Adv. Appl. Prob. **48,** 972–988 (2016)
*doi:*10.1017/apr.2016.61

CRITERION FOR UNLIMITED GROWTH OF CRITICAL MULTIDIMENSIONAL STOCHASTIC MODELS

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Abstract

We give a criterion for unlimited growth with positive probability for a large class of multidimensional stochastic models. As a by-product, we recover the necessary and sufficient conditions for recurrence and transience for critical multitype Galton–Watson with immigration processes and also significantly improve some results on multitype size-dependent Galton–Watson processes.

Keywords: Lyapunov function; martingale; stochastic difference equation; multitype size-dependent Galton–Watson process; critical multitype Galton–Watson process with immigration;

2010 Mathematics Subject Classification: Primary 60J42
Secondary 60J80; 60J10

1. Introduction

We study conditions on possible unlimited growth for sequences of random vectors $\boldsymbol{X}_n$, taking values in $\mathbb{R}_+^d$, which verify the stochastic difference equation

$$\boldsymbol{X}_{n+1} = \boldsymbol{X}_n M + g(\boldsymbol{X}_n) + \xi_n, \qquad n \in \mathbb{N}, \tag{1}$$

where M is a nonnegative primitive $d \times d$ matrix, $g\colon \mathbb{R}_+^d \to \mathbb{R}_+^d$ is a function such that $\|g(\boldsymbol{x})\| = o(\|\boldsymbol{x}\|)$ when $\|\boldsymbol{x}\|$ tends to ∞, and (ξ_n) is a sequence of random vectors (taking values in $\mathbb{R}^d$) such that almost surely (a.s.)

$$\mathbb{E}(\xi_n \mid \mathcal{F}_n) = 0,$$

where $\{\mathcal{F}_n,\ n \in \mathbb{N}\}$ is the natural filtration associated to $(\boldsymbol{X}_n)$. We assume that $\boldsymbol{X}_0 \in \mathbb{R}_+^d$ and that random vectors ξ_n are such that for all n, $\boldsymbol{X}_n$ takes values in $\mathbb{R}_+^d$ a.s.

The Perron–Frobenius Theorem [12, pp. 3–4] states that M has a positive Perron root ρ (which is also the spectral radius of M). We call $\boldsymbol{X}_n$ 'subcritical' if $\rho < 1$, 'supercritical' if $\rho > 1$, and 'critical' if $\rho = 1$. In the 'subcritical' case, one has $\mathbb{P}(\|\boldsymbol{X}_n\| \to \infty \text{ as } n \to \infty) = 0$ since $\|\boldsymbol{X}_n\|$ is bounded in mean. In many applications, one has $\mathbb{P}(\|\boldsymbol{X}_n\| \to \infty \text{ as } n \to \infty) > 0$ in the 'supercritical' case. This is well known for the multitype Galton–Watson process with immigration, for instance. However, this is not necessarily the case in our general framework. For example, if $g(\boldsymbol{X}_n) = 0$ and $\xi_n = \boldsymbol{X}_n M_n$ with M_n independent and identically distributed random matrices such that $\mathbb{P}(M_n = M) = \mathbb{P}(M_n = -M) = \frac{1}{2}$, then $\mathbb{P}(\boldsymbol{X}_n \to 0) = 1$.

In this paper, we focus on the 'critical' case, henceforth $\rho = 1$. We define the normalized right and left eigenvectors u and v associated to ρ in such a way that $vu = u^\top u = 1$.

Received 26 March 2015; revision received 19 January 2016.
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We assume that the sequence $(\boldsymbol{X}_n)$ obeys a weak form of the Markov property. More precisely, we assume that $\mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n)$ is a function of $\boldsymbol{X}_n$ and will use the notation

$$\sigma^2(\boldsymbol{X}_n) = \mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n).$$

The process $(\boldsymbol{X}_n)$ need not be a Markov chain because the law of ξ_n may depend on $(\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_n)$. However, all our examples are Markov chains.

The $d = 1$ case is well understood. The interesting phenomenon is the fact that whether the growth is unlimited depends on both the 'drift' (i.e. $g(\boldsymbol{X}_n)$) and the 'variance' $\sigma^2(\boldsymbol{X}_n)$. This was first noted by Lamperti [10] whose result was generalized by Kersting [6]. But, to the best of the author's knowledge, there is no criterion when $d > 1$. Only particular examples were studied. For instance, Klebaner [8], [9] gave sufficient conditions for unlimited growth or extinction for state-dependent multitype Galton–Watson processes. However, we can build some simple processes which do not satisfy his conditions. Gonzalez *et al.* [1] also gave conditions for unlimited growth in the supercritical case. Jagers and Sagitov [4] investigated population-size-dependent demographic processes that are particular cases of multidimensional growth models. Moreover in the critical case, they restricted themselves to bounded reproduction and bounded 'drift'.

Our aim in this paper is to obtain a criterion in any finite dimension that is analogous to the one in dimension one, which is our main result. The strategy of the proof is the same as in Kersting [6]. We shall illustrate our criterion with several classes of examples, notably the one studied by Klebaner [9] for which we obtain a complete picture (except for a very special case).

Under technical assumptions on functions g and σ^2, we prove in this paper that the process stays bounded almost surely if

$$\limsup_{r\to+\infty} \frac{2rg(rv)u}{\sigma^2(rv)} < 1, \tag{2}$$

while it tends to ∞ with positive probability if

$$\liminf_{r\to+\infty} \frac{2rg(rv)u}{\sigma^2(rv)} > 1. \tag{3}$$

This criterion is reminiscent of the criterion in Kersting [6] for unidimensional models. In fact, the matrix M preserves the component of $\boldsymbol{X}_n$ along the direction v whereas it contracts along others directions.

In Section 2 we state our main result and its proof. In Section 3 we apply it in order to recover a recurrence-transience criterion for critical multitype Galton–Watson processes with immigration and to improve a criterion of almost sure extinction for state-dependent multitype Galton–Watson processes. In the last section, we prove some lemmas which are used in the proof of Theorem 1.

2. Criterion for unlimited growth

2.1. Assumptions

For a row vector $\boldsymbol{x}$, let $\boldsymbol{y} = \boldsymbol{x}(I - uv)$. We assume that there exists a real number α such that $-1 < \alpha < 1$, some positive real numbers c_i, d_i and some real-valued functions f_i and h_i defined on $\mathbb{R}^d$, $i \in \{1, 2\}$, such that

(A1) $g(\boldsymbol{x})u = c_1(\boldsymbol{x}u)^\alpha + h_1(\boldsymbol{y}) + f_1(\boldsymbol{x})$ and $\sigma^2(\boldsymbol{x}) = d_1(\boldsymbol{x}u)^{1+\alpha} + h_2(\boldsymbol{y}) + f_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d_+$, with $h_1 \equiv 0$ if $\alpha \le 0$ and

https://doi.org/10.1017/apr.2016.61 Published online by Cambridge University Press

- $|h_1(\boldsymbol{y})| \le c_2 \|\boldsymbol{y}\|^{\alpha}$,
- $|h_2(\boldsymbol{y})| \le d_2 \|\boldsymbol{y}\|^{1+\alpha}$,
- $f_1(\boldsymbol{x}) = o((\boldsymbol{x}u)^{\alpha})$ when $\|\boldsymbol{x}\| \to +\infty$, and
- $f_2(\boldsymbol{x}) = o((\boldsymbol{x}u)^{1+\alpha})$ when $\|\boldsymbol{x}\| \to +\infty$,

where '$\|\cdot\|$' denotes the Euclidean norm.

We assume that there exist $\delta > 0$ and $A_1 > 0$ such that, for all $n \in \mathbb{N}$ and for all $\boldsymbol{X}_n \in \mathbb{R}_+^d$,

(A2) $\mathbb{E}((\|\xi_n\|)^{2+\delta} \mid \mathcal{F}_n) \le A_1 \sigma^{2+\delta}(\boldsymbol{X}_n)$.

We also need the following condition of unboundedness:

(A3) for all $C > 0$, there exists $n \in \mathbb{N}$ such that $\mathbb{P}(\boldsymbol{X}_n u \ge C) > 0$.

