Probability in the Engineering and Informational Sciences, **27**, 2013, 553–588.
doi:10.1017/S0269964813000259

ON DYNAMIC PROPORTIONAL MEAN RESIDUAL LIFE MODEL

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Recently, proportional mean residual life model has received a lot of attention after the importance of the model was discussed by Zahedi [17]. In this paper, we define dynamic proportional mean residual life model and study its properties for different aging classes. The closure of this model under different stochastic orders is also discussed. Many examples are presented to illustrate different properties of the model.

1. INTRODUCTION

In the areas of reliability and survival analysis, mean residual life (MRL) is a very well-known and central concept, and plays a very important role. To model lifetime data in parametric or non-parametric manner, the lifetime distributions having decreasing, increasing, bathtub-shaped or upside down bathtub-shaped MRL functions are used. The MRL and the hazard rate functions are commonly used to characterize the lifetime of a system. The hazard rate function gives the instantaneous failure rate at any point of time, whereas the MRL function summarizes the entire residual life. Let X be the lifetime of a system. Then the MRL of the system at age t is defined as the expectation of $X_t = [X - t|X > t]$, the remaining lifetime of the system after t, that is,

$$m_X(t) = E[X - t|X > t] = \frac{\int_t^\infty \bar{F}_X(u)du}{\bar{F}_X(t)}, \tag{1.1}$$

for $t \in [0, \infty)$ such that $\bar{F}_X(t) > 0$. It is to be mentioned here that if the support of X is $[a, b]$, then $\bar{F}_X(t) = 0$ for all $t > b$. In such a case, Eq. (1.1) does not define $m_X(t)$ for $t > b$.

 0269-9648/13 $25.00

But it is customary, in such cases, to define $m_X(t) = 0$ for $t > b$; see, for instance, Shaked and Shanthikumar [16, pp. 81–82]. Thus, $m_X(t) \geqslant 0$ for all $t \geqslant 0$.

Failure time data have been modeled by using Cox's proportional hazards (PH) model; see Cox [2]. Based on the assumption of proportional mean residual life (PMRL) functions, the concept of *Proportional Mean Residual Life Model*, parallel to Cox's PH model, was introduced by Zahedi [17] (see also Lam [7], Oakes [14], Oakes and Dasu [15]) as

$$m_{X^*}(t) = c\, m_X(t), \quad \text{for all } t \geqslant 0. \tag{1.2}$$

The following interpretation for the PMRL model has been given recently by Nanda, Bhattacharjee, and Balakrishnan [12]. Let a series system be formed with k components, of which one component has lifetime distribution F and the other $k-1$ components have i.i.d. life distribution, which is the equilibrium distribution corresponding to F. An equilibrium distribution is obtained as a limiting distribution of a renewal process. Then the MRL function of the system so formed and the MRL function corresponding to F are proportional with constant of proportionality $c = 1/k$. This paper deals with the properties of the PMRL model when the constant of proportionality depends on time. Gupta, Gupta, and Gupta [4] proposed the *Proportional Reversed Hazard (PRH) Model* to analyze failure time data. Gupta and Wu [6] have studied some properties of the PRH model. Mi [9] has shown that if a component has a bathtub-shaped failure rate function, then MRL is unimodal, but the converse does not hold. Later, Ghai and Mi [3] developed sufficient conditions for the unimodal MRL to imply that the failure rate function has a bathtub shape. The PMRL model and its implications have been discussed by Gupta and Kirmani [5]. Nanda, Bhattacharjee, and Alam [10] have discussed some properties of the PMRL model in the context of reliability theory. The PMRL model has been extended to a regression model with explanatory variables by Magulury and Zhang [8]. If we replace c in Eq. (1.2) by some non-negative function of t, say $c(t)$, then the corresponding equation becomes

$$m_{X^*}(t) = c(t) m_X(t), \quad \text{for all } t \geqslant 0. \tag{1.3}$$

Let X and X^* be the lifetimes of two systems. Sometimes, the MRL functions of X and X^* may not be proportional over the whole time interval $[0, \infty)$, but they may be proportional differently in different intervals. To be more specific, if $m_X(\cdot)$ and $m_{X^*}(\cdot)$ are the respective MRL functions of X and X^*, then they may be related as

$$m_{X^*}(t) = \begin{cases} a_1 m_X(t), & 0 \leqslant t < t_1, \\ a_2 m_X(t), & t_1 \leqslant t < t_2, \\ a_3 m_X(t), & t_2 \leqslant t < t_3, \\ \vdots & \end{cases}$$

where $a_1, a_2, a_3, \ldots$ are constants, and $t_1 \leqslant t_2 \leqslant \ldots \in [0, \infty)$. This type of model may be called piecewise PMRL model, along the lines of piecewise PH model (cf. Nanda and Das [13]). In a more general case, when the intervals $[t_{i-1}, t_i)$ become smaller and smaller, we get a model of the form Eq. (1.3). This model may be called Dynamic Proportional Mean Residual Life (DPMRL) model. If we consider the DPMRL model in Eq. (1.3), one can proceed in the following way to determine whether the proportionality coefficient in Eq. (1.3) is indeed a function of t. Given the data on both X and X^*, one could proceed parametrically by assuming specific distributional forms for X and X^*, such as Weibull, log-normal, etc. Then, from the given data on X and X^*, we can individually fit the parametric distributions through the maximum likelihood approach with which we can then determine the estimates of the individual MRL functions, $m_X(t)$ and $m_{X^*}(t)$, for $t > 0$. Then, the function

$(m_{X^*}(t))/(m_X(t))$, when plotted as a function of $t > 0$, should give nearly a horizontal line as an evidence to a constant proportionality coefficient. Significant departures from a horizontal line will support the fact that the proportionality coefficient is indeed a function of t. One could alternatively proceed in a non-parametric way by using the given data, and then proceed exactly as above. In fact, the plot of $(m_{X^*}(t))/(m_X(t))$, for $t > 0$, could also provide in addition some ideas for some suitable choice of the function $c(t)$.

For the benefit of the readers, with respect to the choice of the function $c(t)$, we have described above a graphical method based on both parametric and non-parametric approaches. Another interpretation of the DPMRL model may be given in terms of regression model having time-dependent covariates as follows.

Let the model postulate that the MRL function $m_X(t)$ for a system with covariate vector $\underline{z}$ is of the form $m(t;\underline{z}) = m_0(t)\exp[\underline{\beta}'\underline{z}]$, where $m_0(t)$ is the baseline MRL function and $\underline{\beta}$ is the vector of parameters. If the covariate vector $\underline{z}$ depends on t, then we can write $\underline{z} = \underline{z}(t)$, and the model reduces to $m(t;\underline{z}) = m_0(t)\exp[\underline{\beta}'\underline{z}(t)]$. By considering $\exp[\underline{\beta}'\underline{z}(t)] = c(t)$, the above regression model (with time-dependent covariates) reduces to the DPMRL model mentioned above. It is worth mentioning here that $c(t)$ could also be used in a regression model with time-varying slopes. It is well known that regression model is used in different situations, especially when forecasting is required. In reliability analysis, the reliability of any product/system depends on a number of factors which again may vary with time. In such situations, time-dependent regression model may be used. It needs to be pointed out here that the dynamic proportional hazard rate and reversed hazard rate models have been studied by Nanda and Das [13]. In case of a time-dependent covariate model with left censored data, the observed values can be analyzed using the properties of the covariate function (which is $c(t)$ in this model) to get some idea of the past scenario.

From the general model defined in Eq. (1.3), we can always obtain results for a model with specific $c(t)$, viz., $c(t)$ to be linear, quadratic, and so on, provided the function $c(t)$ satisfies the required condition(s) specific to results that follow. The objective of the paper is to study when specific properties of one random variable X carry over to another random variable X^*, in terms of properties of the ratio of their MRL functions.

The rest of this paper is organized as follows. In Section 2, the properties of some aging classes under the DPMRL model are discussed with some illustrative examples. Section 3 deals with the closure property of the DPMRL model under different stochastic orderings. Section 5 tells about the data analysis while Section 5 presents some concluding remarks.

Throughout the paper, *increasing* and *decreasing* properties of a function are not used in the strict sense. For any twice differentiable function $g(t)$, we write $g'(t)$ and $g''(t)$ to denote the first and the second derivatives of $g(t)$ with respect to t, respectively, and $a \overset{\text{sign}}{=} b$ means that a and b have the same sign. Although the existence of MRL function does not need absolute continuity of the distribution function, that of failure rate function needs the distribution to be absolutely continuous. Thus, throughout the discussion, wherever failure rate function is used, we assume that the distribution is absolutely continuous.

2. PROPERTIES OF SOME AGING CLASSES

We start this section by stating a lemma due to Nanda et al. [10].

LEMMA 2.1: *In order that $m_{X^*}(t)$ in Eq. (1.3) is an MRL function of some non-negative random variable X^*, $c(t)$ must satisfy the following conditions:*

(i) $0 \leqslant c(t) < \infty$, *for all* $t \geqslant 0$*;*

(ii) $c(0) > 0$*;*

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(iii) $c(t)$ *is a continuous function of* $t \geqslant 0$*;*

(iv) $t + c(t)m_X(t)$ *is increasing in* $t \geqslant 0$*;*

(v) if there does not exist any t_0 *with* $m_X(t_0) = 0$*, then* $\int_0^\infty (1/(c(t)m_X(t)))\,dt = \infty$*.*

The failure rate function of the random variable $r_X(t)$ is defined as the ratio of the probability density function to its survival function. The following theorem gives some conditions on $c(t)$ under which the increasing in failure rate (IFR)/decreasing in failure rate (DFR) property is preserved for the random variables X and X^*. Keep in mind that a random variable X with failure rate function $r_X(\cdot)$ is said to be IFR (resp. DFR) if $r_X(t)$ is increasing (resp. decreasing) in t. It can be noted that

$$\begin{aligned} r_{X^*}(t) &= \frac{1 + m'_{X^*}(t)}{m_{X^*}(t)} \\ &= \left(\frac{1}{c(t)} - 1\right)\left(\frac{1}{m_X(t)}\right) + \frac{c'(t)}{c(t)} + r_X(t). \end{aligned} \tag{2.1}$$

Theorem 2.1: *If the random variable* X *is IFR (resp. DFR), then the random variable* X^* *satisfying Eq. (1.3) is IFR (resp. DFR) provided, for all* $t > 0$,

(i) $0 < c(t) \leqslant 1$*;*

(ii) $c(t)$ *is decreasing (resp. increasing) and logconvex (resp. logconcave).*

By taking $c(t) = c$, a constant, we have the following corollary, which has been proved by Nanda et al. [10].

Corollary 2.1: *If* X *is IFR (resp. DFR) and* $0 < c < 1$*, then* X^* *is IFR (resp. DFR).*

The following lemma will be used in sequel.

Lemma 2.2: *For* $t \geqslant 0$, $0 < a \leqslant c$ *and* $0 \leqslant b \leqslant d$*, define* $c(t) = (a + bt)/(c + dt)$*. Suppose* X *has an exponential distribution with mean* $\frac{1}{\lambda}$, $\lambda > max\left(0, (ad - bc)/c^2\right)$*. Then:*

(i) $c(t)$ *satisfies all the conditions of Lemma 2.1;*

(ii) $c(t)$ *is decreasing (resp. increasing) and logconvex (resp. logconcave) if* $ad - bc \geqslant$ $(resp. \leqslant)0$*.*

We now present an application of Theorem 2.1.

Example 2.1: *Let us consider an exponential random variable with mean* $1/2$*. Take* $c(t) = (1 + t)/(1 + 2t)$*, for all* $t \geqslant 0$*. Then, by Lemma 2.2, we have* $c(t)$ *to be decreasing and logconvex. Thus, by Theorem 2.1,* X^* *is IFR.*

The following counterexample shows that the condition in *(ii)* of Theorem 2.1 is not necessary.

Counterexample 2.1: *Let* X *be a random variable having an Erlang distribution with probability density function* $f_X(t) = 4te^{-2t}, t \geqslant 0$*. Clearly,* X *is IFR. Take* $c(t) = ((1 + t)/(2 + t)^{1/4})$*, for all* $t \geqslant 0$*. Then,* $c(t)$ *is a non-negative continuous function lying in* $(0, 1)$*, and* $c(t)$ *satisfies all the conditions of Lemma 2.1. Now, one can verify that* $c(t)$ *is increasing and logconcave in* $t \geqslant 0$*. But,* X^* *can be shown to be IFR.*

The following theorem shows that the DPMRL model preserves the increasing failure rate in average (IFRA)/decreasing failure rate in average (DFRA) property under certain conditions on $c(t)$. It is useful to remind that a random variable X with failure rate function $r_X(\cdot)$ is said to be IFRA (resp. DFRA) if $(1/t)\int_0^t r_X(u)du$ is increasing (resp. decreasing) in $t > 0$.

THEOREM 2.2: *If the random variable X is IFRA (resp. DFRA), then the random variable X^* defined in Eq. (1.3) is IFRA (resp. DFRA) provided*

(i) $\frac{1-c(t)}{c(t)m_X(t)}$ *is increasing (resp. decreasing) in t;*

(ii) $c(t)$ is logconvex (resp. logconcave).

PROOF: Let X be IFRA. Note, from Eq. (2.1), that

$$\frac{1}{t}\int_0^t r_{X^*}(u)du = a(t) + \frac{1}{t}\int_0^t r_X(u)du, say, \tag{2.2}$$

where

$$a(t) = \frac{1}{t}\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \frac{1}{t}(\ln c(t) - \ln c(0)).$$

Differentiating $a(t)$ with respect to t, we obtain

$$a'(t) = \frac{1}{t^2}\left(\frac{t(1-c(t))}{c(t)m_X(t)} - \int_0^t \frac{1-c(u)}{c(u)m_X(u)}du\right) + \frac{1}{t^2}\left(\frac{t\ c'(t)}{c(t)} - \ln c(t) + \ln c(0)\right) \geqslant 0$$

for all $t \geqslant 0$. The inequality follows because the first term within parentheses in the above expression is non-negative due to condition (i), and to see that the second term within parentheses is non-negative, define $h(t) = (t\ c'(t)/c(t)) - \ln c(t)$, which, by condition (ii), is increasing in t. This gives that $t(c'(t)/c(t)) - \ln c(t) + \ln c(0) \geqslant 0$ for all $t \geqslant 0$. Again, since X is given to be IFRA, we have that Eq. (2.2) is increasing in t, giving that X^* is IFRA. The proof for the other case follows in an analogous manner. ■

We now present an application of Theorem 2.2.

EXAMPLE 2.2: *Let us take, for $t \geqslant 0$, $c(t) = (1 + 2t)/(1 + 3t)$, and X as an exponential random variable with mean $\frac{1}{2}$. Then, all the required conditions are satisfied by $c(t)$, and so by Theorem 2.2, X^* is IFRA.*

The following counterexample shows that condition (i) of Theorem 2.2 is not necessary.

