}$$

$$\ll X^{-\frac{1}{2}+\kappa(M-1)}\log(X). \tag{i}$$

To bound $S_G^{\nu\neq 0,(2)}(X,\kappa,\alpha)$ first note that since $d_0\xi_0 \ll X^{\frac{1}{6}+\kappa+\alpha}$ we have $d_0 \ll X^{\frac{1}{6}+\kappa+\alpha}$. Then,

$$S_G^{\nu\neq 0,(2)}(X,\kappa,\alpha) \ll \sum_{\substack{d_0 \ll X^{\frac{1}{6}+\kappa+\alpha} \\ X^{\frac{1}{2}+\kappa} \gg d_0 l_0 f^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{d_0(l_0 f^2)^2}$$

$$\times \left(1 + \min\left\{\left(\frac{X^{\frac{1}{2}+\kappa}}{d_0^2 l_0 f^2}\right)^{M+1}, \left(\frac{X^{\frac{1}{6}+\kappa+\alpha}}{d_0}\right)^{M+1}\right\}\right)$$

$$\ll \log(X) + \sum_{\substack{d_0 \ll X^{\frac{1}{6}+\kappa+\alpha} \\ l_0 f^2 \gg \lfloor \frac{X^{\frac{1}{4}-\kappa}}{d_0}\rfloor}} \frac{1}{d_0(l_0 f^2)^2}$$

$$\times \min\left\{\left(\frac{X^{\frac{1}{2}+\kappa}}{d_0^2 l_0 f^2}\right)^{M+1}, \left(\frac{X^{\frac{1}{6}+\kappa+\alpha}}{d_0}\right)^{M+1}\right\}$$

$$\ll \log(X) + X^{(\frac{1}{6}+\kappa+\alpha)(M+1)}(X^{\kappa-\frac{1}{4}} + X^{\alpha-\frac{1}{3}})$$

$$\leqslant X^{(\frac{1}{6}+\kappa+\alpha)(M+1)-\frac{1}{4}+\kappa}. \tag{ii}$$

Where, we used the assumption that $2\kappa+\alpha < \frac{1}{12}$ therefore $\frac{1}{4}-\kappa < \frac{1}{3}-\alpha$. Finally, substituting (i) and (ii) into $(\circ)$ gives,

$$S_G^{\nu\neq 0}(X,\kappa,\alpha) \ll \|G\|_{M,1} X^{\frac{3-M}{2}}\log(X)(X^{-\frac{1}{2}+\kappa(M-1)}\log(X) + X^{(\frac{1}{6}+\kappa+\alpha)(M+1)-\frac{1}{4}+\kappa})$$

$$\ll \|G\|_{M,1} X^{\frac{17-4M}{12}+(M+1)(\kappa+\alpha)+\kappa}\log(X). \qquad \square$$

Corollary 4.15. *Let* $\kappa, \alpha > 0$ *such that* $2\kappa+\alpha < \frac{1}{12}$. *Then for every* $\epsilon > 0$ *we have*

$$S(X,\kappa,\alpha) = O\big(X^{\frac{11}{12}+\kappa+\alpha+\frac{11\epsilon}{6}}\big),$$

where the implied constant depends only on $k, \kappa, \alpha,$ *and* ϵ.

Proof. We start with bounding each individual $T_k(n)$. Although there are better bounds (in particular, the Ramanujan conjecture $|\operatorname{tr}(T_k(n))| = O_\epsilon(n^\epsilon)$ is known in this case thanks to Deligne [6], and see [13] for an excellent survey of bounds toward the Ramanujan conjecture in general) in order to keep the proof self contained, and use only the trace

formula we refer to [2, Theorem 1.1] which, translated to the setting of the current paper, tells us that $\operatorname{tr}(T_k(p)) = O(p^{\frac{k}{4}})$ for any prime p. Then, by the recursion relations the $T_k(p^k)$'s satisfy (cf. [15, (73) on p. 102]) it is straightforward to see that $\operatorname{tr}(T_k(n)) = O_\epsilon(n^{\frac{1}{4}+\epsilon})$.

