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CRYSTALS AND DE RHAM–WITT CONNECTIONS

SPENCER BLOCH

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA (bloch@math.uchicago.edu)

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Abstract For a smooth variety X over a perfect field k in characteristic p, an equivalence of category is established between the category of crystals on $X/W(k)$ and the category of p-adically nilpotent, integrable de Rham–Witt connections.

Keywords: crystal; de Rham–Witt cohomology

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Let me make one thing crystal clear...

R. Nixon

1. Introduction

Let X be a smooth variety over a perfect field k of characteristic $p > 0$. Let $W(k)$ denote the ring of p-Witt vectors over k. It is a complete discrete valuation ring with $W(k)/pW(k) = k$. Recall (see [**I1**, **I2**]) that one has defined a pro-sheaf of $W(k)$-algebras

$$W_X = \{W_{n,X}\}_{n=1,2,\dots}, \tag{1.1}$$

where $W_{n,X}$ is just the sheafified functor of p-Witt vectors of length n, applied to $\mathcal{O}_X$. One has a surjection of algebras $W_n \twoheadrightarrow W_1$, $(r_1, r_2, \dots) \mapsto r_1$, and the formula

$$(0, r_2, \dots) \cdot (0, s_1, \dots) = p(0, r_1 s_1, \dots)$$

gives a canonical divided power structure on the kernel. It then follows that, for $U = \mathrm{Spec}(R) \subset X$ an affine open, the embedding

$$U \hookrightarrow \mathrm{Spec}(\Gamma(U, W_{n,X}))$$

with the above divided powers represents an object in the crystalline site of X/W. By definition, a *crystal* $\mathbb{E}$ on X/W is a sheaf of $\mathcal{O}_{X,\mathrm{crys}}$-modules on the crystalline site, where $\mathcal{O}_{X,\mathrm{crys}}(U \hookrightarrow V) := \Gamma(V, \mathcal{O}_V)$. The rule which associates to $\mathbb{E}$ the Zariski sheaf

$$E := \{U \mapsto \mathbb{E}(U \hookrightarrow \mathrm{Spec}(\Gamma(U, W_{n,X})))_{n=1,2,\dots}\} \tag{1.2}$$

defines a functor from crystals to W_X-modules.

In fact, E has more structure. Quite generally, if $U \hookrightarrow V$ is an object in the crystalline site and if $V^{(2)} \subset V \times_{\operatorname{Spec} W(k)} V$ is the first infinitesimal neighbourhood of the diagonal, we have a diagram

$$\begin{array}{ccc} U & \longrightarrow & V^{(2)} \\ \Big\| & & \mathrm{pr}_1 \Big\downarrow \ \Big\downarrow \mathrm{pr}_2 \\ U & \longrightarrow & V \end{array}$$

From the crystalline property, we deduce an isomorphism,

$$\mathrm{pr}_1^* \, \mathbb{E}(U \hookrightarrow V) \cong \mathrm{pr}_2^* \, \mathbb{E}(U \hookrightarrow V), \tag{1.3}$$

i.e. a connection on the $\mathcal{O}_V$-module. This connection is integrable (see [**Be**]).

Recall that the de Rham–Witt complex is a complex of pro-sheaves for the Zariski topology on X,

$$\{W_{n,X} \xrightarrow{\mathrm{d}} W_n\Omega^1_X \to \cdots \to W_n\Omega^m_X\}_{n=1,2,\dots}, \tag{1.4}$$

where $m = \dim X$. At level n, the complex is a quotient of the de Rham complex of the sheaf of rings $W_{n,X}$. In particular, the integrable connection on the W-module E deduced from (1.3) permits one to couple E to $W\Omega^\bullet$,

$$E \xrightarrow{\nabla} E \otimes_W W\Omega^1_X \xrightarrow{\nabla} E \otimes_W W\Omega^2_X \xrightarrow{\nabla} \cdots. \tag{1.5}$$

We refer to ∇ as a *de Rham–Witt connection* on E. More generally, for a W-module E, a de Rham–Witt connection on E will be a map $\nabla : E \to E \otimes_W W\Omega^1_X$ satisfying the Leibniz rule $\nabla(we) = w\nabla(e) + e \otimes \mathrm{d}w$. The connection is *integrable* if $\nabla^2 = 0$ as in (1.5). Connections arising from crystals are integrable.

The functor $\mathbb{E} \mapsto (E, \nabla)$ was studied in [**E**]. Let $u_n : (X/W_n)_{\mathrm{crys}} \to X_{\mathrm{zar}}$ be the natural map. Assume the crystal $\mathbb{E}$ is locally free as an $\mathcal{O}_{X,\mathrm{crys}}$-module. (This is the only case to be considered in the present note.) The main result in [**E**, Theorem 2.1] is an isomorphism in the derived category of Zariski sheaves on X,

$$\mathbb{R}u_{n,*}(\mathbb{E}) \cong \{E \otimes W_n \to E \otimes W_n\Omega^1_X \to \cdots\}. \tag{1.6}$$