COUNTEREXAMPLE 2.2: *Let X be a random variable with mean residual life function $m_X(t) = e^{-t}$, $t \geqslant 0$. Then the corresponding failure rate function is $r_X(t) = e^t - 1$, for all $t \geqslant 0$. Clearly, X is IFRA. Take $c(t) = e^{t/2}$, for all $t \geqslant 0$, which satisfies all the conditions of Lemma 2.1. Again, for $t \geqslant 0$, $\ln c(t) = t/2$ is convex. Further, for $t \geqslant 0$, $(1 - c(t))/(c(t)m_X(t)) = e^{t/2} - e^t$, which is decreasing in t. Thus, $c(t)$ satisfies condition (ii) of Theorem 2.2, but not condition (i). But, for $t \geqslant 0$, we have, from Eq. (2.2),*

$$\frac{1}{t}\int_0^t r_{X^*}(u)du = \frac{2}{t}(e^{t/2} - 1) - \frac{1}{2}$$

to be increasing in $t \geqslant 0$. Hence, condition (i) in the theorem is a sufficient condition but not necessary.

The following counterexample shows that condition (ii) of Theorem 2.2 cannot be dropped.

COUNTEREXAMPLE 2.3: *Let X be a gamma random variable with probability density function $f_X(t) = te^{-t}$, $t \geqslant 0$. Clearly, X is IFRA. Let*

$$c(t) = \begin{cases} \dfrac{1+t}{4+3t}, & 0 \leqslant t \leqslant 1, \\ \dfrac{2}{7t}, & t \geqslant 1. \end{cases}$$

Then, $c(t)$ satisfies condition (i) of Theorem 2.2, but not condition (ii). Now, by Eq. (2.2), we have, for $t \geqslant 1$,

$$\frac{1}{t}\int_0^t r_{X^*}(u)du = \frac{19}{4t} - \frac{9}{t}\ln 3 + \frac{1}{t}\ln 2 + \frac{7t}{4} - \frac{7}{2} + \frac{8}{t}\ln(2+t) - \frac{1}{t}\ln\frac{7t}{8} - \frac{1}{t}\ln(1+t)$$
$$= \alpha(t), \ say,$$

and we see that $\alpha(1) = 2.0349$, $\alpha(1.2) = 1.9531$, and $\alpha(1.5) = 2.0514$. Therefore, $\alpha(t)$ is not monotone. Hence, condition (ii) of Theorem 2.2 cannot be dropped.

By taking $c(t) = c$, a constant, we have the following corollary.

COROLLARY 2.2: *If X is IFRA (resp. DFRA) as well as DMRL (resp. IMRL), then X^* is IFRA (resp. DFRA), provided $0 < c \leqslant 1$.*

Note that a random variable X is said to be decreasing in mean residual life (DMRL) (resp. increasing in mean residual life (IMRL)) if $m_X(t)$ is decreasing (resp. increasing) in t. The following counterexample shows that the condition "X is DMRL" in Corollary 2.2 is not a necessary condition.

COUNTEREXAMPLE 2.4: *Let X be a random variable with failure rate function given by*

$$r_X(t) = \begin{cases} \dfrac{1+4t}{1+t^2}, & 0 \leqslant t \leqslant 1, \\ \dfrac{5}{2}, & t \geqslant 1. \end{cases}$$

Take $c = 1/2$. We see that

$$\frac{1}{t}\int_0^t r_X(u)du = \begin{cases} \dfrac{1}{t}[\tan^{-1} t + 2\ln(1+t^2)], & 0 \leqslant t \leqslant 1, \\ \dfrac{1}{t}\left[\dfrac{\pi}{4} + 2\ln 2 + \dfrac{5}{2}(t-1)\right], & t \geqslant 1, \end{cases}$$

which is increasing in t. Hence, X is IFRA. A tedious calculation shows that

$$m_X(t) = \begin{cases} \dfrac{\frac{e^{-\tan^{-1} t}}{10}\left(5 - \frac{t^2+4t-1}{1+t^2}\right) - \frac{e^{-(\pi/4)}}{5}}{\frac{e^{-\tan^{-1} t}}{(1+t^2)^2}}, & 0 \leqslant t \leqslant 1, \\ \dfrac{2}{5}, & t \geqslant 1, \end{cases}$$

which is not monotone. Therefore, X is IFRA but not DMRL. It can be shown that X^* is IFRA. Hence, the condition "X is DMRL" is a sufficient condition but not necessary.

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The following counterexample shows that the condition "X is IMRL" in Corollary 2.2 is not necessary.

COUNTEREXAMPLE 2.5: *Let X be a random variable with hazard rate function*

$$r_X(t) = \begin{cases} \left(\dfrac{1-t}{1+t}\right)^2, & 0 \leqslant t \leqslant 2, \\ \dfrac{1}{9}, & t \geqslant 2. \end{cases}$$

Then, X is DFRA but not IMRL. Now take $c = 1/2$. Then, for $0 \leqslant t \leqslant 2$,

$$\frac{1}{t}\int_0^t r_{X^*}(u)du$$

$$= \begin{cases} \dfrac{1}{t}\displaystyle\int_0^t \frac{(1+x)^4 \exp\left[-\left(x + \frac{4x}{1+x}\right)\right]}{\int_x^\infty (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right] du} dx + 1 + \frac{4}{t}\left(\frac{t}{1+t} - \ln(1+t)\right), & 0 \leqslant t \leqslant 2, \\ \dfrac{1}{t}\left(\displaystyle\int_0^2 \frac{(1+x)^4 \exp\left[-\left(x + \frac{4x}{1+x}\right)\right]}{\int_x^\infty (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right] du} dx + \frac{1}{9}(t-2)\right) + \dfrac{40 - 36\ln 3}{9t} + \frac{1}{9}, & t \geqslant 2, \end{cases}$$

which can be shown to be decreasing in t. Thus, X^* is DFRA. Hence, the condition "X is IMRL" in Corollary 2.2 is not a necessary condition.

The following lemma, which gives a relationship between $\bar{F}_X$ and m_X, is well known.

LEMMA 2.3: *For a non-negative random variable X with $m_X(t) > 0$, for all $t \geqslant 0$, we have $\bar{F}_X(t) = (E(X))/(m_X(t)) \exp\left(-\int_0^t (dx/m_X(x))\right)$.*

The following theorem shows that the model in Eq. (1.3) preserves the new better than used (NBU)/new worse than used (NWU) property under certain conditions. It is useful to remind that a random variable X is said to have NBU (resp. NWU) property if

$$\bar{F}_X(t+x) \leqslant \text{(resp. } \geqslant\text{)}$$
$$\bar{F}_X(t)\bar{F}_X(x), \text{ for all } x, t \geqslant 0.$$

THEOREM 2.3: *If a random variable X is NBU (resp. NWU), then the random variable X^* defined in Eq. (1.3) is NBU (resp. NWU) provided*

(i) $c(t+x)c(0) \geqslant$ *(resp.* $\leqslant$*)* $c(t)c(x)$*, for all* $x, t \geqslant 0$*;*

(ii) $0 < c(t) \leqslant 1$*, for all* $t \geqslant 0$*;*

(iii) $(1 - c(t))/(c(t)m_X(t))$ *is increasing (resp. decreasing) in* $t \geqslant 0$*.*

PROOF: Note that X^* is NBU, if and only if

$$\frac{E(X^*)}{m_{X^*}(t+x)} \exp\left(-\int_0^{t+x} \frac{du}{m_{X^*}(u)}\right) \leqslant \frac{E(X^*)}{m_{X^*}(x)}$$
$$\times \exp\left(-\int_0^x \frac{du}{m_{X^*}(u)}\right) \frac{E(X^*)}{m_{X^*}(t)} \exp\left(-\int_0^t \frac{du}{m_{X^*}(u)}\right),$$

or, equivalently,

$$\left(\frac{c(x)c(t)}{c(t+x)}\right)\left(\frac{m_X(x)m_X(t)}{m_X(t+x)}\right) - c(0)E(X) \times \exp\left(\int_x^{t+x}\frac{du}{c(u)m_X(u)} - \int_0^t \frac{du}{c(u)m_X(u)}\right) \leqslant 0, \tag{2.3}$$

which holds if

$$\frac{c(t)c(x)}{c(t+x)}\exp\left(\int_x^{t+x}\frac{du}{m_X(u)} - \int_0^t \frac{du}{m_X(u)}\right) - c(0)\exp\left(\int_x^{t+x}\frac{du}{c(u)m_X(u)} - \int_0^t \frac{du}{c(u)m_X(u)}\right) \leqslant 0,$$

since X is NBU. It is easy to see that the above inequality holds if the three conditions in Theorem 2.3 hold. The proof for the other case follows in an analogous manner. ■

We now present an application of Theorem 2.3.

Example 2.3: *Let us take, for* $t \geqslant 0$, $c(t) = (1+2t)/(1+3t)$, *and* X *as an exponential random variable with mean* $\frac{1}{2}$. *It is then easy to verify that* $c(t)$ *satisfies all the conditions of Theorem 2.3. Hence,* X^* *is NBU.*

The following counterexample shows that condition (i) in Theorem 2.3 is not necessary.

Counterexample 2.6: *Let* X *be the random variable as considered in Counterexample 2.3 and* $c(t) = (1+t)/(4+3t)$, *for all* $t \geqslant 0$. *Then, it can be shown that all the conditions of Lemma 2.1 are satisfied. Also, we can verify that conditions (ii) and (iii) of Theorem 2.3 are satisfied, while condition (i) is not, but* X^* *is NBU.*

The following counterexample shows that condition (ii) in Theorem 2.3 is not necessary.

Counterexample 2.7: *Let* X *be an exponential random variable with mean* $1/2$. *Take* $c(t) = 2/(1+t)$, *for* $t \geqslant 0$. *Then, we can verify that* $c(t)$ *satisfies conditions (i) and (iii) of Theorem 2.3, while condition (ii) is not satisfied, but* X^* *is NBU. This shows that condition (ii) of Theorem 2.3 is not a necessary condition.*

The following counterexample shows that condition (iii) in Theorem 2.3 cannot be dropped.

Counterexample 2.8: *Let* X *be an exponential random variable with mean* $1/2$. *Take*

$$c(t) = \begin{cases} \dfrac{1+2t}{1+3t}, & 0 \leqslant t \leqslant 1, \\ \dfrac{2+t}{3+t}, & t \geqslant 1. \end{cases}$$

Then, we can verify that conditions (i) and (ii) of Theorem 2.3 are satisfied, but condition (iii) is not satisfied. It may be noted that the LHS of Eq. (2.3), for $x = t = 5$, *is* 8.094872, *showing that* X^* *is not NBU. Hence, condition (iii) of Theorem 2.3 cannot be dropped.*

By taking $c(t) = c$, a constant, we have the following corollary.

Corollary 2.3: *If X is NBU (resp. NWU) and DMRL (resp. IMRL), and $0 < c \leqslant 1$, then X^* is NBU (resp. NWU).*

The following counterexample shows that the condition "X is DMRL" in Corollary 2.3 is not a necessary condition.

Counterexample 2.9: *Let X be a random variable with survival function*

$$\bar{F}_X(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 2, \\ \exp(2-t), & 2 \leqslant t \leqslant 3, \\ \exp(-1), & 3 \leqslant t \leqslant 4, \\ \exp(7-2t), & t \geqslant 4. \end{cases}$$

Clearly, X is NBU (see, e.g., Bryson and Siddiqui [1]), but not DMRL. Now, by taking $c = 1/2$, and using some extensive algebra, one can see that X^* is NBU. Hence, the condition "X is DMRL" in Corollary 2.3 is not necessary.

The following counterexample shows that the condition "X is IMRL" in Corollary 2.3 is not a necessary condition.

Counterexample 2.10: *Let X be a random variable with survival function*

$$\bar{F}_X(t) = \begin{cases} \dfrac{1+t^2}{(1+t)^3} \exp\left[t - \dfrac{t^2}{2}\right], & 0 \leqslant t \leqslant 1, \\ \dfrac{1}{4} \exp\left[1 - \dfrac{t}{2}\right], & t \geqslant 1. \end{cases}$$

It can be shown that X is NWU but not IMRL. Take $c = 1/2$. Then, by some algebraic calculation, we can show that X^* is NWU. Hence, the condition "X is IMRL" in Corollary 2.3 is not necessary.

The following theorem shows that the model in Eq. (1.3) preserves the harmonically new better than used in expectation (HNBUE)/harmonically new worse than used in expectation (HNWUE) property. It is useful to remind that a random variable X is said to be HNBUE (resp. HNWUE) if, for all t,

$$\int_t^\infty \bar{F}_X(x)dx \leqslant (resp. \geqslant) \exp\left[-\frac{t}{\int_0^\infty \bar{F}_X(x)dx}\right] \int_0^\infty \bar{F}_X(x)dx.$$

Theorem 2.4: *If a random variable X is HNBUE (resp. HNWUE), then the random variable X^* defined in Eq. (1.3) is HNBUE (resp. HNWUE) provided*

(i) $c(0) \geqslant (resp. \leqslant) 1$;

(ii) $\phi(u) \stackrel{\text{def}}{=} \int_0^u \frac{c(x)-1}{c(x)m_X(x)} dx - \ln c(u) \leqslant (resp. \geqslant) 0$, *for all* $u \geqslant 0$.

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Proof: By Lemma 2.3, we have X^* to be HNBUE, if and only if

$$\int_t^\infty \frac{E(X)}{c(u)m_X(u)} \exp\left(-\int_0^u \frac{dx}{c(x)m_X(x)}\right) du - E(X) \exp\left(-\frac{t}{c(0)E(X)}\right) \leqslant 0. \tag{2.4}$$

Now, it is sufficient to show that

$$\exp\left(\frac{t}{c(0)E(X)}\right) \int_t^\infty \frac{E(X)}{c(u)m_X(u)} \exp\left(-\int_0^u \frac{dx}{c(x)m_X(x)}\right) du$$
$$\leqslant \exp\left(\frac{t}{E(X)}\right) \int_t^\infty \frac{E(X)}{m_X(u)} \exp\left(-\int_0^u \frac{dx}{m_X(x)}\right) du. \tag{2.5}$$

This is because the expression in the RHS of Eq. (2.5) is $\leqslant E(X)$, which follows from Eq. (2.4), since X is HNBUE. Again, Eq. (2.5) holds if

(i) $c(0) \geqslant 1$, and
(ii) $\int_0^u \frac{c(x)-1}{c(x)m_X(x)} dx \leqslant \ln c(u)$.

Hence, the result follows. ■

Remark 2.1: In particular, if $c(0) = 1$, then condition *(ii)* of Theorem 2.4 holds if

$$\frac{c(x)-1}{c'(x)} \leqslant m_X(x), \tag{2.6}$$

for all $x \geqslant 0$.

We now present an application of Theorem 2.4.

Example 2.4: *Let X be a random variable with mean residual life function*

$$m_X(t) = \begin{cases} \dfrac{2+t}{1+t}, & 0 \leqslant t \leqslant 1, \\ \dfrac{3t}{2}, & t \geqslant 1. \end{cases}$$

Take $c(t) = 1 + t$, for all $t \geqslant 0$. It is easy to see that $c(0) = 1$ and $c(t)$ satisfies all the conditions of Lemma 2.1, and also satisfies Eq. (2.6). Hence, by Theorem 2.4, X^* is HNBUE.

The following counterexample shows that condition *(ii)* in Theorem 2.4 cannot be dropped.