Using this it is now straightforward to prove the theorem by choosing a specific mollifier $G(x)$. Let $\phi \in C_c^\infty((-1,1))$ such that it is normalized by $\int \phi = 1$, and let $\mathbf{1}_{[\frac{1}{2},1)}$ be the characteristic function of the interval $[\frac{1}{2},1)$. Let $1 \geqslant \delta \geqslant 0$ set $\phi_{Y^\delta}(x) = Y^{1-\delta}\phi(xY^{1-\delta})$. Using $\phi_{Y^\delta}(x)$ define $G_{Y^\delta}(x) := \mathbf{1}_{[\frac{1}{2},1)} * \phi_{Y^\delta}(x)$, i.e.,

$$G_{Y^\delta}(x) = Y^{1-\delta}\int \mathbf{1}_{[\frac{1}{2},1)}(x-y)\phi(yY^{1-\delta})\,dy. \tag{32}$$

Then it is straightforward to see that

$$G_{Y^\delta}\left(\frac{x}{Y}\right) = \begin{cases} 1 & \frac{Y}{2}+Y^\delta \leqslant x \leqslant Y-Y^\delta \\ O(1) & \left|x-\frac{Y}{2}\right| < Y^\delta \quad \text{or} \quad |x-Y| < Y^\delta \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

and the

$$\|G_{Y^\delta}(x)\|_{M,1} = O(Y^{M(1-\delta)}). \tag{34}$$

Then, by (33) and the bound $\operatorname{tr}(T_k(n)) = O(n^{\frac{1}{4}+\epsilon})$ we get

$$S(X,\kappa,\alpha) = \sum_{j=1}^{\log(X)} \sum_{2^{j-1}\leqslant n < 2^j} \sqrt{n} \sum_{\substack{X^{\frac{1}{2}+\kappa} \gg lf^2 \gg X^{\frac{1}{4}-\kappa}}} \frac{1}{l^2 f^3} \sum_{\substack{\xi \in \mathbb{Z}\backslash\{0\} \\ lf^2\xi \ll X^{\frac{1}{2}+\kappa} \\ \xi \ll X^{\frac{1}{6}+\kappa+\alpha}}} Kl_{l,f}(\xi,n)I_{l,f}(\xi,n)$$

$$\ll \sum_{j=1}^{\log(X)} \left\{ S_{G_{2^{j\delta}}}(2^j,\kappa,\alpha) + 2^{j\delta+\frac{1}{4}+\epsilon} \right\}, \tag{$*$}$$

where the implied constant depends only on k, δ and ϵ. By Theorem 4.14 for every $M > 0$, the sub-sums, $S_{G_{2^{j\delta}}}(2^j,\kappa,\alpha)$, satisfy

$$S_{G_{2^{j\delta}}}(2^j,\kappa,\alpha) = O(\|G_{2^{j\delta}}\|_{M,1}j(2^j)^{\frac{17}{12}+2\kappa+\alpha-M(\frac{1}{3}-\kappa-\alpha)}),$$

where the implied constant depends only on k, κ, α, and M. Substituting this bound in ($*$) and using (34) we get

$$S(X,\kappa,\alpha) = O\left(\sum_{j=1}^{\log(X)} j2^{jM(1-\delta)}(2^j)^{\frac{17}{12}+2\kappa+\alpha-M(\frac{1}{3}-\kappa-\alpha)} + 2^{j\delta+\frac{1}{4}+\epsilon} \right)$$

$$= O\left(X^{M(\frac{2}{3}+\kappa+\alpha-\delta)+\frac{17}{12}+2\kappa+\alpha}\log(X) + X^{\delta+\frac{1}{4}+\epsilon} \right),$$

where the implied constant depends only on k, ϵ, M, and δ. Finally choosing $\delta = \frac{2}{3}+\kappa+\alpha+\frac{5\epsilon}{6}$ and $M = \frac{2}{\epsilon}$ the corollary follows. $\qquad\square$

5. Local analysis

In this section, we derive bounds on the Fourier transforms and character sums that appear after the Poisson summation on the *n*-sum.