Finally, we need two more assumptions on function g and σ^2 to obtain possible unlimited growth for $\boldsymbol{X}_n$. Firstly, that $g(\boldsymbol{x})u$ is bounded away from 0:

(A4) there exists $s_1 > 0$ such that for all $a, b > 0$ such that $s_1 < a < b < \infty$, if $\boldsymbol{x}u \in (a, b)$ then there exists $\varepsilon > 0$ such that $g(\boldsymbol{x})u > \varepsilon$,

and secondly, that σ^2 is not infinite:

(A5) for all $a > 0$, $\sup_{\|\boldsymbol{x}\| < a} \sigma^2(\boldsymbol{x}) < \infty$.

2.2. Main theorem

We now give the criterion of unlimited growth for $\boldsymbol{X}_n$.

Theorem 1. (Unlimited growth criterion.) *We assume (A1) and (A2) hold.*

(i) *If $c_1 < d_1/2$ then $\mathbb{P}(\|\boldsymbol{X}_n\| \to +\infty) = 0$.*

(ii) *If $c_1 > d_1/2$ and (A3), (A4), and (A5) hold then $\mathbb{P}(\|\boldsymbol{X}_n\| \to +\infty) > 0$.*

Compared to (2) and (3), we give a criterion in the special case where g has a dominant term in $(\boldsymbol{x}u)^{\alpha}$. This may seem restrictive, nevertheless most of the applications deal with $\alpha = 0$, which means that g is bounded by a constant. The $c_1 = d_1/2$ case remains unexplored except for critical multitype Galton–Watson processes with immigration under some moment assumptions (see Remark 1 in Section 3).

2.3. Proof of the theorem

The strategy of the proof of the theorem consists in showing that there exist an integer k and a real-valued function L such that

$$\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \le L(\boldsymbol{X}_n u)$$

when $\boldsymbol{X}_n u$ is larger than some constant. Then we build a supermartingale and proceed by using the martingale convergence theorem.

Before proving the theorem, we state two key lemmas providing us with a Lyapunov function. The proofs involve some technical computations and are deferred to Section 4.

Lemma 1. *Let us assume that (A1) and (A2) hold. If $c_1 < d_1/2$, then there exists $s > 0$ and $k \in \mathbb{N}^*$ such that*

$$\mathbb{E}(\log(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \le \log(\boldsymbol{X}_n u) \quad \textit{if } \boldsymbol{X}_n u > s.$$

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Lemma 2. *Let us assume that (A1) and (A2) hold. If $c_1 > d_1/2$, then there exists $s > 0$ and $k \in \mathbb{N}^*$ such that*

$$\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \leq L(\boldsymbol{X}_n u) \quad \text{if } \boldsymbol{X}_n u > s,$$

with $L(x) = (\log x)^{-1}$.

Proof of Theorem 1. Without loss of generality, we assume that, for every $n \in \mathbb{N}$, $\boldsymbol{X}_n u \geq 3$ a.s. (otherwise consider $\boldsymbol{X}_n + 3v$ instead of $\boldsymbol{X}_n$).

(i) We start by considering the case where $c_1 < d_1/2$. Following [6], let us assume that $\boldsymbol{X}_n u \to +\infty$ with positive probability. Let $U_n = \boldsymbol{X}_{nk} u$, then $U_n \to +\infty$ with positive probability, too. Thus, there is a positive integer T such that

$$\mathbb{P}\Big(\inf_{n \geq T} U_n > s,\ U_n \to +\infty\Big) > 0. \tag{4}$$

Let $\tau = \inf\{n \geq T : U_n \leq s\}$ with the convention that $\tau = +\infty$ if $\inf_{n \geq T} U_n > s$. Let

$$V_n = \begin{cases} \log(U_{n+T}) & \text{if } n + T \leq \tau, \\ \log(U_\tau) & \text{otherwise.} \end{cases}$$

Since (V_n) is a positive supermartingale by Lemma 1, it converges a.s. and we obtain a contradiction with (4).

(ii) We now turn to the case where $c_1 > d_1/2$. Let $s > 0$ be large enough, such that the statement of Lemma 2 holds. Let A, B, C, and D be four sets defined as follows:

- $A = \{\limsup_{n\to\infty} \boldsymbol{X}_n u \leq s\}$,
- $B = \{\limsup_{n\to\infty} \boldsymbol{X}_n u < \infty$ and there exists $i \in \{0, \ldots, k-1\}$, $\boldsymbol{X}_{nk+i} u \to R_i$ as $n \to \infty$ with $s < R_i < \infty\}$,
- $C = \{$there exists $i \in \{0, \ldots, k-1\}$, $\boldsymbol{X}_{nk+i} u \to \infty$ as $n \to \infty$ and $\limsup_{n\to\infty} \boldsymbol{X}_{nk+i+1} u < \infty\}$,
- $D = \{\boldsymbol{X}_n u \to \infty$ as $n \to \infty\}$.

We want to prove that $\mathbb{P}(D) > 0$. We first prove that $\mathbb{P}(A \cup B \cup C \cup D) = 1$ (step 1). Secondly, that $\mathbb{P}(A) < 1$ (step 2). Thirdly, that $\mathbb{P}(B) = 0$ (step 3) and, finally, that $\mathbb{P}(C) = 0$ (step 4).

Step 1. Let $i \in \{0, \ldots, k-1\}$. Then $U_{i,n} = \min(L(\boldsymbol{X}_{nk+i} u), L(s))$ is a nonnegative bounded supermartingale which converges a.s. and in mean by Lemma 2. Therefore, either $\boldsymbol{X}_{nk+i} u$ converges to a number greater than s, possibly ∞, or $\limsup_{n\to\infty} \boldsymbol{X}_{nk+i} u \leq s$. So, $\mathbb{P}(A \cup B \cup C \cup D) = 1$.

Step 2. Let us assume that $\mathbb{P}(A) = 1$. Then, for all $i \in \{0, \ldots, k-1\}$, $U_{i,n}$ converges to $L(s)$. Since $\mathbb{E}(\min(L(\boldsymbol{X}_{n+kl} u), L(s))) \leq \mathbb{E}(\min(L(\boldsymbol{X}_n u), L(s)))$, for all $l \in \mathbb{N}$, we obtain

$$\mathbb{E}(\min(L(\boldsymbol{X}_n u), L(s))) \geq L(s).$$

But, by definition, $\min(L(\boldsymbol{X}_n u), L(s)) \leq L(s)$, therefore, $\min(L(\boldsymbol{X}_n u), L(s)) = L(s)$ a.s. or $\boldsymbol{X}_n u \leq s$ for all n, which contradicts (A3).

Thus, $\mathbb{P}(B \cup C \cup D) > 0$.

Step 3. Without loss of generality, let us assume that set B holds with $i = 0$ and let $R = 2\sup \boldsymbol{X}_n u$.

By (A4), there exists $s_1 < R_0$ and $\varepsilon > 0$ such that, for all $\boldsymbol{x}$, such that $\boldsymbol{x}u \in (s_1, R)$, we have $g(\boldsymbol{x})u > \varepsilon$.

Since $\boldsymbol{X}_{nk}u$ converges to R_0, there exists N_0 such that, for all $n \geq N_0$, $\boldsymbol{X}_{nk}u \in (s_1, R)$.

Thus, we can choose N such that, for all $n \geq N$, $\boldsymbol{X}_n u \leq R$ and $g(\boldsymbol{X}_N)u \neq 0$.

Consider now

$$A_n = \sum_{l=0}^{n} g(\boldsymbol{X}_{N+l})u, \qquad M_n = \sum_{l=0}^{n} A_l^{-1}\xi_{N+l}u.$$

One can check that M_n is a martingale.