Taking $\mathbb{H}^0$, it follows easily that the functor $\mathbb{E} \mapsto (E, \nabla)$ is fully faithful. Indeed, for locally free crystals $\mathbb{E}_i$, $i = 1, 2$,

$$\mathbb{R}u_{n,*} \, \underline{\mathrm{Hom}}(\mathbb{E}_1, \mathbb{E}_2) \cong \{E_1^\vee \otimes E_2 \otimes W_n \to E_1^\vee \otimes E_2 \otimes W_n\Omega^1 \to \cdots\},$$

so

$$\begin{aligned} \mathrm{Hom}_{\mathrm{crys}}(\mathbb{E}_1, \mathbb{E}_2) &\cong \varprojlim_n \mathbb{H}^0 \mathbb{R}u_{n,*}(\mathbb{E}_1^\vee \otimes \mathbb{E}_2) \\ &\cong \varprojlim_n \ker(H^0(X, E_1^\vee \otimes E_2 \otimes W_n) \to H^0(X, E_1^\vee \otimes E_2 \otimes W_n\Omega^1)) \\ &\cong \varprojlim_n \mathrm{Hom}_\nabla(E_1 \otimes W_n, E_2 \otimes W_n) \\ &\cong \mathrm{Hom}_\nabla(E_1, E_2). \end{aligned}$$

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The main result in the following is that this functor is an equivalence of category. More precisely, (E, ∇) is said to be quasi-nilpotent if the connection (note that $W_1\Omega^\bullet_X = \Omega^\bullet_X$ is the de Rham complex over X) $E \otimes \mathcal{O}_X \to E \otimes \Omega^1_X$ is quasi-nilpotent in the sense of [**Be**, Definitions 4.10 and 4.14].

Theorem 1.1. *The functor $\mathbb{E} \mapsto (E, \nabla)$ defines an equivalence of category between the category of locally free crystals on X and the category of locally free W_X-modules E with a quasi-nilpotent, integrable connection ∇.*

The Witt vector sheaf W_X has a canonical Frobenius endomorphism σ given by raising coordinates to the pth power. The differential $\mathrm{d}\sigma : W\Omega^1_X \to W\Omega^1_X$ is defined. In fact, $\mathrm{d}\sigma = pF$, where $F : W\Omega^1_X \to W\Omega^1_X$ is the Frobenius endomorphism [**I2**]. Let (E, ∇) be as above. A Frobenius structure on (E, ∇) is, by definition, a map $\Phi : \sigma^*E \to E$ that is horizontal for the connection. When $\mathbb{E}$ is an F-crystal [**K**], then (E, ∇) has a Frobenius structure. As a consequence of Theorem 1.1, one gets the following result.

Corollary 1.2. *The functor $\mathbb{E} \mapsto (E, \nabla)$ defines an equivalence of category between the category of locally free F-crystals on X and the category of locally free W_X-modules E with a quasi-nilpotent, integrable connection ∇ and a Frobenius structure $\Phi : \sigma^*E \to E$.*

The de Rham–Witt complex is globally defined and functorial in X. Heretofore, much of the work on crystals has involved choosing local liftings of X and Frobenius, and then studying connections on the local lifting. My hope is that the possibility of doing this canonically will simplify and clarify the picture.

It seems natural to ask whether one has some sort of $\mathcal{D}$-module interpretation of de Rham–Witt connections. The naive idea of defining derivations to be $\mathrm{Hom}_W(W\Omega^1, W)$ does not seem to work, but perhaps there is something more clever.

2. The structure of $W\Omega^\bullet$ over a polynomial ring

Much of the material in this section is taken from [**I2**]. Let $A = k[T_1, \dots, T_n]$ be a polynomial ring where k is a perfect field as above. Let K be the quotient field of $W(k)$, and let

$$C := \bigcup_{r \geqslant 0} K[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}].$$

An element $x \in \Omega^m_{C/K}$ can be written uniquely in the form

$$x = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} a_{i_1,\dots,i_m}(T)\, \mathrm{d}\log(T_{i_1}) \wedge \cdots \wedge \mathrm{d}\log(T_{i_m}), \tag{2.1}$$

where $a_I \in C$ is divisible by $(T_{i_1} \dots T_{i_m})^{p^{-r}}$ for some $r \geqslant 0$. We say that such an x is *integral* if the $a_I(T)$ have coefficients in $W(k) \subset K$. Define a subcomplex $E^\bullet \subset \Omega^\bullet_C$ by

$$E^i = \{x \in \Omega^i_C \mid x,\ \mathrm{d}x \text{ integral}\}. \tag{2.2}$$

For example, since $\mathrm{d}T^{p^{-r}} = p^{-r}T^{p^{-r}}\, \mathrm{d}\log(T)$, it follows that

$$E^0 = \bigcup_{r \geqslant 0} p^r W[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}]. \tag{2.3}$$

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The subcomplex $E^\bullet$ is compatible with the natural grading on $\Omega^\bullet_C$, where the $I = (i_1, \dots, i_n)$-graded piece is spanned as a K-vector space by

$$T_1^{i_1} \cdot T_1^{i_2} \cdot \dots \cdot T_n^{i_n} \, \mathrm{d}\log(T_{j_1}) \wedge \dots \wedge \mathrm{d}\log(T_{j_r})$$

for varying $\{j_1, \dots, j_r\}$. We write $E^\bullet(I)$ for the corresponding grading.