Counterexample 2.11: *Let X be a random variable with mean residual life function $m_X(t) = 1/(3+t)$, $t \geqslant 0$. Clearly, X is HNBUE. Take $c(t) = 2 + t$, for all $t \geqslant 0$. Then, one can verify that the function $\phi(t)$ defined in Theorem 2.4 crosses the X-axis. Hence, $c(t)$ does not satisfy condition (ii) of Theorem 2.4. Again, one can see that Eq. (2.4) is not satisfied, showing that condition (ii) of Theorem 2.4 cannot be dropped.*

Remark 2.2: If $c(t) = c$ for all $t \geqslant 0$, then it has been shown by Nanda et al. [10] that X^* is HNBUE if and only if X is HNBUE.

The following theorem shows that the model in Eq. (1.3) preserves the new better than used in failure rate (NBUFR)/new worse than used in failure rate (NWUFR) property. It is useful to remind that a random variable X is said to be NBUFR (resp. NWUFR) if, for all $t > 0$, $r_X(t) \geqslant$ (resp. $\leqslant$) $r_X(0)$.

THEOREM 2.5: *If the random variable X is NBUFR (resp. NWUFR), then the random variable X^* satisfying Eq. (1.3) is NBUFR (resp. NWUFR), provided, for all $t > 0$,*

(i) $\frac{1-c(t)}{c(t)m_X(t)} \geqslant$ *(resp.* $\leqslant$*)* $\frac{1-c(0)}{c(0)m_X(0)}$*, for all* $t \geqslant 0$*;*

(ii) $\frac{c'(t)}{c(t)} \geqslant$ *(resp.* $\leqslant$*)* $\frac{c'(0)}{c(0)}$*, for all* $t \geqslant 0$.

PROOF: X^* is NBUFR if $r_{X^*}(t) \geqslant r_{X^*}(0)$, which, from Eq. (2.1), is equivalent to

$$\frac{1-c(t)}{c(t)m_X(t)} - \frac{1-c(0)}{c(0)m_X(0)} + \frac{c'(t)}{c(t)} - \frac{c'(0)}{c(0)} + r_X(t) - r_X(0) \geqslant 0,$$

which follows readily from the conditions. The proof for the other case follows in an analogous manner. ■

We now present an application of Theorem 2.5.

EXAMPLE 2.5: *Let us reconsider Example 2.2. Clearly, $c(t)$ satisfies all the conditions of Lemma 2.2 and also satisfies conditions (i) and (ii) of Theorem 2.5. Hence, by Theorem 2.5, X^* is NBUFR.*

The following counterexample shows that condition (i) in Theorem 2.5 is a sufficient condition but not necessary.

COUNTEREXAMPLE 2.12: *Let X be a random variable having gamma distribution with probability density function given by $f_X(t) = te^{-t}$, $t \geqslant 0$. Clearly, X is NBUFR. Take $c(t) = (1+t^2)/(2+t)$, for all $t \geqslant 0$. We see that $c(t)$ is a non-negative continuous function and $c(0) = 1/2(> 0)$. Again, for all $t \geqslant 0$, $t + c(t)m_X(t) = t + (1+t^2)/(1+t)$ is increasing in t. Again, for all $t \geqslant 0$,*

$$\int_0^t \frac{du}{c(u)m_X(u)} = \tan^{-1} t + \frac{1}{2}\ln(1+t^2),$$

which diverges as $t \to \infty$. Therefore, $c(t)$ satisfies all the conditions of Lemma 2.1. Now, for $t \geqslant 0$,

$$\frac{1-c(t)}{c(t)m_X(t)} - \frac{1-c(0)}{c(0)m_X(0)} = \frac{(1+t)(1+t-t^2)}{(2+t)(1+t^2)} - \frac{1}{2}$$
$$= g_1(t), \text{ say.}$$

We see that $g_1(1/2) = 1/10$ and $g_1(1) = -1/6$. Therefore, condition (i) of Theorem 2.5 is not satisfied. Again, for $t \geqslant 0$,

$$\frac{c'(t)}{c(t)} - \frac{c'(0)}{c(0)} = \frac{t^3 + 4t^2 + 9t}{(2+t)(1+t^2)} \geqslant 0.$$

Therefore, condition (ii) of Theorem 2.5 is satisfied. Now, for $t \geqslant 0$,

$$r_{X^*}(t) - r_{X^*}(0) = \frac{2t^3 + 8t^2 + 8t}{(1+t)(2+t)(1+t^2)} \geqslant 0,$$

showing that X^* is NBUFR. This shows that condition (i) of Theorem 2.5 is a sufficient condition but not necessary.

The following counterexample shows that condition (ii) in Theorem 2.5 cannot be dropped.

Counterexample 2.13: *Let us reconsider Counterexample 2.3. Then, we can verify that condition (i) of Theorem 2.5 is satisfied, but condition (ii) is not. Now, we see that, for* $0 \leqslant t < 1$, $r_{X^*}(t) - r_{X^*}(0) \geqslant 0$, *and for* $t = 1.1$, $r_{X^*}(t) - r_{X^*}(0) = -0.2046$. *Thus,* X^* *is not NBUFR. Hence, condition (ii) of Theorem 2.5 cannot be dropped.*

Corollary 2.4: *If the random variable X is NBUFR (resp. NWUFR), then the random variable* X^* *satisfying Eq. (1.3) is NBUFR (resp. NWUFR), provided,*

(i) $\frac{1-c(t)}{c(t)m_X(t)}$ *is increasing (resp. decreasing) in t;*

(ii) $c(t)$ *is logconvex (resp. logconcave),*

for all $t > 0$.

The following counterexample shows that condition (i) in Corollary 2.4 is not a necessary condition.

Counterexample 2.14: *Let us reconsider Counterexample 2.2. Then, we can verify that condition (ii) of Corollary 2.4 is satisfied, but condition (i) is not. Now, we see that, for* $t \geqslant 0$,

$$\begin{aligned} r_{X^*}(t) - r_{X^*}(0) &= \frac{1-c(t)}{c(t)m_X(t)} - \frac{1-c(0)}{c(0)m_X(0)} + \frac{c'(t)}{c(t)} - \frac{c'(0)}{c(0)} + r_X(t) - r_X(0) \\ &= e^{t/2} - e^t + \frac{\frac{e^{t/2}}{2}}{e^{t/2}} - \frac{1}{2} + e^t - 1 \\ &= e^{t/2} - 1 \geqslant 0. \end{aligned}$$

Therefore, X^* *is NBUFR. This shows that condition (i) of Corollary 2.4 is a sufficient condition but not necessary.*

Remark 2.3: Counterexample 2.13 can be used to show that condition (ii) of Corollary 2.4 cannot be dropped.

By taking $c(t) = c$, a constant, we obtain the following corollary.

Corollary 2.5: *If X is NBUFR (resp. NWUFR) as well as DMRL (resp. IMRL), and* $0 < c \leqslant 1$, *then* X^* *is NBUFR (resp. NWUFR).*

The following counterexample shows that the condition "X is DMRL" in Corollary 2.5 is not a necessary condition.

Counterexample 2.15: *Let us reconsider Counterexample 2.4. Clearly, X is not DMRL. We see that*

$$r_X(t) - r_X(0) = \begin{cases} \dfrac{4t - t^2}{1+t^2}, & 0 \leqslant t \leqslant 1, \\ \dfrac{3}{2}, & t \geqslant 1. \end{cases}$$

https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press

This shows that X is NBUFR. Take $c = 1/2$. Now,

$$r_{X^*}(t) - r_{X^*}(0) = \begin{cases} \dfrac{\frac{e^{-\tan^{-1} t}}{(1+t^2)^2}}{\frac{e^{-\tan^{-1} t}}{10}\left(5 - \frac{t^2+4t-1}{1+t^2}\right) - \frac{e^{-(\pi/4)}}{5}} - \dfrac{5}{3 - e^{-(\pi/4)}} + \dfrac{1+4t}{1+t^2} - 1, & 0 \leqslant t \leqslant 1, \\ 2.0346, & t \geqslant 1, \end{cases}$$

which can be shown to be positive. Therefore, X^* is NBUFR. Hence, the condition "X is DMRL" is a sufficient condition but not necessary.

The following counterexample shows that the condition "X is IMRL" in Corollary 2.5 is not a necessary condition.

COUNTEREXAMPLE 2.16: *Let us reconsider Counterexample 2.5. Clearly, X is not IMRL. We see that*

$$r_X(t) - r_X(0) = \begin{cases} \left(\dfrac{1-t}{1+t}\right)^2 - 1, & 0 \leqslant t \leqslant 2, \\ -\dfrac{8}{9} & t \geqslant 2, \end{cases}$$

which is negative. Therefore, X is NWUFR. Take $c = 1/2$. Now, we can see that

$$r_{X^*}(t) - r_{X^*}(0) = \begin{cases} \dfrac{(1+t)^4 \exp\left[-\left(t + \frac{4t}{1+t}\right)\right]}{\int_t^2 (1+u)^4 \exp\left[-\left(u + \frac{4u}{1+u}\right)\right] du + 3^6 \exp\left[-\frac{42}{9}\right]} \\ -\dfrac{1}{\int_0^2 (1+u)^4 \exp\left[-\left(u + \frac{4u}{1+u}\right)\right] du + 3^6 \exp\left[-\frac{42}{9}\right]} + \left(\dfrac{1-t}{1+t}\right)^2 - 1, & 0 \leqslant t \leqslant 2, \\ -\dfrac{7}{9} - \dfrac{1}{\int_0^2 (1+u)^4 \exp\left[-\left(u + \frac{4u}{1+u}\right)\right] du + 3^6 \exp\left[-\frac{42}{9}\right]}, & t \geqslant 2, \end{cases}$$

which can be shown to be negative. Hence, the condition "X is IMRL" is a sufficient condition but not necessary.

The following theorem shows that the model given in Eq. (1.3) preserves the new better than used in failure rate average (NBAFR)/new worse than used in failure rate average (NWAFR) property. It is useful to remind that a random variable X is said to be NBAFR (resp. NWAFR) if $r_X(0) \leqslant$ (resp. $\geqslant$)$\frac{1}{t}\int_0^t r_X(u)du$, for all $t > 0$.

THEOREM 2.6: *If the random variable X is NBAFR (resp. NWAFR), then the random variable X^* satisfying Eq. (1.3) is NBAFR (resp. NWAFR), provided*

(i) $\frac{1-c(t)}{c(t)m_X(t)} \geqslant$ *(resp.* $\leqslant$*)* $\frac{1-c(0)}{c(0)m_X(0)}$, *for all* $t > 0$;

(ii) $\frac{c'(t)}{c(t)} \geqslant$ *(resp.* $\leqslant$*)* $\frac{c'(0)}{c(0)}$, *for all* $t \geqslant 0$.

Proof: Let X be NBAFR. Then, from Eq. (2.2), we have

$$\frac{1}{t}\int_0^t r_{X^*}(u)du = \frac{1}{t}\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \frac{1}{t}\int_0^t \frac{c'(u)}{c(u)}du + \frac{1}{t}\int_0^t r_X(u)du$$

and $r_{X^*}(0) = (1-c(0))/(c(0)m_X(0)) + (c'(0)/c(0)) + r_X(0)$. Now, X^* is NBAFR if $r_{X^*}(0) \leqslant (1/t)\int_0^t r_{X^*}(u)du$, which is equivalent to

$$\frac{1}{t}\int_0^t \left(\frac{1-c(u)}{c(u)m_X(u)} - \frac{1-c(0)}{c(0)m_X(0)}\right)du + \frac{1}{t}\int_0^t \left(\frac{c'(u)}{c(u)} - \frac{c'(0)}{c(0)}\right)du \\ + \frac{1}{t}\int_0^t r_X(u)du - r_X(0) \geqslant 0.$$

The inequality follows readily from the conditions. The proof for the other case follows in an analogous manner. ■

Remark 2.4: Example 2.5 can be considered as an application of Theorem 2.6.

The following counterexample shows that condition (i) in Theorem 2.6 is not a necessary condition.

Counterexample 2.17: *Let us consider Counterexample 2.12. Clearly, X is NBAFR. We can verify that condition (ii) of Theorem 2.6 is satisfied, but condition (i) is not. Now, for $t \geqslant 0$,*

$$\frac{1}{t}\int_0^t r_{X^*}(u)du - r_{X^*}(0) = \frac{1}{t}\left[\tan^{-1}t + \frac{3}{2}\ln(1+t^2) - \ln(1+t)\right] \\ \geqslant 0.$$

Here in calculating the inverse, the principal value has been taken. Therefore, X^* is NBAFR. Hence, condition (i) in Theorem 2.6 is a sufficient condition but not necessary.

The following counterexample shows that condition (ii) in Theorem 2.6 is not necessary.

Counterexample 2.18: *Let us reconsider Counterexample 2.3. Then, we can verify that condition (i) of Theorem 2.6 is satisfied, but condition (ii) is not. Now, we see that*

$$\frac{1}{t}\int_0^t r_{X^*}(u)du - r_{X^*}(0) = \begin{cases} \frac{5}{4} - \frac{1}{t}\ln[(2+t)(4+3t)] + \frac{1}{t}\ln 8, & 0 \leqslant t < 1, \\ \frac{19}{4t} + \frac{7t}{4} - \frac{21}{4} + \frac{1}{t}\left[\ln\frac{16}{21t(1+t)} + 8\ln\frac{2+t}{3}\right], & t > 1, \end{cases}$$

which can be shown to be positive. Hence, condition (ii) of Theorem 2.6 is not necessary.

Corollary 2.6: *If the random variable X is NBAFR (resp. NWAFR), then the random variable X^* satisfying Eq. (1.3) is NBAFR (resp. NWAFR), provided,*

(i) $\frac{1-c(t)}{c(t)m_X(t)}$ *is increasing (resp. decreasing) in t;*

(ii) $c(t)$ *is logconvex (resp. logconcave), for all $t > 0$.*

The following counterexample shows that condition (i) in Corollary 2.6 is a sufficient condition but not necessary.

COUNTEREXAMPLE 2.19: *Let X be the random variable as given in Counterexample 2.2. Then it has been shown in Counterexample 2.2 that all the conditions of Lemma 2.1 are satisfied. Again, one can verify that condition (ii) of Corollary 2.6 is satisfied, but condition (i) is not. Now, we see that, for $t \geqslant 0$,*

$$\begin{aligned}\frac{1}{t}\int_0^t r_{X^*}(u)du - r_{X^*}(0) &= \frac{2}{t}(e^{t/2}-1)-1\\ &\geqslant 0.\end{aligned}$$

Hence, condition (i) of Corollary 2.6 is a sufficient condition but not necessary.

Remark 2.5: Counterexample 2.18 can be used to show that condition (ii) of Corollary 2.6 is a sufficient condition but not necessary.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 2.7: *If X is NBAFR (resp. NWAFR) as well as DMRL (resp. IMRL), and $0 < c \leqslant 1$, then X^* is NBAFR (resp. NWAFR).*

The following counterexample shows that the condition "X is DMRL" in Corollary 2.7 is not a necessary condition.