Further, by (A4) and (A5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}((M_{n+1} - M_n)^2 \mid \mathcal{F}_{N+n+1}) &= \sum_{n=1}^{\infty} A_n^{-2}\sigma^2(\boldsymbol{X}_{N+n}) \\ &\leq C\sum_{n=1}^{\infty} A_n^{-2} \\ &\leq C'\sum_{n=1}^{\infty} A_n^{-2}(A_n - A_{n-1}) \\ &\leq C'\int_{A_1}^{\infty} t^{-2}\,\mathrm{d}t \\ &< \infty.\end{aligned}$$

By a martingale convergence theorem, M_n converges a.s. Since $\boldsymbol{X}_{nk}u$ converges to R, $A_n \to +\infty$, and by Kronecker's lemma, we have

$$\sum_{l=1}^{n} \xi_l u = o(A_n).$$

For $n \geq N$, we obtain the contradiction

$$\boldsymbol{X}_{n+1}u = \boldsymbol{X}_N u + \sum_{l=N}^{n} g(\boldsymbol{X}_l)u + \sum_{l=N}^{n} \xi_l u = A_n + o(A_n) \to +\infty.$$

Thus, $\mathbb{P}(B) = 0$, so with positive probability, there exists $i \in \{0, \ldots, k-1\}$ such that $\boldsymbol{X}_{nk+i}u$ tends to ∞. Let us prove that this implies that $\boldsymbol{X}_n u$ tends to ∞.

Step 4. Without loss of generality, let us assume that $\boldsymbol{X}_{nk}u$ tends to ∞.
Let $\gamma = 2/(1-\alpha)$, $r > s$ and n such that $n^\gamma > 2r$.
Let $\Gamma_n = [n^\gamma, (n+1)^\gamma)$ be a sequence of intervals and $N(l)$ be an increasing sequence of stopping times defined by

$$N(l) = \inf\{n > N(l-1) \text{ such that } \boldsymbol{X}_{nk}u \in [l^\gamma, \infty)\}.$$

By Markov's inequality, we obtain

$$\begin{aligned}\mathbb{P}(\boldsymbol{X}_{N(n)k+1}u \le 2r \mid \boldsymbol{X}_{N(n)k}u \ge n^{\gamma}) &\le \sup_{l\ge n} \mathbb{P}(\xi_{N(n)k}u \le (2r - l^{\gamma}) \mid \boldsymbol{X}_{N(n)k}u \in \Gamma_l) \\ &\le \sup_{l\ge n} \mathbb{P}((\xi_{N(n)k}u)^2 \ge (l^{\gamma} - 2r)^2 \mid \boldsymbol{X}_{N(n)k}u \in \Gamma_l) \\ &\le \sup_{l\ge n} \frac{K l^{\gamma(\alpha+1)}}{l^{2\gamma}} \\ &\le K' \frac{1}{n^2}.\end{aligned}$$

Hence, by the Borel–Cantelli lemma, for any $r > 0$ sufficiently large,

$$\mathbb{P}\Big(\limsup_{n\to\infty} \boldsymbol{X}_{nk+1}u \le r \mid \boldsymbol{X}_{nk}u \to \infty\Big) = 0.$$

Thus, $\boldsymbol{X}_{nk+1}u$ converges to ∞ so $\mathbb{P}(C) = 0$.

Since $\mathbb{P}(A \cup B \cup C \cup D) = 1$ and $\mathbb{P}(B) = \mathbb{P}(C) = 0$; thus, $\mathbb{P}(A \cup D) = 1$. Since $\mathbb{P}(A) < 1$, we have the desired result

$$\mathbb{P}(D) = \mathbb{P}(\boldsymbol{X}_n u \to \infty) > 0. \qquad \square$$

3. Applications

Our applications focus on the $\alpha = 0$ case. This is because we consider population models with finite variance of number of offsprings per individual. Thus, σ has to be of the order of $\boldsymbol{x}u$ and g of the order of a constant. Note also that all models here are Markov chains, although our result is applicable to processes that need not be Markov chains. In the particular case of irreducible Markov chains, the process has an unlimited growth with positive probability if and only if the chain is transient. Conversely, it does not tend to ∞ a.s. if and only if the chain is recurrent.

3.1. Multitype Galton–Watson process with immigration

A first class of processes governed by the stochastic difference equation (1) is given by critical multitype Galton–Watson processes with immigration. Kawazu [5] gave a criterion of recurrence and transience that he proved by using generating functions. We recover here the same result.

Let (Z_n) be a critical multitype Galton–Watson process with immigration with d-types. At generation n, the kth individual of type-i, $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, (Z_n)_i\}$, gives birth to $X_{i,j,k,n}$ individuals of type-j, $j \in \{1, \dots, d\}$. The random vectors $(X_{i,j,k,n})_{j\in\{1,\dots,d\}}$ with $i \in \{1, \dots, d\}$, $k \ge 1$ and $n \in \mathbb{N}$ are independent with distribution depending only on i. For ease of notation, we write $X_{i,j}$ for $X_{i,j,1,1}$.

We assume that, for all $i, j \in \{1, \dots, d\}$, $\mathbb{P}(X_{i,j} = 0) > 0$. Let $M = (\mathbb{E}(X_{i,j}))_{i,j}$ be the mean matrix. We assume that M is a nonnegative primitive matrix. Since the process is critical, the largest eigenvalue of M is 1. Let u (respectively v) the right (respectively the left) eigenvector corresponding to this eigenvalue. At each generation n, $A_n \in \mathbb{N}^d$ individuals immigrate. The random variables A_n are independent and identically distributed, with $\mathbb{P}(A_1 = (0, \dots, 0)) > 0$, $\mathbb{E}(A_1) = a$, and $\operatorname{var}(A_1 u) = \tau^2$. The random variables A_n are also independent of all variables $X_{i,j,k,n}$. Therefore, we have $\mathbb{E}(Z_{n+1}) = \mathbb{E}(Z_n)M + a$.

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We assume that there exists $\delta > 0$ such that, for $(i, j) \in \{1, \dots, d\}^2$,

$$\mathbb{E}(X_{i,j}^{2+\delta}) < +\infty \quad \text{and} \quad \mathbb{E}((A_1 u)^{2+\delta}) < +\infty. \tag{5}$$

Let $\Gamma_i = (\operatorname{cov}(X_{i,j}, X_{i,j'}))_{j,j' \in \{1,\dots,d\}}$ be the matrix of the covariances of offspring distributions. Let

$$V(z) = \sum_{i=1}^{d} z_i \Gamma_i \quad \text{for } z \in \mathbb{R}^d.$$

We obtain the stochastic difference equation

$$Z_{n+1} = Z_n M + a + \xi_n$$

with

$$\xi_n = \Bigg(\sum_{i=1}^{d} \sum_{k=1}^{(Z_n)_i} \{X_{i,j,k,n} - \mathbb{E}(X_{i,j,k,n})\} + A_n e_j - \mathbb{E}(A_n e_j) \Bigg)_{1 \le j \le d},$$

where the $(e_j)_{j \in \{1,\dots,d\}}$ are the standard unit vectors and

$$\mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n) = u^\top V(Z_n) u + \tau^2.$$

Proposition 1. *The process* (Z_n) *is*

- *recurrent if* $2au < u^\top V(v) u$,
- *transient if* $2au > u^\top V(v) u$.

Remark 1. Kawazu [5] obtained the same criterion under weaker assumptions: he did not require $\operatorname{var}(A_1 u) < +\infty$ and (5). He also proved that the process is null recurrent when $2au = u^\top V(v) u$ if $\mathbb{E}(X_{i,j}^2 \log(X_{i,j})) < +\infty$ and $\mathbb{E}(A_1 \log(A_1)) < +\infty$.

Proof of Proposition 1. Firstly, note that

$$\mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n) = u^\top V(Z_n) u + \tau^2 = (Z_n u) u^\top V(v) u + u^\top V(Z_n(I - uv)) u + \tau^2,$$

then recurrence and transience depend on the sign of $2au - u^\top V(v) u$.

Since (A1) is verified with $\alpha = 0$, $c_1 = au$, $h_1 = 0$, $d_1 = u^\top V(v) u$, $h_2(\mathbf{y}) = u^\top V(\mathbf{y}) u$, $f_1 = 0$, and $f_2 = \tau^2$, and (A3), (A4), and (A5) are also verified, we just have to check (A2) to apply Theorem 1.