The main point [**I1**, § 2, p. 550] is that $W\Omega^\bullet_A$ is a suitable completion of $E^\bullet$. Indeed, define an operator F on $\Omega^\bullet_C$ from the ring automorphism $F^* : C \cong C$, $F^*(T) = T^p$, $F^*|_{W(k)} =$ Frobenius via the formula

$$F\Big(\sum a_I(T) \, \mathrm{d}\log(T_{j_1}) \wedge \dots \wedge \mathrm{d}\log(T_{j_r})\Big) = \sum F^*(a_I) \, \mathrm{d}\log(T_{j_1}) \wedge \dots \wedge \mathrm{d}\log(T_{j_r}).$$

Then F stabilizes $E^\bullet$, as does $V := pF^{-1}$. Define $\mathrm{fil}^r E^i := V^r E^i + \mathrm{d}(V^r E^{i-1})$. One shows that $W_r \Omega^i_A = E^i / \mathrm{fil}^r E^i$.

It will be convenient to write

$$\mathcal{E}^i := E^i / pE^i, \tag{2.4}$$

so, for example, $\mathcal{E}^0$ is an augmented A-algebra,

$$\mathcal{E}^0 = A \oplus \mathcal{J}. \tag{2.5}$$

Here, $\mathcal{J} \subset \mathcal{E}^0$ is generated over k by the classes $p^s T^I$, where $I = (i_1, \dots, i_n)$, with $i_j \in \mathbb{Z}[1/p]$, $i_j \geqslant 0$, $s := \max_j\{-\mathrm{ord}_p(i_j)\} > 0$.

Lemma 2.1. *There is a split-exact sequence*

$$0 \to \mathcal{J} \xrightarrow{\mathrm{d}} \mathcal{E}^1 / \mathcal{J}\mathcal{E}^1 \to \Omega^1_A \to 0.$$

Proof. Here, d is induced from the differential $\mathcal{E}^0 \to \mathcal{E}^1$. The definition of the maps and the splitting are clear. It remains to show exactness.

One has a grading $\mathcal{E}^\bullet = \oplus \mathcal{E}^\bullet(I)$ as above.

We claim that

$$\mathcal{E}^1 = \Omega^1_A + \mathcal{J}\mathcal{E}^1 + \mathrm{d}\mathcal{J}. \tag{2.6}$$

To see this, let $T^I \sum_j b_j \, \mathrm{d}\log(T_j) \in E^1(I)$ represent an element in $\mathcal{E}^1(I)$. In other words, writing $I = (i_1, \dots, i_n)$, we have $b_j, i_r b_j - i_j b_r \in W(k)$. Let $s = s(I) := \max(-\mathrm{ord}_p(i_j)) > 0$ as above, and write $\iota = p^s(i_1, \dots, i_n)$. Let $f := \min(\mathrm{ord}_p(b_j))$, and write $\beta = p^{-f}(b_1, \dots, b_n)$. Write $\bar{\iota}$ and $\bar{\beta}$ for the images of these vectors mod p in k^n. By construction, these vectors are non-zero. We have

$$\bar{\beta} \wedge \bar{\iota} = p^{s-f}(\dots, b_j i_r - b_r i_j, \dots). \tag{2.7}$$

If $f < s$, we conclude that $\beta \equiv c\iota \bmod p$ for some $c \in W(k)$. This implies that

$$T^I \sum_j b_j \, \mathrm{d}\log(T_j) \equiv p^f c \cdot \mathrm{d}(p^s T^I) \mod p^{f+1}. \tag{2.8}$$

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(Note that the right-hand side lies in $\mathrm{d}\mathcal{J}$.) Continuing in this way, we may assume that $p^s|b_j$ for all j, which implies that

$$T^I \sum_j b_j \,\mathrm{d}\log(T_j) = \sum_j p^s T^{I-\{i_j\}}(p^{-s} b_j T_j^{i_j} \,\mathrm{d}\log(T_j)) \in \mathcal{J}\mathcal{E}^1. \tag{2.9}$$

The claim (2.6) follows. In particular, we see that $\Omega^1_{\mathcal{E}^0} \twoheadrightarrow \mathcal{E}^1$, from which it follows that $\Omega^1_A \oplus \mathcal{J} \twoheadrightarrow \mathcal{E}^1/\mathcal{J}\mathcal{E}^1$.

It remains to show that the map d in the exact sequence is injective. This can be done one graded piece at a time, so it suffices to show $\mathrm{d}(p^s T^I) \not\equiv 0 \bmod \mathcal{J}\mathcal{E}^1$. By the assumption on s, $\mathrm{d}(p^s T^I) = T^I \sum_j p^s i_j \,\mathrm{d}\log(T_j)$ and $p^s i_j$ is a unit for some j. On the other hand, elements in the I-graded piece of $\mathcal{J}\mathcal{E}^1$ are of the form $T^I \sum c_j \,\mathrm{d}\log(T_j)$, where all the $c_j \equiv 0 \bmod p$ (cf. (2.9)). It follows that $\mathrm{d} : \mathcal{J} \hookrightarrow \mathcal{E}^1/\mathcal{J}\mathcal{E}^1$, proving the lemma. □

In what follows we shall need detailed information about the de Rham–Witt complex. For a careful exposition of the de Rham–Witt complex, see [**I1**, §§ 1, 2].