COUNTEREXAMPLE 2.20: *Let us reconsider Counterexample 2.4. Clearly, X is not DMRL. We see that*

$$\frac{1}{t}\int_0^t r_X(u)du - r_X(0) = \begin{cases}\dfrac{1}{t}[\tan^{-1}t + 2\ln(1+t^2)] - 1, & 0 \leqslant t \leqslant 1,\\ \dfrac{1}{t}\left[\dfrac{\pi}{4} + 2\ln 2 + \dfrac{5}{2}(t-1)\right] - 1, & t \geqslant 1.\end{cases}$$

Hence, X is NBAFR. Now, by taking $c = 1/2$, we can see that

$$\begin{aligned}&\frac{1}{t}\int_0^t r_{X^*}(u)du - r_{X^*}(0)\\ &= \begin{cases}\dfrac{1}{t}\displaystyle\int_0^t \frac{\frac{e^{-\tan^{-1}u}}{(1+u^2)^2}}{\frac{e^{-\tan^{-1}u}}{10}\left(5-\frac{u^2+4u-1}{1+u^2}\right)-\frac{e^{-(\pi/4)}}{5}}du - \dfrac{5}{3-e^{-(\pi/4)}}\\ +\dfrac{1}{t}[\tan^{-1}t + 2\ln(1+t^2)] - 1, & 0 \leqslant t \leqslant 1,\\ \frac{1}{t}\displaystyle\int_0^1 \frac{\frac{e^{-\tan^{-1}u}}{(1+u^2)^2}}{\frac{e^{-\tan^{-1}u}}{10}\left(5-\frac{u^2+4u-1}{1+u^2}\right)-\frac{e^{-(\pi/4)}}{5}}du + \dfrac{5(t-1)}{t}\\ +\dfrac{1}{t}\left[\dfrac{\pi}{4} + \ln 4\right] - \dfrac{5}{3-e^{-(\pi/4)}} - 1, & t \geqslant 1,\end{cases}\end{aligned}$$

which can be shown to be positive. Thus, $(1/t)\int_0^t r_{X^*}(u)du \geqslant r_{X^*}(0)$, for all $t \geqslant 0$. Hence, the condition "X is DMRL" is a sufficient condition but not necessary.

https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press

The following counterexample shows that the condition "X is IMRL" in Corollary 2.7 is not a necessary condition.

COUNTEREXAMPLE 2.21: *Let us reconsider Counterexample 2.5. Clearly, X is not IMRL. Now,*

$$\frac{1}{t}\int_0^t r_X(x)dx - r_X(0) = \frac{4}{t}\left[\frac{t}{1+t} - \ln(1+t)\right], \quad 0 \leqslant t \leqslant 2,$$
$$\frac{1}{t}\left[\frac{14}{3} - 4\ln 3 - \frac{8t+2}{9}\right], \quad t \geqslant 2.$$

This shows that X is NWAFR. Now,

$$\frac{1}{t}\int_0^t r_{X^*}(x)dx - r_{X^*}(0)$$
$$= \begin{cases} \frac{1}{t}\int_0^t \frac{(1+x)^4 \exp\left[-\left(x+\frac{4x}{1+x}\right)\right]}{\int_x^2 (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right]du + 3^6\exp\left[-\frac{42}{9}\right]}dx \\ -\frac{1}{\int_0^2 (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right]du + 3^6\exp\left[-\frac{42}{9}\right]} \\ +\frac{4}{t}\left(\frac{t}{1+t} - \ln(1+t)\right), & 0 \leqslant t \leqslant 2, \\ \frac{1}{t}\int_0^2 \frac{(1+x)^4 \exp\left[-\left(x+\frac{4x}{1+x}\right)\right]}{\int_x^2 (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right]du + 3^6\exp\left[-\frac{42}{9}\right]}dx \\ -\frac{1}{\int_0^2 (1+u)^4 \exp\left[-\left(u+\frac{4u}{1+u}\right)\right]du + 3^6\exp\left[-\frac{42}{9}\right]} \\ +\frac{38-36\ln 3}{9t} - \frac{7}{9}, & t \geqslant 2, \end{cases}$$

which can be shown to be negative. Thus, $\frac{1}{t}\int_0^t r_X(x)dx \leqslant r_X(0)$, for all $t \geqslant 0$. Hence, the condition "X is IMRL" is a sufficient condition but not necessary.

The following theorem shows that the model in Eq. (1.3) preserves the new better than used in convex ordering (NBUC)/new worse than used in convex ordering (NWUC) property. It is useful to remind that a random variable X is said to be NBUC (resp. NWUC) if $\int_t^\infty \bar{F}_X(x+u)du \leqslant (resp. \geqslant) \bar{F}_X(x)\int_t^\infty \bar{F}_X(u)du$ for all $x, t \geqslant 0$.

THEOREM 2.7: *If the random variable X is NBUC (resp. NWUC), then the random variable X^* satisfying Eq. (1.3) is NBUC (resp. NWUC), provided*

(i) $c(t) \leqslant (resp. \geqslant)\ c(0)$, for all $t > 0$;
(ii) $\frac{1-c(t)}{c(t)m_X(t)}$ is increasing (resp. decreasing) in $t \geqslant 0$.

PROOF: By Lemma 2.3, we have X^* to be NBUC if and only if

$$\int_t^\infty \bar{F}_{X^*}(x+u)du \leqslant \bar{F}_{X^*}(x)\int_t^\infty \bar{F}_{X^*}(u)du, \quad \text{for all } x,t \geqslant 0,$$

which is equivalent to

$$\frac{c(x)m_X(x)}{c(0)E(X)} - \left(\exp\left[-\int_0^x \frac{dv}{c(v)m_X(v)}\right]\right)\frac{\exp\left[-\int_0^t \frac{dv}{c(v)m_X(v)}\right]}{\exp\left[-\int_0^{x+t} \frac{dv}{c(v)m_X(v)}\right]} \leqslant 0. \tag{2.7}$$

Since X is NBUC, we have

$$\frac{m_X(x)}{E(X)}\left(\exp\left[-\int_0^{x+t}\frac{dv}{m_X(v)}\right]\right) - \left(\exp\left[-\int_0^x \frac{dv}{m_X(v)}\right]\right)\left(\exp\left[-\int_0^t \frac{dv}{m_X(v)}\right]\right) \leqslant 0. \tag{2.8}$$

Now, in order for Eq. (2.7) to hold, it is sufficient to show that

$$c(x)\frac{\exp\left[-\int_0^t \frac{dv}{m_X(v)}\right]}{\exp\left[-\int_x^{x+t}\frac{dv}{m_X(v)}\right]} - c(0)\frac{\exp\left[-\int_0^t \frac{dv}{c(v)m_X(v)}\right]}{\exp\left[-\int_x^{x+t}\frac{dv}{c(v)m_X(v)}\right]} \leqslant 0.$$

This holds if the conditions of the theorem are satisfied. ■

We now present an application of Theorem 2.7.

EXAMPLE 2.6: *Let us consider an exponential random variable with mean* $1/2$*. Take, for* $t \geqslant 0$, $c(t) = (1+2t)/(1+3t)$*. Then, clearly* $c(t)$ *satisfies all the conditions of Lemma 2.2. Moreover,* $c(t) \leqslant c(0)$ *and* $(1-c(t))/(c(t)m_X(t))$ *is increasing in* t*. Hence, by Theorem 2.7,* X^* *is NBUC.*

The following counterexample shows that condition (i) in Theorem 2.7 is not a necessary condition.

COUNTEREXAMPLE 2.22: *By considering the random variable* X *with* $m_X(t) = 1/(1+t)$, $t \geqslant 0$, *and*

$$c(t) = \begin{cases} \dfrac{1+t}{2+t}, & 0 \leqslant t \leqslant 1, \\ \dfrac{1+t}{1+2t}, & t \geqslant 1, \end{cases}$$

it can be shown that condition (i) in Theorem 2.7 is not necessary.

The following counterexample shows that condition (ii) in Theorem 2.7 cannot be dropped.

COUNTEREXAMPLE 2.23: *Let us reconsider Counterexample 2.8. Then, clearly* $c(t)$ *satisfies all the conditions of Lemma 2.2. Note that* $c(t) \leqslant c(0)$ *for all* $t \geqslant 0$*. Again,* $(1-c(t))/(c(t)m_X(t))$ *is increasing in* $t \in [0,1]$ *and decreasing in* $t \geqslant 1$*. Therefore,* $c(t)$ *does not satisfy condition (ii) of Theorem 2.7. Moreover, we see that for* $x = 3$ *and* $t = 3$*, the LHS of Eq. (2.7) is* 0.246*. Hence, condition (ii) of Theorem 2.7 cannot be dropped.*

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 2.8: *If X is NBUC (resp. NWUC) as well as DMRL (resp. IMRL), and $0 < c \leqslant 1$, then X^* is NBUC (NWUC).*

The following counterexample shows that the condition "X is DMRL" in Corollary 2.8 is not a necessary condition.

COUNTEREXAMPLE 2.24: *Let X be a random variable as considered in Counterexample 2.9. Then, one can verify that X is NBUC, but not DMRL. Now, through extensive algebra, one can see that X^* is NBUC.*

Remark 2.6: The fact that the condition "X is IMRL" in Corollary 2.8 is not a necessary condition can be shown by taking X to be a random variable as considered in Counterexample 2.10.

3. CLOSURE OF SOME STOCHASTIC ORDERS

In this section, we discuss the condition(s) under which X and X^* satisfying Eq. (1.3) have some stochastic order relations.

The following theorem shows that the model in Eq. (1.3) preserves the usual stochastic ordering. It is useful to remind that a random variable X is said to be larger than another random variable Y in stochastic ordering (written as $X \geqslant_{ST} Y$) if $\bar{F}_X(t) \geqslant \bar{F}_Y(t)$ for all t. It is easy to see that it is equivalent to the condition that $\int_0^t r_X(u)du \leqslant \int_0^t r_Y(u)du$, for all $t > 0$.

THEOREM 3.1: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). Then, $X \geqslant_{ST} X^*$, if and only if*

$$\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \ln c(t) - \ln c(0) \geqslant 0 \tag{3.1}$$

for all $t \geqslant 0$.

PROOF: $X \geqslant_{ST} X^*$, if and only if, for all $t > 0$,

$$\frac{1}{t}\int_0^t r_X(u)du \leqslant \frac{1}{t}\int_0^t r_{X^*}(u)du,$$

which, by Eq. (2.1), reduces to

$$\int_0^t r_X(u)du \leqslant \int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \int_0^t \frac{c'(u)}{c(u)}du + \int_0^t r_X(u)du.$$

This can equivalently be written as

$$\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \ln c(t) - \ln c(0) \geqslant 0$$

for all $t > 0$. ∎

https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press

We now present an application of Theorem 3.1.

EXAMPLE 3.1: *Let us consider Counterexample 2.12, and* $c(t) = (1+t)/(2+t)$, $t \geqslant 0$. *Then, we can easily verify that Eq. (3.1) is satisfied. Hence, by Theorem 3.1,* $X \geqslant_{ST} X^*$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 3.1: $X \geqslant_{ST} X^*$, *if and only if* $0 < c \leqslant 1$, *and* $X^* \geqslant_{ST} X$, *if and only if* $c \geqslant 1$.

The following theorem gives the condition under which the model in Eq. (1.3) preserves the hazard rate ordering. The proof is immediate from Eq. (2.1). It is useful to remind that a random variable X is said to be larger than another random variable Y in hazard rate ordering (written as $X \geqslant_{HR} Y$) if, for all $t \geqslant 0$, $r_X(t) \leqslant r_Y(t)$.

THEOREM 3.2: *Let* X *and* X^* *be two non-negative absolutely continuous random variables satisfying Eq. (1.3). Then,* $X \geqslant_{HR} (\leqslant_{HR}) X^*$, *if and only if* $1 - c(t) + c'(t)m_X(t) \geqslant (\leqslant)\ 0$, *for all* $t \geqslant 0$.

Remark 3.1: $X \geqslant_{HR} X^*$ if $c(t)$ is increasing in t, and lies between $[0,1]$ for all $t \geqslant 0$, and $X^* \geqslant_{HR} X$ if $c(t)$ is decreasing with respect to t and is $\geqslant 1$ for all $t \geqslant 0$. For example, if $c(t) = \frac{t}{1+t}$, $t \geqslant 0$, then $X \geqslant_{HR} X^*$.

We now present an application of Theorem 3.2.

EXAMPLE 3.2: *From Example 3.1, we see, for* $t \geqslant 0$, *that* $1 - c(t) + c'(t)m_X(t) = 1/(1+t) > 0$. *Hence, by Theorem 3.2, we have* $X \geqslant_{HR} X^*$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 3.2: $X \geqslant_{HR} X^*$, *if and only if* $0 < c \leqslant 1$, *and* $X^* \geqslant_{HR} X$ *if and only if* $c \geqslant 1$.

Remark 3.2: It is easy to see that, for $0 \leqslant c(t) \leqslant 1$, $X \geqslant_{MRL} X^*$, and, for $c(t) \geqslant 1$, $X \leqslant_{MRL} X^*$.

The following theorem gives the condition under which the model in Eq. (1.3) preserves the harmonic average mean residual ordering. It is useful to remind that a random variable X is said to be larger than another random variable Y in harmonic average mean residual ordering (written as $X \geqslant_{HAMR} Y$) if, for all $t > 0$, $\int_0^t (dx/m_X(x)) \leqslant \int_0^t (dx/m_Y(x))$.

THEOREM 3.3: *Let* X *and* X^* *be two non-negative absolutely continuous random variables satisfying Eq. (1.3). Then:*

(i) $X \geqslant_{HAMR} X^*$, *if* $0 < c(t) \leqslant 1$, *for all* $t > 0$;
(ii) $X^* \geqslant_{HAMR} X$, *if* $c(t) \geqslant 1$, *for all* $t > 0$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 3.3: $X \geqslant_{HAMR} X^*$, *if and only if* $0 < c \leqslant 1$, *and* $X^* \geqslant_{HAMR} X$, *if and only if* $c \geqslant 1$.

The following theorem shows that the model in Eq. (1.3) preserves the up hazard rate ordering. It is useful to remind that a random variable X is said to be smaller than another random variable Y in up hazard rate order (written as $X \leqslant_{HR\uparrow} Y$) if $X - x \leqslant_{HR} Y$ for all $x \geqslant 0$, or equivalently, if $(\bar{F}_X(t+x)/\bar{F}_Y(t))$ is decreasing in t.

THEOREM 3.4: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). Then, $X \leqslant_{HR\uparrow} X^*$, if and only if*

$$r_X(t) - r_X(t+x) + \frac{1-c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)} \leqslant 0. \tag{3.2}$$

PROOF: $X \leqslant_{HR\uparrow} X^*$, if and only if, for all $x \geqslant 0$,

$$\int_0^t r_{X^*}(u)du - \int_0^{t+x} r_X(u)du \text{ is decreasing in } t.$$

This, by Eq. (2.1), is equivalent to the fact that, for all $x, t \geqslant 0$,

$$\int_0^t r_X(u)du - \int_0^{t+x} r_X(u)du + \int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \int_0^t \frac{c'(u)}{c(u)}du \text{ is decreasing in } t.$$

This shows that Eq. (3.2) holds for all $x, t \geqslant 0$, which is the required condition. ■

We now present an application of Theorem 3.4.