Let $l \in \mathbb{N}^*$ and $(U_k)_{k \in \{1,\dots,l\}}$ be some random variables independent with zero mean and such that $\mathbb{E}(|U_k|^{2+\delta}) < +\infty$ for all $k \in \{1, \dots, l\}$. We can apply both the Marcinkiewicz–Zygmund [11, p. 108] and Hölder inequalities, i.e. there exists $R > 0$ such that

$$\mathbb{E}\Bigg(\Bigg| \sum_{k=1}^{l} U_k \Bigg|^{2+\delta} \Bigg) \le R\, \mathbb{E}\Bigg(\Bigg(\sum_{k=1}^{l} U_k^2 \Bigg)^{1+(\delta/2)} \Bigg) \le R l^{\delta/2} \mathbb{E}\Bigg(\sum_{k=1}^{l} |U_k|^{2+\delta} \Bigg).$$

Since there are three sums in $\|\xi_n\|$, we now apply three times the latter inequality to verify that

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(A2) holds, i.e.

$$
\begin{aligned}
\mathbb{E}(\|\xi_n\|^{2+\delta} \mid \mathcal{F}_n) &\le 2^{2+\delta}\mathbb{E}\biggl(\biggl(\sum_{j=1}^{d}\biggl(\sum_{i=1}^{d}\sum_{k=1}^{(Z_n)_i}\{X_{i,j,k,n}-\mathbb{E}(X_{i,j,k,n})\}\biggr)^2\biggr)^{1+(\delta/2)} \biggm| \mathcal{F}_n\biggr) \\
&\quad + 2^{2+\delta}\mathbb{E}(\|A_n - a\|^{2+\delta} \mid \mathcal{F}_n) \\
&\le 2^{2+\delta}d^{\delta/2}\mathbb{E}\biggl(\biggl(\sum_{j=1}^{d}\biggl|\sum_{i=1}^{d}\sum_{k=1}^{(Z_n)_i}\{X_{i,j,k,n}-\mathbb{E}(X_{i,j,k,n})\}\biggr|^{2+\delta}\biggr) \biggm| \mathcal{F}_n\biggr) \\
&\quad + 2^{2+\delta}\mathbb{E}(\|A_n - a\|^{2+\delta} \mid \mathcal{F}_n) \\
&\le Rd^{\delta}\mathbb{E}\biggl(\sum_{j=1}^{d}\sum_{i=1}^{d}\biggl|\sum_{k=1}^{(Z_n)_i}\{X_{i,j,k,n}-\mathbb{E}(X_{i,j,k,n})\}\biggr|^{2+\delta} \biggm| \mathcal{F}_n\biggr) \\
&\quad + 2^{2+\delta}\mathbb{E}(\|A_n - a\|^{2+\delta} \mid \mathcal{F}_n) \\
&\le R^2d^{\delta}\mathbb{E}\biggl(\sum_{j=1}^{d}\sum_{i=1}^{d}(Z_n)_i^{\delta/2}\sum_{k=1}^{(Z_n)_i}|X_{i,j,k,n}-\mathbb{E}(X_{i,j,k,n})|^{2+\delta} \biggm| \mathcal{F}_n\biggr) \\
&\quad + 2^{2+\delta}\mathbb{E}(\|A_n - a\|^{2+\delta} \mid \mathcal{F}_n).
\end{aligned}
$$

We now apply (5) to obtain, for sufficiently large $\|Z_n\|$,

$$
\mathbb{E}(\|\xi_n\|^{2+\delta} \mid \mathcal{F}_n) \le C\biggl(\sum_{i=1}^{d}(Z_n)_i^{(2+\delta)/2}\biggr) + D \le C'\sigma^{2+\delta}(Z_n). \qquad \square
$$

3.2. State-dependent multitype Galton–Watson processes

State-dependent Galton–Watson processes were first introduced by Klebaner in [7] and Höpfner in [2]. Höpfner compared the probability generating functions of these processes with those of critical Galton–Watson processes with immigration to obtain a criterion of extinction. However, this idea seems difficult to be transferred to the multitype case. Basically this is because we have to alter the transitions of the Galton–Watson with immigration process for an infinite number of states and, thus, we may change the nature of the process (recurrent or transient). Klebaner [8], [9] defined multitype state-dependent Galton–Watson processes for which he only gave sufficient conditions for extinction. In particular, he could not treat some range of a parameter. In this subsection, we obtain a criterion to infer whether there is a.s. extinction or survival with positive probability (except in a very special case).

Following [9], we define a discrete-time state-dependent multitype Galton–Watson process with d types Z_n by

$$
Z_{n+1} = \biggl(\sum_{i=1}^{d}\sum_{k=1}^{(Z_n)_i} X_{i,j,k,n}(Z_n)\biggr)_{j\in\{1,\dots,d\}},
$$

where $X_{i,j,k,n}(z)$ is the number of type-j offspring of the kth type-i parent when the process is in the state z in time n. Given $Z_n = z$, the kth parent of type-i has a random vector of offspring

$$
(X_{i,1,k,n}(z), \dots, X_{i,d,k,n}(z)), \qquad k = 1, \dots, z_i.
$$

For each $n \in \mathbb{N}$, the offspring vectors of distinct parents ($k = 1, \dots, z_i$, $i = 1, \dots, d$) are independent. Moreover, for a fixed parental type-i, the offspring vectors are identically

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distributed for all n and k, with distribution depending at most on the state z. For the sake of notation clarity, we write $X_{i,j}$ for $X_{i,j,1,1}$. Let

$$M(z) = (\mathbb{E}(X_{i,j}(z)))_{i,j\in\{1,\dots,d\}}$$

be the mean matrix.

We assume that

$$M(z) = M + C(z),$$

where M is a nonnegative primitive matrix with Perron root 1 and corresponding right and left eigenvectors u and v, with $vu = u^\top u = 1$, and $C(z)$ is a nonnegative matrix and we let

$$g(z) = zC(z).$$

We assume that

$$\lim_{\|z\|\to+\infty} g(z) = D \in \mathbb{R}_+^d.$$

Let $\Gamma_i(z) = (\operatorname{cov}(X_{i,j}(z), X_{i,j'}(z)))_{j,j'\in\{1,\dots,d\}}$ be the matrix of the covariances of offspring distributions when the population size is in the state z. We assume that, for all $i \in \{1,\dots,d\}$, $\Gamma_i(z)$ converges to Γ_i when $\|z\|$ converges to ∞.

Let

$$\tilde{V}(z) = \sum_{i=1}^{d} z_i \Gamma_i(z)$$

be the conditional dispersion matrix of the next generation when the population is in the state z. We also introduce the quantity

$$V(z) = \sum_{i=1}^{d} z_i \Gamma_i.$$

Then (Z_n) satisfies the stochastic difference equation

$$Z_{n+1} = Z_n M + g(Z_n) + \xi_n,$$

with

$$\xi_n = \Bigg(\sum_{i=1}^{d} \sum_{k=1}^{(Z_n)_i} \{X_{i,j,k,n}(Z_n) - \mathbb{E}(X_{i,j,k,n}(Z_n))\}\Bigg)_{j\in\{1,\dots,d\}}.$$

One can easily check that $\mathbb{E}(\xi_n u \mid \mathcal{F}_n) = 0$ and

$$\mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n) = u^\top \tilde{V}(Z_n)u = (Z_n u)u^\top V(v)u + u^\top V(Z_n(I-uv))u + f(Z_n),$$

where the function f is such that $f(x) = o(\|x\|)$ when $\|x\|$ tends to ∞.

We assume that there exist $\delta > 0$ and $K > 0$ such that, for all $i, j \in \{1,\dots,d\}$ and $z \in \mathbb{R}_+^d$,

$$\mathbb{E}(X_{i,j}(z)^{2+\delta}) < K.$$

As in the previous example, the assumption (A2) is a consequence of the Marcinkiewicz–Zygmund and Hölder inequalities.

We make the usual assumptions when one has in mind a population process: 0 is an absorbing state and all states in $\mathbb{N}^d \setminus \{0\}$ communicate.

Theorem 2. *If*

$$\frac{2Du}{u^\top V(v)u} < 1$$

then the process becomes extinct a.s.

If

$$\frac{2Du}{u^\top V(v)u} > 1$$

then the process survives with positive probability.

We cannot treat the $2Du/u^\top V(v)u = 1$ case.

We now illustrate this result by the following example.