5.1. Archimedean analysis

We begin with a technical lemma that will be useful for the rest of this section. For what follows let us fix two positive integers l and f, and let X denote an independent parameter as in the previous sections.

Lemma 5.1. *Let* $\Phi \in \mathcal{S}(\mathbb{R}), \xi, \alpha \in \mathbb{Z}$, *and* $v \in \mathbb{Z}\setminus\{0\}$. *Let,*

$$V_{l,f,\alpha}(\xi, v, X) := \int G\left(\frac{y}{X}\right) y^{\frac{\alpha}{2}} \Phi\left(\frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}}\right) e\left(\frac{-(x\xi\sqrt{4y}+yv)}{4lf^2}\right) dy.$$

Then, for any $G(x) \in C_c^\infty([\frac{1}{4}, \frac{5}{4}])$ *and* $M, N \in \mathbb{N}$ *we have*

$$V_{l,f,\alpha}(\xi, v, X) = O\left(\|G\|_{M,1} \frac{X^{1+\frac{\alpha}{2}}}{v^M X^{\frac{M}{2}}} \left[\left(\frac{lf^2}{\sqrt{X}}\right)^M + \xi^M\right] \left(\frac{\sqrt{X}\sqrt{1-x^2}}{lf^2}\right)^N\right),$$

where the implied constant depends only on Φ, α, M, *and* N.

Proof. First using the change of variables $y \mapsto Xy$ and then applying integration by parts M-times (keeping in mind that G is compactly supported) gives,

$$V_{l,f,\alpha}(\xi, v, X) = X^{1+\frac{\alpha}{2}} \left(\frac{4lf^2}{-2\pi i X v}\right)^M \int \frac{d^M}{dy^M}$$

$$\times \left\{G(y) y^{\frac{\alpha}{2}} \Phi\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}}\right) e\left(\frac{-x\xi\sqrt{4Xy}}{4lf^2}\right)\right\} e\left(\frac{-Xyv}{4lf^2}\right) dy. \qquad (35)$$

The Mth derivative above is a combination of derivatives (of orders $\leqslant M$) of G, $y^{\frac{\alpha}{2}}$, $\Phi\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}}\right)$, and the exponential. Note that since $G(y)$ is compactly supported away from $y = 0$ the negative powers of y that appear in the derivative are bounded uniformly depending only on M and cause no problem. The only point we need to pay attention is the derivatives of Φ. To that end, note that since Φ decays faster than any polynomial, for any $\beta_1, \beta_2 \in \mathbb{N}$ we have

$$\left(\frac{lf^2}{\sqrt{X}\sqrt{1-x^2}}\right)^{\beta_1} \Phi^{(\beta_2)}\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}}\right) \ll_{\Phi, N_1, \beta_1, \beta_2} \left(\frac{\sqrt{X}\sqrt{1-x^2}}{lf^2}\right)^N.$$

Using this bound we then get

$$
\frac{d^M}{dy^M} \left\{ G(y) \, y^{\frac{\alpha}{2}} \Phi\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}} \right) e\left(\frac{-x\xi\sqrt{4Xy}}{4lf^2} \right) \right\}
$$
$$
\ll \left(\frac{\sqrt{X}\sqrt{1-x^2}}{lf^2} \right)^N \left(1 + \left(\frac{\xi\sqrt{X}}{lf^2} \right)^M \right), \tag{36}
$$

where the implied constant depends only on Φ, M, N, and $\|G\|_{M,1}$. Combining (35) with (36) finishes the proof. $\qquad \square$

For the next corollary let us introduce the following notation,

$$
J_{l,f}(\xi, \nu, X) := \iint \sqrt{y} \, G\left(\frac{y}{X} \right) \theta_\infty(x)
$$
$$
\times \left\{ F\left(\frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}} \right) + \frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}} H\left(\frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}} \right) \right\}
$$
$$
\times e\left(\frac{-(x\xi\sqrt{4y} + y\nu)}{4lf^2} \right) dx \, dy. \tag{37}
$$