EXAMPLE 3.3: *Let X be a random variable with mean residual life function $m_X(t) = 1/(2+t)$, $t \geqslant 0$. Take $c(t) = 2+t$, for all $t \geqslant 0$. Clearly, $c(t)$ satisfies all the conditions of Lemma 2.1. Now, for $t \geqslant 0$,*

$$r_X(t) - r_X(t+x) + \frac{1-c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)} = -x - \frac{x}{(2+t)(2+t+x)} - \frac{1+t}{2+t} - t < 0.$$

Hence, by Theorem 3.4, $X \leqslant_{HR\uparrow} X^*$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 3.4: *$X \leqslant_{HR\uparrow} X^*$ if and only if $r_X(t) - r_X(t+x) + (1-c)/(c\, m_X(t)) \leqslant 0$.*

The following theorem shows that the model in Eq. (1.3) preserves the down hazard rate ordering. It is useful to remind that a random variable X is said to be smaller than another random variable Y in down hazard rate order (written as $X \leqslant_{HR\downarrow} Y$) if $X \leqslant_{HR} [Y - x | Y > x]$, for all $x \geqslant 0$, or equivalently, if $(\bar{F}_X(t)/\bar{F}_Y(t+x))$ is decreasing in $t \geqslant 0$, for all $x \geqslant 0$.

THEOREM 3.5: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). Then, $X \leqslant_{HR\downarrow} X^*$, if and only if*

$$r_X(t+x) - r_X(t) + \frac{1-c(t+x)}{c(t+x)m_X(t+x)} + \frac{c'(t+x)}{c(t+x)} \leqslant 0. \tag{3.3}$$

Proof: $(\bar{F}_X(t)/\bar{F}_{X^*}(t+x))$ is decreasing in $t \geqslant 0$, if and only if, for all $x \geqslant 0$,

$$\left[\int_0^{t+x} r_{X^*}(u)du - \int_0^t r_X(u)du\right] \text{ is decreasing in } t \geqslant 0,$$

for all $x \geqslant 0$. Upon using Eq. (2.1), the above expression reduces to the fact that, for all $x \geqslant 0$,

$$\left[\int_0^{t+x} r_X(u)du - \int_0^t r_X(u)du + \int_0^{t+x} \frac{1-c(u)}{c(u)m_X(u)}du + \int_0^{t+x} \frac{c'(u)}{c(u)}du\right] \text{ is decreasing in } t \geqslant 0.$$

Differentiation of the above expression gives the required condition. ■

Example 3.4: *Let X be a random variable with mean residual life function $m_X(t) = 1+t$, $t \geqslant 0$. Take $c(t) = (2+t)/(1+t)$, for all $t \geqslant 0$. Clearly, $c(t)$ satisfies all the conditions of Lemma 2.1. Now, for $t \geqslant 0$,*

$$\begin{aligned} r_X(t+x) - r_X(t) + \frac{1-c(t+x)}{c(t+x)m_X(t+x)} + \frac{c'(t+x)}{c(t+x)} \\ = -\frac{2x}{(1+t)(1+t+x)} - \frac{2}{(1+t+x)(2+t+x)} < 0. \end{aligned}$$

Hence, by Theorem 3.5, $X \leqslant_{HR\downarrow} X^*$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

Corollary 3.5: *$X \leqslant_{HR\downarrow} X^*$, if and only if $r_X(t+x) - r_X(t) + (1-c)/(c\, m_X(t+x)) \leqslant 0$.*

The following theorem shows that the model in Eq. (1.3) preserves the increasing convex ordering. It is useful to remind that a random variable X is said to be larger than another random variable Y in increasing convex ordering (written as $X \geqslant_{ICX} Y$) if $\int_t^\infty \bar{F}_X(x)dx \geqslant \int_t^\infty \bar{F}_Y(x)dx$, for all $t \geqslant 0$.

Theorem 3.6: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). Then, $X \geqslant_{ICX} X^*$, if and only if $\int_0^t (1-c(x))/(c(x)m_X(x))dx - \ln c(0) \geqslant 0$, for all $t \geqslant 0$.*

Proof: $X \geqslant_{ICX} X^*$, if and only if, for all $t \geqslant 0$,

$$\int_t^\infty \bar{F}_X(x)dx \geqslant \int_t^\infty \bar{F}_{X^*}(x)dx,$$

which, by Lemma 2.3, is equivalent to

$$m_X(t)\frac{E(X)}{m_X(t)} \exp\left[-\int_0^t \frac{dx}{m_X(x)}\right] \geqslant c(t)m_X(t)\frac{c(0)E(X)}{c(t)m_X(t)} \exp\left[-\int_0^t \frac{dx}{c(x)m_X(x)}\right].$$

Now, the required result follows upon simplification. ■

We now present an application of Theorem 3.6.

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Example 3.5: *Let us reconsider Example 3.1. Then, we can easily verify that, for all $t \geqslant 0$,*

$$\int_0^t \frac{1-c(x)}{c(x)m_X(x)}dx - \ln c(0) \geqslant 0.$$

Hence, by Theorem 3.6, $X \geqslant_{ICX} X^*$.

By taking $c(t) = c$, a constant, we obtain the following corollary.

Corollary 3.6: *$X \geqslant_{ICX} X^*$, if and only if $\int_0^t (1-c)/(c\, m_X(x))dx \geqslant \ln c$, for all $t \geqslant 0$.*

The following theorem shows that the model in Eq. (1.3) preserves the increasing concave ordering. It is useful to remind that a random variable X is said to be larger than another random variable Y in increasing concave ordering (written as $X \geqslant_{ICV} Y$) if $\int_0^t \bar{F}_X(x)dx \geqslant \int_0^t \bar{F}_Y(x)dx$, for all $t \geqslant 0$.

Theorem 3.7: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). Then, $X \geqslant_{ICV} X^*$, if and only if*

$$1 - \exp\left[-\int_0^t \frac{dx}{m_X(x)}\right] \geqslant c(0)\left(1 - \exp\left[-\int_0^t \frac{dx}{c(x)m_X(x)}\right]\right), \quad \textit{for all } t \geqslant 0.$$

Proof: $X \geqslant_{ICV} X^*$, if and only if, for all $t \geqslant 0$,

$$\int_0^t \bar{F}_X(x)dx \geqslant \int_0^t \bar{F}_{X^*}(x)dx,$$

which reduces to

$$E(X) - \int_t^\infty \bar{F}_X(x)dx \geqslant E(X^*) - \int_0^t \bar{F}_{X^*}(x)dx,$$

which, by Lemma 2.3, reduces further to

$$E(X) - m_X(t)\frac{E(X)}{m_X(t)} \exp\left[-\int_0^t \frac{dx}{m_X(x)}\right]$$
$$\geqslant c(0)E(X) - c(t)m_X(t)\frac{c(0)E(X)}{c(t)m_X(t)} \exp\left[-\int_0^t \frac{dx}{c(x)m_X(x)}\right].$$

Now, the required result follows upon simplification. ■

We now present an application of Theorem 3.7.

Example 3.6: *Let us take, for $t \geqslant 0$, $c(t) = 1 + t$, and X as a standard exponential random variable. Then, we can verify that $c(t)$ satisfies all the conditions of Lemma 2.1 and the condition in Theorem 3.7. Hence, we have $X \geqslant_{ICV} X^*$.*

By taking $c(t) = c$, a constant, we obtain the following corollary.

Corollary 3.7: *$X \geqslant_{ICV} X^*$, if and only if $(1 - \exp[-\int_0^t (dx/m_X(x))]) \geqslant c(1 - \exp[-\int_0^t (dx/(c\, m_X(x)))])$, for all $t \geqslant 0$.*

The following theorem shows that the model in Eq. (1.3) preserves the aging intensity ordering. It is useful to remind that a random variable X is said to be larger than another random variable Y in aging intensity ordering ($X \geqslant_{AI} Y$) if, for all $t \geqslant 0$, $(r_X(t)/\int_0^t r_X(u)du) \leqslant (r_Y(t)/\int_0^t r_Y(u)du)$.

Before proceeding to prove the theorem, we give a lemma related to the aging intensity function of a random variable X. The proof of the following lemma is adapted from Nanda, Bhattacharjee, and Alam [11].

Lemma 3.1: *If $r_X(t)$ is strictly increasing in t, then $(\ln \bar{F}_X(t)/r_X(t)) < -t$ for all $t \geqslant 0$.*

Proof: If $r_X(t)$ is strictly increasing in t, then

$$\frac{1}{t} < r_X(t), \text{ giving } \frac{-t\, r_X(t)}{\ln \bar{F}_X(t)} > 1.$$

Hence the result follows. ■

Theorem 3.8: *Let X and X^* be two absolutely continuous random variables satisfying Eq. (1.3). If, for all $t \geqslant 0$,*

(i) $(1 - c(t))/(c(t)m_X(t)) + (c'(t)/c(t)) \geqslant 0$;
(ii) *X is DFR;*
(iii) $(1 - c(t))/(c(t)m_X(t))$ *is increasing in t;*
(iv) $c''(t)c(t) - [c'(t)]^2 \geqslant 0$.

Then, $X \geqslant_{AI} X^*$.

Proof: Note that $X \geqslant_{AI} X^*$, if and only if, for all $t > 0$, $(r_X(t)/\int_0^t r_X(u)du) \leqslant (r_{X^*}(t)/\int_0^t r_{X^*}(u)du)$, which, by Eq. (2.1), reduces to

$$\left[\int_0^t \frac{1 - c(u)}{c(u)m_X(u)}du + \ln\left(\frac{c(t)}{c(0)}\right)\right] + \frac{\ln \bar{F}_X(t)}{r_X(t)}\left(\frac{1 - c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)}\right) \leqslant 0. \quad \textbf{(3.4)}$$

Note, from Theorem 2.1 of Nanda et al. [11], that

$$\begin{aligned}
&\left[\int_0^t \frac{1 - c(u)}{c(u)m_X(u)}du + \ln\left(\frac{c(t)}{c(0)}\right)\right] + \frac{\ln \bar{F}_X(t)}{r_X(t)}\left(\frac{1 - c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)}\right)\\
&\leqslant \left[\int_0^t \frac{1 - c(u)}{c(u)m_X(u)}du + \ln\left(\frac{c(t)}{c(0)}\right)\right] - t\left(\frac{1 - c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)}\right).\\
&\leqslant \ln\left(\frac{c(t)}{c(0)}\right) - \frac{t\, c'(t)}{c(t)} \leqslant 0.
\end{aligned}$$

The second inequality follows from condition (iii) of the theorem, whereas the last inequality follows from condition (iv). ■

We now present an application of Theorem 3.8.

Example 3.7: *Let X be an exponential random variable with mean $1/2$. Let $c(t) = (1 + t)/(2 + 3t)$, for all $t \geqslant 0$. Then, $c(t)$ satisfies all the conditions of Theorem 3.8. Hence, $X \geqslant_{AI} X^*$.*

The following counterexample shows that condition (i) of Theorem 3.8 is not a necessary condition.

Counterexample 3.1: *Let X be an exponential random variable with mean $1/2$. Let $c(t)=(1+2t)/(1+3t)$, for all $t\geqslant 0$. Then, $(1-c(t))/(c(t)m_X(t))+(c'(t)/c(t))=(2t+6t^2-1)/((1+2t)(1+3t))$, which is negative for $t=0$ and positive for $t=1$. Hence, $c(t)$ does not satisfy condition (i) of Theorem 3.8. We can easily verify that all other conditions of Theorem 3.8 are satisfied. But, from Eq. (3.4), we have*

$$\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du+\ln\frac{c(t)}{c(0)}+\frac{\ln\bar{F}_X(t)}{r_X(t)}\left(\frac{1-c(t)}{c(t)m_X(t)}+\frac{c'(t)}{c(t)}\right)$$
$$=\frac{2t+3t^2}{(1+2t)(1+3t)}+\ln\left(\frac{\sqrt{1+2t}}{1+3t}\right)\leqslant 0,$$

for all $t\geqslant 0$. Hence, condition (i) of Theorem 3.8 is not necessary.

The following counterexample shows that condition (ii) of Theorem 3.8 is a sufficient condition but not necessary.

Counterexample 3.2: *Let X be the random variable having the failure rate function $r_X(t)=t/(1+t)$, $t\geqslant 0$. Then the corresponding mean residual life function is given by $m_X(t)=(2+t)/(1+t)$, $t\geqslant 0$. Clearly, X is IFR. Take $c(t)=1/(2+t)$, for all $t\geqslant 0$. We see that $c(t)$ satisfies all the conditions of Lemma 2.1. Now, for $t\geqslant 0$,*

$$\frac{1-c(t)}{c(t)m_X(t)}+\frac{c'(t)}{c(t)}=\frac{(1+t)^2-1}{2+t}\geqslant 0.$$

Therefore, condition (i) of Theorem 3.8 is satisfied. Now, for $t\geqslant 0$,

$$\frac{1-c(t)}{c(t)m_X(t)}=\frac{(1+t)^2}{2+t}$$

is increasing in t. Thus, condition (iii) of Theorem 3.8 is satisfied. Lastly, for $t\geqslant 0$,

$$c''(t)c(t)-[c'(t)]^2=\frac{1}{(2+t)^4}>0.$$

Thus, $c(t)$ satisfies all the conditions of Theorem 3.8 other than (iii). Now, for $t\geqslant 0$,

$$\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du+\ln\frac{c(t)}{c(0)}+\frac{\ln\bar{F}_X(t)}{r_X(t)}\left(\frac{1-c(t)}{c(t)m_X(t)}+\frac{c'(t)}{c(t)}\right)$$
$$=2t+\frac{t^2}{2}-2t+\ln\left(\frac{2+t}{2}\right)-\ln\left(\frac{2+t}{2}\right)+\frac{(1+t)(\ln(1+t)-t)((1+t)^2-1)}{t(2+t)}$$
$$=\frac{t^2}{2}+\frac{(1+t)(\ln(1+t)-t)((1+t)^2-1)}{t(2+t)}$$
$$\leqslant 0.$$

Thus, condition (ii) of Theorem 3.8 is a sufficient condition but not necessary.

The following counterexample shows that condition (iii) of Theorem 3.8 cannot be relaxed.

COUNTEREXAMPLE 3.3: *Let X be a random variable having mean residual life function $m_X(t) = 1 + t$, $t \geqslant 0$. Then the corresponding failure rate function is $r_X(t) = 2/(1 + t)$, $t \geqslant 0$. Clearly, X is DFR. Take $c(t) = 1/(3 + t)$, for all $t \geqslant 0$, which satisfies all the conditions of Lemma 2.1. Now, one can verify that $c(t)$ satisfies conditions (i), (ii) and (iv) but not (iii) of Theorem 3.8. Now, for $0 \leqslant t \leqslant 1$,*

$$\begin{aligned}
&\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \ln\frac{c(t)}{c(0)} + \frac{\ln \bar{F}_X(t)}{r_X(t)}\left(\frac{1-c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)}\right)\\
&= t + \ln\left(\frac{3}{3+t}\right) - \frac{2+3t-t^2}{3+t}\ln(1+t)\\
&= h_1(t),\ \textit{say}.
\end{aligned}$$

We see that $h_1(0) = -0.030654 < 0$ and $h_1(2) = 0.61032 > 0$. Hence, condition (iii) of Theorem 3.8 cannot be dropped.

The following counterexample shows that condition (iv) of Theorem 3.8 is a sufficient condition but not necessary.