Example 1. We take the example of a two-type cell division process from [9]. We recall that $X_{i,j,k,n}(z)$ is the number of children of type-j for the kth parent of type-i at generation n when the population is at state z. Again, we write $X_{i,j}(z)$ for $X_{i,j,1,1}(z)$.

We assume that $X_{i,j}(z)$ take values 0 or 1 with probabilities $p_{i,j}(z)$ and that $\mathbb{P}(X_{i,1}(z) = 0, X_{i,2}(z) = 0) > 0$, $i \in \{1, 2\}$. Let $b_i(z) = \mathbb{P}(X_{i,1}(z) = 1, X_{i,2}(z) = 1)$, $i \in \{1, 2\}$ and $a_{i,j}(x)$, $i, j \in \{1, 2\}$, be arbitrary functions nonvanishing for $x > 0$, such that

$$M(z) = \begin{pmatrix} p & 1-p \\ p' & 1-p' \end{pmatrix} + \begin{bmatrix} \dfrac{c_1 a_{1,1}(z_1)}{z_1 a_{1,1}(z_1) + z_2 a_{2,1}(z_2)} & \dfrac{c_2 a_{1,2}(z_1)}{z_1 a_{1,2}(z_1) + z_2 a_{2,2}(z_2)} \\ \dfrac{c_1 a_{2,1}(z_2)}{z_1 a_{1,1}(z_1) + z_2 a_{2,1}(z_2)} & \dfrac{c_2 a_{2,2}(z_2)}{z_1 a_{1,2}(z_1) + z_2 a_{2,2}(z_2)} \end{bmatrix},$$

where $p, p' \in (0, 1)$ and $c_1, c_2 > 0$. We assume that $b_i(z) \sim b_i$ when $\|z\|$ tends to ∞.

With the previous notation, we have

- $u = 1/\sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v = \sqrt{2}\big(p'/(1-p+p') \quad (1-p)/(1-p+p')\big)$,
- $Du = (c_1 + c_2)/\sqrt{2}$,
- and

$$V(z) = z_1 \begin{pmatrix} p(1-p) & b_1 - p(1-p) \\ b_1 - p(1-p) & p(1-p) \end{pmatrix} + z_2 \begin{pmatrix} p'(1-p') & b_2 - p'(1-p') \\ b_2 - p'(1-p') & p'(1-p') \end{pmatrix}.$$

Corollary 1. *If*

$$c_1 + c_2 < \frac{p'}{1-p+p'} b_1 + \frac{1-p}{1-p+p'} b_2$$

then the process becomes extinct a.s.

If

$$c_1 + c_2 > \frac{p'}{1-p+p'} b_1 + \frac{1-p}{1-p+p'} b_2$$

then the process survives with positive probability.

In [9], Klebaner proved almost sure extinction if $c_1 + c_2 < \min(b_1, b_2)$ and survival with positive probability if $c_1 + c_2 > \max(b_1, b_2)$. Thus, we have improved his result since we prove that the critical value for $c_1 + c_2$ is $(p'/(1-p+p'))b_1 + ((1-p)/(1-p+p'))b_2$. Except for the equality case, we obtain a complete picture of the fate of the process.

4. Proof of Lemmas 1 and 2

In this section we prove Lemmas 1 and 2. The proof is based upon the following result. Let

$$Y_n = X_n - (X_n u)v$$

be the population vector minus the contribution along the eigenvector v. For later convenience, we set $\Delta_{n,k} = X_{n+k}u - X_n u$.

Lemma 3. *Let us assume that (A1) and (A2) hold. There exist $c_2' \geq 0$ and $d_2' > 0$ such that, for all integers $n, k \geq 1$ and for all $\varepsilon > 0$,*

$$|\mathbb{E}(\Delta_{n,k} \mid \mathcal{F}_n) - c_1 k (X_n u)^{\alpha}| \leq c_2' \|Y_n\|^{\alpha} + o((X_n u)^{\alpha}), \tag{6}$$

$$|\mathbb{E}(\Delta_{n,k}^2 \mid \mathcal{F}_n) - k d_1 (X_n u)^{1+\alpha}| \leq d_2' \|Y_n\|^{1+\alpha} + o((X_n u)^{1+\alpha}), \tag{7}$$

$$\mathbb{E}(|\Delta_{n,k}|^2 \mathbf{1}_{\{\Delta_{n,k} \geq \varepsilon X_n u\}} \mid \mathcal{F}_n) = \mathcal{O}((X_n u)^{1+\alpha+((\alpha-1)/2)\delta}), \tag{8}$$

with $c_2' = 0$ if $\alpha \leq 0$.

The proof of this lemma is based upon two technical lemmas that we state and prove first.

Lemma 4. *Let us assume that (A1) and (A2) hold. For all $\alpha \in (-1, 1)$ and $n, i \in \mathbb{N}$,*

$$\mathbb{E}((X_{n+i}u)^{\alpha} \mid \mathcal{F}_n) = (X_n u)^{\alpha} + o((X_n u)^{\alpha}),$$

and

$$\mathbb{E}((X_{n+i}u)^{1+\alpha} \mid \mathcal{F}_n) = (X_n u)^{1+\alpha} + o((X_n u)^{1+\alpha}).$$

Proof. We first prove that

$$\mathbb{E}((X_{n+i}u)^{\gamma} \mid \mathcal{F}_n) = (X_n u)^{\gamma} + o((X_n u)^{\gamma})$$

for all $\gamma \in [0, 2[$ whatever the value of α.

The result is obvious if $\gamma = 0$. We first deal with the case where $0 < \gamma \leq 1$. Then for all positive real r, $(1+r)^{\gamma} \leq 1 + \gamma r$, we obtain the upper bound

$$\begin{aligned} \mathbb{E}((X_{n+1}u)^{\gamma} \mid \mathcal{F}_n) &\leq \mathbb{E}(X_{n+1}u \mid \mathcal{F}_n)^{\gamma} \\ &\leq (X_n u + g(X_n)u)^{\gamma} \\ &\leq (X_n u)^{\gamma} + \gamma g(X_n) u (X_n u)^{\gamma-1}. \end{aligned}$$

By using the inequality $(1+r)^{\gamma} \geq 1 - |r|^{\gamma}$, that holds for all $r \geq -1$, we obtain the lower bound

$$\begin{aligned} \mathbb{E}((X_{n+1}u)^{\gamma} \mid \mathcal{F}_n) &\geq \mathbb{E}((X_n u)^{\gamma} - |g(X_n)u + \xi_n u|^{\gamma} \mid \mathcal{F}_n) \\ &\geq (X_n u)^{\gamma} - 2^{\gamma}(g(X_n)u)^{\gamma} - 2^{\gamma}\mathbb{E}(|\xi_n u|^{\gamma} \mid \mathcal{F}_n). \end{aligned}$$

Since $g(X_n)u = \mathcal{O}((X_n u)^{\alpha})$ and using

$$\mathbb{E}(|\xi_n u|^{\gamma} \mid \mathcal{F}_n) \leq \mathbb{E}(|\xi_n u|^2 \mid \mathcal{F}_n)^{\gamma/2} = \mathcal{O}((X_n u)^{(1+\alpha)\gamma/2}),$$

we obtain

$$\mathbb{E}((X_{n+1}u)^{\gamma} \mid \mathcal{F}_n) = (X_n u)^{\gamma} + o((X_n u)^{\gamma}).$$

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We now deal with the case where $\gamma > 1$. Since, for all real $r \geq -1$,

$$(1+r)^\gamma \leq 1 + 2^{\gamma-1}|r|^\gamma + 2^\gamma |r|,$$

we obtain

$$\begin{aligned}
&\mathbb{E}((X_{n+1}u)^\gamma \mid \mathcal{F}_n) \\
&\quad \leq (X_n u)^\gamma + 2^{\gamma-1}\mathbb{E}(|g(X_n)u + \xi_n u|^\gamma \mid \mathcal{F}_n) + 2^\gamma (X_n u)^{\gamma-1}\mathbb{E}(|g(X_n)u + \xi_n u| \mid \mathcal{F}_n) \\
&\quad \leq (X_n u)^\gamma + \mathcal{O}((X_n u)^{(1+\alpha)\gamma/2}) + \mathcal{O}((X_n u)^{\gamma+(\alpha-1)/2}).
\end{aligned}$$

The lower bound is an easy consequence of Jensen's inequality, i.e.

$$\mathbb{E}((X_{n+1}u)^\gamma \mid \mathcal{F}_n) \geq \mathbb{E}(X_{n+1}u \mid \mathcal{F}_n)^\gamma \geq (X_n u)^\gamma.$$

We have proved that

$$\mathbb{E}((X_{n+1}u)^\gamma \mid \mathcal{F}_n) = (X_n u)^\gamma + o((X_n u)^\gamma) \quad \text{for } \gamma \in [0, 2). \tag{9}$$

We will prove that

$$\mathbb{E}(f(X_{n+1}u) \mid \mathcal{F}_n) = o((X_n u)^\gamma), \tag{10}$$

if f is a real-valued function such that $f(r) = o(r^\gamma)$ when r tends to ∞. We recall that $f(r) = o(r^\gamma)$ if and only if, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|f(r)| \leq \varepsilon r^\gamma + C_\varepsilon$ since $\gamma > 0$.