Proposition 5.2. *Let* $\xi, \alpha \in \mathbb{Z}$, *and* $\nu \in \mathbb{Z}\backslash\{0\}$. *Then, for any* $G(x) \in C_c^\infty([\frac{1}{4}, \frac{5}{4}])$ *and* $M, N \in \mathbb{N}$ *we have*

$$
J_{l,f}(\xi, 0, X) \ll \|G\|_1 \frac{X^{\frac{N+3}{2}}}{(lf^2)^N \xi^{N+2}},
$$
$$
J_{l,f}(\xi, \nu, X) \ll \|G\|_{M,1} \frac{X^{\frac{N-M+3}{2}}}{\nu^M (lf^2)^N} \left[\left(\frac{lf^2}{\sqrt{X}} \right)^M + \xi^M \right],
$$

where the implied constant depends only on θ_∞, F, M, *and* N.

Proof. We first remark that because $G(y)$ is compactly supported away from $y = 0$ we always have $y > 0$.

• $\nu = 0$. By [2, Corollary 4.8], for any $N \geqslant 0$ and $y \neq 0$, we have

$$
\int \theta_\infty(x) \left\{ F\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}} \right) + \frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}} H\left(\frac{lf^2}{\sqrt{4Xy}\sqrt{1-x^2}} \right) \right\}
$$
$$
\times e\left(\frac{-x\xi\sqrt{4Xy}}{4lf^2} \right) dx \ll \left(\frac{\sqrt{Xy}}{lf^2\xi} \right)^N \frac{1}{\xi^2}.
$$

Using the change of variables $y \mapsto Xy$ and using the above bound gives,

$$
J_{l,f}(\xi, 0, X) \ll \frac{X^{\frac{N+3}{2}}}{(lf^2)^N \xi^{N+2}} \int G(y) y^{\frac{N+1}{2}} \, dy \ll \|G\|_1 \frac{X^{\frac{N+3}{2}}}{(lf^2)^N \xi^{N+2}},
$$

where the implied constants depend only on θ_∞, F, and N. This finishes the proof of the case $\nu = 0$.

- $\nu \neq 0$. Let,

$$J_{l,f}(\xi, \nu, X) = J_{l,f}^1(\xi, \nu, X) + J_{l,f}^2(\xi, \nu, X),$$

where

$$J_{l,f}^1(\xi, \nu, X) := \iint G\left(\frac{y}{X}\right) \sqrt{y}\theta_\infty(x)F\left(\frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}}\right) e\left(\frac{-(x\xi\sqrt{4y}+y\nu)}{4lf^2}\right) dx\, dy,$$

$$J_{l,f}^2(\xi, \nu, X) := \frac{lf^2}{2} \iint G\left(\frac{y}{X}\right) \frac{\theta_\infty(x)}{\sqrt{1-x^2}} H\left(\frac{lf^2}{\sqrt{4y}\sqrt{1-x^2}}\right) e\left(\frac{-(x\xi\sqrt{4y}+y\nu)}{4lf^2}\right) dx\, dy,$$

By [2, Lemma 4.5] both F and H are in $\mathcal{S}(\mathbb{R})$. Moreover recall that $\theta_\infty(x)$ and $G(y)$ are both compactly supported. Therefore, the double integrals in $J_{l,f}^1(\xi, \nu, X)$ and $J_{l,f}^2(\xi, \nu, X)$ are both absolutely convergent, and hence we can interchange the order of integration in both. Doing so and using Lemma 5.1 in the y-integrals (take $\alpha = 1$ in $J_{l,f}^1(\xi, \nu, X)$ and $\alpha = 0$ in $J_{l,f}^2(\xi, \nu, X)$) gives, for any $M, N_0, N_1 \in \mathbb{N}$,

$$J_{l,f}^1(\xi, \nu, X) \ll \|G\|_{M,1} \frac{X^{\frac{N_0-M+3}{2}}}{\nu^M (lf^2)^{N_0}} \left[\left(\frac{lf^2}{\sqrt{X}}\right)^M + \xi^M\right] \int \theta_\infty(x)(1-x^2)^{\frac{N_0}{2}}\, dx,$$

$$J_{l,f}^2(\xi, \nu, X) \ll \|G\|_{M,1} \frac{X^{1+\frac{N_1-M}{2}}}{\nu^M (lf^2)^{N_1-1}} \left[\left(\frac{lf^2}{\sqrt{X}}\right)^M + \xi^M\right] \int \theta_\infty(x)(1-x^2)^{\frac{N_1-1}{2}}\, dx.$$