COUNTEREXAMPLE 3.4: *Let X be an exponential random variable with mean $1/2$. Take*

$$c(t) = \begin{cases} \dfrac{\sqrt{2-t}}{2}, & 0 \leqslant t \leqslant 1,\\[2mm] \dfrac{1}{2\sqrt{t}}, & t \geqslant 1. \end{cases}$$

Then, clearly $c(t)$ satisfies all the conditions of Lemma 2.1. Further, $c(t)$ satisfies all the conditions of Theorem 3.8 except condition (iv). Now,

$$\begin{aligned}
&\int_0^t \frac{1-c(u)}{c(u)m_X(u)}du + \ln\left(\frac{c(t)}{c(0)}\right) + \frac{\ln \bar{F}_X(t)}{r_X(t)}\left(\frac{1-c(t)}{c(t)m_X(t)} + \frac{c'(t)}{c(t)}\right)\\
&= \begin{cases} 8\sqrt{2} - 8\sqrt{2-t} - 2t + \ln\left(\dfrac{\sqrt{2-t}}{2}\right) - t\left(\dfrac{2(2-\sqrt{2-t})}{\sqrt{2-t}} - \dfrac{1}{2(2-t)}\right) - \ln\dfrac{\sqrt{2}}{2}, & 0 \leqslant t \leqslant 1,\\[2mm] 8\sqrt{2} - \dfrac{61}{6} - \dfrac{4}{3}t^{3/2} - \dfrac{1}{2}\ln(2t), & t \geqslant 1, \end{cases}
\end{aligned}$$

which is negative. Hence, condition (iv) of Theorem 3.8 is a sufficient condition but not necessary.

By taking $c(t) = c$, a constant, we obtain the following corollary.

COROLLARY 3.8: *If $c(t) = c$, for all t. Then, $X \geqslant_{AI} X^*$ if X is DMRL and $0 < c \leqslant 1$.*

PROOF: Note that $X \geqslant_{AI} X^*$, if and only if, for all $t > 0$,

$$r_X(t)\int_0^t \left(\frac{1-c}{c\,m_X(u)} + r_X(u)\right)du \leqslant \left(\frac{1-c}{c\,m_X(t)} + r_X(t)\right)\int_0^t r_X(u)du.$$

Since $0 < c \leqslant 1$, it is sufficient to prove that

$$r_X(t)\int_0^t \frac{du}{m_X(u)} \leqslant \frac{1}{m_X(t)}\int_0^t r_X(u)du.$$

Note, from Lemma 3.1, that $tr_X(t) \leqslant \int_0^t r_X(u)du$. Therefore, it is sufficient to show that

$$\int_0^t \left(\frac{1}{m_X(u)} - \frac{1}{m_X(t)}\right) du \leqslant 0.$$

This holds if X is DMRL. ■

The following counterexample shows that the condition "X is DMRL" of Corollary 3.8 cannot be dropped.

COUNTEREXAMPLE 3.5: *Let us reconsider Counterexample 2.4. Clearly, X is not DMRL. Take $c = 1/2$. Then, for all $0 \leqslant t \leqslant 1$,*

$$\begin{aligned} &r_X(t)\int_0^t r_{X^*}(u)du - r_{X^*}(t)\int_0^t r_X(u)du \\ &= \left(\frac{1+4t}{1+t^2}\right)\int_0^t \frac{\frac{e^{-\tan^{-1}u}}{(1+u^2)^2}}{\frac{e^{-\tan^{-1}u}}{10}\left(5 - \frac{u^2+4u-1}{1+u^2}\right) - \frac{e^{-(\pi/4)}}{5}} du \\ &\quad - \frac{\frac{e^{-\tan^{-1}t}}{(1+t^2)^2}}{\frac{e^{-\tan^{-1}t}}{10}\left(5 - \frac{t^2+4t-1}{1+t^2}\right) - \frac{e^{-(\pi/4)}}{5}} \int_0^t \frac{1+4u}{1+u^2} du \\ &= h_2(t), \mathit{say}. \end{aligned}$$

We see that $h_2(0.5) = 0.4869 > 0$. Hence, the condition "X is DMRL" of Corollary 3.8 cannot be dropped.

4. DATA ANALYSIS

Let us draw a random sample of size 500 from a gamma distribution having shape parameter 2 and scale parameter 1. Call them t_i, $i = 1, 2, \ldots, 500$. The corresponding MRL is given by $m_X(t_i) = (2 + t_i)/(1 + t_i)$, $t_i \geqslant 0$. Let $m_{X^*}(t_i)$ be the MRL corresponding to the random variable X^*, given by $m_{X^*}(t_i) = t^\delta m_X(t)$, with $\delta = 1/2$. Further let ϵ_i be the normally distributed error in $m_{X^*}(t_i)$, having mean zero and variance $(0.002)^2$. Suppose that $x_i = \ln m_X(t_i)$ and $y_i = \ln m_{X^*}(t_i)$, so that $y_i = \delta \ln t_i + x_i + \epsilon_i$. Let $z_i = y_i - x_i = \delta \ln t_i + \epsilon_i$. By using the method of least square, we find $\hat{\delta} = (\sum z_i \ln t_i)/(\sum (\ln t_i)^2)$, which is the estimated value of δ. Thus, $m_{X^*}(t_i) = t_i^{\hat{\delta}} m_X(t_i)$, for all $t_i \geqslant 0$. Then by plotting $(m_{X^*}(t_i)/m_X(t_i)) = t_i^{\hat{\delta}}$ against the corresponding t_i we get a non-linear curve as shown in Figure 1. The data and the calculated values are given in the Appendix.

5. CONCLUDING REMARKS

Here, the results corresponding to proportional MRL model have been generalized to a situation in which the MRL function may not be proportional throughout, but may be piecewise proportional. This concept has also been generalized wherein the constant of proportionality is a continuous function of time, which is appropriate in the context of regression models with time-dependent covariates. Under this proposed model, we have developed conditions under which different aging properties such as IFR, IFRA, NBU, HNBUE, NBUFR, NBAFR, and

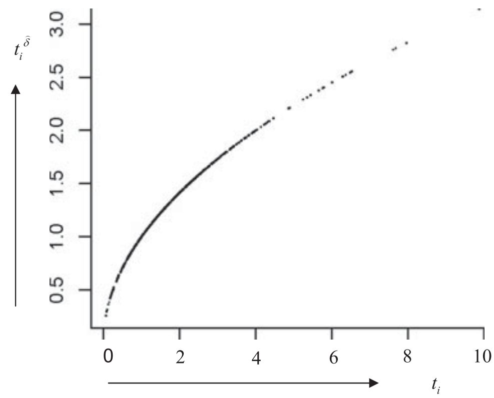

Figure 1. Plot of $t_i^{\hat{\delta}}$ against t_i.

NBUC, and their duals are transmitted from one random variable to another random variable. We have also stated conditions under which different stochastic orders such as the usual stochastic order, hazard rate order, HAMR order, shifted hazard rate order, increasing convex, and increasing concave orders, and ageing intensity order are preserved between the two random variables of the proposed model. While concluding the study, a natural question that arises is under what condition(s) the ILR (increasing in likelihood ratio)/DLR (decreasing in likelihood ratio) properties or the LR (likelihood ratio) ordering will be preserved for the proposed model. Unfortunately, we did not get any simple expression for the condition(s) on $c(t)$ to answer this question, and so this question remains open!

Acknowledgments

The authors thank Prof. Debasis Sengupta, Indian Statistical Institute, Kolkata, for comments and suggestions on an earlier draft. The fruitful suggestions of the referee are also acknowledged by the authors.

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https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press

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APPENDIX

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
5.39267605	1.156429	0.1453368	2.761897e-06	0.987860454
5.91254102	1.144665	0.1351117	3.798430e-06	1.023653315
2.41893748	1.292489	0.2565695	−3.403743e-06	0.698230242
0.88812198	1.529627	0.4250238	7.798126e-06	0.365708491
2.36161283	1.297476	0.2604210	8.192017e-06	0.690101622
1.42235417	1.412822	0.3455888	4.847862e-06	0.521750334
1.51660378	1.397361	0.3345854	−2.848878e-06	0.542819294
3.70971156	1.212327	0.1925418	−7.961779e-07	0.848018109
1.93384854	1.340849	0.2933032	1.610179e-06	0.623060813
0.15080843	1.868954	0.6253791	−2.906611e-06	−0.320496273
2.29688998	1.303316	0.2649119	5.263351e-07	0.680690445
0.29804066	1.770392	0.5712009	−9.808312e-07	−0.034062725
0.52087177	1.657518	0.5053211	−3.290089e-07	0.179195053
1.00108022	1.499730	0.4052852	−3.360785e-06	0.405821611
3.69401960	1.213037	0.1931272	−3.340104e-06	0.846481412
1.84648486	1.351310	0.3010749	−4.642671e-06	0.607712092
0.18768838	1.841972	0.6108366	1.143997e-06	−0.225648418
1.28332095	1.437959	0.3632245	2.049002e-06	0.487952109
1.50864757	1.398621	0.3354869	−1.339926e-06	0.541092326
1.20212482	1.454107	0.3743919	1.483620e-06	0.466438694
0.96425942	1.509098	0.4115119	−1.001251e-06	0.393313482
0.69089364	1.591403	0.4646162	−1.823504e-06	0.279729660
3.58997150	1.217866	0.1971004	2.275828e-06	0.836174782
0.68223981	1.594446	0.4665261	1.862598e-06	0.275340905
1.77509241	1.360348	0.3077408	7.186277e-07	0.594667774
3.09937305	1.243940	0.2182836	−1.096241e-06	0.783882382
4.04525648	1.198206	0.1808254	−1.081103e-06	0.879596817
0.44601109	1.691558	0.5256498	−1.634463e-06	0.121942412
2.20205342	1.312300	0.2717810	−1.988234e-06	0.666474179
3.90723268	1.203781	0.1854673	−1.485521e-06	0.866880503
5.52860838	1.153172	0.1425164	−7.518800e-06	0.997476954
2.76506464	1.265600	0.2355461	8.014346e-06	0.744086094
1.67241629	1.374193	0.3178668	−2.817284e-06	0.574998707
1.39167191	1.418118	0.3493303	3.508282e-06	0.514586750
0.63071032	1.613230	0.4782382	−3.512939e-06	0.247780378
0.78292258	1.560877	0.4452477	−2.412079e-06	0.322884603
0.22135646	1.818762	0.5981559	1.422586e-06	−0.155833113
5.91073934	1.144702	0.1351446	1.623914e-06	1.023531704
3.25560521	1.234984	0.2110582	−9.641555e-07	0.801246311
0.15502152	1.865785	0.6236817	4.474973e-06	−0.308409459
0.49496268	1.668913	0.5121725	−2.095650e-06	0.160533978
0.13950990	1.877570	0.6299785	5.073772e-06	−0.354826247
2.78794438	1.263995	0.2342777	1.306915e-06	0.746931257

(*continued*)

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
4.14606211	1.194323	0.1775798	1.325252e-06	0.888660609
1.11397063	1.473043	0.3873306	3.117104e-07	0.441296347
0.94156460	1.515049	0.4154475	1.010611e-06	0.385342326
3.35267438	1.229744	0.2068059	1.297404e-06	0.811686337
1.58422248	1.386964	0.3271169	−3.240850e-06	0.557160515
0.17045616	1.854368	0.6175438	−3.965489e-06	−0.267098721
1.20311362	1.453903	0.3742517	9.697112e-06	0.466717837
1.07273424	1.482455	0.3936992	3.148536e-06	0.428807697
4.91306806	1.169117	0.1562487	−2.102506e-06	0.952195916
0.91679134	1.521705	0.4198315	3.914037e-06	0.376397770
0.64024922	1.609663	0.4760251	−6.071647e-06	0.253070162
0.22715145	1.814895	0.5960278	−4.638165e-06	−0.145045985
0.38775005	1.720591	0.5426678	−3.565902e-06	0.068967015
1.77698413	1.360103	0.3075603	4.188713e-07	0.595019577
2.18150039	1.314317	0.2733172	4.877692e-06	0.663328545
2.78001466	1.264549	0.2347157	3.514702e-07	0.745944177
0.77248012	1.564181	0.4473625	3.179495e-06	0.318291183
1.96841231	1.336880	0.2903389	1.050577e-06	0.628953552
1.79592963	1.357663	0.3057647	7.737442e-06	0.598533819
1.99510374	1.333878	0.2880907	−1.071802e-06	0.633437630
0.71933508	1.581620	0.4584498	4.377665e-06	0.293740141
1.46476053	1.405719	0.3405489	−2.840982e-06	0.531391908
1.46672680	1.405396	0.3403188	6.954420e-06	0.531842353
0.47786377	1.676652	0.5167992	3.093161e-06	0.147587462
1.91002917	1.343639	0.2953817	1.774261e-06	0.618942760
0.20017741	1.833210	0.6060686	6.073715e-06	−0.198200942
1.51792078	1.397153	0.3344367	−1.902850e-07	0.543107208
2.37477714	1.296316	0.2595263	−3.526884e-06	0.691974562
1.29050439	1.436585	0.3622688	−5.886686e-07	0.489784794
0.75076262	1.571180	0.4518267	−4.401428e-06	0.308489431
4.58079350	1.179186	0.1648244	3.501827e-06	0.925763981
3.55964197	1.219315	0.1982896	2.687068e-07	0.833119860
1.85592191	1.350150	0.3002154	−3.752451e-06	0.609402451
0.91290185	1.522766	0.4205284	−7.189428e-07	0.374964228
0.64538892	1.607759	0.4748413	2.242711e-06	0.255892472
1.86652952	1.348854	0.2992553	−2.515490e-06	0.611293170
2.80294801	1.262954	0.2334533	−3.240231e-06	0.748785961
0.86843480	1.535207	0.4286654	−3.936518e-06	0.358130118
1.48144711	1.402991	0.3386061	−2.202147e-07	0.535115615
1.50112117	1.399821	0.3363442	−1.513809e-06	0.539448776
1.19658640	1.455252	0.3751790	−5.763068e-06	0.464909626
3.77609042	1.209376	0.1901047	−2.486329e-06	0.854446858
1.55690521	1.391098	0.3300932	−2.064438e-06	0.551441161
1.79859221	1.357323	0.3055140	−1.188194e-06	0.599014956
0.82777116	1.547114	0.4363915	1.660588e-06	0.341883938
5.65457208	1.150273	0.1399990	3.683228e-06	1.006234879
0.76100349	1.567858	0.4497104	−3.152380e-07	0.313151403
2.30856941	1.302245	0.2640900	2.865074e-06	0.682406911
1.55626837	1.391195	0.3301633	4.676927e-06	0.551313376
3.77216102	1.209549	0.1902473	4.333649e-06	0.854075650
1.74484689	1.364319	0.3106554	1.238826e-06	0.588990080
1.53567923	1.394372	0.3324439	−8.605601e-06	0.546921668
0.21100158	1.825763	0.6019978	7.334180e-06	−0.175939656
0.81055247	1.552318	0.4397490	−2.471155e-06	0.334726970
1.02193096	1.494577	0.4018430	−1.647786e-06	0.412688361
3.96277325	1.201500	0.1835710	−7.277099e-07	0.872042296
0.90168116	1.525851	0.4225520	3.136072e-06	0.370807948
0.89631289	1.527339	0.4235271	7.763399e-06	0.368801996
2.34009949	1.299392	0.2618967	1.377840e-06	0.686994778
5.65248220	1.150320	0.1400400	2.984000e-06	1.006090390

(*continued*)