Corollary 2.2. *Let $\mathcal{W}(A) = W(A)/pW(A)$ and let*

$$\mathcal{I}_A := \ker(\mathcal{W}(A) \to A), \qquad \mathcal{W}\Omega^1_A = W\Omega^1_A/pW\Omega^1_A.$$

Then there is a split-exact sequence

$$0 \to \mathcal{I}_A \xrightarrow{\mathrm{d}} \mathcal{W}\Omega^1_A/\mathcal{I}_A\mathcal{W}\Omega^1_A \to \Omega^1_A \to 0. \tag{2.10}$$

Proof. Using the lemma and the identity

$$W_r\Omega^i_A = E^i/(V^n E^i + \mathrm{d}(V^n E^{i-1})),$$

one reduces to showing

$$(V^n\mathcal{E}^1 + \mathcal{J}\mathcal{E}^1) \cap \mathrm{d}\mathcal{J} = (0).$$

A non-zero element in $\mathrm{d}\mathcal{J}$ of graded degree I is represented by an element

$$T^I \sum_j a_j \,\mathrm{d}\log(T_j) \in E^1,$$

where one at least of the $a_j \in W(k)$ is a unit. On the other hand, elements in $V^n\mathcal{E}^1 + \mathcal{J}\mathcal{E}^1$ are represented by elements where all the a_j are divisible by p. (For example, $V(T^I \sum_j b_j \,\mathrm{d}\log(T_j)) = pT^{I/p} \sum b_j \,\mathrm{d}\log(T_j)$.) The corollary follows from this. □

Lemma 2.3. *The composition*

$$\mathcal{J}\mathcal{E}^1 \xrightarrow{\mathrm{d}} \mathcal{E}^2 \to \mathcal{E}^2/(\Omega^2_A + \mathcal{J}\mathcal{E}^2 + V\mathcal{E}^2)$$

is injective.

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Proof. We can work with I-graded pieces for I non-integral and ignore Ω^2_A. The map sends

$$\sum \tau_\mu w^\mu \mapsto \sum \mathrm{d}\tau_\mu \wedge w^\mu \quad \mathrm{mod}\, \mathcal{J}\mathcal{E}^2, \quad \tau_\mu \in \mathcal{J}, \quad w^\mu \in \mathcal{E}^1. \tag{2.11}$$

If we lift elements of $\mathcal{J}\mathcal{E}^2 + V\mathcal{E}^2$ to E^2 and expand as

$$T^I \sum_{q,r} c_{q,r}\, \mathrm{d}\log(T_q) \wedge \mathrm{d}\log(T_r),$$

we find all coefficients $c_{q,r} \equiv 0 \bmod p$. (Indeed, J is generated by elements $p^s T^I$ for $s = s(I) > 0$, and $V = pF^{*-1}$.) On the other hand, if we lift $\sum \tau_\mu w^\mu$ to an element $\omega \in E^1$ and expand $\mathrm{d}\omega = T^I \sum_{q,r} C_{q,r}\, \mathrm{d}\log(T_q) \wedge \mathrm{d}\log(T_r)$, then this class is trivial in $\mathcal{E}^2$ if and only if the $C_{q,r}$ are divisible by p. Indeed, the condition for a form to lie in $E^\bullet$ is that both it and its differential have integral coefficients. Since $\mathrm{d}\omega$ is closed and since the $E^\bullet$ have no p-torsion, the assertion follows. But notice the expansion of $\omega \in E^1$ also has all coefficients divisible by p because the τ_μ do, so if the coefficients of $\mathrm{d}\omega$ are divisible by p, then ω/p is integral, so $\omega \equiv 0 \bmod p$. □

3. The structure of $W\Omega^\bullet_R$ for general R

Let $\mathcal{R}$ be a commutative ring in which every rational prime $\ell \neq p$ is invertible. We write $W(\mathcal{R})$ for the ring of p-Witt vectors with coefficients in $\mathcal{R}$. Recall (see [**I1**]) that an element $a \in W(\mathcal{R})$ has Witt coordinates $a = (a_0, a_1, \dots)$. The *ghost* coordinates

$$w_n(a) := \sum_{j=0}^{n} p^j a_j^{p^{n-j}}, \quad n = 0, 1, \dots,$$

define ring homomorphisms $W(\mathcal{R}) \to \mathcal{R}$. The Frobenius $F : W(\mathcal{R}) \to W(\mathcal{R})$ (respectively, $F_n : W_n(\mathcal{R}) \to W_{n-1}(\mathcal{R})$) is a ring homomorphism satisfying $w_n \circ F = w_{n+1}$. When $p\mathcal{R} = 0$, $F(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$.

Lemma 3.1. *Let S be a $W(k)$-algebra and assume S has no p-torsion. Let $f : S \to S$ be a ring homomorphism lifting the pth power Frobenius map on S/pS. Write $W(S) = \varprojlim W_n(S)$ for the ring of p-Witt vectors on S. Then there exists a unique ring homomorphism $\rho = \rho_f : S \to W(S)$ such that $\rho \circ f = F \circ \rho$. One has $w_n \circ \rho = f^n$, $n = 0, 1, \dots$.*

Proof. The referee suggests this lemma is due to Cartier. For a proof, see [**L**, VII, § 4]. □

With notation as above, let $R = S/pS$. We will be more interested in the composed map, which we also denote by $\rho = \rho_f$,

$$\rho = \rho_f : S \to W(S) \to W(R). \tag{3.1}$$

(This composed map still depends on the choice of f lifting Frobenius.) Write

$$\mathcal{W}(R) = \varprojlim \mathcal{W}_n(R) = W(R)/pW(R), \qquad \mathcal{W}\Omega^r_R = W\Omega^r_R / pW\Omega^r_R. \tag{3.2}$$