Finally choosing $N_0 = N$ and $N_1 = 1 + N$, which guarantees that the x-integrals converge, finishes the proof. $\qquad\square$

5.2. Non-Archimedean analysis

Let us first introduce some notation that will be used throughout the calculations. Let p be a prime. For any integer $A \in \mathbb{Z}$ let $v_p(A)$ denote the p-adic valuation of A. In what follows, we denote the 'p-part' and the 'prime to p-part' of A respectively by $A_{(p)}$ and $A^{(p)}$. They are defined by,

$$A_{(p)} := q^{v_p(A)} \quad \text{and} \quad A^{(p)} := \frac{A}{A_{(p)}}.$$

For an integer $A \in \mathbb{Z}\backslash\{0\}$, let $\mathrm{rad}(A)$ denote the radical of A. i.e.,

$$\mathrm{rad}(A) = \prod_{\substack{p|A \\ p-\text{prime}}} p.$$

Finally, let us introduce the character sums that will be the focus of this section:

$$\omega_{l,f}(\xi, \nu) := \sum_{b \bmod 4lf^2} Kl_{l,f}(\xi, \nu)e\left(\frac{b\nu}{4lf^2}\right). \tag{38}$$

Lemma 5.3. *Let $k_1, k_2 \in \mathbb{N}$. Then,*

$$\omega_{l,f}(\xi, v) = \prod_p \omega_{p^{v_p(l)}, p^{v_p(f)}} \left(((4lf^2)^{(p)})^{-1}\xi, ((4lf^2)^{(p)})^{-1}v \right).$$

Proof. The proof follows from the Chinese remainder theorem. The details are exactly the same as in [2, Lemma B.1]. $\qquad\square$

Lemma 5.3 reduces the calculation of $\omega_{l,f}(\xi, v)$ to $\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta)$, where $\alpha, \beta \in \mathbb{Z}$.

Lemma 5.4. *Let p be an odd prime, $m \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{Z}$. Then,*

$$\sum_{a \bmod p^m} e\left(\frac{2a\alpha + a^2\beta}{p^m} \right) = \begin{cases} p^m & \beta = 0, \quad v_p(\alpha) \geqslant m \\ 0 & \beta = 0, \quad v_p(\alpha) < m \\ p^{\frac{m+\min\{m, v_p(\delta)\}}{2}} \eta(p^{m-v_p(\delta)}) & \\ \quad \times \left(\frac{p^{-v_p(\delta)}\beta}{p^{m-v_p(\delta)}} \right) e\left(\frac{-\beta(\alpha_0\beta_0^{-1})^2}{p^m} \right) & \beta \neq 0, \end{cases}$$

where

$$\delta = \gcd(\alpha, \beta), \quad \alpha = \delta\alpha_0, \quad \beta = \delta\beta_0,$$

and for any $n \in \mathbb{N}$

$$\bar{\eta}(n) = \frac{1 + i^n}{1 + i}$$

($\bar{\eta}(n)$ denoting the complex conjugate of $\eta(n)$). Finally, we emphasize that if $v_p(\beta) > v_p(\gcd(\alpha, \beta))$ the right hand side is 0 because of the appearance of the Jacobi symbol, $\left(\frac{p^{-v_p(\delta)}\beta}{p^{m-v_p(\delta)}} \right)$.

Proof. First, note that if $\beta = 0$ the sum is a complete character sum over a and is 0 unless $v_p(\alpha) \geqslant m$, in which case it is p^m. Also, the sum is trivially p^m if $v_p(\delta) \geqslant m$, so for the rest of the calculation we assume that $\beta \neq 0$ and $v_p(\delta) < m$.