Let f be a real-valued function such that $f(r) = o(r^\gamma)$, $\varepsilon > 0$ and $C_\varepsilon > 0$ such that $|f(r)| \leq \varepsilon r^\gamma + C_\varepsilon$. By (9), we have

$$\begin{aligned}
\mathbb{E}(|f(X_{n+1}u)| \mid \mathcal{F}_n) &\leq \mathbb{E}(\varepsilon(X_{n+1}u)^\gamma + C_\varepsilon \mid \mathcal{F}_n) \\
&\leq \varepsilon(X_n u)^\gamma + C_\varepsilon + \varepsilon o((X_n u)^\gamma) \\
&\leq 2\varepsilon(X_n u)^\gamma + C_\varepsilon + C_1.
\end{aligned}$$

Thus, we obtain (10). Since we have (9) and (10), the result follows by induction.

We end the proof with the $-1 < \alpha < 0$ case. The lower bound is again a consequence of Jensen's inequality, i.e.

$$\mathbb{E}((X_{n+1}u)^\alpha \mid \mathcal{F}_n) \geq (X_n u + g(X_n)u)^\alpha \geq (X_n u)^\alpha + \alpha g(X_n)u(X_n u)^{\alpha-1}.$$

For the upper bound, we first majorize the probability that $X_{n+1}u$ is smaller than $X_n u/2$ by Markov's inequality, i.e.

$$\begin{aligned}
\mathbb{P}\bigg(X_{n+1}u \leq \frac{X_n u}{2} \;\bigg|\; \mathcal{F}_n\bigg) &= \mathbb{P}\bigg(\xi_n u \leq -\frac{X_n u}{2} - g(X_n)u \;\bigg|\; \mathcal{F}_n\bigg) \\
&\leq \mathbb{P}\bigg(\xi_n u \leq -\frac{X_n u}{2} \;\bigg|\; \mathcal{F}_n\bigg) \\
&\leq \mathbb{P}\bigg((\xi_n u)^2 \geq \frac{(X_n u)^2}{4} \;\bigg|\; \mathcal{F}_n\bigg) \\
&\leq \frac{4\mathbb{E}((\xi_n u)^2 \mid \mathcal{F}_n)}{(X_n u)^2} \\
&\leq K(X_n u)^{\alpha-1}.
\end{aligned}$$

Therefore, since, for all $r > -\frac{1}{2}$, $(1+r)^\alpha \le 1 + 4^{-\alpha}|r|$, we obtain

$$\begin{aligned}\mathbb{E}((\boldsymbol{X}_{n+1}u)^\alpha \mid \mathcal{F}_n) &= \mathbb{E}((\mathbf{1}_{\{\boldsymbol{X}_{n+1}u \le \boldsymbol{X}_n u/2\}} + \mathbf{1}_{\{\boldsymbol{X}_{n+1}u > \boldsymbol{X}_n u/2\}})(\boldsymbol{X}_{n+1}u)^\alpha \mid \mathcal{F}_n) \\ &\le \mathbb{P}\bigg(\boldsymbol{X}_{n+1}u \le \frac{\boldsymbol{X}_n u}{2} \biggm| \mathcal{F}_n\bigg) \\ &\quad + \mathbb{E}\bigg((\boldsymbol{X}_n u)^\alpha \bigg(1 + 4^{-\alpha}\frac{|g(\boldsymbol{X}_n)u + \xi_n u|}{\boldsymbol{X}_n u}\bigg) \biggm| \mathcal{F}_n\bigg) \\ &\le (\boldsymbol{X}_n u)^\alpha + \mathcal{O}((\boldsymbol{X}_n u)^{\alpha-1}) + \mathcal{O}((\boldsymbol{X}_n u)^{(3\alpha-1)/2}).\end{aligned}$$

We conclude in the same way as above by using the fact that $f(r) = o(r^\alpha)$ if and only if, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|f(r)| \le \varepsilon r^\alpha + C_\varepsilon r^{(-1+\alpha)/2}$. □

Lemma 5. *Let us assume that (A1) and (A2) hold. For all* $\gamma \in [0, 2]$*, there exists* $C \ge 0$ *such that, for all* $k, n \in \mathbb{N}$,

$$\sum_{i=0}^{k-1} \mathbb{E}(\|\boldsymbol{Y}_{n+i}\|^\gamma \mid \mathcal{F}_n) \le C\|\boldsymbol{Y}_n\|^\gamma + o((\boldsymbol{X}_n u)^\gamma).$$

Proof. We first write a recurrence relation for $\boldsymbol{Y}_n$ by (1), i.e.

$$\begin{aligned}\boldsymbol{Y}_{n+1} &= \boldsymbol{Y}_n M + g(\boldsymbol{X}_n)(I - uv) + \xi_n(I - uv) \\ &= \boldsymbol{Y}_n(M - uv) + g(\boldsymbol{X}_n)(I - uv) + \xi_n(I - uv)\end{aligned}$$

since $\boldsymbol{Y}_n u = 0$. Hence,

$$\boldsymbol{Y}_{n+i} = \boldsymbol{Y}_n(M - uv)^i + \sum_{j=0}^{i-1}(g(\boldsymbol{X}_{n+j}) + \xi_{n+j})(I - uv)(M - uv)^{i-1-j}.$$

The Perron–Frobenius theorem states that the spectral radius λ of $M - uv$ is less than 1. Let $\lambda_1 \in (\lambda, 1)$. We recall that by Gelfand's formula (see [3, p. 349]), we have

$$\lambda = \lim_{m\to\infty} \|(M - uv)^m\|^{1/m}$$

for any matrix norm. Consequently, there exists a constant C which does not depend on i such that we obtain

$$\|\boldsymbol{Y}_{n+i}\| \le C\bigg(\lambda_1^i\|\boldsymbol{Y}_n\| + \sum_{j=0}^{i-1}(\|g(\boldsymbol{X}_{n+j})\| + \|\xi_{n+j}\|)\bigg).$$

Hence, by (14) below, we obtain

$$\|\boldsymbol{Y}_{n+i}\|^\gamma \le C2^\gamma\lambda_1^{\gamma i}\|\boldsymbol{Y}_n\|^\gamma + C2^\gamma\sum_{j=0}^{i-1} i^\gamma(\|g(\boldsymbol{X}_{n+j})\|^\gamma + \|\xi_{n+j}\|^\gamma).$$

Since

$$\mathbb{E}(\|g(\boldsymbol{X}_{n+j})\|^\gamma \mid \mathcal{F}_n) = \mathcal{O}((\boldsymbol{X}_n u)^{\alpha\gamma}) \quad \text{and} \quad \mathbb{E}(\|\xi_{n+j}\|^\gamma \mid \mathcal{F}_n) = \mathcal{O}((\boldsymbol{X}_n u)^{(1+\alpha)\gamma/2}),$$

by (A2), we obtain by summation

$$\sum_{i=0}^{k-1} \mathbb{E}(\|\boldsymbol{Y}_{n+i}\|^\gamma \mid \mathcal{F}_n) \le C2^\gamma \frac{1}{1-\lambda_1^\gamma}\|\boldsymbol{Y}_n\|^\gamma + o((\boldsymbol{X}_n u)^\gamma).$$ □

Proof of Lemma 3. We first prove (6). The proof is an easy consequence of Lemma 4, Lemma 5, and (A1), i.e.

$$\begin{aligned}\mathbb{E}(\Delta_{n,k} \mid \mathcal{F}_n) &\le \mathbb{E}\bigg(\sum_{i=0}^{k-1} g(\boldsymbol{X}_{n+i})u \,\bigg|\, \mathcal{F}_n\bigg)\\ &\le \mathbb{E}\bigg(\sum_{i=0}^{k-1} c_1(\boldsymbol{X}_{n+i}u)^{\alpha} + c_2\|\boldsymbol{Y}_{n+i}\|^{\alpha} + f_1(\boldsymbol{X}_{n+i}) \,\bigg|\, \mathcal{F}_n\bigg)\\ &\le kc_1(\boldsymbol{X}_n u)^{\alpha} + C'\|\boldsymbol{Y}_n\|^{\alpha} + o((\boldsymbol{X}_n u)^{\alpha}).\end{aligned}$$

The same proof with $-c_2$ instead of c_2 gives the lower bound.