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Let $\mathcal{I} \subset \mathcal{W}$ be the kernel of the projection $\mathcal{W} \to R$. Let

$$\sigma := \rho \mod p : R \to \mathcal{W}. \tag{3.3}$$

We have a split-exact sequence

$$0 \to \mathcal{I} \to \mathcal{W} \overset{\overset{\sigma}{\longleftarrow}}{\longrightarrow} R \to 0. \tag{3.4}$$

Note that $\mathcal{I}^2 = (0)$, because $\mathcal{I} = V\mathcal{W}$ and $V(x)V(y) = pV(xy)$. In particular, $\mathcal{I}$ has a natural R-module structure, which is independent of the choice of σ. (Recall that σ depends on the choice of f.) Let

$$\mathrm{d} : \mathcal{I} \to \mathcal{W}\Omega^1_R \tag{3.5}$$

be induced by the de Rham–Witt differential.

Lemma 3.2. *With notation as above, assume R is smooth over k. There is a split-exact sequence of R-modules,*

$$0 \to I \xrightarrow{\mathrm{d}} \mathcal{W}\Omega^1_R/\mathcal{I}\mathcal{W}\Omega^1_R \overset{\overset{\sigma}{\longleftarrow}}{\longrightarrow} \Omega^1_R \to 0.$$

Proof. The maps are all defined. To check exactness, it suffices to work locally on $\mathrm{Spec}(R)$, so we may assume that R is an étale algebra over a polynomial ring $A := k[T_1, \ldots, T_n]$. One knows (see [**I1**, Proposition 1.14, p. 549]) that, in this case, $W_m\Omega^i_R \cong W_m\Omega^i_A \otimes_{W_m(A)} W_m(R)$, which implies

$$W_m\Omega^i_R/\mathcal{I}_R W_m\Omega^i_R \cong W_m\Omega^i_A \otimes_{W_m(A)} R \cong W_m\Omega^i_A/\mathcal{I}_A W_m\Omega^i_A \otimes R. \tag{3.6}$$

Let $I_{m,R} = \ker(W_m(R) \to R)$. Since $R = R^pA$ and $V(r^pa) = rV(a)$, we see that

$$I_{m,R} = I_{m,A} \otimes_A R. \tag{3.7}$$

Also, of course, $\Omega^1_R = \Omega^1_A \otimes_A R$. Combining these identities, exactness in Lemma 3.2 reduces to the case $R = A = k[T_1, \ldots, T_n]$. This is Corollary 2.2. □

4. Descent modulo p

Let R be a smooth k-algebra as above, and let (M, ∇) be an integrable de Rham–Witt connection. We assume $M \cong W(R)^r$ as a $W(R)$-module. (For the complicated calculations that follow, it is convenient to work with $W(R)$-modules rather than pro-objects of $W_n(R)$-modules. Because our W-modules are assumed to be locally free in Theorem 1.1, this is legitimate. I do not know what is true more generally.)

Write $\mathcal{M} = M/pM$ and let $\nabla = \nabla_{\mathcal{M}}$ denote the corresponding connection on $\mathcal{M}$ as well. Define

$$(N_0, \Xi_0) := (M, \nabla) \otimes_{W(R)} R = (\mathcal{M}, \nabla) \otimes_{\mathcal{W}} R, \tag{4.1}$$

where $\mathcal{W} = W(R)/pW(R)$.

Assume we are given a flat, p-adically complete $W(k)$-algebra S, an isomorphism of k-algebras $R \cong S/pS$, and a lifting $f : S \to S$ of Frobenius. Let $\sigma : R \to \mathcal{W}$ be the

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corresponding map (3.3). Corresponding to σ, we have the direct sum decomposition from Lemma 3.2,

$$\mathcal{W}\Omega^1_R/\mathcal{I}\mathcal{W}\Omega^1_R \cong \Omega^1_R \oplus \mathcal{I}. \tag{4.2}$$

Define a map $\nabla_{\mathcal{I}} : \mathcal{M} \to N_0 \otimes_R \mathcal{I}$ to be the composition

$$\begin{aligned}\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{W}} \mathcal{W}\Omega^1_R \to \mathcal{M} \otimes_{\mathcal{W}} (\mathcal{W}\Omega^1_R/\mathcal{I}\mathcal{W}\Omega^1_R) \\ \cong \mathcal{M} \otimes_{\mathcal{W}} R \otimes_R (\Omega^1_R \oplus \mathcal{I}) \xrightarrow{\text{proj}} N_0 \otimes_R \mathcal{I}.\end{aligned} \tag{4.3}$$

Let $\pi : \mathcal{M} \to N_0 = \mathcal{M}/\mathcal{I}\mathcal{M}$ be the reduction map, and let

$$\theta = (\pi, \nabla_{\mathcal{I}}) : \mathcal{M} \to N_0 \oplus (N_0 \otimes_R \mathcal{I}) \cong N_0 \otimes_{R,\sigma} \mathcal{W}. \tag{4.4}$$

Proposition 4.1. *The map θ is a horizontal isomorphism of connections*

$$(\mathcal{M}, \nabla_{\mathcal{M}}) \cong (N_0, \Xi_0) \otimes_{R,\sigma} \mathcal{W}.$$

Proof. Note first that θ is a homomorphism of $\mathcal{W}$-modules. In other words, given $x \in R$, $m \in \mathcal{M}$ and $\iota \in \mathcal{I}$, we have $\theta(\sigma(x)m) = x\theta(m)$ and $\theta(\iota m) = \pi(m) \otimes \iota$. In particular, θ is the identity $\bmod \mathcal{I}$, so it is an isomorphism.