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
2.90055559	1.256374	0.2282296	4.355754e-06	0.760685089
1.46267476	1.406063	0.3407933	−7.157892e-06	0.530919518
0.88477807	1.530566	0.4256379	−3.532677e-06	0.364425141
0.58231577	1.631985	0.4897971	1.783100e-06	0.219427706
4.70063003	1.175419	0.1616249	4.011737e-06	0.935477142
3.00245893	1.249846	0.2230207	7.300827e-06	0.772743772
4.96814568	1.167556	0.1549129	1.021554e-05	0.956446423
4.32752507	1.187704	0.1720224	1.359361e-06	0.904521645
3.79811403	1.208415	0.1893098	1.361858e-06	0.856563447
0.77936322	1.561999	0.4459663	−6.365391e-06	0.321320880
0.84024420	1.543406	0.4339918	3.479093e-06	0.346963874
2.68301141	1.271517	0.2402106	6.394159e-06	0.733686937
3.22295519	1.236801	0.2125282	2.498598e-06	0.797680054
4.95811257	1.167838	0.1551545	−2.969389e-06	0.955664107
1.24961212	1.444521	0.3677778	4.738083e-06	0.479199170
1.81492954	1.355249	0.3039850	−3.159907e-06	0.602005128
1.42286651	1.412734	0.3455270	1.740730e-06	0.521865507
1.99498150	1.333892	0.2881009	1.658113e-06	0.633419940
2.00506261	1.332772	0.2872608	6.632289e-07	0.635099117
1.83512823	1.352718	0.3021157	2.890622e-06	0.605675773
2.98619243	1.250866	0.2238361	−4.608061e-06	0.770831044
0.94073356	1.515269	0.4155930	−2.209622e-06	0.385043164
1.35637572	1.424381	0.3537370	2.431386e-06	0.506147556
1.26078740	1.442324	0.3662555	5.622709e-06	0.482129387
2.95538038	1.252820	0.2253972	−5.763971e-06	0.767205077
0.20080589	1.832774	0.6058307	5.731485e-06	−0.196871855
0.31862396	1.758366	0.5643852	−1.511233e-06	−0.007488200
2.04185751	1.328746	0.2842360	−2.236064e-06	0.641163747
0.86949519	1.534904	0.4284677	1.708097e-06	0.358548151
1.21502376	1.451462	0.3725716	6.326849e-08	0.469953498
2.81619241	1.262041	0.2327305	1.365224e-06	0.750424732
1.33898838	1.427535	0.3559494	2.045452e-06	0.501908600
2.75085265	1.266606	0.2363409	−2.077218e-06	0.742294301
0.45623481	1.686702	0.5227754	−8.554982e-06	0.130393014
0.56211148	1.640159	0.4947933	4.414529e-06	0.206770170
2.74596362	1.266954	0.2366156	−4.129848e-06	0.741677500
2.40606654	1.293594	0.2574242	−8.378999e-06	0.696412490
1.46620275	1.405482	0.3403801	−4.992163e-06	0.531713023
2.15932356	1.316523	0.2749945	1.419907e-06	0.659893437
5.55437892	1.152570	0.1419940	−1.439876e-05	0.999272928
1.63458025	1.379567	0.3217698	−3.053689e-06	0.567459726
0.35588887	1.737524	0.5524609	−2.416977e-06	0.035890072
0.81745807	1.550219	0.4383962	−1.119970e-06	0.337617269
1.20165923	1.454203	0.3744579	−4.573487e-06	0.466304987
1.35403445	1.424803	0.3540333	1.130123e-06	0.505578729
2.77101517	1.265181	0.2352149	−4.585198e-06	0.744817150
0.33786883	1.747457	0.5581618	−2.670425e-06	0.015610393
2.11592552	1.320932	0.2783375	−2.355580e-06	0.653081299
2.89629272	1.256654	0.2284528	9.675940e-06	0.760178264
1.30407793	1.434013	0.3604769	1.014011e-06	0.493226003
1.06619565	1.483981	0.3947285	4.588016e-07	0.426777406
2.67509580	1.272102	0.2406705	9.464567e-07	0.732663997
3.76905372	1.209685	0.1903602	1.046699e-06	0.853773194
2.23196997	1.309409	0.2695757	7.157242e-06	0.671025202
0.99111872	1.502230	0.4069508	4.480992e-07	0.402490791
0.83297370	1.545562	0.4353873	7.989732e-07	0.344011510
0.88632361	1.530132	0.4253538	1.001837e-05	0.365027275
1.42318423	1.412680	0.3454887	1.427323e-06	0.521938522
2.15516523	1.316941	0.2753113	−1.273558e-06	0.659243751

(continued)

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
1.03169115	1.492201	0.4002521	−1.554333e-06	0.415850195
6.03882213	1.142069	0.1328417	−2.275366e-06	1.031943941
1.17210811	1.460382	0.3786982	7.312862e-06	0.458107490
0.56173225	1.640315	0.4948881	−1.992200e-06	0.206521094
0.47470804	1.678100	0.5176624	−2.477234e-06	0.145132258
0.40377494	1.712365	0.5378754	−9.233087e-06	0.084417352
1.59646099	1.385140	0.3258009	−5.129818e-06	0.559690463
1.36794165	1.422308	0.3522807	3.159102e-06	0.508937438
2.47952200	1.287396	0.2526214	1.374378e-06	0.706655691
0.73810943	1.575338	0.4544697	1.715439e-06	0.302639827
3.40618977	1.226953	0.2045342	7.036379e-07	0.817332085
1.68855534	1.371947	0.3162309	7.288093e-06	0.578174829
2.17305686	1.315154	0.2739534	−5.179531e-06	0.662015649
0.59967207	1.625128	0.4855867	5.535882e-07	0.229901048
1.38933491	1.418527	0.3496187	−1.693335e-06	0.514029546
1.08926926	1.478636	0.3911202	−3.687697e-06	0.433870055
2.39759815	1.294326	0.2579898	3.154407e-07	0.695223809
2.32810329	1.300471	0.2627268	1.241058e-06	0.685255038
2.90880929	1.255832	0.2277986	9.125242e-08	0.761670608
1.83522184	1.352706	0.3021071	−1.072434e-06	0.605688706
3.25000751	1.235294	0.2113088	3.212892e-07	0.800637732
2.66504185	1.272848	0.2412571	3.254085e-06	0.731370186
0.22299289	1.817666	0.5975534	−1.111393e-05	−0.152765409
5.08312863	1.164389	0.1521966	−3.341363e-06	0.965156697
3.30727381	1.232165	0.2087731	−5.843489e-06	0.806829389
1.20419942	1.453679	0.3740979	2.462541e-06	0.467007845
2.90975976	1.255770	0.2277491	−5.452493e-06	0.761778890
2.41367692	1.292939	0.2569181	−7.582603e-06	0.697486182
1.82616471	1.353836	0.3029424	−4.566949e-06	0.604046776
0.96289970	1.509450	0.4117456	4.384165e-07	0.392843023
1.43033892	1.411465	0.3446283	−4.706855e-06	0.523579350
1.04707693	1.488501	0.3977699	5.788098e-06	0.420776849
3.26419114	1.234511	0.2106750	−5.649685e-06	0.802175340
0.09098296	1.916605	0.6505552	−3.106330e-06	−0.547989439
0.49091592	1.670729	0.5132598	3.434188e-06	0.157522072
1.98264588	1.335273	0.2891356	3.948444e-06	0.631355684
5.01412982	1.166275	0.1538150	−3.581579e-07	0.959944577
1.95021518	1.338958	0.2918919	−5.785832e-06	0.625856015
3.33846456	1.230496	0.2074176	−2.518197e-06	0.810170554
0.17849021	1.848543	0.6143979	1.277220e-06	−0.247211543
1.33250978	1.428723	0.3567809	2.326716e-06	0.500315295
0.81463885	1.551074	0.4389475	−4.632342e-06	0.336437665
1.34541181	1.426364	0.3551288	−4.991170e-06	0.503473881
0.77552042	1.563215	0.4467447	−5.256470e-06	0.319628954
1.58058412	1.387509	0.3275102	−9.893146e-07	0.556406421
1.68783918	1.372046	0.3163031	2.944636e-06	0.578030610
0.70580956	1.586232	0.4613613	4.149190e-06	0.287160573
4.60302158	1.178475	0.1642213	−4.277593e-06	0.927573539
6.33626010	1.136309	0.1277855	3.317798e-07	1.050930181
2.86299834	1.258866	0.2302115	−8.898527e-06	0.756137360
3.92245804	1.203151	0.1849436	−5.785010e-06	0.868297034
1.08980307	1.478514	0.3910375	−5.321757e-06	0.434030704
1.68801713	1.372021	0.3162852	9.266659e-06	0.578071694
4.19560240	1.192470	0.1760272	3.201796e-07	0.893045960
1.41074912	1.414809	0.3469944	−4.942675e-07	0.519054345
4.30178898	1.188616	0.1727893	−3.880498e-06	0.902300860
1.38770715	1.418812	0.3498198	3.798177e-06	0.513650006
1.75686839	1.362730	0.3094903	4.173050e-06	0.591260965
0.36631197	1.731897	0.5492175	2.490190e-06	0.047085018

(*continued*)

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
1.36467555	1.422891	0.3526907	−5.026146e-06	0.508144042
0.58272553	1.631821	0.4896969	−6.889884e-06	0.219670482
0.06777411	1.936528	0.6608965	4.791616e-06	−0.684886211
3.11454514	1.243040	0.2175602	−5.818577e-06	0.785595919
0.70158259	1.587688	0.4622790	−4.962944e-07	0.285070167
2.71275327	1.269342	0.2384986	−1.283164e-06	0.737479333
3.00820292	1.249488	0.2227342	−6.481994e-07	0.773404945
1.85992695	1.349659	0.2998522	5.284837e-06	0.610126067
1.73342528	1.365841	0.3117706	9.030417e-07	0.586821207
3.11752132	1.242865	0.2174188	−2.895605e-06	0.785935067
1.90990426	1.343654	0.2953927	1.772616e-06	0.618917492
1.36634046	1.422593	0.3524816	2.339201e-06	0.508551908
0.28390651	1.778873	0.5759800	−8.403158e-07	−0.053576014
1.98823731	1.334645	0.2886657	−1.568049e-09	0.632289906
2.24558147	1.308111	0.2685843	5.850052e-06	0.673072361
1.16667739	1.461536	0.3794881	−6.973788e-07	0.456567298
1.42535479	1.412311	0.3452272	−2.773662e-06	0.522434841
1.68480986	1.372466	0.3166090	2.067032e-06	0.577437440
1.75728024	1.362676	0.3094506	7.084556e-06	0.591341314
1.35115208	1.425323	0.3543987	−1.005261e-05	0.504867493
6.30259418	1.136938	0.1283384	4.281545e-06	1.048823309
3.14960900	1.240987	0.2159067	5.389539e-07	0.789546375
3.30739775	1.232159	0.2087677	−6.512558e-06	0.806842036
0.39992944	1.714322	0.5390175	1.021484e-05	0.080794143
4.79748983	1.172488	0.1591284	2.835727e-06	0.943177616
2.32268369	1.300962	0.2631036	1.564711e-06	0.684466842
5.53509253	1.153020	0.1423846	−3.398640e-06	0.997935349
0.30000296	1.769229	0.5705439	4.056353e-06	−0.031433538
1.99228649	1.334193	0.2883263	−3.101905e-06	0.632964697
2.92555082	1.254741	0.2269294	−5.702401e-07	0.763670251
0.33057092	1.751557	0.5605052	3.720981e-06	0.007041841
0.58405668	1.631291	0.4893714	−4.416123e-06	0.220488407
2.55612389	1.281205	0.2478011	−2.156148e-06	0.717044927
0.27886046	1.781946	0.5777061	−6.216799e-06	−0.060821992
1.98154199	1.335397	0.2892286	−3.490829e-06	0.631162737
0.76398265	1.566899	0.4490985	2.586148e-06	0.314495994
0.52834587	1.654302	0.5033793	−2.827848e-07	0.184376901
1.68802033	1.372021	0.3162848	−5.414678e-06	0.578057637
2.12238493	1.320268	0.2778348	1.171251e-06	0.654106134
3.29009267	1.233095	0.2095274	2.589235e-06	0.804987884
2.15491761	1.316965	0.2753302	−4.011082e-06	0.659202452
1.48525824	1.402373	0.3381656	3.097824e-07	0.535960207
0.53607193	1.651011	0.5013879	−1.161886e-06	0.189643309
2.02673856	1.330389	0.2854711	3.465027e-06	0.638688507
2.73314136	1.267871	0.2373391	−4.245995e-06	0.740060619
2.43861814	1.290814	0.2552734	1.773392e-06	0.700990956
1.84455072	1.351549	0.3012516	−1.522249e-06	0.607367958
4.20033829	1.192295	0.1758802	−1.330786e-06	0.893461377
0.54857764	1.645754	0.4981986	2.582641e-06	0.197987908
0.64718987	1.607095	0.4744279	3.592149e-06	0.256873736
2.19496470	1.312993	0.2723089	−1.217005e-05	0.665379699
1.96386621	1.337397	0.2907253	−2.926740e-06	0.628179913
2.06629003	1.326127	0.2822627	1.119879e-07	0.645140156
0.74180354	1.574118	0.4536948	−2.643098e-08	0.304359389
1.26522307	1.441458	0.3656548	−4.342197e-06	0.483274723
1.61267079	1.382750	0.3240743	3.146822e-06	0.563023327
2.85157945	1.259634	0.2308210	−2.089470e-06	0.754755428
0.33220971	1.750633	0.5599772	−7.834172e-06	0.008974926
1.54014553	1.393678	0.3319465	2.559644e-07	0.547885173