We now prove (7). As for (6), the main point is to show that d_2' does not depend on k.

By means of Lemma 4, Lemma 5, and (A1), we obtain

$$\begin{aligned}\mathbb{E}(|\Delta_{n,k}|^2 \mid \mathcal{F}_n) &\le \mathbb{E}\bigg(\bigg(\sum_{i=0}^{k-1}\{g(\boldsymbol{X}_{n+i})u + \xi_{n+i}u\}\bigg)^2 \,\bigg|\, \mathcal{F}_n\bigg)\\ &\le \mathbb{E}\bigg(\sum_{i=0}^{k-1}(\xi_{n+i}u)^2 + \bigg(\sum_{i=0}^{k-1} g(\boldsymbol{X}_{n+i})u\bigg)^2 \,\bigg|\, \mathcal{F}_n\bigg)\\ &\quad + 2\mathbb{E}\bigg(\bigg(\sum_{i=0}^{k-1}\xi_{n+i}u\bigg)\bigg(\sum_{i=0}^{k-1} g(\boldsymbol{X}_{n+i})u\bigg) \,\bigg|\, \mathcal{F}_n\bigg)\\ &\le kd_1(\boldsymbol{X}_n u)^{1+\alpha} + d_2\|\boldsymbol{Y}_n\|^{1+\alpha} + o((\boldsymbol{X}_n u)^{1+\alpha})\\ &\quad + \mathcal{O}((\boldsymbol{X}_n u)^{2\alpha}) + \mathcal{O}((\boldsymbol{X}_n u)^{\alpha(1+\alpha)/2}),\end{aligned}$$

and the proof for the lower bound is similar. We conclude with the proof of (8).

By Markov's inequality, we have

$$\begin{aligned}\mathbb{E}(|\Delta_{n,k}|^2\mathbf{1}_{\{\Delta_{n,k}\ge\varepsilon\boldsymbol{X}_n u\}} \mid \mathcal{F}_n) &\le \mathbb{E}(|\Delta_{n,k}|^2\mathbf{1}_{\{(\Delta_{n,k})^{\delta}\ge(\varepsilon\boldsymbol{X}_n u)^{\delta}\}} \mid \mathcal{F}_n)\\ &\le \mathbb{E}\bigg(\frac{|\Delta_{n,k}|^{2+\delta}}{(\varepsilon\boldsymbol{X}_n u)^{\delta}} \,\bigg|\, \mathcal{F}_n\bigg)\\ &\le \frac{(2k)^{2+\delta}}{(\varepsilon\boldsymbol{X}_n u)^{\delta}}\mathbb{E}\bigg(\sum_{i=0}^{k-1}|g(\boldsymbol{X}_{n+i})u|^{2+\delta} + |\xi_{n+i}u|^{2+\delta} \,\bigg|\, \mathcal{F}_n\bigg).\end{aligned}$$

Since $\mathbb{E}(|g(\boldsymbol{X}_{n+i})u|^{2+\delta} \mid \mathcal{F}_n) = \mathcal{O}((\boldsymbol{X}_n u)^{2\alpha+\alpha\delta})$ by Lemma 4 and $\mathbb{E}(|\xi_{n+i}u|^{2+\delta} \mid \mathcal{F}_n) = \mathcal{O}((\boldsymbol{X}_n u)^{1+\alpha+((1+\alpha)/2)\delta})$ by (A2), we obtain

$$\begin{aligned}\mathbb{E}(|\Delta_{n,k}|^2\mathbf{1}_{\{\Delta_{n,k}\ge\varepsilon\boldsymbol{X}_n u\}} \mid \mathcal{F}_n) &\le \frac{(2k)^{2+\delta}}{(\varepsilon\boldsymbol{X}_n u)^{\delta}}\Big(e_1(k)(\boldsymbol{X}_n u)^{2\alpha+\alpha\delta} + e_2(k)(\boldsymbol{X}_n u)^{1+\alpha+((1+\alpha)/2)\delta}\Big)\\ &\le e_1'(k,\varepsilon)(\boldsymbol{X}_n u)^{1+\alpha+((\alpha-1)/2)\delta},\end{aligned}$$

which is the desired inequality. □

We now prove Lemmas 1 and 2.

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Proof of Lemma 1. We first recall an inequality proved in [6]. If $\varepsilon > 0$, $x > 0$, and $h > -x$, then

$$\log(x+h) \le \log x + \frac{h}{x} - \frac{h^2 \mathbf{1}_{\{h \le \varepsilon x\}}}{2(1+\varepsilon)x^2}. \tag{11}$$

Let $k \in \mathbb{N}$ and $\varepsilon > 0$, both to be fixed later on. We apply (11) with $x = \boldsymbol{X}_n u$ and $h = \Delta_{n,k}$ to obtain

$$\begin{aligned}
&\mathbb{E}(\log(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \\
&\quad \le \log(\boldsymbol{X}_n u) + \frac{\mathbb{E}(\Delta_{n,k} \mid \mathcal{F}_n)}{\boldsymbol{X}_n u} - \frac{\mathbb{E}(|\Delta_{n,k}|^2 \mid \mathcal{F}_n)}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2} + \frac{\mathbb{E}(|\Delta_{n,k}|^2 \mathbf{1}_{\{\Delta_{n,k} > \varepsilon \boldsymbol{X}_n u\}} \mid \mathcal{F}_n)}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2}.
\end{aligned}$$

Using (6)–(8) from Lemma 3, we obtain

$$\begin{aligned}
&\mathbb{E}(\log(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \\
&\quad \le \log(\boldsymbol{X}_n u) + \frac{c_1 k(\boldsymbol{X}_n u)^\alpha + c_2' \|\boldsymbol{Y}_n\|^\alpha + o((\boldsymbol{X}_n u)^\alpha)}{\boldsymbol{X}_n u} \\
&\qquad - \frac{k d_1 (\boldsymbol{X}_n u)^{1+\alpha} - d_2' \|\boldsymbol{Y}_n\|^{1+\alpha} + o((\boldsymbol{X}_n u)^{1+\alpha})}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2} + \frac{\mathcal{O}((\boldsymbol{X}_n u)^{1+\alpha+((\alpha-1)/2)\delta})}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2}.
\end{aligned}$$

By the Perron–Frobenius theorem [12], all coordinates of u are positive. Therefore, by definition of $\boldsymbol{Y}_n$, there exists $b > 0$ such that, for every n,

$$\|\boldsymbol{Y}_n\| \le b \boldsymbol{X}_n u. \tag{12}$$

We obtain

$$\begin{aligned}
&\mathbb{E}(\log(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \\
&\quad \le \log(\boldsymbol{X}_n u) + \frac{c_1 k(\boldsymbol{X}_n u)^\alpha + c_2' b^\alpha (\boldsymbol{X}_n u)^\alpha + o((\boldsymbol{X}_n u)^\alpha)}{\boldsymbol{X}_n u} \\
&\qquad - \frac{k d_1 (\boldsymbol{X}_n u)^{1+\alpha} - d_2' b^{1+\alpha} (\boldsymbol{X}_n u)^{1+\alpha} + o((\boldsymbol{X}_n u)^{1+\alpha})}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2} + \frac{\mathcal{O}((\boldsymbol{X}_n u)^{1+\alpha+((\alpha-1)/2)\delta})}{2(1+\varepsilon)(\boldsymbol{X}_n u)^2}.
\end{aligned}$$