We must show θ is compatible with connections, i.e. the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\theta} & N_0 \otimes_{(R,\sigma)} \mathcal{W} \\ \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle \Xi_0 \otimes 1 + 1 \otimes \mathrm{d}} \\ \mathcal{M} \otimes_{\mathcal{W}} \mathcal{W}\Omega^1_R & \xrightarrow{\theta \otimes 1} & N_0 \otimes_{(R,\sigma)} \mathcal{W}\Omega^1_R \end{array} \tag{4.5}$$

commutes.

Identify $N_0 = \ker(\nabla_{\mathcal{I}}) \subset \mathcal{M}$. Let $\{m_i\}$ be an R-basis for the free module N_0. By assumption,

$$\nabla(m_\ell) = \sum_i m_i \otimes \eta^i_\ell + \sum_{i,\mu} m_i \otimes \tau_\mu w^{i,\mu}_\ell, \tag{4.6}$$

where $\eta^i_\ell \in \Omega^1_R \overset{\sigma}{\hookrightarrow} \mathcal{W}\Omega^1_R$, τ_μ runs through an R-basis of $\mathcal{I}$, and $w^{i,\mu}_\ell \in \mathcal{W}\Omega^1_R$. It follows from integrability that

$$0 = \nabla^2(m_\ell) \equiv \sum_{i,\mu} m_i \otimes \mathrm{d}\tau_\mu \wedge w^{i,\mu}_\ell \quad \bmod N_0 \otimes (\Omega^2_R + \mathcal{I}\mathcal{W}\Omega^2_R). \tag{4.7}$$

I claim that the composition map

$$\mathcal{I}\mathcal{W}\Omega^1_R \xrightarrow{\mathrm{d}} \mathcal{W}\Omega^2_R \to \mathcal{W}\Omega^2_R/(\Omega^2_R + \mathcal{I}\mathcal{W}\Omega^2_R) \tag{4.8}$$

is injective. Granting this, it follows from (4.6) that

$$\nabla(m_\ell) = \sum_i m_i \otimes \eta^i_\ell \in N_0 \otimes_R \Omega^1_R, \tag{4.9}$$

which shows that the diagram (4.5) commutes on the m_ℓ. Since these form a $\mathcal{W}$-basis for $\mathcal{M}$, the diagram commutes, proving the proposition.

To see that (4.8) is injective, one can localize R and assume R is étale over a polynomial ring A. The assertion then follows by tensoring with R from the corresponding assertion for A. This is Lemma 2.3. □

5. Descent modulo p^N; proof of Theorem 1.1

We keep the same notations as above. In particular, we view $\Omega^\bullet_R \subset \mathcal{W}\Omega^\bullet_R$ via the map $\sigma : R \hookrightarrow \mathcal{W}$. Define $\mathcal{C}^\bullet = \mathcal{W}\Omega^\bullet_R/\Omega^\bullet_R$. We define a decreasing filtrations $P^*\mathcal{W}\Omega^\bullet_R$ and $P^j\mathcal{C}^\bullet$ by

$$P^j\mathcal{W}\Omega^i_R = \mathrm{Image}(\mathcal{W}\Omega^{i-j}_R \wedge \Omega^j_R) \subset \mathcal{W}\Omega^i_R, \qquad P^j\mathcal{C}^i = P^j\mathcal{W}\Omega^i_R/\Omega^i_R. \tag{5.1}$$

Lemma 5.1. *We have $H^i(gr^j_P\mathcal{C}^\bullet) = (0)$ for $i = 0, 1$ and all j.*

Proof. Again, as above, we reduce to the case where $R = A = k[T_1, \dots, T_n]$ is a polynomial ring and we work with I-graded pieces. For I integral, we are in $\Omega^\bullet_A$, so $\mathcal{C}^\bullet(I) = (0)$. Assume $s = s(I) = \max\{-\mathrm{ord}_p(i_j)\} > 0$. Then

$$H^0(gr^0_P\mathcal{C}^\bullet(I)) = \ker(\mathrm{d} : \mathcal{C}^0(I) \to \mathcal{C}^1(I)/P^1\mathcal{C}^1(I)). \tag{5.2}$$

Suppose $-\mathrm{ord}_p(i_j) = s(I)$. Then $\mathrm{d}(p^sT^I)$ contains the term $T^I \mathrm{d}\log(T_j)$, which does not lie in $P^1\mathcal{C}^1(I)$, from which it follows that this kernel is zero.