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
0.75029003	1.571334	0.4519249	1.311966e-07	0.308277276
2.71192700	1.269402	0.2385458	−4.338886e-06	0.737371193
1.78768590	1.358720	0.3065434	5.457392e-06	0.597009879
1.21956429	1.450539	0.3719351	−1.567064e-07	0.471181798
3.06865201	1.245782	0.2197632	−3.025496e-06	0.780379328
0.39862167	1.714990	0.5394070	1.245883e-05	0.079548241
2.78953830	1.263884	0.2341898	−4.027767e-08	0.747127836
1.52286542	1.396375	0.3338794	5.785374e-08	0.544176281
2.42190143	1.292235	0.2563734	1.819985e-06	0.698651714
1.16637239	1.461601	0.3795325	1.239956e-06	0.456482962
4.22716047	1.191308	0.1750522	−3.397054e-06	0.895814091
4.16668795	1.193548	0.1769300	5.421376e-07	0.890491314
6.78184928	1.128504	0.1208930	−4.539024e-06	1.078013374
0.47242920	1.679150	0.5182876	4.494022e-06	0.143358398
2.23163606	1.309441	0.2696002	4.628460e-06	0.670972281
1.26416183	1.441665	0.3657984	4.643850e-06	0.483007682
0.85022505	1.540475	0.4320907	2.968697e-06	0.350966545
1.93906333	1.340244	0.2928520	1.145605e-06	0.623955690
1.43286040	1.411039	0.3443262	−5.643709e-06	0.524156884
1.38231614	1.419760	0.3504875	3.099804e-06	0.512370868
1.77988258	1.359727	0.3072843	−9.539229e-07	0.595557001
1.20945167	1.452601	0.3733557	1.758368e-06	0.468441037
1.52470520	1.396086	0.3336725	3.736743e-07	0.544573412
3.26525250	1.234453	0.2106277	3.671426e-06	0.802299939
6.41598692	1.134844	0.1264950	1.770191e-06	1.055893224
0.90411969	1.525177	0.4221105	−8.908594e-06	0.371704843
4.15238322	1.194085	0.1773802	4.109246e-06	0.889225482
0.92474221	1.519550	0.4184143	−1.677877e-06	0.379292489
1.20472258	1.453572	0.3740238	−3.433703e-07	0.467148155
1.15722867	1.463558	0.3808703	2.037419e-06	0.453886336
3.03202558	1.248014	0.2215537	2.564993e-06	0.776171746
0.42328543	1.702600	0.5321564	3.200427e-06	0.102305277
4.91086007	1.169180	0.1563027	−4.174743e-06	0.952023121
1.09763346	1.476728	0.3898286	−2.470734e-06	0.436404394
3.59933563	1.217423	0.1967361	3.837785e-06	0.837114566
2.77679231	1.264775	0.2348942	3.428473e-06	0.745545839
2.06325446	1.326450	0.2825063	−3.472722e-06	0.644645156
1.02605990	1.493569	0.4011684	4.826844e-07	0.414031986
1.49191361	1.401298	0.3373990	2.970188e-07	0.537429062
2.02560323	1.330513	0.2855643	−2.684391e-06	0.638495374
2.54739863	1.281897	0.2483408	1.190419e-06	0.715878322
1.29959699	1.434859	0.3610665	5.165570e-06	0.492098722
2.66762749	1.272656	0.2411059	−1.000487e-06	0.731699666
0.81990425	1.549479	0.4379190	1.165003e-05	0.338646832
1.65687366	1.376382	0.3194585	1.052374e-06	0.571925772
5.07823262	1.164522	0.1523103	1.202887e-06	0.964793129
1.82279191	1.354259	0.3032546	2.714156e-06	0.603441973
2.46523775	1.288580	0.2535412	4.332210e-06	0.704689675
5.49146142	1.154049	0.1432762	8.393878e-07	0.994874257
1.91016418	1.343623	0.2953699	−1.730434e-06	0.618962731
1.19631990	1.455307	0.3752169	2.543710e-06	0.464844521
0.33617056	1.748407	0.5587053	2.214379e-06	0.013639241
0.50108245	1.666186	0.5105371	−3.203477e-06	0.165041623
2.74103097	1.267306	0.2368934	7.339242e-06	0.741067780
1.59500216	1.385356	0.3259572	−6.205882e-06	0.559388583
2.94113725	1.253734	0.2261262	−1.718051e-07	0.765524185
3.17556005	1.239489	0.2146991	7.564082e-07	0.792441824
3.37519894	1.228561	0.2058436	−3.478311e-06	0.814067250
0.53492419	1.651498	0.5016827	5.263285e-06	0.188872878

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
0.53054163	1.653363	0.5028117	6.125714e-07	0.185883872
5.28555839	1.159095	0.1476394	−4.151769e-06	0.980124386
1.80955231	1.355929	0.3044865	3.527070e-08	0.601026299
1.77281019	1.360645	0.3079588	−6.626705e-06	0.594235174
0.65369423	1.604707	0.4729410	5.566797e-06	0.260388786
0.84851202	1.540976	0.4324158	3.243739e-07	0.350280567
0.42436722	1.702066	0.5318429	5.793592e-06	0.103270625
1.75108404	1.363493	0.3100498	1.175306e-06	0.590168545
0.59677289	1.626263	0.4862848	−8.303876e-06	0.228167195
4.34114564	1.187226	0.1716193	−7.760080e-07	0.905687650
1.04030848	1.490122	0.3988580	5.617623e-06	0.418622233
0.31166252	1.762391	0.5666715	−2.065127e-06	−0.016247695
0.63316030	1.612310	0.4776678	6.145691e-06	0.249158108
0.40111531	1.713717	0.5386648	−4.640521e-06	0.081906962
0.66197727	1.601693	0.4710612	6.122644e-06	0.264805255
3.80995724	1.207902	0.1888850	2.145030e-06	0.857696145
2.22767365	1.309821	0.2698902	1.600165e-06	0.670370749
1.18562295	1.457535	0.3767470	1.817504e-06	0.461882952
0.65241021	1.605177	0.4732338	9.386326e-06	0.259702291
0.41060325	1.708917	0.5358596	7.625430e-07	0.090796408
2.75565556	1.266265	0.2360717	8.934064e-07	0.742900277
1.07813952	1.481200	0.3928523	−2.927377e-06	0.430467848
1.78318944	1.359300	0.3069699	−2.687035e-06	0.596168975
0.38730446	1.720822	0.5428023	−2.187780e-06	0.068527982
2.54976423	1.281709	0.2481942	−4.501669e-06	0.716190174
0.29033103	1.774995	0.5737976	−1.848471e-06	−0.044571052
1.84710516	1.351234	0.3010182	−6.062592e-06	0.607821969
0.77965380	1.561907	0.4459075	5.546275e-06	0.321460431
1.03162689	1.492216	0.4002625	−8.216042e-07	0.415830216
2.84877943	1.259823	0.2309709	−5.159212e-06	0.754411100
2.77449095	1.264936	0.2350218	5.105997e-07	0.745255991
1.09308744	1.477763	0.3905295	−1.526983e-06	0.435031120
3.80808476	1.207983	0.1889520	−4.324701e-06	0.857510908
1.19091750	1.456430	0.3759881	5.187765e-07	0.463350614
1.80272451	1.356796	0.3051258	5.687982e-06	0.599781059
3.02599295	1.248386	0.2218515	9.868867e-07	0.775472090
1.55043129	1.392091	0.3308066	−2.867277e-06	0.550070312
1.74392765	1.364441	0.3107449	2.226287e-06	0.588817037
2.18276373	1.314192	0.2732223	5.942586e-06	0.663524153
0.48984040	1.671213	0.5135496	−3.324252e-06	0.156708462
4.57238631	1.179456	0.1650536	4.732519e-06	0.925075947
0.25138849	1.799112	0.5872934	−4.205648e-07	−0.103084903
1.27462427	1.439633	0.3643883	2.059401e-06	0.485716034
2.36870826	1.296850	0.2599380	−3.763392e-06	0.691106628
2.19533525	1.312956	0.2722812	−8.832314e-07	0.665447743
2.07139127	1.325585	0.2818541	−2.101101e-06	0.645962280
0.97922045	1.505249	0.4089586	7.159088e-07	0.398460090
4.76485981	1.173465	0.1599607	−3.328806e-06	0.940591432
3.28092294	1.233594	0.2099323	5.678733e-06	0.804000318
2.31508015	1.301652	0.2636341	3.707372e-06	0.683359959
3.72843664	1.211486	0.1918480	−3.905024e-06	0.849838630
0.42143555	1.703514	0.5326933	−1.807408e-07	0.100648864
1.31466275	1.432028	0.3590919	−2.209288e-06	0.495879763
2.03238514	1.329773	0.2850086	1.528930e-06	0.639615117
3.62776042	1.216087	0.1956385	−2.842909e-07	0.839945991
2.20163277	1.312341	0.2718123	−1.491172e-06	0.666410420
3.55472030	1.219552	0.1984839	4.518269e-06	0.832626657
1.67955657	1.373196	0.3171409	2.839439e-08	0.576405859
0.96777635	1.508188	0.4109088	3.378801e-06	0.394535068

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Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
2.16678562	1.315778	0.2744278	3.401895e-06	0.661053627
1.12975694	1.469537	0.3849475	2.087232e-06	0.445950831
7.32918539	1.120060	0.1133820	−2.470542e-06	1.109311759
1.81076593	1.355775	0.3043732	−8.038775e-06	0.601240104
2.39913916	1.294192	0.2578867	−6.975173e-06	0.695434684
1.23660628	1.447106	0.3695657	−1.204881e-06	0.475749848
2.82203879	1.261640	0.2324128	−7.838642e-07	0.751141844
1.50216914	1.399653	0.3362245	7.017385e-06	0.539686616
2.23168317	1.309436	0.2695967	8.931896e-07	0.670975656
1.07334831	1.482312	0.3936028	6.498490e-07	0.428994943
0.57490175	1.634960	0.4916185	1.294260e-06	0.214841724
1.07345460	1.482287	0.3935861	−5.784767e-06	0.429021340
0.42451464	1.701993	0.5318002	3.773900e-06	0.103399586
2.31416932	1.301735	0.2636978	1.997134e-06	0.683225182
4.70824293	1.175185	0.1614258	3.740303e-07	0.936083568
1.27192289	1.440156	0.3647513	−1.186116e-06	0.485015026
1.21882741	1.450689	0.3720383	5.617063e-06	0.470988517
1.12579164	1.470413	0.3855433	2.629676e-06	0.444789170
2.32183846	1.301038	0.2631625	−2.373473e-06	0.684339781
0.35533491	1.737825	0.5526343	−1.044007e-07	0.035286971
1.90827188	1.343847	0.2955363	6.949054e-06	0.618642231
1.98071887	1.335490	0.2892979	2.948133e-06	0.631030791
2.15123811	1.317336	0.2756112	−3.980719e-07	0.658632576
1.91028481	1.343609	0.2953593	−3.566676e-06	0.618981870
2.63627820	1.275006	0.2429512	−3.714209e-06	0.727631607
1.23219025	1.447990	0.3701767	4.055813e-07	0.474573769
2.50178500	1.285569	0.2512011	1.110746e-06	0.709704496
2.45405584	1.289515	0.2542660	−6.075099e-06	0.703130935
0.67007698	1.598775	0.4692376	−2.835200e-06	0.269053406
2.24365499	1.308294	0.2687241	4.774879e-07	0.672777738
1.94342000	1.339741	0.2924762	8.288867e-06	0.624709140
4.58003987	1.179210	0.1648449	1.448658e-06	0.925700183
0.77216702	1.564281	0.4474262	9.145973e-08	0.318149119
0.98637047	1.503431	0.4077497	6.261227e-06	0.400894302
1.53611878	1.394303	0.3323949	1.245158e-06	0.547025589
2.02645477	1.330420	0.2854944	8.412888e-06	0.638646724
0.64447423	1.608097	0.4750516	−3.634901e-06	0.255387694
2.06354103	1.326420	0.2824833	5.341109e-06	0.644700391
2.82203460	1.261641	0.2324131	3.307648e-06	0.751145421
3.52526431	1.220982	0.1996551	4.396780e-06	0.829637203
0.69552889	1.589786	0.4635997	−1.120929e-07	0.282058237
0.91009412	1.523534	0.4210329	6.583076e-07	0.373929937
5.15276572	1.162529	0.1505974	−1.156573e-05	0.970352641
0.55399344	1.643503	0.4968302	1.384611e-06	0.201530329
1.12397866	1.470815	0.3858163	5.128225e-06	0.444258856
5.75783974	1.147976	0.1380006	−6.691997e-06	1.013275125
2.71602924	1.269104	0.2383115	−4.063147e-06	0.737892917
2.52451486	1.283727	0.2497676	7.760933e-06	0.712799776
0.79046053	1.558516	0.4437338	2.139719e-06	0.326166144
2.10496226	1.322065	0.2791950	2.702095e-06	0.651346460
4.66734139	1.176450	0.1625011	2.329736e-06	0.932798208
1.99677294	1.333692	0.2879512	−1.696990e-06	0.633715725
0.54964792	1.645308	0.4979275	2.758646e-06	0.198691609
0.24907622	1.800592	0.5881153	−4.427145e-06	−0.106887288
0.35986709	1.735366	0.5512183	−3.437985e-06	0.040204644
0.14330274	1.874659	0.6284268	−3.924239e-06	−0.342975073
2.41930804	1.292457	0.2565449	6.524953e-07	0.698286364
2.34399824	1.299043	0.2616280	7.903447e-06	0.687564972
1.45343872	1.407591	0.3418799	−5.056729e-06	0.528840948

(*continued*)

https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press

Appendix *continued*

t_i	$m_X(t_i)$	x_i	ϵ_i	y_i
5.54397186	1.152812	0.1422045	4.360502e-06	0.998564470
0.45236993	1.688530	0.5238582	−5.253238e-06	0.127225466
1.32149977	1.430756	0.3582030	−6.581913e-07	0.497585972
0.95758602	1.510833	0.4126613	1.988330e-06	0.390993435
0.56463235	1.639128	0.4941643	6.261272e-06	0.208380284
2.47333998	1.287907	0.2530187	−6.379410e-06	0.705797016
0.52850496	1.654234	0.5033381	3.021005e-06	0.184489572
3.03664173	1.247731	0.2213264	3.285014e-06	0.776705836
2.74597922	1.266953	0.2366147	7.670747e-06	0.741691264
0.42849369	1.700038	0.5306507	−3.286973e-06	0.106907736
0.60111491	1.624565	0.4852400	−2.709398e-06	0.230752665
0.39395807	1.717382	0.5408009	1.383456e-06	0.075046846
3.55397659	1.219588	0.1985133	−8.618222e-07	0.832546056
1.37236102	1.421521	0.3517274	1.513078e-06	0.509995255
1.99344148	1.334064	0.2882297	5.162201e-07	0.633161446
8.01885485	1.110879	0.1051514	1.021878e-05	1.146059465
2.00409313	1.332879	0.2873414	1.789134e-06	0.634939001
2.08371746	1.324284	0.2808719	−4.329152e-06	0.647944319
6.25161910	1.137900	0.1291847	4.503924e-06	1.045609415
3.27969672	1.233661	0.2099865	1.508382e-06	0.803863495
2.23646124	1.308979	0.2692478	−3.277294e-06	0.671691913
1.68775702	1.372057	0.3163114	−2.504968e-06	0.578009111
2.25098173	1.307599	0.2681929	−4.842342e-06	0.673871311
1.61243247	1.382785	0.3240996	1.960830e-06	0.562973498
4.43442695	1.184012	0.1689087	−1.750562e-06	0.913606155
1.81197898	1.355621	0.3042600	9.216207e-07	0.601470699
3.87240130	1.205238	0.1866767	−5.137326e-06	0.863609000
2.54150559	1.282366	0.2487066	5.480099e-07	0.715085517
6.33427600	1.136346	0.1278179	2.943902e-06	1.050808653
4.70775435	1.175200	0.1614386	1.808085e-06	0.936045874
0.39339763	1.717670	0.5409689	2.640318e-06	0.074504303
1.44303377	1.409327	0.3431124	5.533036e-06	0.526491752
1.65557986	1.376566	0.3195917	−2.936749e-06	0.571664417
2.15466064	1.316991	0.2753498	−4.519764e-06	0.659161920
0.13201116	1.883384	0.6330699	−2.848489e-07	−0.379364778
1.25637882	1.443188	0.3668546	−4.212358e-06	0.480967153
3.85472646	1.205985	0.1872965	2.222431e-06	0.861948765
0.93557807	1.516642	0.4164984	−2.452985e-07	0.383202777
1.25724375	1.443018	0.3667369	2.308689e-06	0.481200093
4.05627201	1.197774	0.1804650	−2.931468e-06	0.880594209
1.57129872	1.388909	0.3285182	−2.062173e-06	0.554467389
0.72544027	1.579562	0.4571477	1.203415e-06	0.296660668
4.97182975	1.167453	0.1548243	−3.500191e-06	0.956714802

We see that $\sum z_i \ln t_i = 214.4555$, $\sum(\ln t_i)^2 = 428.9111$. Then $\hat{\delta} = \frac{\sum z_i \ln t_i}{\sum (\ln t_i)^2} = 0.4999999$.

https://doi.org/10.1017/S0269964813000259 Published online by Cambridge University Press