Let $\delta = \delta_0 p^{v_p(\delta)}$ and let $a = a_0 + a_1 p^{m-v_p(\delta)}$, where a_0 and a_1 are running modulo $p^{m-v_p(\delta)}$ and $p^{v_p(\delta)}$ respectively. Then,

$$\sum_{a \bmod p^m} e\left(\frac{2a\alpha + a^2\beta}{p^m} \right) = p^{v_p(\delta)} \sum_{a_0 \bmod p^{m-v_p(\delta)}} e\left(\frac{\delta_0(2a_0\alpha_0 + a_0^2\beta_0)}{p^{m-v_p(\delta)}} \right). \tag{39}$$

- *Claim*: $v_p(\beta_0) > 0 \Rightarrow (39) = 0$.

  Proof. Suppose $v_p(\beta_0) > 0$ and let $a_0 = a_2 + a_3 p^{m-v_p(\delta)-v_p(\beta_0)}$., where a_2 and a_3 are running modulo $p^{m-v_p(\delta)-v_p(\beta_0)}$ and $p^{v_p(\beta_0)}$ respectively. Then,

$$(39) = p^{v_p(\delta)} \sum_{a_2 \bmod p^{m-v_p(\delta)-v_p(\beta_0)}} e\left(\frac{\delta_0(2a_2\alpha_0 + a_2^2\beta_0)}{p^{m-v_p(d)}} \right) \sum_{a_3 \bmod p^{v_p(\beta_0)}} e\left(\frac{2\delta_0 a_3\alpha_0}{p^{v_p(\beta_0)}} \right).$$

Since $\gcd(\alpha_0, \beta_0) = 1$, $v_p(\beta_0) > 0 \Rightarrow v_p(\alpha_0) = 0$. We also have $v_p(2\delta_0) = 0$ (recall that $p \neq 2$). Therefore, the last sum over a_3 vanishes. $\qquad\square$

Furthermore, again by the claim above we can assume that $v_p(\beta_0) = 0$, otherwise (39) is 0. Then,

$$
\begin{aligned}
(39) &= p^{v_p(\delta)} \sum_{a_0 \bmod p^{m-v_p(\delta)}} e\left(\frac{\delta_0\beta_0(2a_0\alpha_0\beta_0^{-1} + a_0^2)}{p^{m-v_p(\delta)}}\right) \\
&= p^{v_p(\delta)} \sum_{a_0 \bmod p^{m-v_p(\delta)}} e\left(\frac{\delta_0\beta_0((a_0 + \alpha_0\beta_0^{-1})^2 - (\alpha_0\beta_0^{-1})^2)}{p^{m-v_p(\delta)}}\right) \\
&= p^{v_p(\delta)} e\left(\frac{-\beta(\alpha_0\beta_0^{-1})^2}{p^m}\right) \sum_{a_0 \bmod p^{m-v_p(\delta)}} e\left(\frac{\delta_0\beta_0 a_0^2}{p^{m-v_p(\delta)}}\right) \\
&= p^{\frac{m+v_p(\delta)}{2}} \eta(p^{m-v_p(\delta)}) \left(\frac{\delta_0\beta_0}{p^{m-v_p(\delta)}}\right) e\left(\frac{-\beta(\alpha_0/\beta_0)^2}{p^m}\right).
\end{aligned}
$$

Note that in the last line we used the explicit calculation of the Gauss sum (cf. [9, Theorem 3.4]). □

Lemma 5.5. *Let p be an odd prime, $m \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{Z}$. Then,*

$$
\sum_{b \bmod p^m} \left(\frac{b}{p^m}\right) e\left(\frac{b\beta}{p^m}\right) = \begin{cases} \phi(p^m) & v_p(\beta) \geqslant m, \quad m \equiv 0 \bmod 2 \\ -p^{m-1} & v_p(\beta) = m-1, \quad m \equiv 0 \bmod 2 \\ \left(\dfrac{p^{-v_p(\beta)}\beta}{p}\right) \eta(p) p^{m-\frac{1}{2}} & v_p(\beta) = m-1, \quad m \equiv 1 \bmod 2 \\ 0 & otherwise. \end{cases}
$$