For H^1, there are two complexes to consider

$$\mathcal{C}^0(I) \to \mathcal{C}^1(I)/P^1\mathcal{C}^1(I) \to \mathcal{C}^2(I)/P^1\mathcal{C}^2(I), \tag{5.3}$$

$$0 \to P^1\mathcal{C}^1(I) \to P^1\mathcal{C}^2(I)/P^2\mathcal{C}^2(I). \tag{5.4}$$

We write $s = s(I) > 0$ and note that $P^m\mathcal{C}^m(I)$ consists of elements of the form $T^I \sum_{|J|=m} a_J \mathrm{d}\log(T_J)$ with $\mathrm{ord}_p(a_J) \geqslant s > 0$. (Here, of course, $\mathrm{d}\log(T_J) := \mathrm{d}\log(T_{j_1}) \wedge \mathrm{d}\log(T_{j_2}) \wedge \cdots$.) In particular, if $w \in P^1\mathcal{C}^1(I)$ and $\mathrm{d}w \in P^2\mathcal{C}^2(I)$, then w/p is integral, so $w \equiv 0 \bmod p$. It follows that the complex (5.4) has $H^1 = (0)$.

Finally, we consider the complex (5.3). We start with $w \in \mathcal{C}^1(I)$ represented by $T^I \sum v_j \mathrm{d}\log(T_j) \in E^1$. Integrality means that

$$v_j \in W(k) \quad \forall j, \qquad x_{m\ell} := v_\ell i_m - v_m i_\ell \in W(k) \quad \forall m, \ell. \tag{5.5}$$

The assumption $\mathrm{d}w \in P^1\mathcal{C}^2$ means that

$$\sum x_{m\ell} \,\mathrm{d}\log(T_{m\ell}) = \sum_\nu \biggl(\sum_\mu y_{\mu\nu} \,\mathrm{d}\log(T_\mu)\biggr) \wedge \mathrm{d}\log(T_\nu), \tag{5.6}$$

where

$$y_{\mu\nu} \in W(k), \quad y_{\mu\nu}i_\tau - y_{\tau\nu}i_\mu \in W(k) \quad \forall \mu, \nu, \tau. \tag{5.7}$$

Furthermore, if $y_{\mu\nu} \neq 0$, then because of the denominators in the $\mathrm{d}\log$ terms, we must have $i_\mu i_\nu \neq 0$. Thus we can define $\theta_{ab} := y_{ab}i_a^{-1}$ with the understanding that $\theta_{ab} = 0$ if $y_{ab} = 0$. Note that, by (5.7),

$$\theta_{ab} - \theta_{cb} \in i_a^{-1}i_b^{-b}W(k). \tag{5.8}$$

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Fix ϵ so that $\operatorname{ord}_p(i_\epsilon) = -s = -s(I) < 0$. Define $w' := T^I \sum_\ell \theta_{\epsilon\ell} \operatorname{d}\log(T_\ell)$. Because $p^s | \theta_{\epsilon\ell}$, we see that $w' \in P^1\mathcal{C}^1$. We have (defining $z_{\ell m}$)

$$\begin{aligned}
&\operatorname{d}(w - w') \\
&\quad = T^I \sum_{m,\ell} ((v_\ell - \theta_{\epsilon\ell}) i_m - (v_m - \theta_{\epsilon m}) i_\ell) \operatorname{d}\log(T_{\ell m}) \\
&\quad = T^I \sum_{m,\ell} ((v_\ell - \theta_{m\ell} + (\theta_{m\ell} - \theta_{\epsilon\ell})) i_m - (v_m - \theta_{\ell m} + (\theta_{\ell m} - \theta_{\epsilon m})) i_\ell) \operatorname{d}\log(T_{\ell m}) \\
&\quad = T^I \sum_{m,\ell} (x_{m\ell} - (y_{m\ell} - y_{\ell m}) + p^s z_{m\ell}) \operatorname{d}\log(T_{m\ell}) \\
&\quad = p^s T^I \sum_{m,\ell} z_{m\ell} \operatorname{d}\log(T_{m\ell}).
\end{aligned} \tag{5.9}$$

Here, $z_{m\ell} \in W(k)$.

Replacing w by $w - w'$, it follows from the above calculation that we may suppose that

$$v_\ell i_m - v_m i_\ell \in p^s W(k). \tag{5.10}$$

Recall we had chosen ϵ, so that $-\operatorname{ord}_p(i_\epsilon) = s = s(I)$. Since $v_\ell \in W(k)$, we can write

$$v_\epsilon = \mathrm{cp}^s\, i_\epsilon, \quad c \in W(k). \tag{5.11}$$

Then, for any m,

$$v_m = \mathrm{cp}^s\, i_m - i_\epsilon^{-1} p^s x, \quad x \in W(k). \tag{5.12}$$

In particular, $\mathrm{cp}^s\, T^I \in \mathcal{C}^0(I)$, and

$$\operatorname{d}(\mathrm{cp}^s\, T^I) = w + p^{2s} T^I \sum_j b_j \operatorname{d}\log(T_j), \quad b_j \in W(k). \tag{5.13}$$

It follows that $w = \operatorname{d}(\mathrm{cp}^s\, T^I) \in \mathcal{C}^1(I)$. □

Our final task will be to 'lift' (so to speak) our mod p descent to a descent mod p^N. We keep the notation from § 4. Recall that we have chosen a lifting S of R and a Frobenius $f : S \to S$ which induces $\rho : S \hookrightarrow W(R)$. We view S as a subring of $W(R)$ via ρ. We fix a trivialization $M = W(R)^{\oplus r}$, and we write G for the connection matrix. G is an $r \times r$ matrix with entries in $W\Omega^1_R$. We assume inductively that the entries of G lie in $\Omega^1_S + p^s W\Omega^1_R$ for some $s \geqslant 1$,

$$G = \beta + p^s \gamma. \tag{5.14}$$

Lemma 5.2. *There exists a matrix $U_s \in \operatorname{Mat}_r(W(R))$ such that the coordinate change $I + p^s U_s$ leads to a connection matrix with entries in $\Omega^1_S + p^{s+1} W\Omega^1_R$. The matrix U_s is unique modulo p.*