We first choose $\varepsilon > 0$ such that $c_1 < d_1/2(1+\varepsilon)$. We now choose k such that

$$k\left(c_1 - \frac{d_1}{2(1+\varepsilon)}\right) + c_2' b^\alpha + \frac{d_2' b^{1+\alpha}}{2(1+\varepsilon)} < 0.$$

Thus, there exists $s > 0$ such that

$$\mathbb{E}(\log(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \le \log(\boldsymbol{X}_n u) \quad \text{if } \boldsymbol{X}_n u > s. \qquad \square$$

Proof of Lemma 2. We recall another inequality proved in [6]. For $x \ge 3$, let

$$L(x) = (\log x)^{-1}.$$

There exists $C_2 > 0$ such that, for any $x \ge 3$, $h > 3 - x$ and $0 < \delta \le 1$, then

$$L(x+h) \le L(x) + L'(x)h + \frac{L''(x)h^2}{2} + C_2 \frac{|h|^{2+\delta}}{(\log x)^2 x^{2+\delta}} + \mathbf{1}_{\{h \le -x/2\}}. \tag{13}$$

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As in the first case, we prove that $\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \le L(\boldsymbol{X}_n u)$ for some fixed k and large enough $\boldsymbol{X}_n u$.

We apply (13) with $x = \boldsymbol{X}_n u$, $h = \Delta_{n,k}$, and k an integer to be fixed later on to obtain

$$\begin{aligned}
&\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n)\\
&\quad \le L(\boldsymbol{X}_n u) - \frac{\mathbb{E}(\Delta_{n,k} \mid \mathcal{F}_n)}{(\boldsymbol{X}_n u)(\log(\boldsymbol{X}_n u))^2} + \frac{\mathbb{E}(|\Delta_{n,k}|^2 \mid \mathcal{F}_n)}{2(\boldsymbol{X}_n u)^2(\log(\boldsymbol{X}_n u))^2} + \frac{2\mathbb{E}(|\Delta_{n,k}|^2 \mid \mathcal{F}_n)}{2(\boldsymbol{X}_n u)^2(\log(\boldsymbol{X}_n u))^3}\\
&\qquad + C_2 \frac{\mathbb{E}(|\Delta_{n,k}|^{2+\delta} \mid \mathcal{F}_n)}{(\log(\boldsymbol{X}_n u))^2(\boldsymbol{X}_n u)^{2+\delta}} + \mathbb{E}(\mathbf{1}_{\{\Delta_{n,k} \le -\boldsymbol{X}_n u/2\}} \mid \mathcal{F}_n).
\end{aligned}$$

We start with the estimate

$$\mathbb{E}(\mathbf{1}_{\{\Delta_{n,k} \le -\boldsymbol{X}_n u/2\}} \mid \mathcal{F}_n) \le \mathbb{E}(\mathbf{1}_{\{2^{2+\delta}|\Delta_{n,k}|^{2+\delta}/(\boldsymbol{X}_n u)^{2+\delta} \ge 1\}} \mid \mathcal{F}_n) \le \mathbb{E}\bigg(2^{2+\delta}\frac{|\Delta_{n,k}|^{2+\delta}}{(\boldsymbol{X}_n u)^{2+\delta}} \;\bigg|\; \mathcal{F}_n\bigg),$$

that follows easily from Markov's inequality. We now use the basic inequality

$$(a+b)^{2+\delta} \le 2^{2+\delta}(a^{2+\delta} + b^{2+\delta}), \qquad a, b > 0, \tag{14}$$

and the facts (resulting from (A1) and (A2)) that there exist some positive real numbers A and B such that

$$\mathbb{E}(|g(\boldsymbol{X}_{n+i})u|^{2+\delta} \mid \mathcal{F}_n) \le A(\boldsymbol{X}_n u)^{\alpha(2+\delta)},$$

and

$$\mathbb{E}(|\xi_{n+i}u|^{2+\delta} \mid \mathcal{F}_n) \le B(\boldsymbol{X}_n u)^{((\alpha+1)/2)(2+\delta)},$$

in order to obtain the upper bound

$$\begin{aligned}
\mathbb{E}(|\Delta_{n,k}|^{2+\delta} \mid \mathcal{F}_n) &\le (2k)^{2+\delta}\mathbb{E}\bigg(\sum_{i=0}^{k-1} |g(\boldsymbol{X}_{n+i})u|^{2+\delta} + |\xi_{n+i}u|^{2+\delta} \;\bigg|\; \mathcal{F}_n\bigg)\\
&\le C_3(k)(\boldsymbol{X}_n u)^{((\alpha+1)/2)(2+\delta)}.
\end{aligned}$$

Therefore, there exists $C_4(k)$ such that

$$\mathbb{E}(\mathbf{1}_{\{\Delta_{n,k} \le -\boldsymbol{X}_n u/2\}} \mid \mathcal{F}_n) \le C_4(k)(\boldsymbol{X}_n u)^{((\alpha-1)/2)(2+\delta)}.$$

We use the inequalities (6)–(8) from Lemma 3 and (12) to obtain

$$\begin{aligned}
\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) &\le L(\boldsymbol{X}_n u) - \frac{(c_1 k(\boldsymbol{X}_n u)^\alpha - c_2'\|\boldsymbol{Y}_n\|^\alpha + o((\boldsymbol{X}_n u)^\alpha))}{(\boldsymbol{X}_n u)(\log(\boldsymbol{X}_n u))^2}\\
&\quad + \frac{(kd_1(\boldsymbol{X}_n u)^{1+\alpha} + d_2'\|\boldsymbol{Y}_n\|^{1+\alpha} + o((\boldsymbol{X}_n u)^{1+\alpha}))}{2(\boldsymbol{X}_n u)^2(\log(\boldsymbol{X}_n u))^2}\\
&\quad + \frac{2(kd_1(\boldsymbol{X}_n u)^{1+\alpha} + d_2'\|\boldsymbol{Y}_n\|^{1+\alpha} + o((\boldsymbol{X}_n u)^{1+\alpha}))}{2(\boldsymbol{X}_n u)^2(\log(\boldsymbol{X}_n u))^3}\\
&\quad + C_2\frac{\mathcal{O}((\boldsymbol{X}_n u)^{1+\alpha+((\alpha-1)/2)\delta})}{(\log(\boldsymbol{X}_n u))^2(\boldsymbol{X}_n u)^{2+\delta}} + C_4(k)(\boldsymbol{X}_n u)^{(\alpha-1)(2+\delta)/2}\\
&\le L(\boldsymbol{X}_n u) + \frac{k(d_1/2 - c_1) + b_2'}{(\boldsymbol{X}_n u)^{1-\alpha}(\log(\boldsymbol{X}_n u))^2} + o\bigg(\frac{1}{(\boldsymbol{X}_n u)^{1-\alpha}(\log \boldsymbol{X}_n u)^2}\bigg).
\end{aligned}$$

https://doi.org/10.1017/apr.2016.61 Published online by Cambridge University Press

Since $d_1/2 < c_1$, we first choose k such that

$$k\left(\frac{d_1}{2} - c_1\right) + b_2' < 0,$$

with $b_2' = b^{1+\alpha} d_2'/2 + b^\alpha c_2'$. Then, there exists $s > 0$ such that

$$\mathbb{E}(L(\boldsymbol{X}_{n+k}u) \mid \mathcal{F}_n) \le L(\boldsymbol{X}_n u) \quad \text{if } \boldsymbol{X}_n u > s. \qquad \square$$

Acknowledgements

The author thanks Vincent Bansaye and Jean-René Chazottes for many helpful discussions on the subject of this paper. He is also very grateful to the anonymous referee for his/her careful reading of the original manuscript. This article benefited from the support of the ANR MANEGE (grant number ANR-09-BLAN-0215) and from the Chair 'Modélisation Mathématique et Biodiversité' of Veolia Environnement – Ecole Polytechnique – Museum Nation d'Histoire Naturelle – Fondation X.

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