Proof. Let $b = b_0 + b_1 p$. Then,

$$
\begin{aligned}
\sum_{b \bmod p^m} \left(\frac{b}{p^m}\right) e\left(\frac{b\beta}{p^m}\right) &= \sum_{\substack{b_0 \bmod p \\ b_1 \bmod p^{m-1}}} \left(\frac{b_0}{p^m}\right) e\left(\frac{b_0\beta}{p^m}\right) e\left(\frac{b_1\beta}{p^{m-1}}\right) \\
&= \sum_{b_0 \bmod p} \left(\frac{b_0}{p^m}\right) e\left(\frac{b_0\beta}{p^m}\right) \begin{cases} p^{m-1} & v_p(\beta) \geqslant m-1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

A case by case calculation of the b_0-sum finishes the proof. □

Proposition 5.6. *Let p be an odd prime, $k_1, k_2 \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{Z}$. Then,*

$$
\omega_{p^{k_1}, p^{k_2}}(\alpha, 0) = \begin{cases} p^{k_1 + 2k_2} \phi(p^{k_1}) & v_p(\alpha) \geqslant k_1 + 2k_2, \quad k_1 \equiv 0 \bmod 2 \\ 0 & otherwise, \end{cases}
$$

and for $\beta \neq 0$ we have,

$$\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta) = \tau_p(k_1, k_2, \alpha, \beta) \begin{cases} \phi(p^{k_1}) & v_p(\beta) \geqslant k_1, \quad k_1 \equiv 0 \bmod 2 \\ -p^{k_1 - 1} & v_p(\beta) = k_1 - 1, \quad k_1 \equiv 0 \bmod 2 \\ \left(\dfrac{-p^{-v_p(\beta)}\beta}{p} \right) \\ \quad \times \eta(p) p^{k_1 - \frac{1}{2}} & v_p(\beta) = k_1 - 1, \quad k_1 \equiv 1 \bmod 2 \\ 0 & \text{otherwise,} \end{cases}$$

where α_0, β_0, and δ are as in Lemma 5.4, and

$$\tau_p(k_1, k_2, \alpha, \beta) := {}^{\frac{k_1 + 2k_2 + \min\{v_p(\beta), k_1 + 2k_2\}}{2}} \eta(p^{k_1 + 2k_2 - v_p(\beta)}) \left(\frac{p^{-v_p(\beta)}\beta}{p^{k_1 + 2k_2 - v_p(\beta)}} \right) e\left(\frac{-\beta(\alpha_0 \beta_0^{-1})^2}{p^{k_1 + 2k_2}} \right)$$

if $v_p(\alpha) \geqslant \min\{v_p(\beta), k_1 + 2k_2\}$, and is 0 otherwise.

Proof. First note that since p is odd, the substitution $a \mapsto 2a$ does not change the sum and gives

$$\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta) = \sum_{\substack{a, b \bmod p^{k_1 + 2k_2} \\ a^2 \equiv b \bmod p^{2k_2}}} \left(\frac{(a^2 - b)/p^{2k_2}}{p^{k_1}} \right) e\left(\frac{2a\alpha + b\beta}{p^{k_1 + 2k_2}} \right).$$

For each $a \bmod p^{k_1 + 2k_2}$ the elements $b \bmod p^{k_1 + 2k_2}$ satisfying $a^2 \equiv 4b \bmod p^{2k_2}$ can be parametrized by $b = a^2 + b_0 p^{2k_2}$, where b_0 is running $\bmod p^{k_1}$. Note that with this parametrization $a^2 - b \equiv -b_0 p^{2k_2} \bmod p^{k_1 + 2k_2}$. Using this observation we get,

$$\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta) = \sum_{\substack{a \bmod p^{k_1 + 2k_2} \\ b_0 \bmod p^{k_1}}} \left(\frac{-b_0}{p^{k_1}} \right) e\left(\frac{b_0 \beta}{p^{k_1}} \right) e\left(\frac{2a\alpha + a^2\beta}{p^{k_1 + 2k_2}} \right).$$