Proof. The new connection matrix is

$$(I+p^sU_s)G(I+p^sU_s)^{-1}+p^s\mathrm{d}U_s(I+p^sU_s)^{-1} \equiv \beta+p^s(\gamma+[U_s,\beta]+\mathrm{d}U_s) \mod p^{s+1}. \quad (5.15)$$

The curvature equation $\mathrm{d}G-G^2=0$ yields

$$\mathrm{d}\beta-\beta^2+p^s(\mathrm{d}\gamma-\beta\gamma-\gamma\beta)\equiv 0 \mod p^{2s}. \quad (5.16)$$

Recall that we have defined $(\mathcal{C}^\bullet,\mathrm{d})=(\mathcal{W}\Omega^\bullet_R/\Omega^\bullet_R,\mathrm{d})$. One has an action of $\Omega^\bullet_R$ on $\mathcal{C}^\bullet$ in an obvious sense. Let $b:=\beta \mod p\in \mathrm{Mat}_r(\Omega^1_R)$. Note that $\mathrm{d}b-b^2=0$. Define a complex $(\mathrm{Mat}_r(\mathcal{C}^\bullet),\delta_b)$ by

$$\delta_b(e)=\mathrm{d}e+(-1)^{\deg e}[e,b], \quad (5.17)$$

where the bracket is the graded bracket $[e,b]=eb-(-1)^{\deg e}be$. Note that

$$\begin{aligned}\delta_b^2(e)&=(-1)^{\deg e}(\mathrm{d}[e,b]-[\mathrm{d}e+(-1)^{\deg e}[e,b],b])\\ &=(-1)^{\deg e}(\mathrm{d}e)b+e(\mathrm{d}b)-(\mathrm{d}b)e+b(\mathrm{d}e)-(-1)^{\deg e}(\mathrm{d}e)b\\ &\qquad -b(\mathrm{d}e)-eb^2+(-1)^{\deg e}beb+(-1)^{\deg e+1}beb+b^2e\\ &=0. \end{aligned} \quad (5.18)$$

By (5.16), γ represents an element in $H^1(\mathrm{Mat}_r(\mathcal{C}^\bullet),\delta_b)$. By (5.15), the desired matrix U_s exists if and only if the cohomology class of γ is trivial. If $H^0=(0)$, then U_s is unique mod p.

The filtration $P^*\mathrm{Mat}_r(\mathcal{C}^\bullet):=\mathrm{Mat}_r(P^*\mathcal{C}^\bullet)$ is stabilized by δ_b, and the differential on $gr_P\mathrm{Mat}_r(\mathcal{C}^\bullet)$ is just given by d. Using the exact sequences associated to the filtration, we see for any i that

$$H^i(gr^*_P\mathrm{Mat}_r(\mathcal{C}^\bullet),\mathrm{d})=(0) \quad\Rightarrow\quad H^i(\mathrm{Mat}_r(\mathcal{C}^\bullet),\delta_b)=(0).$$

But the complex $gr^*_P\mathrm{Mat}_r(\mathcal{C}^\bullet),\mathrm{d})$ is just a direct sum of copies of $(gr^*_P\mathcal{C}^\bullet,\mathrm{d})$, so the desired vanishing follows from Lemma 5.1. □

We turn now to the proof of Theorem 1.1. As explained in the introduction (see equation (1.2)), one has a functor $\mathbb{E}\mapsto(E,\nabla)$ associating a de Rham–Witt connection to a crystal on X. By [**E**, II, Théorème 2.1], there is a canonical isomorphism between the cohomologies of $\mathbb{E}$ and (E,∇). In particular, looking at H^0, we see that the above functor is fully faithful. This reduces the problem of essential surjectivity to a local problem (gluing data lifts canonically).

We may assume that $X=\mathrm{Spec}(R)$ and we have S, f lifting R and the Frobenius as above, so $S\hookrightarrow W(R)$. Given an integrable de Rham–Witt connection (M,∇), we have, from Proposition 4.1 and Lemma 5.2, that there exists a canonical descent (N,Ξ) to an S-module with an integrable connection such that $(M,\nabla)=(N,\Xi)\otimes_S W(R)$. Recall that in § 1 we defined (M,∇) to be quasi-nilpotent if the connection $M/VW\cdot M$ on

Spec(R) was quasi-nilpotent. This connection coincides with the connection on N/pN, so it follows that the functor

$$N \mapsto N \otimes_S W(R), \tag{5.19}$$

from quasi-nilpotent connections on S to quasi-nilpotent de Rham–Witt connections, is essentially surjective. One knows (from a theorem of Berthelot; for a precise statement, see [**K**, § 2.4, p. 145]) that the category of crystals on Spec(R) is equivalent to the category of quasi-nilpotent connections on S. The functor associating a de Rham–Witt connection to a crystal is easily identified with (5.19) (use the crystal property). It follows that the functor from crystals to quasi-nilpotent de Rham–Witt connections is essentially surjective, and hence, by the result of Etesse cited above, is an equivalence of category.

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