The proposition now follows from Lemmas 5.4 and 5.5 (One just needs to keep in mind that in Lemma 5.4 the sum vanishes unless $v_p(\alpha) \geqslant \min\{m, v_p(\beta)\}$.). $\qquad\square$

Corollary 5.7. *Let p be an odd prime, $k_1, k_1 \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{Z}$. Then,*

$$\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta) \ll \begin{cases} p^{k_1 + k_2} \sqrt{\gcd(p^{k_1 + 2k_2}, \beta) \gcd(p^{k_1}, \beta)} & p^{k_1 - 1} \mid \beta, \quad \gcd(p^{k_1 + 2k_2}, \beta) \mid \alpha \\ 0 & \text{otherwise,} \end{cases}$$

where the implied constant is absolute.

Proof. First assume $\beta \neq 0$. Then, by the second statement of Proposition 5.6 we know that $\omega_{p^{k_1}, p^{k_2}}(\alpha, \beta)$ is the product of two terms (the first is $\tau_p(k_1, k_2, \alpha, \beta)$, and the second is in the braces). The second one of those terms is 0 unless $p^{k_1 - 1} \mid \beta$, in which case it

is $O(p^{\frac{k_1}{2}} \sqrt{\gcd(\beta, p^{k_1})})$, where the implied constant is absolute. On the other hand, the first term satisfies the following bound:

$$\tau_p(k_1, k_2, \alpha, \beta) \ll \begin{cases} p^{\frac{k_1+2k_2}{2}} \sqrt{\gcd(p^{k_1+2k_2}, \beta)} & v_p(\alpha) \geqslant \min\{v_p(\beta), k_1 + 2k_2\} \\ 0 & \text{otherwise.} \end{cases}$$

These two bounds imply the corollary in the case $\beta \neq 0$. For $\beta = 0$, the bound in the statement is $O(p^{2k_1+2k_2})$ if $v_p(\alpha) \geqslant k_1 + 2k_2$ and 0 otherwise. A comparison of this bound with the first statement of Proposition 5.6 finishes the proof. $\qquad\square$

Lemma 5.8. *Let $k_1, k_1 \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}$. Then,*

$$\omega_{2^{k_1}, 2^{k_2}}(\alpha, \beta) \ll \begin{cases} 2^{k_1+k_2} \sqrt{\gcd(2^{k_1+2k_2}, \beta) \gcd(2^{k_1}, \beta)} & 2^{k_1-1} | \beta, \quad \gcd(2^{k_1+2k_2}, \beta) \mid \alpha \\ 0 & otherwise, \end{cases}$$

where the implied constant is absolute.

Proof. The proof of this case follow the proofs of Proposition 5.6 and Corollary 5.7 verbatim. One just needs to take into account the extra condition $\frac{a^2-4b}{p^{2k_2}} \equiv 0, 1 \bmod 4$ and recall that the Kronecker symbol $\left(\frac{\cdot}{2}\right)$ is periodic $\bmod 8$. These do not bring any new ingredients to the proof but rather a delicate case by case analysis whose details we leave to the reader. $\qquad\square$

Corollary 5.9. *Let $l, f \in \mathbb{Z}_{>0}$ and $\xi, v \in \mathbb{Z}$. Then,*

$$\omega_{l,f}(\xi, v) \ll \begin{cases} \log(lf) lf \sqrt{\gcd(lf^2, v) \gcd(l, v)} & \dfrac{l}{\text{rad}(l)} \mid v, \quad \gcd(lf^2, v) \mid \xi \\ 0 & otherwise, \end{cases}$$

where the implied constant is absolute.

Proof. By Lemma 5.3 it is enough to bound $\omega_{p^{k_1}, p^{k_2}}(\alpha_p, \beta_p)$, where $\alpha_p = ((4lf^2)^{(p)})^{-1}\xi$ and $\beta_p = (4lf^2)^{(p)})^{-1}v$, and to bound $\prod_{p|lf^2} O(1)$. The bound on the character sums follow from Corollary 5.7 and Lemma 5.8. Finally, by the prime number theorem we have $\prod_{p|lf^2} O(1) = O(\log(lf^2)) = O(lf)$. The corollary follows. $\qquad\square$

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